4 Multiplier-Accelerator Models with Random Perturbations

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4.1 Introduction

In his original contribution Samuelson (1939) for the first time presented one of the major structural reasons of possible cyclical behavior in macroeconomic models: the interaction of the multiplier and the accelerator principles which induces a *second order delay equation* of real aggregate output. While he realized that his model could not generate permanent cycles, it was Hicks (1950) in a subsequent extension introducing ceilings and floors showing that permanent "harmonic" fluctuations arise in a natural way under the Multiplier-Accelerator principle. These models have received wide interest within dynamical systems theory, since they supply a wide range of explanations of truly complex business cycle phenomena originating from a linear model with restrictions implying a minimal degree of non-linearity.

As an alternative to such restrictions, the introduction of random perturbations to linear delay systems has also served as an explanation of business cycle phenomena which has mainly been studied within linear time series analysis. The recent development of new techniques from the theory of stochastic dynamical systems allows an extension of results within the dynamic frame work for the Multiplier-Accelerator model. Most importantly, however, these techniques combined with the availability of efficient and fast numerical techniques allow a significantly more detailed insight into the range of qualitative features of the random Multiplier-Accelerator model.

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This chapter takes this new view of random dynamical systems theory to examine the classical Multiplier-Accelerator model when random perturbations are introduced to model parameters. The emphasis of the chapter is on the revelation of the dynamical richness of business cycle scenarios which may occur in such simple economic models.

4.2 Random Dynamical Systems

The traditional description of the dynamic evolution of stochastic economic models is carried out using the mathematical formalization of stochastic processes, i. e. as a family of random variables given a specified exogenous structure of stochastic properties. When the standard tools of stochastic processes are used, the actual evolution of the stochastic data, (the sample paths), is often suppressed in favor of results and characterizations of the evolution of the probabilistic features or the statistical properties of the model. In this case the experimental perspective of the characteristics of a specific sample path, i. e. the empirical observation becomes of secondary importance. In many economic applications, however, as well as from a dynamical systems point of view, it is often natural and desirable to analyze the generation of stochastic orbits directly. This can be done in many situations by modelling the stochastic environment of a dynamical economic system in an explicit fashion.

Consider for example a parameterized dynamical system $F : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$, given by a family of mappings

$$F(\cdot, \xi) : \mathcal{X} \subset \mathbb{R}^n \to \mathcal{X}, \tag{1}$$

where $\xi \in \mathbb{R}^m$ is a vector of parameters which is subjected to random perturbations and x is the vector of endogenous variables. The evolution of x (the orbit) for a *given* value of the parameter $\xi \in \mathbb{R}^m$ is described in the usual way by

$$x_t = F^t_\xi(x_0) \qquad F_\xi \equiv F(\cdot, \xi), \tag{2}$$

i. e. the *dynamics* follow the rules and the description of a deterministic dynamical system once the value of a particular ξ is given. Now let ξ follow a given random path described by $\omega := (\ldots, \xi_{s-2}, \xi_{s-1}, \xi_s, \xi_{s+1}, \ldots)$. Then, the generation of the random path

$$x_{t+1} = F_{\xi_t}(x_t) = F(x_t, \xi_t) \qquad \text{for all} \quad t \tag{3}$$

means that the change of ξ implies choosing at each t a *different mapping*. If $F(\cdot, \xi)$, $\xi \in [\underline{\xi}, \overline{\xi}]$ is a family of contraction mappings with upper and lower bounds $[\underline{\xi}, \overline{\xi}]$, then for any random path ω the associated evolution of x will eventually be trapped in some compact interval $[\underline{x}, \underline{\underline{x}}]$. For a one dimensional system, for example, the dynamic evolution of $\{x_t\}$ can be visualized as in Figure 1 for any initial value x_0 and a given $\omega = (\ldots, \underline{\xi}, \overline{\xi}, \underline{\xi}, \overline{\xi}, \overline{\xi}, \underline{\xi}, \ldots)$.

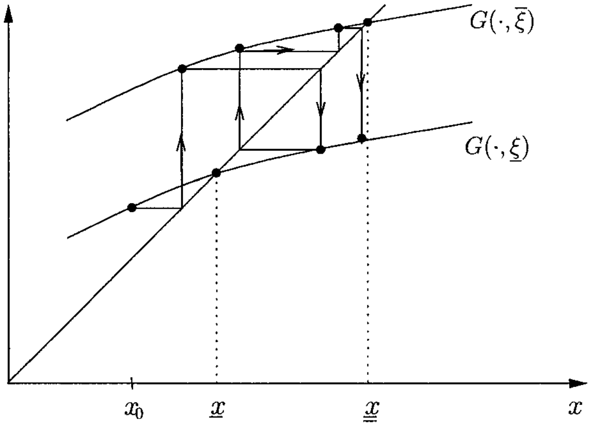


Figure 1: *A random orbit of x for ω.*

Formally a random dynamical system in the sense of Arnold (1998)[1] has two building blocks:

- a model describing a dynamical system perturbed by noise
- and a model of the noise.

1) The exogenous noise process is modelled as a so called *metric dynamical system* known from ergodic theory.

Let $\vartheta : \Omega \longrightarrow \Omega$ be a measurable invertible mapping on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which is measure preserving with respect to $\mathbb{P}$ and whose inverse ϑ^{-1} is again measurable. Assume that $\mathbb{P}$ is ergodic with respect to ϑ and let ϑ^t denote the t-th iterate of the map ϑ. The collection $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta^t\}_{t \in \mathbb{Z}})$ is called an *ergodic metric dynamical system* (for details see Arnold (1998)).

[1] A synthesis of this view of dynamical systems with noise has been developed by many researchers among them Kesten (1973), Brandt (1986), Borovkov (1998), Lasota & Mackey (1994).

Any stationary ergodic process $\{\xi_t\}_{t\in\mathbb{N}}$, $\xi_t : \Omega \longrightarrow \mathbb{R}^m$ can be represented by an ergodic dynamical system. This implies that there exists a measurable map $\xi : \Omega \to \mathbb{R}^m$ such that for each fixed $\omega \in \Omega$, a sample path of the noise process is given by $\xi_t(\omega) = \xi(\vartheta^t\omega)$, $t \in \mathbb{Z}$. Such a process is often referred to as a *real noise process* .

2) The second ingredient is a parameterized family of invertible time-one maps of topological dynamical systems $F : X \times \mathbb{R}^m \longrightarrow X$, $X \subset \mathbb{R}^K$ inducing the *random difference equation* $F : X \times \Omega \longrightarrow X$,

$$x_{t+1} = F(x_t, \xi(\vartheta^t\omega)) \equiv F(\vartheta^t\omega)x_t. \tag{4}$$

For any x_0, the iteration of the map F under the perturbation ω induces a measurable map $\phi : \mathbb{Z} \times \Omega \times X \longrightarrow X$ defined by

$$\phi(t,\omega,x_0) := \begin{cases} (F(\vartheta^{t-1}\omega) \circ \ldots \circ F(\omega))x_0 & \text{if } t > 0 \\ x_0 & \text{if } t = 0 \\ (F(\vartheta^t\omega)^{-1} \circ \ldots \circ F(\vartheta^{-1}\omega)^{-1})x_0 & \text{if } t < 0 \end{cases} \tag{5}$$

such that $x_t = \phi(t,\omega,x_0)$ is the state of the system at time t.

– For any $x_0 \in X$ and any $\omega \in \Omega$, the sequence $\gamma(x_0) := \{x_t\}_{t\in\mathbb{Z}}$ with $x_t = \phi(t,\omega)x_0$ is called an orbit of the random dynamical system ϕ.

– For any t and s one has:

$$\phi(t+s,\omega,x_0) = F(\vartheta^{t+s}\omega) \circ \ldots \circ F(\omega)x_0 \tag{6}$$

$$= \phi(t,\vartheta^t\omega,\phi(s,\omega,x_0)) \tag{7}$$

Many stochastic processes can be described as metric dynamical systems. As an example, consider the representation for a standard i. i. d. process. Let $\{\xi_t\}$ denote a family of independent and identically distributed random variables with values in $W \subset \mathbb{R}^m$, which have the common distribution (measure) λ. Then one has:

- $\Omega := W^{\mathbb{Z}} = \cdots \cdot W \times W \times W \times \cdots$
- $\mathcal{F} = \mathcal{B}(\Omega)$ Borel σ-algebra
- $\omega = (\ldots, \xi_{s-1}, \xi_s, \xi_{s+1}, \ldots)$ with $\omega(s) \equiv \xi_s$

- $\theta : \Omega \rightarrow \Omega$ is the so called shift map with $\omega \mapsto \theta\omega$ and $\theta\omega(s) = \omega(s+1) \equiv \xi_{s+1}$
- $\xi : \Omega \rightarrow \mathbb{R}^m$ is the evaluation map with $\xi(\omega) \equiv \omega(0)$
- $\xi_t = \xi(\theta^t\omega)$
- $\mathbb{P} = \lambda^{\mathbb{Z}}$

4.3 Random Fixed Points

The long run behavior of a random dynamical system is described by random attractors, the random analogue of an attractor of a deterministic dynamical system, the random fixed point being a special case[2].

Definition 4.1
Consider a random dynamical system ϕ induced by the continuous mapping $F : X \times \mathbb{R}^m \longrightarrow X$ with real noise process $\xi_t = \xi \circ \vartheta^t$, $\xi : \Omega \longrightarrow \mathbb{R}^m$ measurable, over the ergodic dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta^t))$.

*A **random fixed point** of ϕ is a random variable $x_* : \Omega \longrightarrow X$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that almost surely*

$$x_*(\vartheta\omega) = \phi(1, \omega, x_*(\omega)) = F(x_*(\omega), \xi(\omega)) \quad \textit{for all} \quad \omega \in \Omega', \tag{8}$$

where $\Omega' \subset \Omega$ is a ϑ-invariant set of full measure, $\mathbb{P}(\Omega') = 1$.

Thus, a random fixed point is a stationary solution of the stochastic difference equation generated by the metric dynamical system. Some implications of the definition can be observed directly. If F is independent of the perturbation ω, then the Definition 4.1 coincides with the one of a deterministic fixed point. Definition 4.1 implies that $x_*(\vartheta^{t+1}\omega) = F(x_*(\vartheta^t\omega), \xi(\vartheta^t\omega))$ for all times t. Therefore, the orbit $\{x_*(\vartheta^t\omega)\}_{t\in\mathbb{N}}$, $\omega \in \Omega$ generated by x_* solves the random difference equation

$$x_{t+1} = F(x_t, \xi_t(\omega)).$$

Stationarity and ergodicity of ϑ implies that the stochastic process $\{x_*(\vartheta^t)\}_{t\in\mathbb{N}}$ is stationary and ergodic.

[2]Schmalfuß (1996, 1998), also Arnold (1998).

The random fixed point x_* induces an invariant distribution $x_*\mathbb{P}$ on $\mathbb{R}^K$ defined by

$$x_*\mathbb{P}(B) \quad := \quad \mathbb{P}\{\omega \in \Omega \,|\, x_*(\omega) \in B\}\,. \tag{9}$$

The invariance of the measure $\mathbb{P}$ under the shift ϑ implies the invariance of $x_*\mathbb{P}$, i. e. $(x_*\vartheta)\mathbb{P}(B) = x_*\mathbb{P}(B)$. If, in addition, $\mathbb{E}||x_*|| < \infty$, then

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} 1_B(x_*(\vartheta^t\omega)) = x_*\mathbb{P}(B) \tag{10}$$

for every $B \in \mathcal{B}(X)$. In other words, the empirical law of an orbit is well defined and it is equal to the distribution $x_*\mathbb{P}$ of x_*. Finally, if the perturbation corresponds to an i. i. d. process the orbit of the fixed point x_* will be an ergodic Markov equilibrium in the usual sense (cf. Duffie, Geanakoplos, Mas-Colell & McLennan 1994). The following definition of a stable random

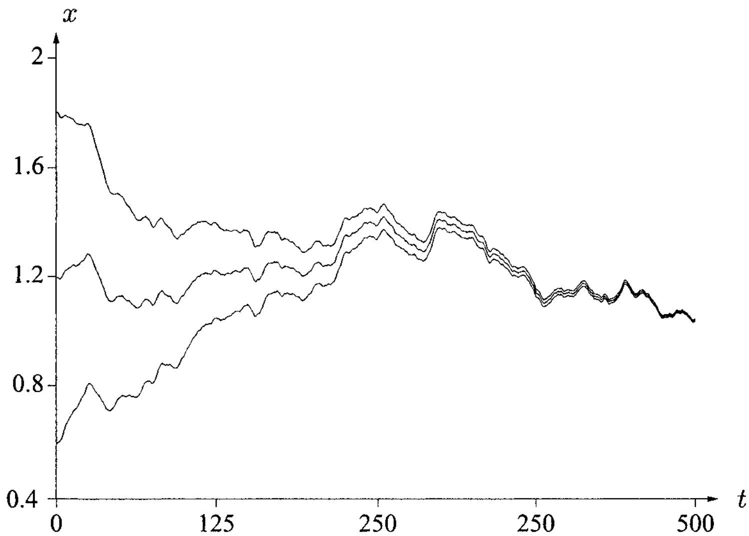


Figure 2: *Asymptotic convergence to a random fixed point.*

fixed point (due to Schmalfuß (1996, 1998)) includes the notion of stability given by Definition 7.4.6 in Arnold (1998).

Definition 4.2
*A random fixed point x_* is called **asymptotically stable** with respect to a norm $\|\cdot\|$, if there exists a random neighborhood $U(\omega) \subset X$, $\omega \in \Omega$ such that $\mathbb{P}$- a.s.*

$$\lim_{t\to\infty} ||\phi(t,\omega,x_0) - x_*(\vartheta^t\omega)|| = 0 \quad \textit{for all } x_0(\omega) \in U(\omega).$$

Figure 2 portrays the convergence property of a random fixed point for the one dimensional growth model for three random orbits associated with different initial conditions and the same noise path.

The following theorem, which is due to Arnold (1998)[3], will be the central result applied to the random Multiplier-Accelerator model supporting the numerical analysis and implying the dynamic and statistical properties to be exhibited. Consider *invertible* affine transformations on $\mathbb{R}^n$ defined by pairs (A, b) where A is an invertible $n \times n$ matrix and $b \in \mathbb{R}^n$. Let $\mathcal{A}$ denote the space of non singular $n \times n$ matrices and assume A, A^{-1}, and b to be bounded.

Theorem 4.1
Let $F_\xi : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible affine random dynamical system with stationary noise process $\{\xi_t\}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume $\xi : \Omega \to (\mathcal{A}, \mathbb{R}^n)$ with $\xi(\omega) = (A(\omega), b(\omega))$, which implies the random difference equation

$$x_{t+1} = A(\vartheta^t\omega)x_t + b(\vartheta^t\omega) \tag{11}$$

and the random dynamical system[4]

$$\phi(t,x,\omega) := \begin{cases} \Phi(t,\omega)\left(x + \sum_{j=0}^{t-1} \Phi(j+1,\omega)^{-1} b(\vartheta^j\omega)\right), & t > 0 \\ x & t = 0 \\ \Phi(t,\omega)\left(x - \sum_{j=t}^{-1} \Phi(j+1,\omega)^{-1} b(\vartheta^j\omega)\right) & t < 0 \end{cases} \tag{12}$$

where

$$\Phi(t,\omega) := \begin{cases} A(\vartheta^{t-1}\omega)\cdots A(\omega), & t > 0 \\ I & t = 0 \\ A^{-1}(\vartheta^t\omega)\cdots A^{-1}(\vartheta^{-1}\omega) & t < 0. \end{cases} \tag{13}$$

[3]Theorem 5.6.5 and Corollary 5.6.6.
[4]See Chapter 5 in Arnold (1998).

1. *There exists a unique random fixed point* $x_* : \Omega \longrightarrow \mathbb{R}^n$ *such that*

$$x_*(\vartheta^{t+1}\omega) = A(\vartheta^t\omega)x_*(\vartheta^t\omega) - b(\vartheta^t\omega) \qquad \mathbb{P}-a.s. \tag{14}$$

2. x_* *induces an invariant distribution* $x_*\mathbb{P} \equiv \mu_*$
3. *with unique support* $supp(\mu_*) = A_*$
4. *if* $A(\omega)$ *are contracting maps,* x_* *is globally attracting, i. e. for any* x_0

$$\lim_{t\longrightarrow\infty} |\phi(t,\omega,x_0) - x_*(\vartheta^t\omega)| = 0 \qquad \mathbb{P}-a.s. \tag{15}$$

and has the explicit form

$$x_*(\omega) := \sum_{t=-\infty}^{-1} \Phi(t+1,\omega)^{-1}\, b(\vartheta^t\omega) \tag{16}$$

4.4 Random Multiplier Accelerator Models

Consider the standard Multiplier-Accelerator model (in the version of Hicks (1950)) defined by the three equations

$$\begin{aligned} C &= m^0 + mY_{-1} \qquad 0 < m < 1 \qquad m_0, v_0 \geq 0 && (17)\\ I &= v^0 + v(Y_{-1} - Y_{-2}) \qquad v > 0 && (18)\\ Y &= C + I && (19) \end{aligned}$$

implying the determination of aggregate real income in each period as

$$Y = (m_0 + v_0) + (m+v)Y_{-1} - vY_{-2}, \tag{20}$$

which is a linear delay equation of order two. Using the form

$$f(y_1,y_2) := (m_0 + v_0) + (m+v)y_2 - vy_1 \tag{21}$$

for the delay map f implies the associated two dimensional affine dynamical system $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$\begin{aligned} F(y_1,y_2) &:= (y_2, f(y_1,y_2)) && (22)\\ &= (y_2, (m_0+v_0) + (m+v)y_2 - vy_1) && (23)\\ &= \begin{pmatrix} 0 & 1 \\ -v & m+v \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ m_0+v_0 \end{pmatrix}. && (24) \end{aligned}$$

The function F has the unique fixed point (stationary state)

$$\bar{y} = \left(\frac{m_0 + v_0}{1 - m}, \frac{m_0 + v_0}{1 - m} \right). \tag{25}$$

The accelerator v has no influence on the steady state $\bar{y}$ while aggregate demand $m_0 + v_0$ does not influence the stability of the steady state. $\bar{y} >> 0$ requires $m < 1$. $\bar{y}$ is asymptotically stable if and only if $0 \leq m < 1$ and $0 \leq v < 1$. From the characteristic equation

$$\chi(\lambda) := \lambda^2 - (m + v)\lambda + v$$

one finds that the eigenvalues $\lambda_{1,2}$ are complex if and only if $m < 2\sqrt{v} - v$. Thus, for stability considerations (the projection into $\mathbb{R}^2$ of) the space of parameters can be partitioned into a complex and into a real region as depicted in Figure 3. Therefore, for $(m, v) \in [0, 1)^2$ the mapping F is a linear contraction with a unique steady state which is either a stable node or a stable focus. The above description shows that the Multiplier-Accelerator

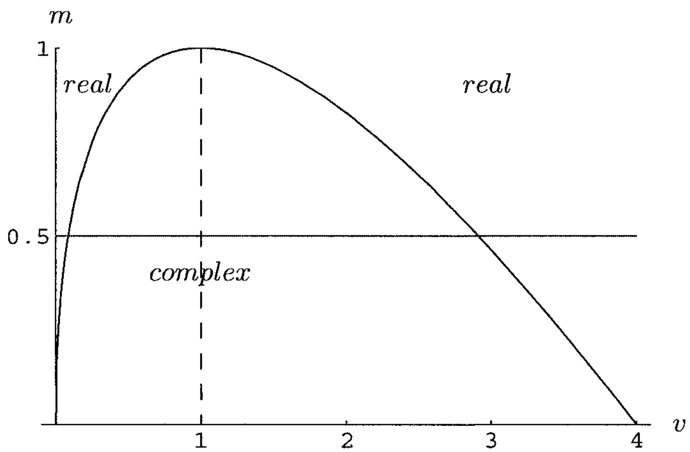


Figure 3: *Regions of eigenvalues in Multiplier-Accelerator Model.*

model consists of a family of affine parameterized maps $F_\mu : \mathbb{R}^2 \to \mathbb{R}^2$ with parameters $\mu \in \mathbb{R}^3_+$. Without restricting economic generality, one may assume $v_0 \equiv 0$ capturing all effects of aggregate demand in the parameter $0 \leq m_0$ and thus restrict the analysis to situations of nonnegative parameter values $\mu := (m_0, m, v) \in \mathbb{R}^3_+$, i. e. aggregate demand, the multiplier, and the accelerator. In most applications, economic reasoning suggests further that the multiplier m takes values only between zero and one and that the accelerator v is restricted to values between 0 and 4. Therefore, for the rest of the analysis define the set of possible parameter values as

$$M := \left\{ (m_0, m, v) \in \mathbb{R}^3 \,|\, 0 \leq m_0 \leq \bar{m}_0, 0 \leq m \leq 1, 0 \leq v \leq 4 \right\}. \quad (26)$$

As a consequence, the *Random Multiplier Accelerator Model* consists of the random family of affine maps $F_\mu : \mathbb{R}^2 \to \mathbb{R}^2$ with an associated (vector valued) stochastic process of parameters $\{\mu_t\}_{t=0}^{\infty}$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which takes values in M, i. e. $\mu_t : \Omega \to M$. More specifically, let $\mu(\omega) \equiv (m_0(\omega), m(\omega), v(\omega))$, and define

$$A(\omega) := \begin{pmatrix} 0 & 1 \\ -v(\omega) & m(\omega) + v(\omega) \end{pmatrix} \qquad \text{and} \qquad b(\omega) := \begin{pmatrix} 0 \\ m_0(\omega) \end{pmatrix}.$$

which implies the random difference equation

$$x_{t+1} = A(\vartheta^t\omega)x_t + b(\vartheta^t\omega)$$

(as in equation (11)) and the random dynamical system as in equation (12). This formulation fits precisely into the mathematical framework presented in Section 4.2. As a consequence, one has the following result for the class of random multiplier accelerator models.

Proposition 4.1
Let the random multiplier accelerator model F_μ be given as in equation (24) and (26) and assume that the random perturbation is described by a stationary and ergodic process $\{\mu_t\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a compact set $\bar{M} := \left\{ (m_0, m, v) \in \bar{M} \,|\, m < 1, v < 1 \right\} \subset M$.

(i) There exists a unique random fixed point $y^* : \Omega \longrightarrow \mathbb{R}^2_+$, given by

$$y^*(\omega) := \sum_{t=-\infty}^{-1} \Phi(t+1, \omega)^{-1}\, b(\vartheta^t\omega), \quad (27)$$

with

$$\Phi(t, \omega) := \begin{cases} A(\vartheta^{t-1}\omega) \cdots A(\omega), & t > 0 \\ I & t = 0 \\ A^{-1}(\vartheta^t\omega) \cdots A^{-1}(\vartheta^{-1}\omega) & t < 0. \end{cases} \quad (28)$$

(ii) y^* is asymptotically stable and induces a unique (stationary) invariant distribution $y^*\mathbb{P}$ on $\mathbb{R}^2$ defined by

$$(y^*\mathbb{P})(B) \quad := \quad \mathbb{P}\{\omega \in \Omega \,|\, y^*(\omega) \in B\} \tag{29}$$

for every $B \in \mathcal{B}(\mathbb{R}^2)$.

(iii) Moreover,

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} 1_B(y^*(\vartheta^t \omega)) = y^*\mathbb{P}(B) = \mathbb{P}\{\omega \in \Omega | y^*(\omega) \in B\} \tag{30}$$

for every $B \in \mathcal{B}(\mathbb{R}^2)$.

Equation (30) states that the empirical law of an orbit is well defined and it is asymptotically equal to the distribution $y^*\mathbb{P}$ of y^*.

The result follows as a direct application of Theorem 4.1. The given noise process can be represented as a real noise process in the sense of Arnold (1998). The assumption that the multiplier m as well as the accelerator v are assumed to be strictly less than one imply that the family of mappings F_μ are contractions. Therefore, existence, uniqueness, and asymptotic stability of the random fixed point y^* follows from Theorem 4.1. While the result here is formulated for the simple two dimensional Multiplier-Accelerator model, the mathematical framework is much more general. It covers the whole class of affine random contraction mappings of finite dimension and not only delay systems. Such random models have unique globally attracting random fix points (stationary solutions). Most importantly, however, these properties hold for very general stationary and ergodic perturbations whether smooth or discrete, including in particular Markov processes and so called Markov switching models. Thus, from a time series perspective, Arnold's result sets a bench mark for the description of the invariance of affine economic models. Therefore, a large spectrum of qualitatively different sample profiles can be shown to appear, all consistent with a unique stationary and asymptotically stable solution. Observe that this was primarily obtained by the *dynamic* features of the construction chosen by the approach given in Arnold (1998).

The major purpose of the remainder of this section is to examine the *dynamic qualitative* properties of some specific random examples using numerical simulations. This will reveal insights into the nature of the recurrence of the stochastic multiplier accelerator model and into the role of the different parameters determining the invariant behavior. This can be done safely (with proper care of the numerical analysis) due to the ergodicity property given in condition (30). In this case, a statistical examination of the long run behavior of one generic sample path suffices to characterize the invariant statistical properties of the model.

¿From an economic point of view the three perturbations correspond to structurally different situations:

1. a perturbation of the additive parameter m_0 corresponding to random exogenous demand in consumption or investment;

2. a perturbation of the multiplicative parameters, $0 \leq m, v \leq 1$, corresponding to random propensities to consume or a random accelerator.

The numerical experiments will use i. i. d. processes only, in spite of the fact that general Markov processes fall under the assumptions of Proposition 4.1 as well. First, the analysis investigates the additive noise situation separately from each of the multiplicative effects. The additive noise will be chosen to be smooth, while the multiplicative and accelerator will be chosen from discrete sets. Mixing these two types reveal some specific and interesting features.

4.5 The Dynamics with Smooth Additive i. i. d. Noise

Consider the random equation (21) with an aggregate demand shock $\xi \geq 0$

$$f_\lambda(y_1, y_2, \xi) := m_0 + \xi + (m + v)y_2 - vy_1 \tag{31}$$

which is distributed uniformly on some compact interval

$$\xi \sim [0, 2\lambda], \qquad \lambda \geq 0, \tag{32}$$

implying a mean $\mathbb{E}\xi = \lambda$ and a variance $\mathbb{V}\xi = \lambda^2/3$. In time series analysis such systems are referred to as a second order autoregressive process, denoted AR(2). Equation (31) induces a parameterized two dimensional random dynamical system $F_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F_\lambda(y_1, y_2, \xi) \;:=\; (y_2, f_\lambda(y_1, y_2, \xi)) \tag{33}$$

$$= \begin{pmatrix} 0 & 1 \\ -v & m+v \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ m_0 + \xi \end{pmatrix} \tag{34}$$

with additive noise, a so called Vector Autoregressive System of order 1, denoted VAR(1) in time series analysis. The characteristics of the stationary distribution are known from the standard time series approach and can be calculated explicitly in this particular case.

Under the hypotheses of Proposition 4.1 the unique stationary solution can be characterized numerically by the limiting statistical behavior of any *single* sample path to be calculated from data. On the other hand, the true moments of the random fixed point y^* can also be calculated given the noise distribution $\xi \sim [0, 2\lambda]$ for any $\lambda \geq 0$.

The stationarity of y^* implies that the first moment $\mathbb{E}y^*$ must satisfy

$$\mathbb{E}y^* = \begin{pmatrix} 0 & 1 \\ -v & m+v \end{pmatrix} \mathbb{E}y^* + \begin{pmatrix} 0 \\ m_0 + \mathbb{E}\xi \end{pmatrix}.$$

Hence, y_1^* and y_2^* have the same mean given by

$$\mathbb{E}\begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} = \left(I - \begin{pmatrix} 0 & 1 \\ -v & m+v \end{pmatrix}\right)^{-1} \begin{pmatrix} 0 \\ m_0 + \mathbb{E}\xi \end{pmatrix} = \frac{m_0 + \lambda}{1-m} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (35)$$

The covariance matrix $\mathbb{C}\text{ov}(y_1^*, y_2^*)$ satisfies

$$\mathbb{C}\text{ov}(y_1^*, y_2^*) = \begin{pmatrix} 0 & 1 \\ -v & m+v \end{pmatrix} \mathbb{C}\text{ov}(y_1^*, y_2^*) \begin{pmatrix} 0 & 1 \\ -v & m+v \end{pmatrix}^T + \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{V}\xi \end{pmatrix}.$$

As the solution one obtains

$$\mathbb{C}\text{ov}(y_1^*, y_2^*) = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \quad (36)$$

with

$$\mathbb{V}y^* = v_{11} = v_{22} = \frac{\lambda^2}{3(1-v)} \frac{1+v}{(1+v)^2 - (m+v)^2} \quad (37)$$

and

$$v_{12} = v_{21} = \frac{m+v}{1+v} v_{22} = \frac{\lambda^2}{3(1-v)} \frac{m+v}{(1+v)^2 - (m+v)^2}. \quad (38)$$

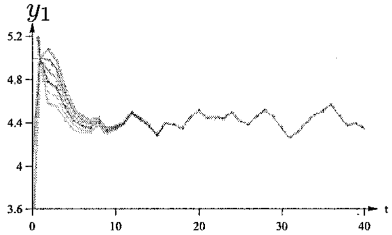
Therefore,

$$0 < v_{12} = v_{21} < v_{11} = v_{22}, \qquad \text{for all } \lambda > 0,\ 0 < m < 1.$$

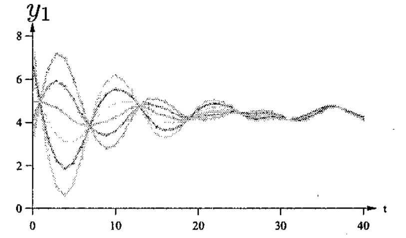
Observe, that the first moment is independent of the accelerator and it depends on (m, λ) only. The multiplier and accelerator together induce a positive cross correlation on the time series. Both correlation coefficients increase as the accelerator increases. Notice in particular, that with a higher accelerator the attractor increases in size, including values of the state variable less than $m_0/(1-m)$ and larger than $(m_0 + 2\lambda)/(1-m)$. For small values of v, the attractor lies inside the cube defined by these two values. Since ξ has a uniform distribution, the attractor as well as the distribution must be symmetric but not uniform. Table 1 shows the list of theoretical and computed values.

To examine the qualitative properties of the (dynamic) invariant behavior, two different cases will be discussed first to examine the role of the accelerator. Choose $m = 0.75$ for the multiplier and consider two values $v = 0.1$ and $v = 0.8$ for the accelerator. $v = 0.1$ implies real eigenvalues such that the associated deterministic fixed point is a stable node implying monotonic convergence without rotation. In contrast, $v = 0.8$ implies complex eigenvalues and a corresponding stable focus in the deterministic case. Most importantly, however, for each pair $0 << (m, v) << 1$, the set valued mapping associating the support of the invariant distribution with each parameter pair (m, v) will have compact images which depend on λ alone and not on the particular noise chosen on $[0, 2\lambda]$. This implies that the attractor i. e. the support of the measure of the random fixed point will be a compact set which depends on the interval $[0, \lambda]$, the support of ξ, but is independent of the particular form of the distribution. In this case, one would expect that under additive noise the complex case exhibits a much stronger rotation of the random orbits in the state space than in the case with real eigenvalues.

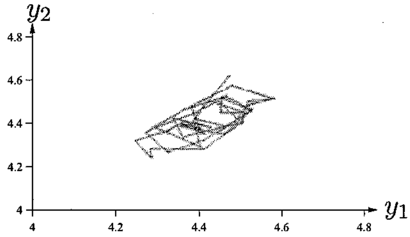
Figure 4 provides time series characteristics for the case $v = 0.1$ (left column) and $v = 0.8$ (right column). All calculations are carried out for the same noise path. Panel (a) and (b) show the convergence to the random fixed point for five different initial values of y_1, while (e) and (f) show typical time windows of the corresponding long run development of the (one dimensional projection of the) random fixed point. Panel (c) and (d) show the first 50 iterates with connecting lines. Observe that, in spite of the fact that for $v = 0.1$ the deterministic fixed points are stable foci, the orbits show a low rotation phenomenon, caused by the stochastic displacement of the mappings. For $v = 0.8$, however, a strong rotation property appears induced by the complex eigenvalues of the matrix.



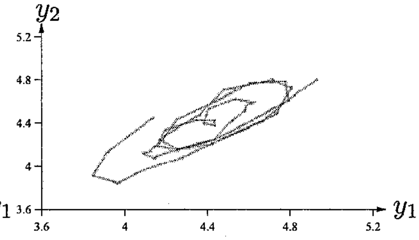
(a) Convergence $v = 0.1$



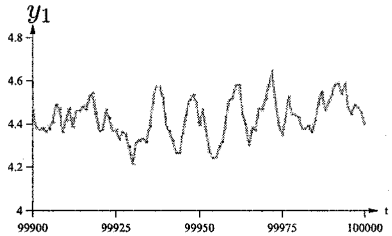
(b) Convergence $v = 0.8$



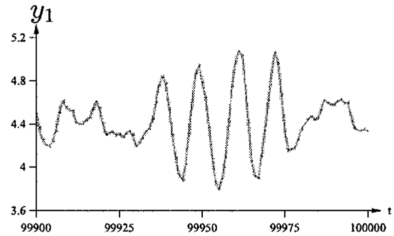
(c) Low rotation: $T = 50, v = 0.1$



(d) High rotation: $T = 50, v = 0.8$



(e) $T = 10^5, v = 0.1$



(f) $T = 10^5, v = 0.8$

Figure 4: *Transients and the role of the accelerator* v ; $m = 0.75$, $\lambda = 0.1$.

The difference in the cyclical behavior becomes even more apparent when the long run of the random fixed point is examined. Panels (a) - (d) of Figure 5 show the two attractors with corresponding relative frequencies (densities). The grey shading of the profile of the invariant distribution indicate equidistant levels of frequencies.

The attractor under low rotation is almost a parallelogram while under high rotation it has an elliptical form. Observe that both are perfectly symmetric with respect to the diagonal which implies that their respective marginal distributions must be identical. Since the noise is strictly additive, the mean is the same while the variance is higher in the case with $v = 0.8$ (see panels (e) and (f)). Table 1 provides numerical results of some of the

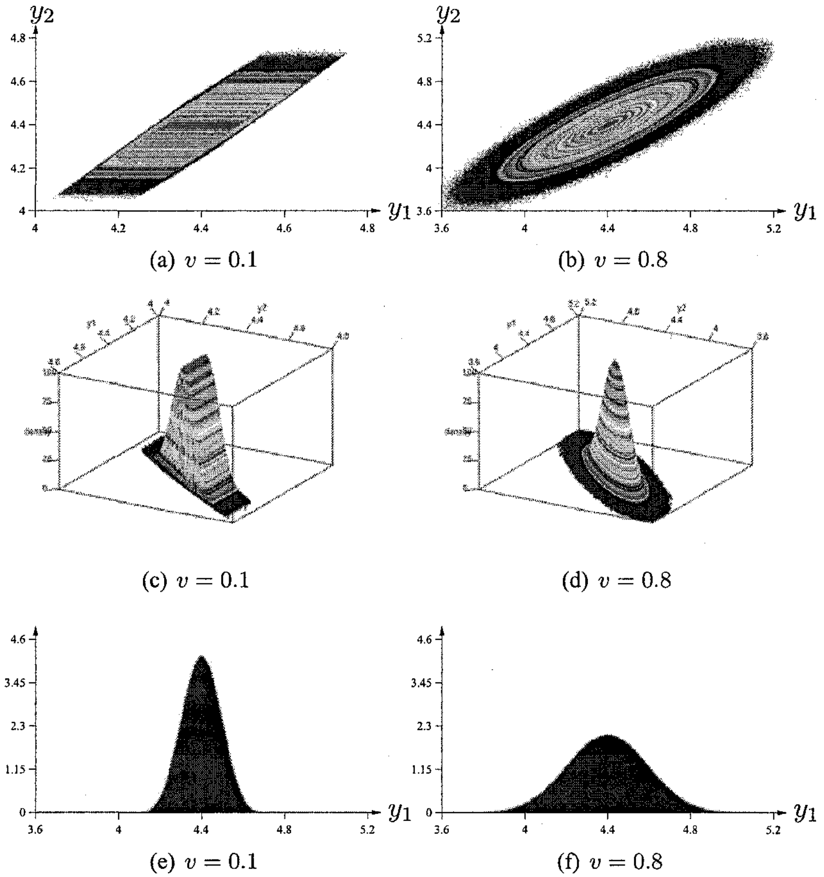


Figure 5: *The role of the accelerator* v *;* $T = 10^7$, $m = 0.75$, $\lambda = 0.1$.

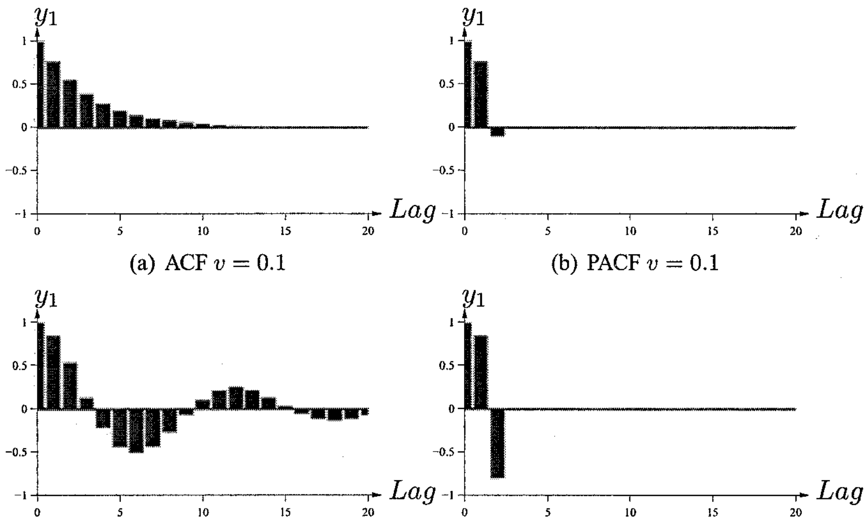
standard statistics for the two cases, confirming the symmetry (low skewness), and the high variance for the situation with $v = 0.8$.

To complete the description of the statistical features, Figure 6 provides data on autocorrelation for a large sample, which confirms the typical characteristics of the autocorrelation functions of an AR(2) for both the high rotation and the low rotation case (see for example Hamilton 1994).

The bifurcation diagram Figure 7 shows the change to the elliptic shape of the support of the invariant distribution and the increasing variance as the accelerator increases.

statistic	$v = 0.1$		$v = 0.8$	
	theoretical	estimate	theoretical	estimate
mean	4.4	4.39986	4.4	4.39985
variance	0.008357	0.00837783	0.0358	0.0360081
stand. dev.	0.09141662	0.0915305	0.189208879	0.189758
skewness	0	0.00297042	0	-0.00375951

Table 1: *Statistics:* $m_0 = 1$, $m = 0.75$; $\lambda = 0.1$.



(a) ACF $v = 0.1$ (b) PACF $v = 0.1$

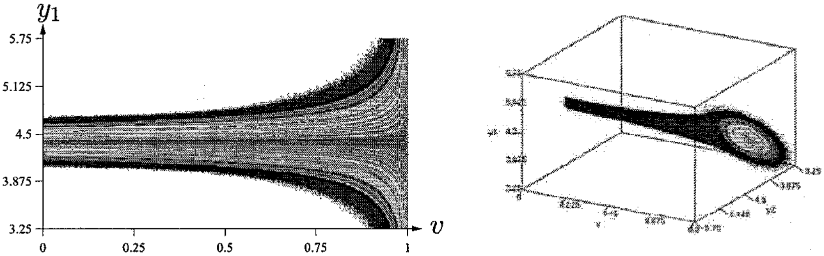
(c) ACF $v = 0.8$ (d) PACF $v = 0.8$

Figure 6: *The role of v on correlation;* $m = 0.75$, $\lambda = 0.1$.

4.6 The Samuelson Model with Mixed Additive Noise

Consider now the situation with mixed discrete/continuous additive noise

$$\begin{aligned} f(y_1, y_2) &:= (m_0 + v_0) + \xi + (m + v)y_2 - vy_1 \\ \xi &\sim [0, 2\lambda], \quad \text{uniformly} \quad \lambda \geq 0 \qquad (39) \\ m_0 &\sim \{0, 1\}, \text{ discrete with equal probability} \end{aligned}$$



(a) $\lambda = 0.1, m = 0.75$ (b) $\lambda = 0.1, m = 0.75$

Figure 7: *Bifurcation of the accelerator* $v \in [0, 1]$*;* $m = 0.75$*,* $\lambda = 0.1$.

describing a discrete switch of aggregate demand plus a small continuous noise, both of which follow an i. i. d. process. According to Proposition 4.1, there exists a unique random fixed point (stationary solution) which is asymptotically stable.

With finite discrete noise only ($\lambda = 0$) the system becomes a so called *Iterated Function System*[5](IFS) which often possesses complex or 'fractal' attractors made up of uncountably many disjoint compact sets of Lebesgue measure zero (Cantor sets). Such attractors are caused by gaps of the images of the finite list of mappings on invariant sets of the state space, i. e. subsets which are left with probability one in finite time. As a consequence the corresponding invariant measures will typically be 'fractal' and without densities. The experiment here is designed to reveal the effect of discrete noise on the attractor and examine the role of additional small smooth noise, to determine to what extent "smoothing by noise" appears.

For the situation described by the system (39), the numerical analysis reveals the following property: there exists $0 << (m, v) << 1$, a pair of values $(m_0^1 < m_0^2)$, and a small level of noise $\lambda > 0$ such that the attractor consists of 2^k self similar disconnected subsets of $\mathbb{R}^2$, for some $k > 1$ (see Figure 8). The invariant measure has 2^k modes and has the same shape on each subset. Thus, the associated random fixed point (stationary solution) moves in a random fashion between the disjoint subsets and not in any specific harmonic or periodic way. The autocorrelation functions are not distinguishable from those of the smooth noise only (Figure 6). For example, panel (c) and (d) of Figure 8 show a 16 piece attractor and the associated histogram with 16 modes for $\lambda = 0.025$. As the continuous noise increases the attractor as well as the measure becomes more smooth with only four

[5]See Barnsley (1988) or Lasota & Mackey (1994).

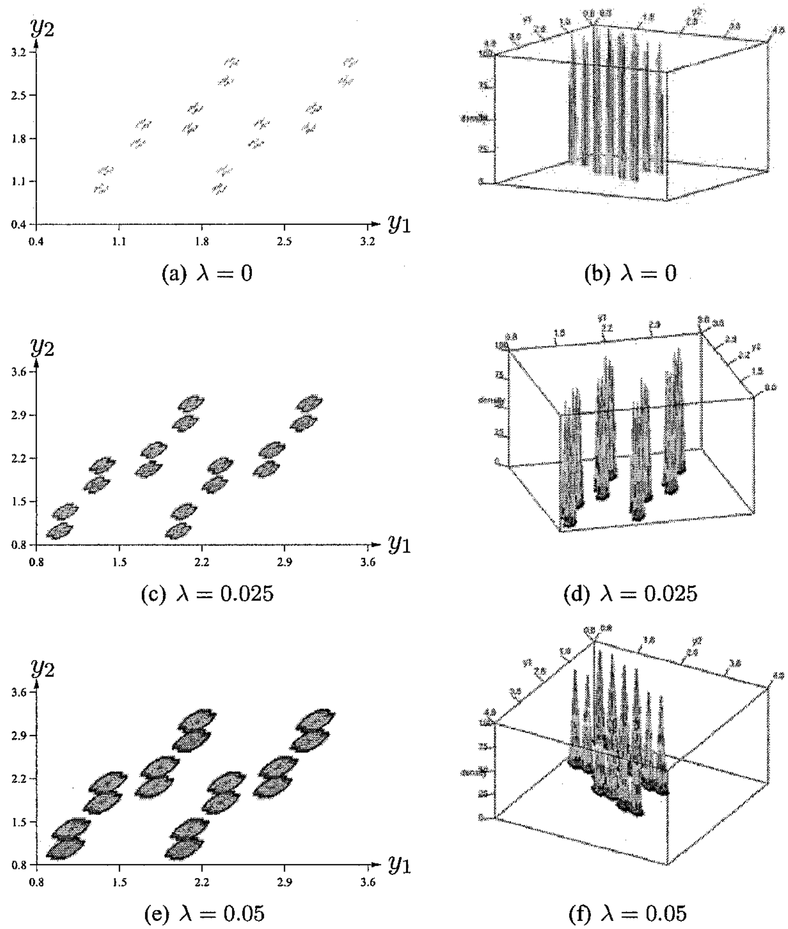
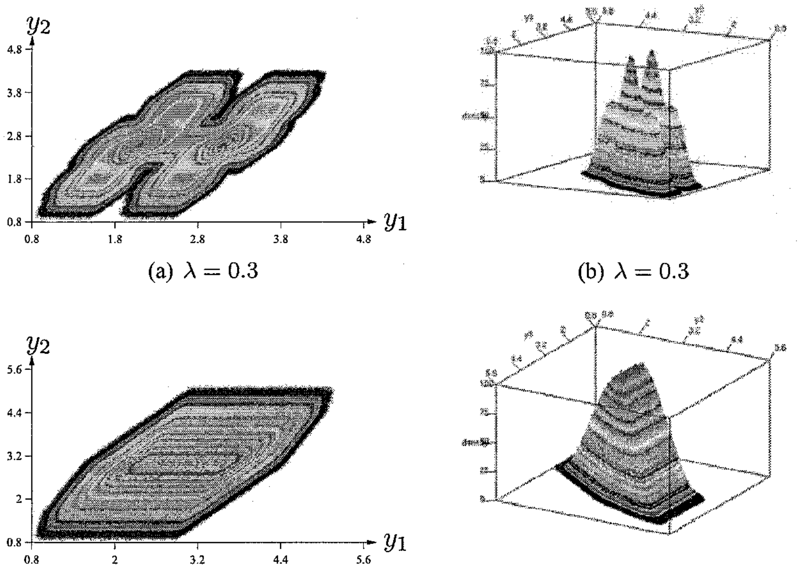


Figure 8: *Low accelerator:* $v = 0.25$; $m_0 \sim \{0, 1\}$; $\xi \sim [0, 2\lambda]$; $m = .5$.

modes. Figures 8 and 9 display the change of the attractor and the invariant measure as the continuous noise increases from $\lambda = 0$ to $\lambda = 0.5$.

The sensitivity of these features with respect to the multiplier and the accelerator is quite different. It is a remarkable fact, that the appearance of the 'gaps' is more frequent for low values of the accelerator. As in the case with smooth additive noise alone, it appears again that the increase in the rotation caused by complex eigenvalues is the reason for this phenomenon. Therefore, a high value of the accelerator may create sufficient rotation by



(a) $\lambda = 0.3$ (b) $\lambda = 0.3$

(c) $\lambda = 0.5$ (d) $\lambda = 0.5$

Figure 9: *Low accelerator:* $v = 0.25$; $m_0 \sim \{0, 1\}$; $\xi \sim [0, 2\lambda]$; $m = .5$.

itself, so that even for $\lambda = 0$, no gaps appear. As a consequence, for all small positive values of λ, the long run behavior induces essentially the same invariant distribution as for $\lambda = 0$, as can be seen in Figure 10.

Figures 11 displays the results of bifurcations of the accelerator under different levels of noise for aggregate demand. The v-bifurcation shows quite clearly the disconnected attractor for low values of the accelerator while its mean remains at the same level. In contrast, any m-bifurcation displays the joint effect of the multiplier on rotation and on the position of the attractor. In both cases, the invariant measure will have multiple modes of different order.

Summarizing the results of the experiments with additive demand shocks, one finds that the attractor may consist of a symmetric collection of disconnected subsets of the state space provided the perturbation is discrete (with small smooth noise) and the accelerator is low. In such a situation, the unique stationary solution fluctuates in a random fashion between the disconnected subsets inducing multi modal invariant distributions on the symmetric disconnected subsets of the attractor without regularity or periodicity. If the smooth additive noise becomes larger or the accelerator becomes large, the

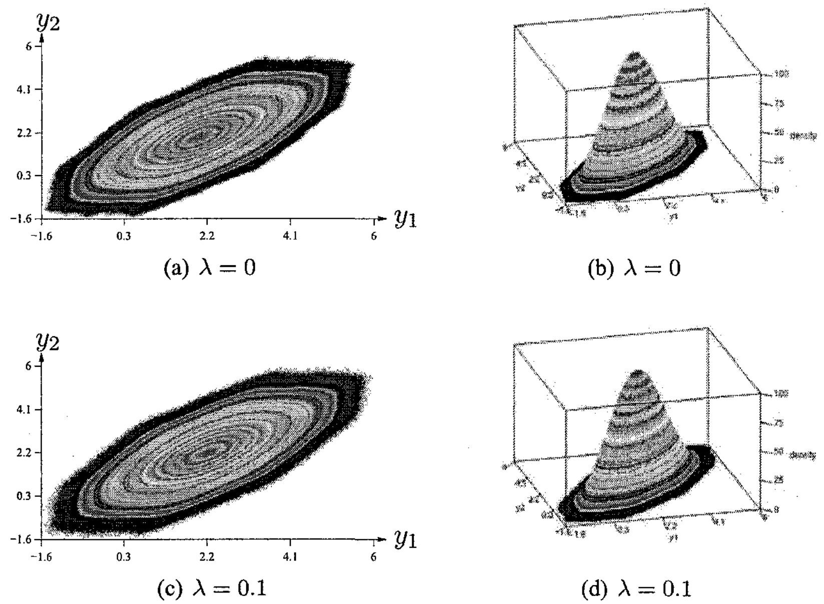


Figure 10: *High accelerator:* $v = 0.75$; $m_0 \sim \{0, 1\}$; $\xi \sim [0, 2\lambda]$; $m = 0.5$.

attractor is always a connected compact set. The multi modality disappears as the noise and/or the accelerator increase. Then, the stationary distribution exhibits the typical features of a VAR(1) model with an AR(2) delay structure with high rotation, an invariant distribution with support similar to an ellipsoid and with positive cross correlation, as presented in section 4.5. Thus, the statistical properties of smooth additive noise with high accelerators may not be distinguishable from those of a mixed perturbation scenario with low accelerators. However, from a time series perspective, much of the regularity of the smooth case is lost. Sample paths will reveal clustering, moment reversion, and slow convergence of moments. From the perspective of time series analysis or estimating procedures, little seems to be known about the theoretical properties of the invariant distributions or methods to estimate parameters of an affine system under discrete noise[6].

[6]For some preliminary results see Böhm & Jungeilges (2004).

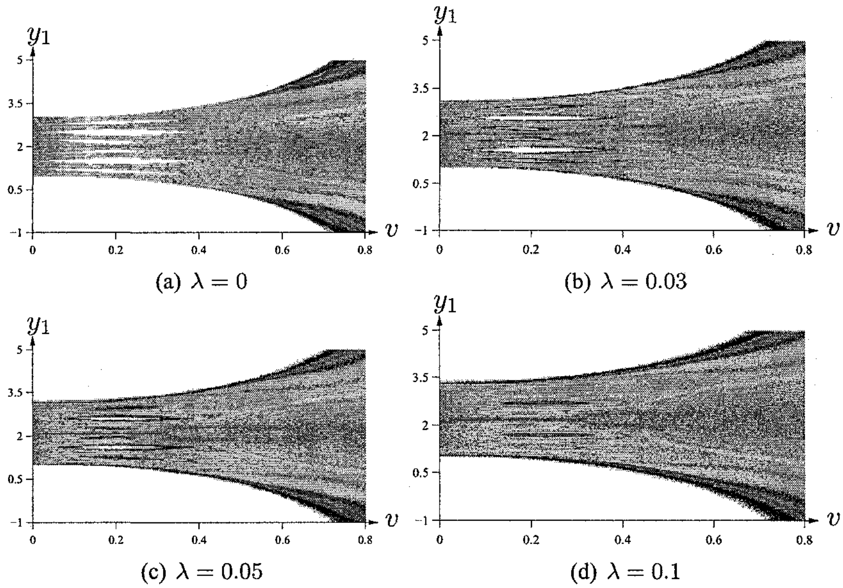


Figure 11: v - *Bifurcations;* $m_0 \sim \{0,1\}$; $\xi \sim [0, 2\lambda]$; $m = 0.5$.

4.7 Random Multiplier and Random Accelerator

Finally, consider a discrete perturbation of the multiplier or the accelerator combined with small additive noise $\xi \sim [0, 2\lambda]$ on aggregate demand. In such a case, the system becomes a Markov switching model and is no longer a VAR(1), since the noise acts in a multiplicative way on the delay equation. Due to Proposition 4.1, there exists a unique asymptotically stable random fixed point (stationary solution) whose statistical properties can be derived from the empirical statistics of a single sample path. The multiplicative random effects change the local stability property of the mappings implying a *random* change of the type of rotation. As a consequence, the attractor will not be symmetric any longer implying also that the stationary solution may show reversion of moments, volatility clustering or alike. However, while the random accelerator leaves the steady state unchanged (for $\lambda = 0$), the random multiplier has both an effect on the rotation and on steady states. Therefore, in the latter case, one would expect larger attractors (higher variance) than with a random accelerator alone, a feature which is confirmed by the numerical experiments.

In general, one finds qualitatively that multiplicative discrete noise reduces the occurrence of "gaps" but it often induces non symmetric attractors.

Discrete random multipliers generate less smooth invariant distributions than discrete random accelerators (compare Figures 13 and 14). Random accelerators increase the rotation inducing more symmetry of the attractor. In the latter case, the data may be indistinguishable from the situation with continuous additive noise (VAR1). In particular, autocorrelations will be indistinguishable.

4.8 Two Special Cases with Discrete Noise

Consider a model with simultaneous discrete switching of the accelerator *and* aggregate demand as characterized by Table 2 while keeping the multiplier constant. Four mappings which involve one real root and three complex roots are chosen with equal probability. The table lists the set of parameters but also the four associated fixed points and their eigenvalues λ_j.

The resulting dynamics, however, leads to an overall low rotation with an asymmetric attractor (see Figure 15). The time series indicates effects of mean reversion and of volatility clustering, while there does not appear any

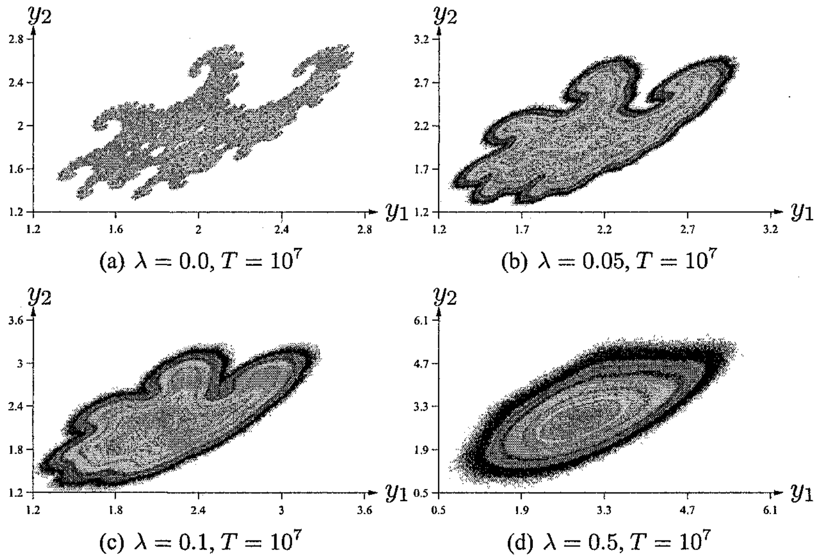
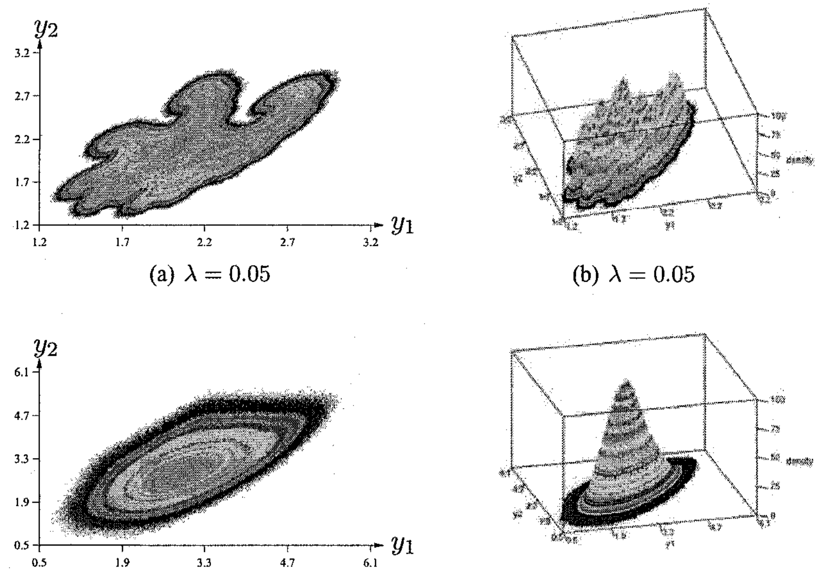


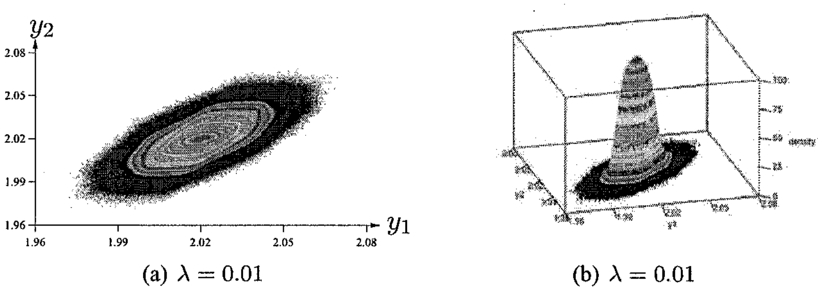
Figure 12: *Random Multiplier:* $m \sim \{0.4, 0.6\}$*;* $m_0+\xi$*;* $\xi \sim [0, 2\lambda]$*;* $v = .5$.



(a) $\lambda = 0.05$ (b) $\lambda = 0.05$

(c) $\lambda = 0.5$ (d) $\lambda = 0.5$

Figure 13: *Random multiplier:* $m \sim \{0.4, 0.6\}$; $v = .5$; $m_0 + \xi$; $\xi \sim [0, 2\lambda]$.



(a) $\lambda = 0.01$ (b) $\lambda = 0.01$

Figure 14: *Random Accelerator:* $v \sim \{0.25, 0.75\}$; $m = 0.5$; $m_0 + \xi$; $\xi \sim [0, 2\lambda]$.

substantial correlation of higher order. The fixed points of the four deterministic mappings are contained in the asymmetric attractor which is stretched out along the diagonal. The invariant distribution is highly skewed with high

i	m_i	v_i	A_i	m_0^i	$\lambda_j(A_i)$	$\bar{y}_i$	p_i
1	.80	$\mathbf{.10}$	$\begin{pmatrix} 0 & 1 \\ -.1 & .9 \end{pmatrix}$	$\begin{pmatrix} 0 \\ \mathbf{.5} \end{pmatrix}$	$\begin{pmatrix} .77 \\ .13 \end{pmatrix}$	$\begin{pmatrix} 2.5 \\ 2.5 \end{pmatrix}$	$\frac{1}{4}$
2	.80	$\mathbf{.31}$	$\begin{pmatrix} 0 & 1 \\ -.31 & 1.1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ \mathbf{1} \end{pmatrix}$	$\begin{pmatrix} .55 \\ .085i \end{pmatrix}$	$\begin{pmatrix} 5 \\ 5 \end{pmatrix}$	$\frac{1}{4}$
3	.80	$\mathbf{.80}$	$\begin{pmatrix} 0 & 1 \\ -.8 & 1.6 \end{pmatrix}$	$\begin{pmatrix} 0 \\ \mathbf{0.5} \end{pmatrix}$	$\begin{pmatrix} .8 \\ .4i \end{pmatrix}$	$\begin{pmatrix} 10 \\ 10 \end{pmatrix}$	$\frac{1}{4}$
4	.80	$\mathbf{1.0}$	$\begin{pmatrix} 0 & 1 \\ -1 & 1.8 \end{pmatrix}$	$\begin{pmatrix} 0 \\ \mathbf{1} \end{pmatrix}$	$\begin{pmatrix} .9 \\ .436i \end{pmatrix}$	$\begin{pmatrix} 15 \\ 15 \end{pmatrix}$	$\frac{1}{4}$

Table 2: *Parameters of Model SAM5.*

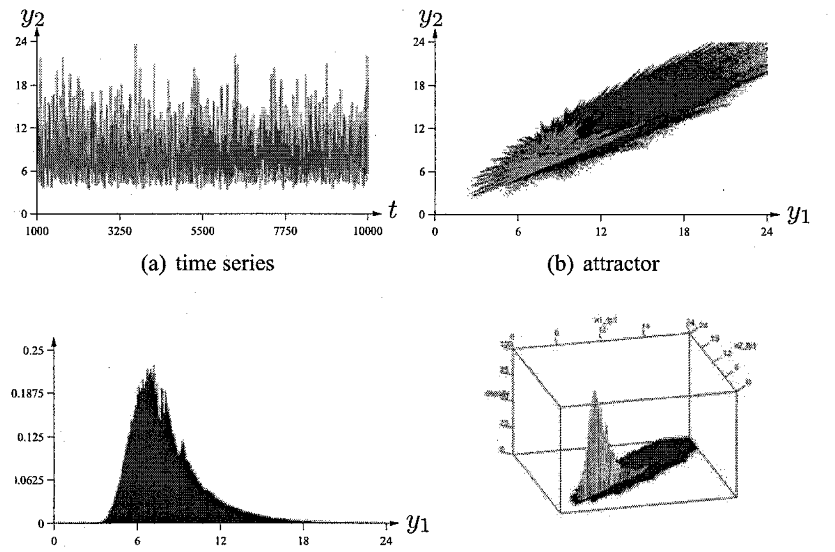
statistic	time series SAM5
mean	8.12587
variance	6.88639
standard deviation	2.62419
skewness	1.41137
kurtosis	2.80578
quantile (0.55)	7.72191

Table 3: *Statistics of Model SAM5.*

frequency occurring near the two lower fixed points and a high kurtosis. Table 3 provides empirical estimates of the basic statistics only. Theoretical values of the true moments seem to be unaccessible and not known for Markov switching models.

Finally, consider the model SAM4 describing a situation with simultaneous discrete switching of the multiplier, the accelerator, and of aggregate demand as given by Table 4.

This corresponds to a pure Markov switching model. The two mappings which are chosen with equal probability have fixed points with complex eigenvalues. The time series also shows the typical moment reversion and clustering as the previous model.



(a) time series (b) attractor

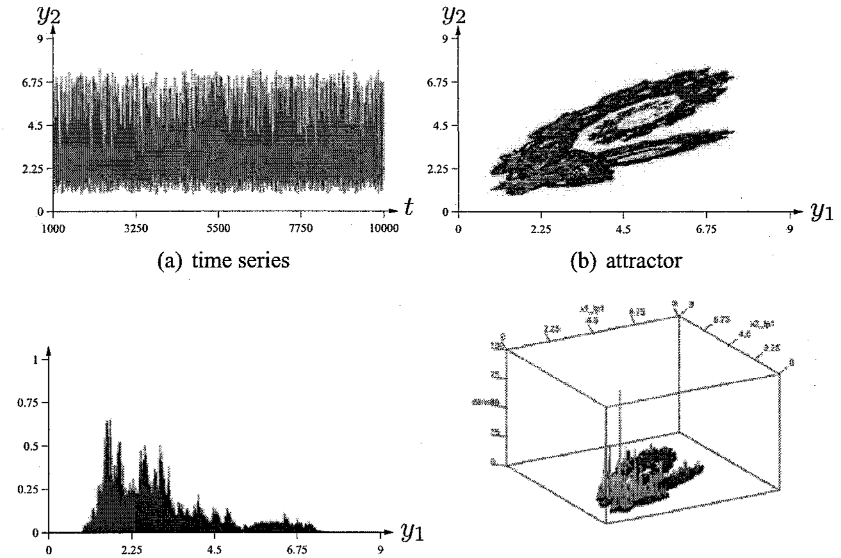
(c) histogram (d) attractor with histogram

Figure 15: *Characteristics of Model SAM5* ($T = 10^6$).

i	m_i	v_i	A_1	m_0^i	$\lambda_j(A_i)$	$\bar{y}_i$	p_i
1	**.40**	**.31**	$\begin{pmatrix} 0 & 1 \\ -.31 & 0.71 \end{pmatrix}$	$\begin{pmatrix} 0 \\ \mathbf{1} \end{pmatrix}$	$\begin{pmatrix} .355 \\ .429i \end{pmatrix}$	$\begin{pmatrix} 1.7 \\ 1.7 \end{pmatrix}$	$\frac{1}{2}$
2	**.60**	**.80**	$\begin{pmatrix} 0 & 1 \\ -.8 & 1.4 \end{pmatrix}$	$\begin{pmatrix} 0 \\ \mathbf{0.5} \end{pmatrix}$	$\begin{pmatrix} .7 \\ .557i \end{pmatrix}$	$\begin{pmatrix} 5 \\ 5 \end{pmatrix}$	$\frac{1}{2}$

Table 4: *Parameters of Model SAM4.*

In contrast to the previous situation SAM5, however, the resulting dynamics shows a high degree of rotation with a less connected attractor than in Model SAM5, which points to a 'fractal' structure. Observe that the two stationary points are $(1.7, 1.7)$ and $(5, 5)$ most likely are not in the attractor. The invariant measure is much less smooth and less skewed. However, the autocorrelation is not distinctly different than in Model SAM5.



(a) time series (b) attractor

(c) histogram (d) attractor with histogram

Figure 16: *Characteristics of Model SAM4 (T = 10^6).*

4.9 Summary and Conclusions

The random multiplier accelerator model can be described as a parameterized family of random affine mappings, induced by a random family of second order delay equations. If the multiplier and the accelerator are restricted to be strictly between zero and one, i. e. the stable case, every Multiplier-Accelerator map is a contraction. Applying a result on existence, uniqueness, and asymptotic stability of a random fixed point for invertible affine random maps due to Arnold (1998), it was shown that for stationary and ergodic compact valued noise processes, the dynamics of the random Multiplier-Accelerator model has a well defined *unique, stationary* and *stable* long run random behavior, satisfying the following properties:

1. (almost all) random orbits/sample paths converge to a unique stationary solution which induces a unique invariant distribution on a unique attractor;

2. time averages converge to the invariant distribution according to the Mean Ergodic Theorem;

statistic	time series SAM4
mean	3.00249
variance	2.0681
standard deviation	1.43809
skewness	1.10593
kurtosis	0.578912
quantile (0.55)	2.8285

Table 5: *Statistics of Model SAM4.*

3. when perturbations are discrete (finite) and i. i. d., the random multiplier-accelerator map corresponds to a Hyperbolic Iterated Function System (IFS).
4. In this case, the unique attractor (the support of the invariant measure) may be a complex ('fractal') set or a Cantor set, and
5. the invariant measure (distribution) may be very complex (with discontinuous distribution functions).

With i. i. d. perturbations, the random multiplier accelerator model belongs to the class of generalized two dimensional Vector Autoregressive Systems of Order 1 (VAR1) including so called Markov switching models. A numerical analysis with different i. i. d. perturbations showed that

1. additive uniform i. i. d. perturbations alone lead to symmetric attractors and distributions
2. on ellipsoidal attractors for high accelerators and on rectangular attractors for low attractors;
3. fractal attractors and distributions under discrete additive noise are more frequent for low than for high accelerators;
4. adding small/continuous additive noise reduces/eliminates the 'fractal' structure of the attractor implying a multi modal invariant distribution on a finite collection of disjoint compact sets which make up the support/attractor;
5. random accelerators as well as random multipliers typically lead to stationary solutions which can show a variety of complex time series phenomena, like moment reversion and clustering;

6. while the attractors and invariant distributions are typically non symmetric;
7. these features seem less prevalent under a discrete random accelerator than under random multipliers or random aggregate demand.

Since the mathematical result is applicable to general invertible affine random dynamical systems, the above features would be expected to appear as properties of unique stationary stable solutions also in random affine delay equations of any finite order as well as in more general affine economic models. Therefore, even with i. i. d. perturbations alone, these models represent a rich structure for interesting complex business cycle features.

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