1 Some Methods for the Global Analysis of Closed Invariant Curves in Two-Dimensional Maps

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1.1 Introduction

It is well known that models of nonlinear oscillators applied to the study of the business cycle can be formulated both as continuous or discrete time dynamic models (see e.g. [23], [33], [34]). However, economic time is often discontinuous (discrete) because decisions in economics cannot be continuously revised. For this reason discrete-time dynamical systems, represented by difference equations or, more properly, by the iterated application of maps, are often a more suitable tool for modelling dynamic economic processes. So, it is useful to study the peculiarities of discrete dynamical systems and their possible applications to the study of self sustained oscillations. This is the main goal of this chapter, where we describe, on the light of some recent results about local and global properties of iterated maps of the plane, some particular routes to the creation/destruction of closed invariant curves, along which self sustained oscillations occur.

In fact, even if in the fifties and sixties the methods for the study of iterated maps were less developed than those for ordinary differential equations, the situation is now rapidly changing because many results have been obtained about discrete dynamical systems (see e.g. [25], [26], [24], [16], [42],[28], [29]). Indeed, the dynamic properties and bifurcations of one dimensional iterated maps are now quite well known, as well as their implications about periodic and chaotic behaviors of their trajectories (see e.g.[15], [40],[41]). Even for two-dimensional maps more and more results can be found in the literature, starting from the pioneering works [25] and [26], (see also [32], [35], [36], [1]). The qualitative methods for the study of discrete dynamical systems are in many aspects similar to those employed in continuous time systems, but important differences are worth to be emphasized. For example, a version of the Andronov-Hopf bifurcation theorem also exists for discrete dynamical systems, known as Neimark-Sacker bifurcation theorem, and it is quite similar to the one in continuous time, with the expected difference that while in the continuous-time case an equilibrium point undergoes an Hopf bifurcation when a pair of eigenvalues cross the line of vanishing real part, in the discrete-time case the Neimark-Sacker bifurcation occurs when a pair of eigenvalues cross the unitary circle of the complex plane. However, remarkable differences can be evidenced, both concerning the kind of motion along the closed invariant curve created at the bifurcation (it is no longer a unique trajectory but the closure of infinitely many distinct trajectories, either periodic or quasiperiodic) and the fate of such invariant curve as the parameters move far from their bifurcation values.

In this chapter, some global bifurcations that cause the creation and destruction of invariant closed curves via global bifurcations are also considered, related with the occurrence of saddle-node or saddle-focus heteroclinic or homoclinic connections and tangles. Some exemplary global bifurcations are shown through numerical explorations and qualitative geometrical explanations.

Indeed, several aspects in the study of the global dynamical properties of two-dimensional discrete dynamical systems are still obscure, and their study often require an interplay between analytical, geometric, numerical and graphical methods. Moreover, the differences between continuous and discrete dynamical systems become particularly evident when the latter are obtained by the iteration of noninvertible maps. A map is invertible if it maps distinct points into distinct points, whereas whenever distinct points which are mapped into the same point exist, then we say it is a noninvertible map. Hence, the geometric action of a noninvertible map can be expressed by saying that it "folds and pleats" the phase space, so that distinct points can be mapped into the same point (see e.g. [36], [3] for recent studies of the properties of noninvertible maps, [13], [12] and the monograph [39] for recent applications in economics). This introduces some peculiar dynamic properties when a business cycle model is represented by a discrete dynamical system obtained by the iteration of a noninvertible map.

1.2 Basic Definitions and Properties of Two-Dimensional Discrete Dynamical Systems

In this section we give some basic definitions and properties concerning two-dimensional discrete dynamical systems[1], represented by the iterated application of a map of the plane

$$x' = T(x), \qquad T : S \to S, \quad S \subseteq \mathbb{R}^2 \tag{1}$$

At any iteration it transforms a point $x \in S$ into a unique point $x' \in S$ called *rank-1 (forward) image* of x. A point x such that $T(x) = x'$ is a *rank-1 preimage* of x'.

If $x \neq y$ implies $T(x) \neq T(y)$ for each x, y in S, then T is an *invertible map* in S, because the inverse mapping $x = T^{-1}(x')$ is uniquely defined; otherwise T is said to be a *noninvertible map*, because points x exist that have several rank-1 preimages, i.e. the inverse relation $x = T^{-1}(x')$ is multivalued. So, noninvertible means "many-to-one", that is distinct points $x \neq y$ may have the same image, $T(x) = T(y) = x'$.

Geometrically, the action of a noninvertible map can be expressed by saying that it "folds and pleats" the space S, so that distinct points are mapped into the same point. This is equivalently stated by saying that several inverses are defined in some points of S, and these inverses "unfold" S.

For a noninvertible map, S can be subdivided into regions Z_k, $k \geq 0$, whose points have k distinct rank-1 preimages. Generally, for a continuous map, as the point x' varies in $\mathbb{R}^2$, pairs of preimages appear or disappear as it crosses the boundaries separating different regions. Hence, such boundaries are characterized by the presence of at least two coincident (merging) preimages. This leads us to the definition of the *critical curves*, one of the distinguishing features of noninvertible maps (see [25] and [36]):

Definition. The *critical curve* LC of a continuous map T is defined as the locus of points having at least two coincident *rank-1* preimages, located on a set LC_{-1}, called *set of merging preimages.*

Portions of LC separate regions Z_k of the phase space characterized by a different number of $rank - 1$ preimages, for example Z_k and Z_{k+2} (this is the standard occurrence for continuous maps). The critical set LC is the generalization of the notion of local extrema (minimum or maximum value)

[1] The reader is addressed to [24], [32], [37], [36] for a more complete treatment.

of a one-dimensional map[2] and the set LC_{-1} is the generalization of local extremum point of a one-dimensional map (i.e. $T(LC_{-1}) = LC)$).

Starting from an initial condition $x_0 \in S$, the (forward) iteration by T uniquely defines a *trajectory*

$$\tau(x_0) = \{x_n = T^n(x_0), \ n = 0, 1, 2, ...\}$$

where T^0 is the identity function and $T^n = T \circ T^{n-1}$. The set of points that form a trajectory is also called *orbit*, however many authors consider these two terms as equivalent.

The simplest orbits are *fixed points*, that is a singleton $\{p^*\}$ such that $T(p^*) = p^*$, so that $T^n(p^*) = p^*$ for all n, and *cycles of period* k, that is a set of k ($k > 1$) distinct periodic points $\{p_1^*, p_2^*, ..., p_k^*\}$ such that $T(p_i^*) = p_{i+1}^*$ for $i = 1, 2, ..., k-1$ and $T(p_k^*) = p_1^*$. Observe that the periodic points of a cycle of period k are fixed points of the map T^k, and a fixed point is a $k-$cycle with $k = 1$.

We recall that a set $E \subset \mathbb{R}^n$ is *invariant* for the map T if it is mapped onto itself, $T(E) = E$. This means that if $x \in E$ then $T(x) \in E$, i.e. E is *trapping,* and each point of E is the forward image of at least one point of E. The simplest examples of invariant sets are the fixed points and the cycles of the map. More generally, the attracting (repelling) sets and the attractors (repellors) of a map are invariant sets.

An *attracting set* A is a closed invariant set such that a neighborhood U of A exists which is strictly mapped into itself and whose trajectories (i.e. the trajectories starting from any point of U) converge to A. A closed invariant set which is not attracting is called a *repelling set* if however close to A there are points whose trajectories goes away from A. An *attractor* (*repellor*) is an attracting (repelling) set containing a dense orbit. An attracting set may contain one or several attractors, coexisting with sets of repelling points, whereas an attractor is an undecomposable set. In the case of a cycle *attractor* (*repellor*) is synonymous of asymptotically stable (unstable). In particular unstable nodes and foci are also called *expanding*.

As the definition suggests, there exist points which converge to an attracting set (or to an attractor) A: The trapping set made up by all such points constitutes *the basin of attraction* of A and it can be obtained considering the union of the preimages of any rank of the neighborhood U (defined above):

$$B(A) = \bigcup_{n=0}^{\infty} T^{-n}(U) \tag{2}$$

[2]This terminology, and notation, originates from the notion of critical point as it is used in the classical works of Julia and Fatou.

where $T^{-1}(x)$ represents the set of all the rank-1 preimages of x and $T^{-n}(x)$ represents the set of all the rank-n preimages of x (i.e., the points mapped into x after n applications of T).

In other words, the basin of an attracting set A is the set of all the points that generate trajectories ultimately belonging to A or to the neighborhood U defined above.

As we are interested in the asymptotic behavior of the trajectories, we also introduce the *$\omega-$limit set* of a point x: A point $q \in \omega(x)$ if there exists an increasing sequence $n_1 < n_2 < ... < n_k...$ such that the points $T^{n_k}(x)$ tend to q as k goes to infinity (clearly such a point q belongs to the limit set of the trajectory $\tau(x)$). The set $\omega(x)$ is invariant and gives an idea of the long run behavior of the trajectory from x.

The same definition can be associated with the backward iterations of T, so obtaining the *$\alpha-$limit set* of x: A point $q \in \alpha(x)$ if there exists an increasing sequence $n_1 < n_2 < ... < n_k...$ such that the points $T_{j_k}^{-n_k}(x)$, for a suitable sequences of inverses j_k in case of a noninvertible map, tend to q as k goes to infinity (clearly such a point q belongs to the limit set of $\bigcup_{n \geq 0} T^{-n}(x)$).

In the particular case of a fixed point p^* of T we define the stable and unstable sets of p^* as

$$W^{st}(p^*) = \left\{ x : \lim_{n \to +\infty} T^n(x) = p^* \right\}$$

$$W^{un}(p^*) = \left\{ x : \lim_{n \to +\infty} T_{j_n}^{-n}(x) = p^* \right\}$$

respectively, where $T_{j_n}^{-n}$ means for a suitable sequence of inverses. This means that the stable set of p^* is the set of points x having p^* as ω-limit set and the unstable set of p^* is given by the points having p^* in their α-limit set.

If p^* is an asymptotically stable fixed point, then its stable set coincides with its basin of attraction, $B(p^*)$, and its unstable set is not empty if the map is noninvertible in p^*. If p^* is an expanding fixed point, then its unstable set is a whole area and its stable set is not empty if the map is noninvertible in p^*.

Other important sets in the study of the global properties of a map T are the stable and unstable sets of an hyperbolic[3] saddle fixed point p^*. Indeed,

[3]A fixed point p^* is said hyperbolic if the jacobian matrix evaluated at p^* has no eigenvalues of unit modulus.

if the map T admits several disjoint attracting sets, the stable sets of some saddles (fixed points or cycles) often play the role of separatrices between basins of attraction.

If p^* is a hyperbolic saddle and T is smooth in a neighborhood U of p^* in which T has a local inverse denoted as T_1^{-1}, the *Stable Manifold Theorem* states the existence of the local stable and unstable sets (defined in such a neighborhood U of p^*) as

$$W_{loc}^S(p^*) = \{x \in U : x_n = T^n(x) \to p^* \text{ and } x_n \in U\}$$
$$W_{loc}^U(p^*) = \left\{x \in U : x_{-n} = T_1^{-n}(x) \to p^* \text{ and } x_{-n} \in U\right\}.$$

The set $W_{loc}^S(p^*)$ (resp. $W_{loc}^U(p^*)$) is a one-dimensional curve as smooth as T, passing through p^* and tangent at p^* to the stable (resp. unstable) eigenspace. Then the global stable and unstable sets are made up, respectively, by all the preimages of any rank and the (forward) images of the points of the local sets, that is:

$$W^S(p^*) = \bigcup_{n\geq 0} T^{-n}\left(W_{loc}^S(p^*)\right) \tag{3}$$

$$W^U(p^*) = \bigcup_{n\geq 0} T\left(W_{loc}^U(p^*)\right). \tag{4}$$

where T^{-n} denotes all the existing preimages of rank-n.

If the map is invertible, the stable and unstable sets of a saddle p^* are invariant manifolds of T. If the map is noninvertible, the stable set of p^* is backward invariant, but it may be strictly mapped into itself (since some of its points may have no preimages), and it may be not connected. The unstable set of p^* is an invariant set, but it may be not backward invariant and (contrarily to what occurs in invertible maps) self intersections are allowed (several examples will be shown in this book).

It is worth to observe that analogous concepts are also given for continuous flows, but the main difference here is that the stable and unstable sets are not trajectories, but union of different trajectories (indeed infinitely many distinct trajectories). A qualitative representation of the local stable and unstable sets, W_{loc}^S and W_{loc}^U, of a saddle fixed point p^* is given in Fig.1, where E^S and E^U are the eigenspaces.

In the following, we shall consider the stable (resp. unstable) set of a saddle as given by the union of two branches merging in p^* denoted by ω_1 and ω_2 (resp α_1 and α_2) because all the points in these branches have p^* as *$\omega-$limit set* (resp. in their *$\alpha-$limit set).*

$$W^S(p^*) = \omega_1 \cup \omega_2 \ , \ \ W^U(p^*) = \alpha_1 \cup \alpha_2$$

The concepts of stable and unstable sets can be easily extended to a cycle of period k, say $\mathcal{C} = \{p_1^*, p_2^*, ..., p_k^*\}$, simply considering the union of the stable (unstable) sets of the points of the cycle considered as k fixed points of the map T^k. For example

$$W^{st}(\mathcal{C}) = \bigcup_{i=1}^{k} W^{st}(p_i^*) \quad , \quad W^{st}(p_i^*) = \left\{ x : \lim_{n \to +\infty} T^{kn}(x) = p_i^* \right\}$$

and analogously for the unstable set. In particular, for a $k-$cycle saddle we

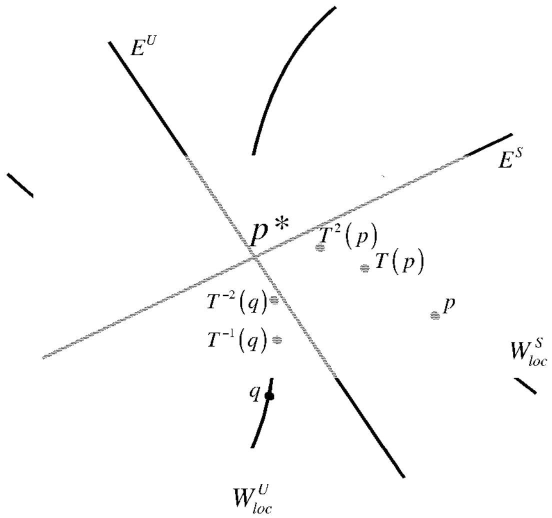

Figure 1: *The local stable and unstable sets of the saddle* p^*.

obtain the stable and unstable sets from (3) and (4) with the map T^k instead of T, that is

$$W^S(\mathcal{C}) = \bigcup_{i=1}^{k} W^S(p_i^*) = \bigcup_{i=1}^{k} (\omega_{1,i} \cup \omega_{2,i})$$

$$W^U(\mathcal{C}) = \bigcup_{i=1}^{k} W^U(p_i^*) = \bigcup_{i=1}^{k} (\alpha_{1,i} \cup \alpha_{2,i})$$

The importance of the stable and unstable sets is related to the fact that they are global concepts, that is, they are not defined only in a neighborhood of the fixed point (or cycle). Thus, being interested in the global properties of the map T, we may study its invariant sets, through a continuous dialogue between analytic, geometric and numerical methods, and focus our attention on the basins of attraction of its attractors and on the stable and unstable sets of some of its saddle points or cycles.

If the map is nonlinear, the stable and unstable sets may intersect, i.e. it may exist a point q such that $q \in W^{st}(p^*) \cap W^{un}(p^*)$, or

$$q \in W^S(p^*) \cap W^U(p^*).$$

Such a point q is a *homoclinic point* and it can be proved that if a homoclinic point exists then infinitely many homoclinic points must also exist, accumulating in a neighborhood of p^*. Intuitively, this can be understood observing that the forward orbit of q and a suitable backward sequence is also made up of homoclinic points, and converge to p^*. The union of the forward orbit and a suitable backward orbit of a homoclinic point q is called a *homoclinic orbit of* p^*, or *orbit homoclinic to* p^*:

$$\tau(q) = \{..., q_{-n}, ..., q_{-2}, q_{-1}, q, q_1, q_2, ..., q_n, ...\}$$

where $q_n = T^n(q)$ and $T^n(q) \to p^*$ while $q_{-n} = T_{j_n}^{-n}(q)$ and $T_{j_n}^{-n}(q) \to p^*$ is a suitable backward orbit. More generally, an *orbit homoclinic to a cycle* approaches the cycle asymptotically both through forward and backward iterations, so that it always belong to the intersection of the stable and unstable sets of the cycle.

The appearance of homoclinic orbits of a saddle point p^* corresponds to a *homoclinic bifurcation* and implies a very complex configuration of W^S and W^U, called *homoclinic tangle*, due to their winding in proximity of p^*. The existence of an homoclinic tangle is often related to a sequence of bifurcations occurring in a suitable parameter range, and qualitatively shown in Fig.2: First, a homoclinic tangency between one branch, say ω_1, of the stable set of the saddle and one branch of the unstable one, say α_1, followed by a transversal crossing between ω_1 and α_1, that gives rise to a homoclinic tangle, and by a second homoclinic tangency of the same stable and unstable branches, occurring at opposite side with respect to the previous one, which closes the sequence. It is worth to recall that in the parameter range in which the manifolds intersect transversely, an invariant set exists such that the restriction of the map to this invariant set is *chaotic,* that is, the restriction is topologically conjugated with the shift map, as stated in the Smale-Birkhoff Theorem (see for example in [24], [35], [42], [9], [32]). Thus we say that the map possesses a *chaotic repellor*, made up of infinitely many (countable) repelling cycles and uncountable aperiodic trajectories. In the case shown in Fig.2 such a chaotic repellor certainly exists after the first homoclinic tangency and disappears after the second one.

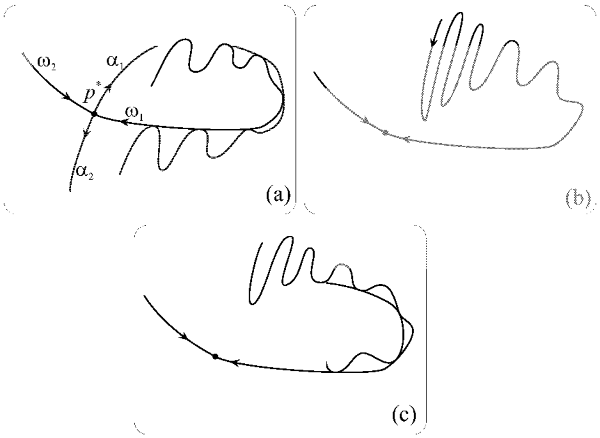

Figure 2: *Homoclinic tangle involving the branches α_1 of the unstable set and ω_1 of the stable one.*

Before and after the homoclinic tangle (i.e. before the first and after the last homoclinic tangencies), the dynamic behavior of the two branches involved in the bifurcation must differ: The invariant set towards which α_1 tends to (or equivalently the ω-limit set of the points of α_1) and the invariant set from which ω_1 comes from (or equivalently the α-limit set of the points of ω_1) before and after the two tangencies are different. Also at the bifurcation value, as in Fig.2a, are different from those of Fig.2c. Thus we can detect the occurrence of such a sequence of bifurcations looking at the asymptotic behavior of W^S and W^U.

We observe that if the saddle is a cycle $\mathcal{C} = \{p_1^*, p_2^*, ..., p_k^*\}$, we may have homoclinic orbits of p_i^*, $i = 1, ..., k$, belonging to the stable and unstable sets of the periodic point p_i^* (considered as fixed points of the map T^k): In such a case we say that there exists points homoclinic to $\mathcal{C}$. But it may also occur that the unstable set $W^U(p_i^*)$ transversely intersects $W^S(p_{i+1}^*)$, $i = 1, ..., k$

and $p_{k+1}^* = p_1^*$: In such a case we have *heteroclinic points* and *heteroclinic tangle* denotes the corresponding configuration of W^S and W^U sets. An example of heteroclinic tangle associated with a saddle cycle of period 4 is qualitatively shown in Fig.3: It involves the internal branches $\alpha_{1,i}$ and $\omega_{1,i}$ which, after a first tangency, transversely intersect each other and then have a second tangency.

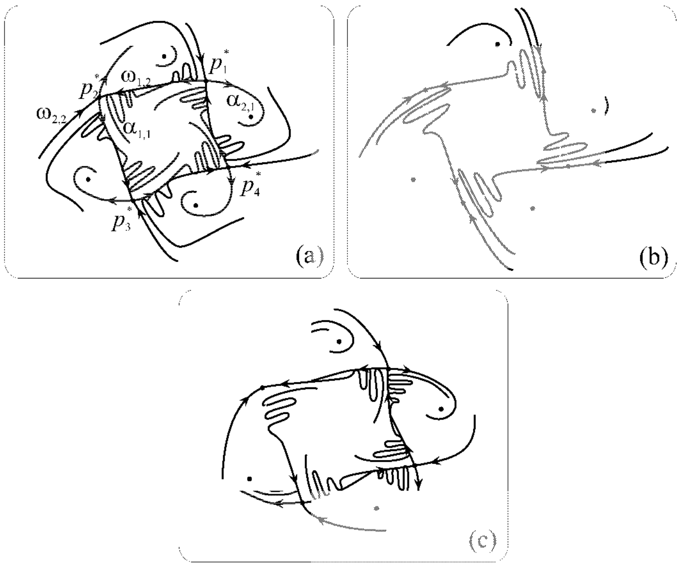

Figure 3: *Heteroclinic tangle associated with a saddle cycle of period 4 (or 4 saddle points of the map T^4).*

Let us also remark that, as in the case of a homoclinic tangle, also in a heteroclinic tangle the asymptotic behavior of the involved branches, before and after the two tangencies, changes. Dynamically, heteroclinic tangles are as important as homoclinic ones since it is possible to prove that also in such cases an invariant set exists on which the restriction of the map is *chaotic*. This homoclinic bifurcation is also called a *cyclical heteroclinic connection* in the sense of Birkhoff (see [10]), who first showed that the same properties occur when the stable and unstable manifolds of a saddle fixed point intersect transversely, or when there are two saddle fixed points, say s_i^* and s_j^*, such that $W^S(s_i^*) \cap W^U(s_j^*) \neq \emptyset$, thus giving cyclical heteroclinic points that form an heteroclinic connection (see also [19]). In such a case, the transverse intersections of $W^U(C)$ and $W^S(C)$ for the saddle cycle $C = \{p_1^*, p_2^*, ..., p_k^*\}$, called *homoclinic points of non simple type* in [10] gives the same properties as the homoclinic points of a saddle fixed point (called *homoclinic points of simple type* in [10]). Thus the occurrence of a transverse homoclinic orbit of a saddle cycle is enough to prove the existence of chaotic dynamics, because it is possible to prove that in the neighborhood of any homoclinic orbit there are infinitely many repelling cycles and an invariant "scrambled set" on which the restriction of the map is chaotic in the sense of Li and Yorke (see for example in [20], [21], [42]).

It is worth to notice that if the map T is noninvertible and p^* is an expanding fixed point of T (i.e., a fixed point such that the Jacobian matrix evaluated at p^* has all the eigenvalues greater than 1 in modulus) then the *stable set of* p^* is given by the preimages of any rank of p^*, if they exist (as defined at the beginning of this section). The existence of a stable set for repelling points is a distinguishing feature of noninvertible maps, because such a set is empty in invertible maps. In fact, for a noninvertible map the only preimage of a fixed point p^* is p^*, as $T(p^*) = p^*$, whereas preimages $p_{-1}^* \neq p^*$ may exist if T is noninvertible, i.e. several rank-1 preimages exist. This implies that for noninvertible maps *homoclinic bifurcations* may also occur for expanding fixed points (repelling nodes and foci), whereas for invertible maps they can only occur for saddles. Another difference between invertible and noninvertible maps is associated with *non connected basins* of attraction, which are only possible for noninvertible maps, whereas they are always simply connected in invertible maps.

1.3 Closed Invariant Curves

The main interest in this chapter is to show some local and global bifurcations related to closed invariant curves in two-dimensional maps, as the dynamics related to such curves is what can be interpreted (in applied models) as cyclical behavior. As we shall see (in later sections and in several examples in later chapters), the appearance/disappearance of closed curves may be related to some global bifurcation. However, the most known mechanism leading to such curves is the Neimark-Sacker bifurcation.

Let us simply recall the properties of a focus fixed point $p^* = (x^*, y^*)$ of a smooth map T, for which the Jacobian matrix DT in p^* has complex-conjugate eigenvalues, assuming that the stability of the fixed point is investigated as a function of one parameter μ. As long as the eigenvalues are in modulus less than one, say for $\mu < \mu_0$, the focus is stable and locally (in a small neighborhood of p^*) the trajectories belong to spirals and tend to the fixed point. When the eigenvalues are in modulus greater than one, say for $\mu > \mu_0$, the focus is unstable (repelling) and locally the trajectories still belong to spirals, however they have a different asymptotic behavior. The crossing of the complex eigenvalue trough the unitary circle, at $\mu = \mu_0$, corresponds to a *Neimark-Sacker bifurcation*. The analytical conditions at which it occurs, and the so called "resonant cases", now belong to standard dynamical results, which can be found in many textbooks, see for example [28], [29], [24], [32], [42]. Let us here briefly recall the main features, which are useful in the study of applied models. A Neimark-Sacker bifurcation is related with closed invariant curves, existing in a small neighborhood of the stable fixed point when the bifurcation is subcritical, or of the unstable fixed point when it is supercritical. The critical case occurs when locally the map behaves as a linear map, that is, the dynamic behavior at the bifurcation value is that of a center, and locally infinitely many closed invariant curves exist (instead of only one, as it occurs before or after the bifurcation value in the subcritical or supercritical case, respectively). Fig.4a qualitatively shows a bifurcation diagram in the subcritical case: A repelling closed invariant curve

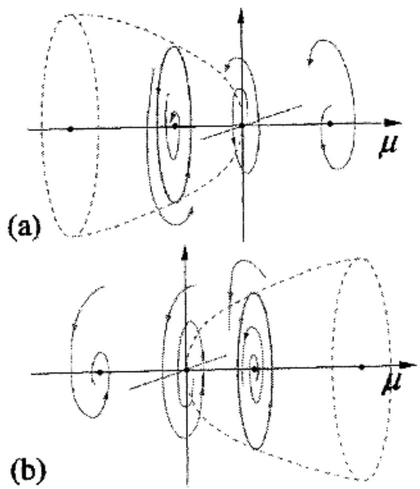

Figure 4: *Qualitative diagram of the Neimark-Sacker bifurcation:* (a) *subcritical case,* (b) *supercritical case.*

Γ exists surrounding the stable fixed point, for $\mu < \mu_0$. As μ increases the repelling closed curve decreases in size and shrinks merging with the fixed point at $\mu = \mu_0$, leaving a repelling focus. It is worth noting that in such a case the closed repelling curve is generally the boundary of the basin of attraction of the stable fixed point. After the bifurcation the fixed point is unstable and the ω-limit set of a point close to it depends on the nonlinearity of the map (it may converge to another attracting set or diverge).

Fig.4b qualitatively shows a bifurcation diagram in the supercritical case: At $\mu = \mu_0$ the fixed point becomes an unstable focus and for $\mu > \mu_0$ an attracting closed invariant curve Γ exists, surrounding the unstable fixed point. Thus the ω-limit set of points close to it is such closed invariant curve.

For μ in a neighborhood of μ_0 the closed invariant curve Γ (stable or unstable) is homeomorphic to a circle, and the restriction of the map to Γ is conjugated with a rotation on the circle. Thus the dynamics on Γ are either periodic or quasiperiodic, depending on the rotation number. Roughly speaking, the rotation number represents the average number of turns of a trajectory around the fixed point. When the rotation number is rational, say m/n, it means that a pair of periodic orbits of period n exists on Γ, and to get the whole periodic orbit a trajectory makes m turns around the fixed point. The dynamics occurring in such a case on Γ are qualitatively shown in Fig.5a in case of a supercritical bifurcation (Γ is attracting): The closed curve is made up by the unstable set of the saddle cycle, and Γ is also called a saddle-(stable) node connection. Instead, Fig.5b shows the subcritical case (Γ is repelling): The closed curve is made up by the stable set of the saddle cycle, and Γ is also called a saddle-(unstable) node connection. When the rotation number is irrational, the trajectories of T on the closed curve Γ are all quasiperiodic. That is, each point on Γ gives rise to a trajectory on the invariant curve which never comes on the same point, and the closure of the trajectory is exactly Γ.

Investigating the bifurcation of a fixed point of T as a function of two parameters, it is quite common to derive the so called *stability triangle*, whose boundaries represent the stability loss due to different properties of the eigenvalues. That is, one side represents a flip-bifurcation (one eigenvalue equal to -1), another side a fold or pitchfork-bifurcation (one eigenvalue equal to +1), and a third side the Neimark-Sacker bifurcation (complex eigenvalues in modulus equal to +1). In the supercritical case, such a portion of bifurcation curves is the starting point of so called "periodicity tongues", or *Arnol'd tongues,* associated with different rational rotation numbers m/n. A peculiar property of such tongues is associated with the summation rule [27]: Between any two tongues with rotation numbers m_1/n_1 and m_2/n_2 there is also a tongue associated with the rotation number $m'/n' = (m_1 + m_2)/(n_1 + n_2)$.

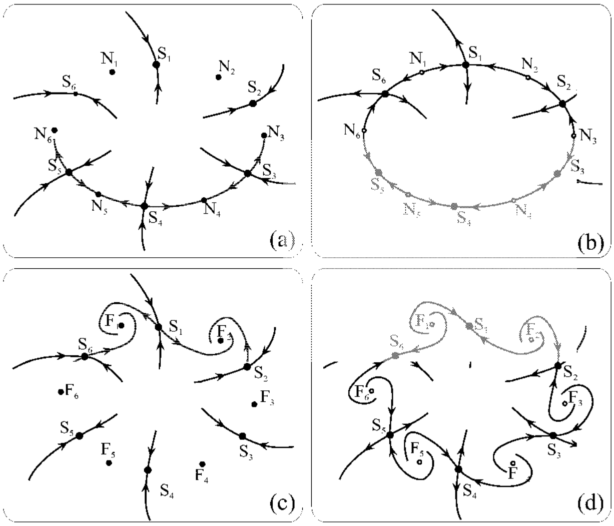

Figure 5: *Dynamics on a closed invariant curve* Γ: (a) *saddle-(stable) node connection,* (b) *saddle-(unstable) node connection,* (c) *saddle-(stable) focus connection,* (d) *saddle-(unstable) focus connection.*

It is clear that properties and bifurcations similar to those described above for a fixed point can occur also for a $k-$cycle of any period $k > 1$, simply considering the k periodic points as fixed points of the map T^k. In such a case the closed invariant curves Γ_k of the map T^k belong to a $k-$cyclical set for the map T. Several examples of bifurcation diagrams and invariant closed curves Γ (cyclical or not), with rational rotation numbers and saddle-connections or with quasiperiodic trajectories, will be shown in later chapters, associated with several business cycles models.

The dynamic evolution of Γ clearly depends on the nonlinearity of the map. Several examples will be given, both in piecewise linear maps (see the next chapter and Chapter 12) and in smooth maps (Chapters 8, 9, 11), together with a survey of possible mechanisms leading to the destruction of a closed curve. We only note here that the destruction may occur in two different ways: Either because the invariant closed curve Γ becomes no longer homeomorphic to a circle, or because the restriction of the map on Γ becomes no longer conjugate with a rigid rotation or an invertible map of the circle. The first case naturally occurs when the cycle node (stable or unstable) on Γ becomes a focus: Fig.5c-d qualitatively represent this case, together with a saddle-focus connection, which may be stable (Fig.5c) or unstable (Fig.5d).

We finally remark that when a pair of parameters are let to vary in a parameter plane outside the stability triangle, from the region close to a supercritical pitchfork (or flip) bifurcation curve towards the region where a supercritical Neimark-Sacker bifurcation occurs, then global bifurcations associated with (attracting and repelling) closed invariant curves must necessarily occur. Some of the mechanisms explaining such global bifurcations are described in the next sections.

1.4 Effects of Critical Curves on Invariant Closed Curves

In this section we consider the transformations of an invariant closed curve, born from a focus fixed point of a noninvertible map of the plane via a supercritical Neimark-Sacker bifurcation, as some parameter is gradually moved away from its bifurcation value. As stated in the previous section, just after the bifurcation an attracting invariant closed curve, say Γ, exists around the unstable focus. It is smooth and homeomorphic to a circle, with radius proportional to the square root of the distance from the bifurcation set in the parameter space (see e.g. [24], p.305).

The dynamics of the iterated map restricted to Γ is conjugate to a map of the circle, and may be characterized by an irrational or a rational rotation number. In the former case, the motion along Γ is non periodic (also called quasiperiodic) and the iterated points are densely distributed along the whole invariant curve, whereas in the latter case, if the rational rotation number has the form m/n, the motion is n-periodic, i.e., an attracting cycle of period n exists embedded into Γ, and the n periodic points are cyclically visited every m turnings around the unstable focus. The latter situation is observed when the parameters are chosen inside a m/n Arnold tongue. The whole curve Γ is covered by the iterated points only in the case of irrational rotation number, otherwise only the periodic points are visited by the asymptotic dynamics, so that it is difficult to see Γ numerically, when the period n is small, even if the closed invariant curve exists (given by the saddle-node connection).

However, the Neimark-Sacker bifurcation theorem only gives local results in the parameter space, in the sense that it says nothing about the changes in the shape, or even the existence, of the invariant curve, as the parameters move away from the bifurcation values. Indeed, the closed invariant curve may suddenly disappear, or drastically change its shape, or evolve into an annular chaotic attractor (a chaotic ring). In the case of a noninvertible map of the plane, important modifications of the shape and global properties of Γ occur due to the folding action of the critical curves.

In order to illustrate this point, let us consider an exemplary case, obtained by using the quadratic map $T:(x,y)\rightarrow(x',y')$ defined by

$$T:\begin{cases} x'=y \\ y'=y-\lambda x+x^2 \end{cases} \tag{5}$$

where λ is a real parameter (see [36] for a more detailed study of this map). Given x' and y', if we solve the algebraic system with respect to the unknowns x and y we obtain

$$T_1^{-1}:\begin{cases} x=\frac{\lambda}{2}-\sqrt{\frac{\lambda^2}{4}+y'-x'} \\ y=x' \end{cases};T_2^{-1}:\begin{cases} x=\frac{\lambda}{2}+\sqrt{\frac{\lambda^2}{4}+y'-x'} \\ y=x' \end{cases} \tag{6}$$

So, a point (x',y') has two distinct rank-1 preimages if $y'>(x'-\lambda^2/4)$, and no preimages if the reverse inequality holds. This means that the map (5) is a Z_0-Z_2 noninvertible map, where Z_0 (region whose points have no preimages) is the half plane $Z_0=\{(x,y)\,|\,y<x-\lambda^2/4\}$ and Z_2 (region whose points have two distinct rank-1 preimages) is the half plane $Z_2=\{(x,y)\,|\,y>x-\lambda^2/4\}$. The line $y=x-\lambda^2/4$, which separates these two regions, is the critical curve LC, i.e. the locus of points having two merging rank-1 preimages, located on the line $x_1=\lambda/2$, that represents LC_{-1}. Any point $(x,y)\in Z_2$ has the two rank-1 preimages symmetrically located at opposite sides with respect to LC_{-1}: $T_1^{-1}(x,y)\in R_1$ and $T_2^{-1}(x,y)\in R_2$, where R_1 is the region defined by $x<\lambda/2$ and R_2 is defined by $x>\lambda/2$. We notice that, being (5) a continuously differentiable map, the line LC_{-1} belongs to the set of points at which the Jacobian determinant vanishes, i.e. $LC_{-1}\subseteq J_0$, where $J_0=\{(x,y)\,|\,\det DT(x,y)=2x-\lambda=0\}$, and the critical curve LC is the image by T of LC_{-1}, i.e. $LC=T(LC_{-1})=T(\{x=\lambda/2\})=\{(x,y)\,|y=x-\lambda^2/4\}$.

The folding action related to the presence of the critical lines can be expressed by saying that the image of any region U separated by LC_{-1} into two portions, say $U_1 \in R_1$ and $U_2 \in R_2$, is folded along LC, in the sense that $T(U_1) \cap T(U_2)$ is a nonempty set included in Z_2. This means that two points $p \in U_1$ and $q \in U_2$, located at opposite sides with respect to LC_{-1}, are mapped in the same side with respect to LC, in the region Z_2. This can be equivalently expressed by stressing the "unfolding" action of T^{-1}, obtained by the application of the two distinct inverses in Z_2 which merge along LC. Indeed, if we consider a region $V \subset Z_2$, then the set of its rank-1 preimages $T_1^{-1}(V)$ and $T_2^{-1}(V)$ is made up of two regions $T_1^{-1}(V) \in R_1$ and $T_2^{-1}(V) \in R_2$, that are disjoint if $V \cap LC = \emptyset$ whereas they merge along LC_{-1} if $V \cap LC \neq \emptyset$.

The map (5) has two fixed points, $O = (0,0)$ and $P = (\lambda, \lambda)$. It is easy to see that O is stable for $0 < \lambda < 1$, and as λ is increased through the bifurcation value $\lambda = 1$ a supercritical Neimark-Sacker bifurcation occurs at which a stable invariant closed curve arises around the unstable focus O, as shown in Fig.6a, obtained for $\lambda = 1.02$. In the situation shown in Fig.6a the other fixed point, P, is a saddle, whose stable set constitutes the boundary that separates the basin of attraction of the closed invariant curve Γ (the white region) from the basin of diverging trajectories, also called basin of infinity (the grey region). Notice that in Fig.6a the invariant curve Γ appears to be smooth and of approximately circular shape, so that the quasi-periodic motion along it is very similar to purely trigonometric oscillations. It can also be noticed that Γ is entirely included in the region R_1, i.e. it has no intersections with LC_{-1}. It is important to remark that just after its creation Γ cannot be too close to LC_{-1}, because at the Neimark-Sacker bifurcation the eigenvalues are complex conjugate and belong to the unit circle of the complex plane, whereas along LC_{-1} one eigenvalue must necessarily be zero being $\det(DT) = 0$ along LC_{-1}. Therefore, intersections between Γ and LC_{-1} are only possible when the parameters are sufficiently far from the Neimark-Sacker bifurcation values.

We now describe the changes of the stable invariant closed curve Γ as the parameter λ is increased. Indeed, as far as the attracting invariant closed curve Γ does not intersect LC_{-1} it can be thought of as entirely contained in one sheet of the Riemann foliation. This means that a neighborhood $U(\Gamma)$ of Γ exists such that not only $T(U) \subset U$ (since Γ is attracting) but a unique inverse exists, say T_1^{-1}, such that $T_1^{-1} : T(U) \to U$. This implies that the curve Γ, as well as the area of the phase plane enclosed by Γ, say $a(\Gamma)$, is both forward invariant (under T) and backward invariant (under T_1^{-1}).

The situation changes when Γ grows up until it has a contact with the set of merging preimages LC_{-1}, and then intersects it, as shown in Fig.6b, obtained for $\lambda = 1.3$. We now describe the consequences of the contact between Γ and LC_{-1}.

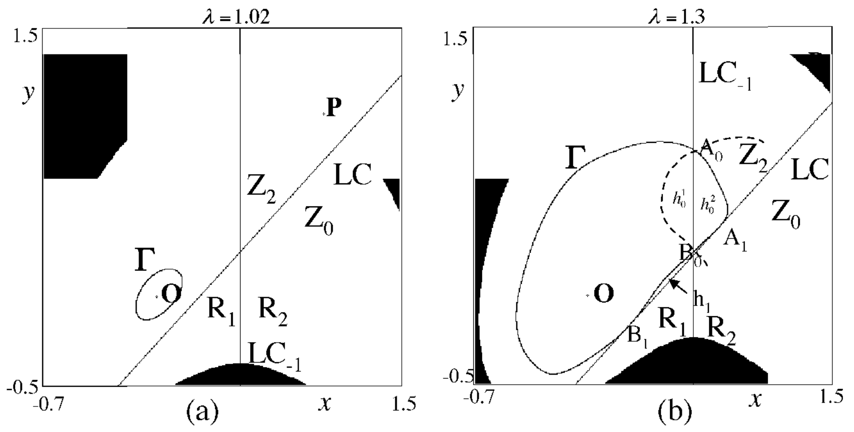

Figure 6: (a) *Just after the supercritical Neimark-Sacker bifurcation of the fixed point O a smooth attracting closed curve* Γ *appears.* (b) *Far away from the bifurcation value, the area inside* Γ *is no longer invariant.*

Let A_0 and B_0 be the two points of intersection between Γ and LC_{-1}, and let R_1 and R_2 the two regions, separated by LC_{-1}, giving the ranges of the two inverses T_1^{-1} and T_2^{-1}, respectively. Then the points $A_1 = T(A_0)$ and $B_1 = T(B_0)$, which must belong both to Γ and to $LC = T(LC_{-1})$, are points of tangential contact between Γ and LC. In fact, the arc $A_0B_0 = \Gamma \cap R_2$ must be mapped by T in the arc $A_1B_1 = T(A_0B_0)$, entirely included in the region Z_2, on one side of LC (i.e. on the side of region Z_2). If we look at the area $a(\Gamma)$, bounded by the invariant curve, it is easy to see that such an area is no longer invariant under application of T. In fact, $T_1^{-1}(A_1B_1)$ gives an arc inside $a(\Gamma)$ but not belonging to the invariant curve, while $A_0B_0 = \Gamma \cap R_2$ is given by $T_2^{-1}(A_1B_1)$. It means that the region h_1, located between the arc A_1B_1 of Γ and LC, is "unfolded" by the action of the two inverses T_1^{-1} and T_2^{-1} in two distinct preimages, located in the regions R_1 and R_2 respectively, represented in Fig.6b by the two portions $h_0^1 = T_1^{-1}(h_1)$ and $h_0^2 = T_2^{-1}(h_1)$ of $a(\Gamma)$ bounded by the two arcs A_0B_0 inside and along Γ respectively. In other words, the two portions h_0^1 and h_0^2 of $a(\Gamma)$ are both "folded" by T along LC *outside* the area $a(\Gamma)$ (as both cover the area h_1 which is outside Γ). This implies that the area $a(\Gamma)$, bounded by Γ, is no longer forward invariant (since some points inside Γ are mapped outside it, and are exactly the points belonging to h_0^1 and h_0^2).

This phenomenon of forward invariance of a closed curve, together with noninvariance of the area inside it, is specific to noninvertible discrete maps, that is, it cannot be observed neither in two-dimensional invertible maps nor in two-dimensional continuous dynamical systems. The property of noninvariance of $a(\Gamma)$ and the creation of convolutions of Γ are two aspects of the same mechanism, related to the fact that curves crossing LC_{-1} are folded along LC and are confined into the region with an higher number of preimages.

Another consequence of the intersection between Γ and LC_{-1} is that for a periodic cycle not belonging to Γ, it may happen that some of the periodic points are inside and the others are outside the invariant curve Γ. In the case of the map (5) this may be observed for example when $\lambda = 1.4014$, because a stable cycle of period 7 coexists with the stable invariant curve Γ (see Fig.7, where the seven periodic points of the stable cycle are labelled as C_1, ..., C_7). As it can be seen in the figure, the periodic point C_1, inside Γ in the region h_0^2, is mapped in the point $C_2 \in h_1$, i.e. outside $a(\Gamma)$.

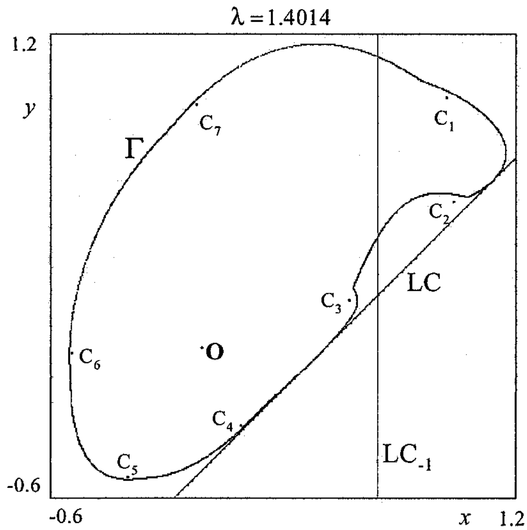

Figure 7: *The periodic point C_1, inside Γ in the region h_0^2, is mapped in the point $C_2 \in h_1$, i.e. outside $a(\Gamma)$.*

As the parameter λ is further increased, the convolutions become more and more pronounced and another phenomenon peculiar of noninvertible maps can be seen, that is the appearance of knots, or loops, or self intersections of the unstable set of the saddle belonging to the closed invariant curve, and such a dynamic situation is soon followed by homoclinic situations (intersections between the stable and unstable sets of the saddle) leading to a chaotic attractor, also called "weakly chaotic ring" in [36] for their particular shape. An example is given in Fig.8a obtained with $\lambda = 1.505$. As emphasized in the enlargement shown in Fig.8b, the attractor is no longer a closed invariant curve, as it includes loops and self-intersections. The mechanisms through which such loops and chaotic rings are created, and the related loss of invariance of Γ have been recently studied by many authors (see e.g. [36], [17] or [18] and references therein), and still have some open problems.

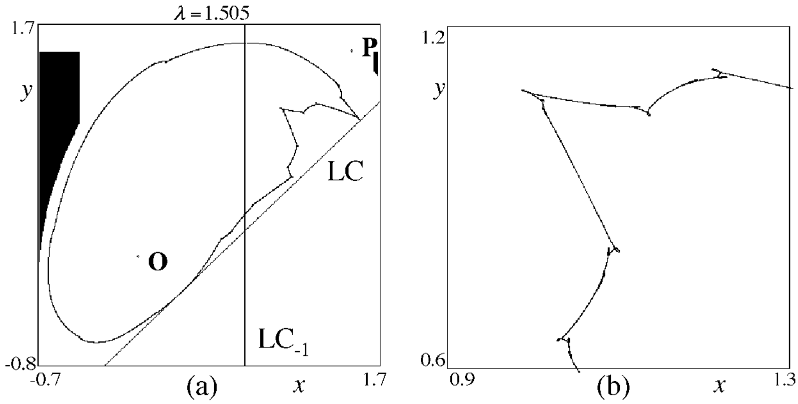

Figure 8: (a) *A "weakly chaotic ring" caused by some homoclinic bifurcation.* (b) *The enlargement shows the loops and the self-intersections of the attractors.*

As the parameter λ is further increased, so that it is more and more far from the Neimark-Sacker bifurcation value, a fully developed chaotic ring is created, like the one shown in Fig.9, obtained for $\lambda = 1.54$, on which the dynamics are characterized by chaotic time series that exhibit some particular time patterns, as shown in Fig.9b. It is worth to notice that in Fig.9a the attractor is very close to the boundary of the basin of diverging trajectories (gray points in the figure). This suggests that a further increase of λ will lead to a contact between the attractor and the boundary of its basin, and this represents a global bifurcation that marks the destruction of the attractor (more properly, it becomes a chaotic repellor after the contact). Such bifurcation is known as final bifurcation, or boundary crisis, and here corresponds to the first homoclinic bifurcation of the saddle fixed point P on the basin boundary. Indeed, its unstable set tends to the attractor while its stable set belongs to the frontier of the basin, thus a contact of the attracting set with the basin boundary also implies a contact between the stable and unstable sets of P. Of course, this contact between an invariant attracting set and its basin boundary may occur at the beginning of the story, i.e. soon after the creation of the closed invariant curve Γ. In other words, even if the Neimark-Sacker bifurcation theorem marks the appearance of Γ, it gives no indications about its survival as the parameters are moved away from their bifurcation values.

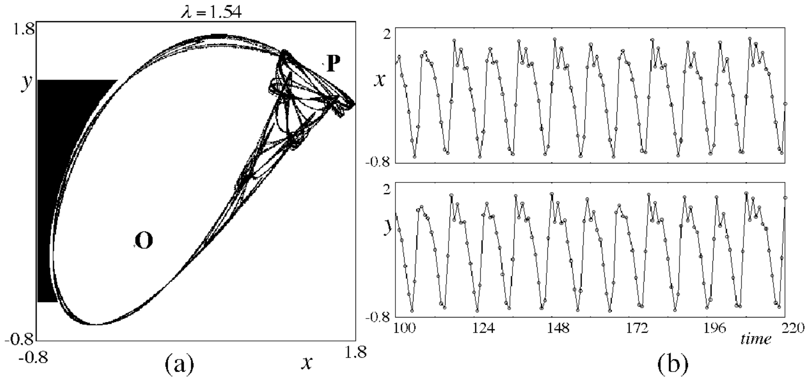

Figure 9: (a) *The fully developed chaotic ring.* (b) *The corresponding chaotic time series.*

To sum up, just after a supercritical Neimark-Sacker bifurcation, the long run dynamics of a discrete dynamical system is characterized by endogenous oscillations that may be quasiperiodic or periodic, converging towards a smooth and attracting closed curve Γ. Then, when the parameters move along a path away from the Neimark-Sacker bifurcation value, the closed invariant curve grows up, i.e. oscillations of increasing amplitude characterize the asymptotic dynamics. Such enlargement of Γ may lead to its disappearance or to some changes of its shape, due to the nonlinearities of the map. If the map is noninvertible, the intersections between Γ and LC_{-1} gives rise to convoluted shapes of the invariant curve, until it is replaced by an annular chaotic attractor.

As usual, sets of parameters are met at which stable cycles are created via a saddle-node bifurcation. The periodic points of these stable cycles may belong to Γ, or may be inside $a(\Gamma)$, or outside $a(\Gamma)$ or, if Γ intersects LC_{-1}, some of the periodic points may be inside and other outside $a(\Gamma)$. Furthermore, several coexisting attractors may be simultaneously present, such as coexisting attracting cycles or quasiperiodic or chaotic attractors together with attracting cycles.

An important property of noninvertible maps is that in any case, segments of the critical curves LC, together with a suitable number of their images $LC_i = T^i(LC)$, may be used to bound a trapping region where all the attracting sets are included. Such trapping sets, also called absorbing areas in [36], act like a bounded vessel inside which the asymptotic dynamics of the bounded trajectories are ultimately confined (see also [3], [12], [39]).

1.5 Invariant Closed Curves and Saddle Connections

In this section we present some global bifurcations involving invariant closed curves, which may be related to the appearance/disappearance of endogenous fluctuations, to qualitative changes in their amplitude and to complex structure in their basins of attraction. These bifurcations are related to the dynamic behavior of the stable and unstable sets of same saddle cycle, so they can be observed both with invertible and noninvertible maps. In the following we restrict our attention to (at least locally) invertible maps.

Before proceeding, it is worth to recall that the bifurcations related to invariant curves are well known in continuous dynamical systems, but in discrete models are still an open problem (see [32]): Here we give some qualitative results obtained by computer assisted proofs, with the awareness that further investigations need for a more complete understanding.

As already stated above, from a local point of view, in a nonlinear discrete map endogenous fluctuations naturally appear when a fixed point is destabilized through a supercritical Neimark-Sacker bifurcation: A stable focus becomes unstable and an attracting closed curve appears around it, becoming wider and wider when the parameters move away from the bifurcation value. Generally this local bifurcation has no global effect, in the sense that after the bifurcation the trajectories of points close to the unstable focus reach the attracting closed curve. However, some recent papers (see, among others, the endogenous business cycle models studied in [38] and in [31] or the cobweb model with predictor selection proposed in [14]) have stressed the importance of homoclinic tangencies and homoclinic tangles of saddles in the transition from local regular to global irregular fluctuations, due to increasing complexity of the attractors. Moreover, if the map T exhibits some multistability phenomena, then the invariant closed curve may interact with other attractors and interesting dynamic phenomena may occur, often associated with homoclinic or heteroclinic tangles.

Different, but still interesting, problems arise when the Neimark-Sacker bifurcation is of subcritical type, that is, when a repelling closed curve coexists with a stable focus, and generally such a repelling closed curve gives the boundary of the basin of attraction of the stable focus. Indeed, a subcritical bifurcation may be seen as a catastrophe phenomenon, in the sense that after its occurrence no attractors exist in the phase space or, if an attractor exists, it is quite far from the bifurcating fixed point. Instead, in the case of a supercritical Neimark-Sacker bifurcation, the phase portrait is completely different: The attracting closed curve which appears after the bifurcation is very small and close to the fixed point.

The dynamical behavior of a subcritical Neimark-Sacker bifurcation is very importand in the economic literature (as well as in other applied models). In fact, the existence of a repelling closed curve which bounds the basin of attraction of the stable fixed point implies that small shocks of the system have no effects on its dynamical behavior, while large enough shocks may lead to another attractor. This requires the coexistence of the fixed point with a different attracting set, and may cause *hysteresis* phenomena. Indeed, in such a case, if a parameter is varied so that a stable focus becomes unstable via a subcritical Neimark-Sacker bifurcation, i.e. a repelling curve shrinks and at the bifurcation merges with the fixed point, leaving a repelling focus, then the trajectories that start close to the fixed point reach the second attractor. In this case, a simple restoration of the previous value of the bifurcation parameter does not permit to move again the state of the system to the stable equilibrium, since the phase point is out of its basin. An example of this situation is the so called "*crater bifurcation*" scenario (see [30]): Two invariant closed curves, one repelling and one attracting, appear surrounding the fixed point when it is still stable. As the parameters move, the attracting closed curve moves away from the fixed point whereas the repelling one, which play the role of separatrix between the basins of attraction, shrinks merging with the fixed point in a subcritical Neimark-Sacker bifurcation. After such a bifurcation, the trajectories, previously converging to the fixed point, are converging to the attracting closed curve (which is quite far from the fixed point). The phase portrait so obtained (unstable focus and attracting closed curve) may suggest that a supercritical Neimark-Sacker bifurcation has occurred, but looking at the amplitude of the fluctuations we obtain the correct understanding of the bifurcation sequence giving rise to it.

When a Neimark-Sacker bifurcation of subcritical type occurs, it is also interesting to study the mechanism which gives rise to the appearance of the repelling closed curve, or to the two closed curves in the case of a crater bifurcation. Such occurrence may be related to the appearance of a pair of cycles (a saddle cycle and a repelling one) on the boundary of the basin of attraction of the fixed point. The heteroclinic connection of these cycles, formed by the stable set of the saddle cycle which comes from the periodic repelling points, constitutes a repelling closed curve. An example of this situation is given in [8]. Sometimes, for example when a crater bifurcation occurs, more complex situations are possible: We shall see that, as in the supercritical case, homoclinic tangencies and homoclinic tangles of saddles play an important role in the mechanism associated with the appearance/disappearance of closed invariant curves.

In continuous dynamical systems one of the mechanism associated with the appearance and disappearance of closed invariant curves involves a *saddle connection*: A branch of the stable set of a saddle point (or cycle) merges with a branch of the unstable one (of the same saddle or a different one), giving rise to an invariant closed curve.

When the involved saddle is a fixed point, the saddle connection can be due to the merging of one branch of the stable set and one of the unstable set, as in Fig.10a: We shall call such a situation *homoclinic loop*. Otherwise, if both the branches of the stable and unstable sets are involved in the saddle connection we obtain an eight-shaped structure that we shall call *double homoclinic loop* (see Fig.10b).

Homoclinic loops and double homoclinic loops can also involve a saddle cycle of period k, being related to the map T^k, but in this case we can also obtain an *heteroclinic loop*: Indeed, the map T^k exhibits k saddles points and a branch of the stable set of a saddle may merge with a branch of another periodic point of the saddle cycle. Stated in other words, if S_i, $i = 1, ..., k$, are the periodic points of the saddle cycle and $\alpha_{1,i} \cup \alpha_{2,i}$ ($\omega_{1,i} \cup \omega_{2,i}$) are the unstable (stable) sets of S_i, then a heteroclinic loop is given by the merging, for example, of the unstable branch $\alpha_{1,i}$ of S_i with the stable branch $\omega_{1,j}$ of a different periodic point S_j. Then each periodic point of the saddle cycle is connected with another one, and an invariant closed curve is so created that connects the periodic points of the saddle cycle. In Fig.10c an heteroclinic loop is shown, related to a pair of saddles (or a saddle cycle of period 2).

All these loops correspond to structurally unstable situations and cause a qualitative change in the dynamic behavior of the dynamical system. Since they cannot be predicted by a local investigation, i.e., a study of the linear approximation of the map, we classify them as global bifurcations. Indeed, we study this kind of bifurcation looking at the asymptotic behavior of the stable and unstable sets of the saddle: If a bifurcation associated with a loop has occurred, before and after the bifurcation the involved branch of the unstable set converges to different attracting sets, and the points of the involved stable branch have a different α-limit set, as well.

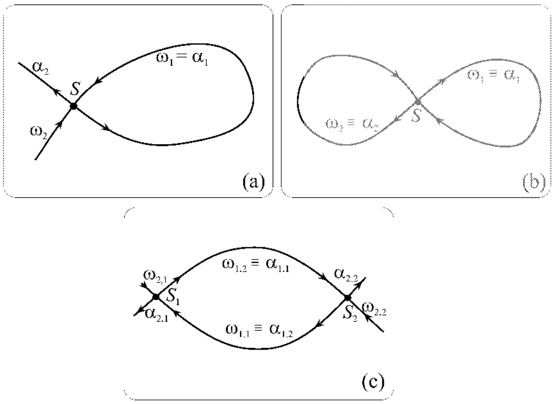

Figure 10: *Saddle connections*: (a) *homoclinic loop,* (b) *double homoclinic loop,* (c) *heteroclinic loop.*

Although homoclinic and heteroclinic loops may also occur in discrete dynamical systems, in this case they are frequently replaced by homoclinic tangles, as described in Section 1.2. That is, a tangency between the unstable branch $W_1^U(S) = \cup\alpha_{1,i}$ with the stable one $W_1^S(S) = \cup\omega_{1,i}$ occurs, followed by transverse crossings of the two manifolds, followed by another tangency of the same manifolds, but on opposite sides.

In the following we shall qualitatively describe some global bifurcations that involve closed invariant curves and may occur in the business cycle models. We first consider global bifurcations causing the appearance/disappearance of closed invariant curves, then the case in which at least a closed invariant curve coexists with some cycle and we shall see as these interact. All the global bifurcations here presented involve homoclinic connections of the periodic points of a saddle cycle.

1.6 Appearance of an Invariant Closed Curve (Homoclinic Loop)

In this section we show a mechanism which may cause the appearance of an invariant closed curve (or cyclical closed invariant curves), already known in the literature, see e.g. [35], [32], [2].

In the simplest starting situation, an attracting set A coexists with a saddle point S^* and a repelling fixed point P^*: A qualitative draft of the global bifurcation is given in Fig.11, where we assume that the attracting set A is a focus fixed point as well as P^*. Initially (see Fig.11a), the unstable set of the saddle converges to the attracting set A, and a branch of it, say α_1, turns around the repelling focus P^*. The α-limit set of the points of the branch ω_1 of the stable set of the saddle is the fixed point P^* and ω_2 comes from the boundary of the basin of attraction of A. After the bifurcation (Fig.11c), we have a bistability situation: The attracting set A coexists with an attracting closed curve Γ_s surrounding the repelling focus. The basins of attraction of A and Γ_s are separated by the stable set of the saddle point S^*. The attracting closed curve Γ_s is the ω-limit set of the points of the unstable branch α_1 and the stable branch ω_1 no longer exits from P^*, coming from the boundary of the set of the feasible trajectories (or the basin boundary of a different attracting set). The changes in the asymptotic behavior of the two branches suggest that the appearance of the curve Γ_s is due to a global bifurcation involving ω_1 and α_1. Indeed, we can conjecture that at the bifurcation the stable branch ω_1 and the unstable branch α_1 merge, giving rise to a *homoclinic loop*, as shown in Fig.11b, whose effect is to create a closed invariant curve. Obviously, this is a schematic representation of the mechanism involved, since we expect that, as usual with discrete maps, the single bifurcation value of the homoclinic loop is replaced by an interval of values associated with an homoclinic tangle between the two branches α_1 and ω_1, as shown in Fig.2: A tangency, followed by transverse crossing, that gives homoclinic points to the saddle S^*, followed by a second tangency between the same manifolds at which the transverse homoclinic points to S^* disappear.

The same mechanism may also give rise to a repelling closed curve Γ_u, but in such a case we start from the coexistence of at least two attractors, say an attracting set A, an attracting fixed point P^* and a saddle S^*, as in Fig.12a, where the attracting set A is a fixed point. The stable set of the saddle separates the basins of attraction of A and P^*. The branch ω_1 of $W^S(S^*)$ turns around P^*. The branch α_1 of the unstable set $W^U(S^*)$ tends to P^* whereas the ω-limit set of the points of the branch α_2 is the attracting set A. After the homoclinic loop, or homoclinic tangle, of the two branches

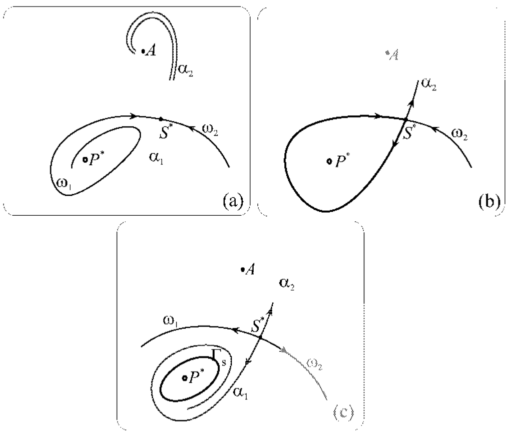

Figure 11: *Qualitative representation of a mechanism leading to the appearance of an attracting closed curve.*

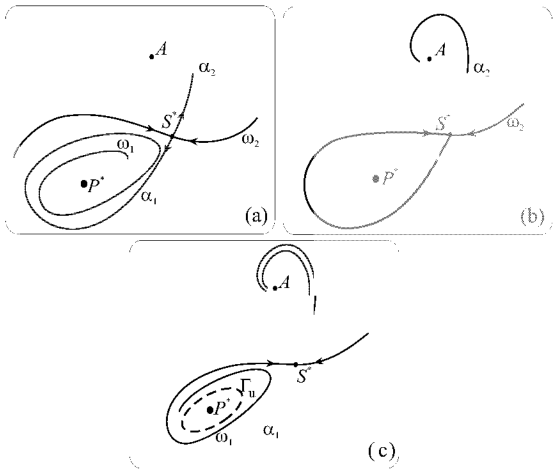

Figure 12: *Qualitative representation of a mechanism leading to a repelling closed curve.*

α_1 and ω_1, shown in Fig.12b, a repelling closed curve Γ_u appears, bounding the basin of attraction of P^* (see Fig.12c). Such a curve is the α-limit set of the points of the branch ω_1 of the set $W^S(S^*)$ and A is the ω-limit set of the points of the whole unstable set $W^U(S^*)$.

It is worth to observe that in the two cases considered above, the appearance of a closed invariant curve is due to a mechanism associated with a homoclinic loop, or tangle, and if the fixed points surrounded by the homoclinic loop is repelling (resp. attracting) then the closed curve which appears is attracting (resp. repelling). The case associated with the attracting fixed point P^* is also interesting because it may explain the appearance of the repelling closed curve involved in the Neimark-Sacker bifurcation of subcritical type.

Clearly the bifurcations described above may involve saddles and attracting or repelling cycles of period k ($k > 1$) instead of fixed points: In such a case the mechanisms previously described occur for the map T^k and lead to k cyclical invariant closed curves, repelling or attracting, for the map T.

1.7 Appearance of Two Invariant Closed Curves (Heteroclinic Loop)

In this section we describe the mechanism that may be associated with the appearance/disappearance of two disjoint invariant closed curves, one attracting and one repelling. This mechanism has been investigated also in [7] and [2], where it was associated with a Neimark-Sacker bifurcation of subcritical type.

It is know that when the map T depends on two parameters, two invariant curves can coexist if a bifurcation of codimension 2 occurs, called *Chenciner bifurcation* or *generalized Hopf bifurcation*; see [32] for mathematical details, and [22] for an application in economics. When such a bifurcation occurs, in the parameter space a curve exists crossing which an attracting closed curve, Γ_s, and a repelling one, Γ_u, appear very close one to each other. The way in which they appear suggests a "saddle-node" bifurcation for closed invariant curves, but it is well known that such a bifurcation, although usual in continuous flows, is an exceptional case in discrete time. Here we shall present a sequence of global bifurcations which give rise to Γ_s and Γ_u and involves two cycles, one of which is a saddle. We shall qualitatively describe this sequence when a saddle cycle and a focus cycle exist, since this is the case effectively observed in our study, and we shall conclude with a conjecture about the situation in which the focus cycle is replaced by a node cycle.

As in the previous section, we start from a situation, shown in Fig.13a, in which only an attracting set A exists (a stable focus in Fig.13a). Moreover, we assume that a pair of cycles of period k, a saddle S and a repelling focus C, exist: The emergence of these two cycles can be due to a standard saddle-node bifurcation, and then the node cycle turns into a focus. The stable

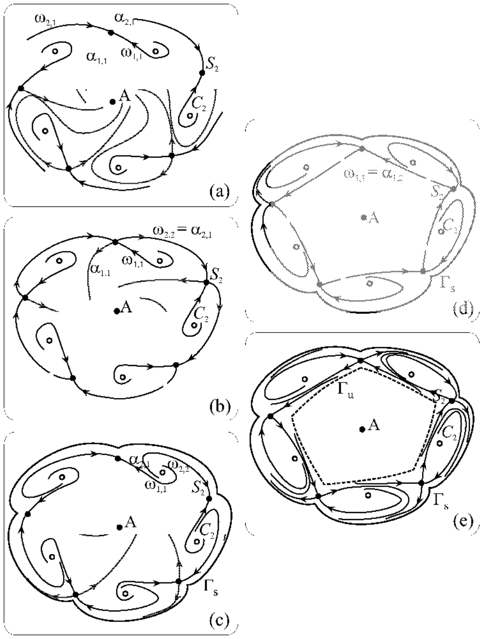

Figure 13: *Qualitative representation of a sequence of global bifurcations leading to the appearance of two closed invariant curves, one attracting and one repelling.*

set $W^S(S)$ of the saddle cycle is such that the outer branch $\omega_2 = \bigcup_{i=1}^{k} \omega_{2,i}$ comes from outside (the boundary of the set of feasible trajectories or from the basin boundary of coexisting attracting sets) whereas the α-limit set of the points of the inner one $\omega_1 = \bigcup_{i=1}^{k} \omega_{1,i}$ is the repelling focus C. The unstable set $W^U(S) = \bigcup_{i=1}^{k} (\alpha_{1,i} \cup \alpha_{2,i})$ reaches the attracting set A: Stated in other words, A is the ω-limit set of the points of the two branches $\alpha_{1,i}$ and $\alpha_{2,i}$, $i = 1, ..., k$. As the parameters are moved, the branches $\omega_2 = \bigcup_{i=1}^{k} \omega_{2,i}$ and $\alpha_2 = \bigcup_{i=1}^{k} \alpha_{2,i}$ are closer and closer and at the bifurcation they merge giving rise to a *heteroclinic loop* (see Fig.13b). More precisely, each stable branch $\omega_{2,i}$ of a periodic point of the saddle merges with the unstable branch $\alpha_{2,j}$ of a different periodic point of the same saddle cycle, giving rise to a closed connection among the periodic points of S. However, as already remarked, this transition may occur via a homoclinic tangle of $W_2^U(S)$ and $W_2^S(S)$, which includes a tangency between the two manifolds, followed by transverse crossings, and a tangency again of $W_2^U(S)$ and $W_2^S(S)$, as qualitatively shown in Fig.3.

After the bifurcation, originated by this structurally unstable situation, an attracting closed curve Γ_s exists as well as a saddle-focus connection made up by the stable set $W^S(S)$, surrounded by Γ_s (see Fig.13c). That a global bifurcation really occurred is proved by the changes in the asymptotic behaviors of the to branches involved in the heteroclinic loop, as it can be seen in the qualitative picture: After the bifurcation the stable set of the saddle constitutes a closed invariant curve (a repelling saddle-focus connection), which did not exist before the bifurcation, while the involved unstable branch of the saddle tends to A before the bifurcations and tends to the attracting closed curve Γ_s after. Thus *two invariant curves* exists after the bifurcation: An attracting one Γ_s and an unstable saddle-focus connection, and a multistability situation between the attracting set A and the closed curve Γ_s is created. Moreover, note that the unstable saddle-focus connection made up by the stable set of S, and connecting the periodic points of S and C, bounds the basin of attraction of A, and separates the two basins of attraction of A and Γ_s.

Such a bifurcation of the outer branches is often followed by a similar bifurcation of the inner ones. In fact, also the inner branches ω_1 of the stable set and α_1 of the unstable one approach each other (as some parameters are changed). At a new bifurcation, each stable branch $\omega_{1,i}$ of a periodic point of S merges with the unstable branch $\alpha_{1,j}$ of a different periodic point of the same saddle cycle, giving rise to a closed connection between the periodic points of S and the periodic points of the cycle C, shown in Fig.13d. The effect of this second *heteroclinic loop,* or more often homoclinic tangle, are shown in Fig.13e: A repelling closed curve Γ_u appears, replacing the saddle-focus connection (and replacing it in the role of separatrix between the basins of attraction of A and Γ_s). Once more, the occurrence of this global bifurcation can be checked observing the behavior of the branches α_1 and ω_1 involved in it.

Summarizing, we have seen that the coexistence of two closed invariant curves, one attracting and one repelling, in discrete maps can be achieved by a double mechanism: Starting from a repelling cycle and a saddle cycle, a first saddle connection (or tangle) causes the appearance of the attracting one associated with an (unstable) heteroclinic connection saddle - repelling cycle that plays the role of separatrix of basins, which is then replaced by the second closed curve, repelling, whose appearance is associated with a second saddle connection (or tangle).

The same mechanism can be observed starting with an attracting $k-$cycle (born together with a saddle), instead of a repelling one, i.e., a situation of bistability due to the coexistence of the attracting set A and a $k-$cycle C. In such a case the sequence of bifurcations takes place in a "reversed" way: First the appearance of a repelling closed curve Γ_u associated with a saddle-attracting cycle connection and then the appearance of an attracting closed curve, replacing the heteroclinic connection. We use the qualitative figure 14 to illustrate such a sequence. At the beginning, the attracting set A (a stable focus in Fig.14a) coexists with an attracting focus cycle C of period k, born as node cycle via saddle-node bifurcation together with a saddle cycle S of the same period. The stable set $W^S(S) = \bigcup_{i=1}^{k} (\omega_{1,i} \cup \omega_{2,i})$ of the saddle cycle separates the basins of attraction of the two attracting sets, A and the cycle C. The unstable set $W^U(S) = \bigcup_{i=1}^{k} (\alpha_{1,i} \cup \alpha_{2,i})$ reaches the attracting sets: More precisely, the outer branches $\alpha_{2,i}$ converge to the cycle C, whereas A is the ω-limit set of the points of the inner branches $\alpha_{1,i}$. Differently from the case previously analyzed, as some parameters are changed first the inner branches $\omega_1 = \bigcup_{i=1}^{k} \omega_{1,i}$ and $\alpha_1 = \bigcup_{i=1}^{k} \alpha_{1,i}$ approach each other, merging at the bifurcation so giving rise to a *heteroclinic loop*, (see Fig.14b), or heteroclinic tangle. This bifurcation gives rise to a repelling closed curve Γ_u (see Fig.14c) which is the α-limit set of the points of the branches $\omega_{1,i}$ of the stable set of the saddle S. Also the asymptotic behavior of the branches $\alpha_{1,i}$ is changed: Indeed with the branches $\alpha_{2,i}$ they give rise to a heteroclinic connection, reaching the periodic points of the attracting cycle C. The effect of this global bifurcation is a change in the basin of attraction of A: After the bifurcation it is bounded by the closed repelling curve Γ_u, so that it has been significantly reduced. Moreover another invariant closed curve exists, made up by the unstable set of the saddle S, which connects the points of the two $k-$cycles.

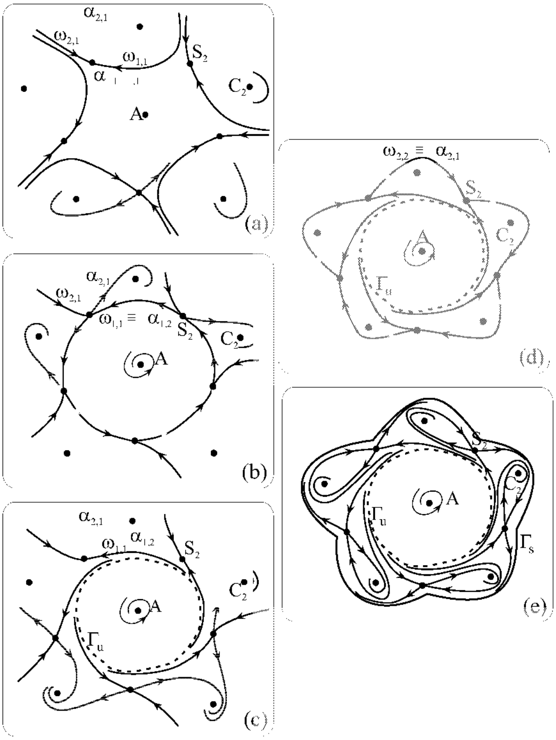

Figure 14: *Qualitative representation of a sequence of global bifurcations leading to the appearance of two repelling closed curves, one repelling and one attracting.*

Stronger effects on the dynamics are obtained after a second heteroclinic loop, made up by the merging of the outer branches, shown in Fig.14d. Indeed, after such a global bifurcation we obtain the coexistence of three attracting sets: The focus cycle C, the set A and an attracting closed curve Γ_s, whose appearance is associated with the heteroclinic loop, or tangle (see Fig.14e).

The repelling closed curve Γ_u bounds the basin of attraction of A; those of Γ_s and C are separated by the stable set of the saddle cycle S. The branches of the unstable set have as ω-limit set the closed curve Γ_s on one side, and the attracting cycle C on the other side.

We remark again that if the cycle involved in the global bifurcation together with the saddle is repelling (attracting) then the closed curve appearing after the first step is attracting (repelling), together with a repelling (attracting) saddle-connection. The second step involves the saddle-connection, after which two invariant closed curves still exist: We simply observe a change in their topological structure.

The global bifurcations arising when cycles and invariant closed curves coexist will be the topic of the next sections. Before that, let us observe that if the repelling (or attracting) focus, considered in our examples, is replaced by a repelling (or attracting) node, then the same sequence of bifurcations can occur and the two curves appear more close to each other. In Fig.15 a

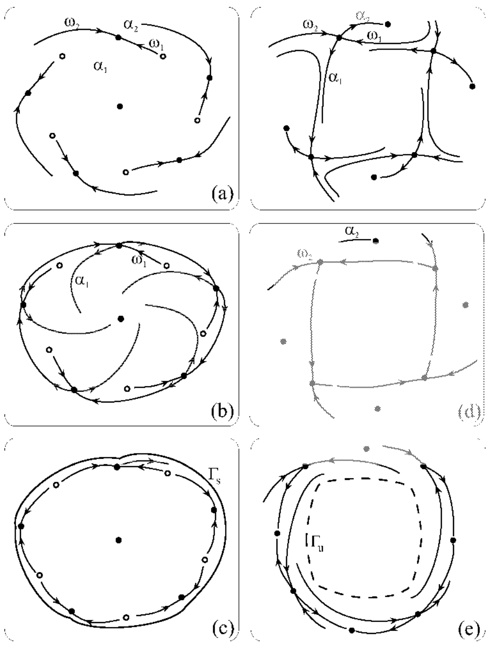

Figure 15: *Qualitative representation of a mechanism leading to two invariant closed curves associated either with a repelling node cycle* (a,b,c) *or an attracting node cycle* (d,e,f).

qualitative draft is given: Fig.15a-b-c refer to the repelling cycle whereas Fig.15d-e-f to the attracting one.

Moreover, if the node cycle is of very high period, then the saddle-node connection appearing at the first step looks like an invariant closed curve: In this case, the phase space recall in its shape that associated with a "saddle-node" bifurcation of invariant closed curves. It is for this reason that we propose this mechanism as a generic sequence of global bifurcations giving rise to two coexisting closed curves. More theoretical studies need to confirm such a conjecture.

1.8 Coexistence of Curves and Cycles and Their Interactions (Heteroclinic Loop)

In this section we show a mechanism that causes the transition from an attracting closed invariant curve, say Γ_a, with a pair of cycles of period k outside it, a saddle S and an attracting one, C, inside a wider attracting closed invariant curve, say Γ_b. This transition takes place via the occurrence of two heteroclinic loops of the saddle S, first with the merging of the unstable branches $W_1^U(S) = \cup\alpha_{1,i}$ and the stable ones $W_1^S(S) = \cup\omega_{1,i}$ and then via the merging of the unstable branches $W_2^U(S) = \cup\alpha_{2,i}$ and the stable ones $W_2^S(S) = \cup\omega_{2,i}$.

Similar bifurcation sequences have been observed in [4] and [5], associated with a two-dimensional map having a fixed point which may lose stability via a supercritical Neimark-Sacker bifurcation and a supercritical pitchfork or flip bifurcation. Examples in economic dynamic modelling can be found, for instance, among Kaldorian discrete-time models (see [11], [6]). Further examples are given in several chapters of this book.

Let us consider the situation described in Fig.16. In Fig.16a we have an attracting closed invariant curve Γ_a (which may also follow from the situation described in Fig.11-13), and a pair of cycles that have been created via a saddle-node bifurcation outside Γ_a. Such external cycles do not form an heteroclinic connection, whereas the stable set of the saddle S bounds the basin of attraction of the related attracting fixed points C_i of the map T^k. The unstable branches $\alpha_{1,i}$ of S_i tend to the attracting curve Γ_a, while the unstable branches $\alpha_{2,i}$ of S_i tend to the attracting cycle.

At the bifurcation (Fig.16b) we may have that the closed invariant curve Γ_a merges with the unstable branches $W_1^U(S) = \cup\alpha_{1,i}$ and with the stable ones $W_1^S(S) = \cup\omega_{1,i}$ as well, in a *heteroclinic loop*, or tangle, of the saddle S, causing the disappearance of the attracting closed invariant curve Γ_a, and leaving another closed invariant curve, see Fig.16c, which is now the heteroclinic connection involving the saddle S and the related attracting cycle C. After the bifurcation of the heteroclinic loop a closed curve still exists, but differently from Γ_a it includes the two cycles on it (Fig.16c).

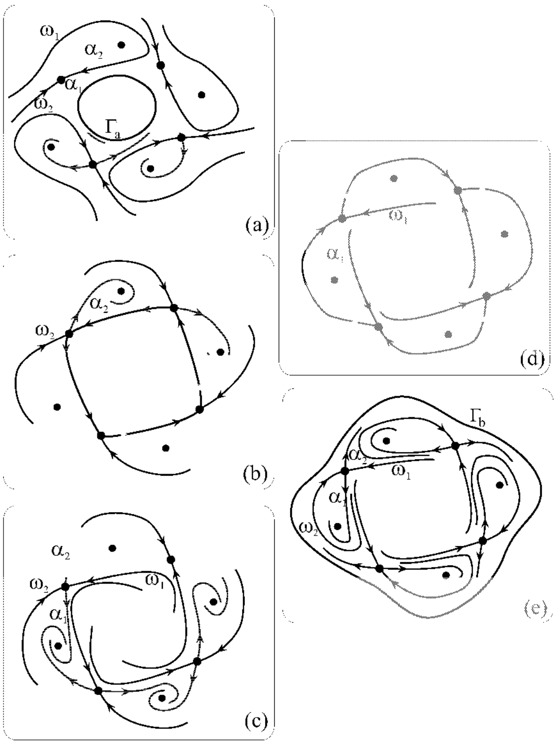

Figure 16: *Qualitative representation of a mechanism causing the transition from an attracting closed invariant curve into a wider one.*

Starting from this situation, a second heteroclinic loop (or tangle) may be formed. The heteroclinic connection turns into a heteroclinic loop in which the unstable branches $W_2^U(S) = \cup\alpha_{2,i}$ merge with the stable ones $W_2^S(S) = \cup\omega_{2,i}$ (see Fig.16d). After the bifurcation a new closed attracting curve exists, say Γ_b, and the two cycles are both inside Γ_b (Fig.16e). The stable set of the saddle S separates the basins of attraction of the k attracting fixed points C_i of the map T^k. The unstable branches $\cup\alpha_{1,i}$ tend to the attracting cycle while the unstable branches $\cup\alpha_{2,i}$ tend to Γ_b.

As mentioned before, in the case of discrete dynamical systems, the dynamic behaviors more frequently observed is such that the heteroclinic loop of Figs.16b-d are replaced by homoclinic tangles. That is, a tangency occurs between the two manifolds involved in the bifurcation, followed by transverse intersections and a tangency again on the opposite side, after which all the homoclinic points of the saddle S, existing during the tangle, are destroyed (several examples are shown in [4] and [5]).

It is worth noticing that all the unstable periodic points associated with the first homoclinic tangle, due to $W_1^U(S) \cap W_1^S(S) \neq \emptyset$, are in the region interior to the set of periodic points of the saddle S, whereas in the strange repellor associated with the second homoclinic tangle, in which $W_2^U(S) \cap W_2^S(S) \neq \emptyset$, all the unstable cycles are "outside" the saddle cycle S. The existence of a strange repellor has noticeable consequences with regard to the trajectories starting on the area occupied by it, since they are characterized by a long chaotic transient.

Notice also that before the first heteroclinic loop (tangle) of Fig.16 we have two distinct attracting sets: Γ_a and the stable $k-$cycle outside it; after the second one of Fig.16, we have again two distinct attractors: Γ_b, which is wider than Γ_a, and the $k-$cycle inside it, while between the two heteroclinic loops only one attractor may survive, that is the k-cycle.

It is plain that this process may be repeated many times. In fact, by a saddle-node bifurcation a new pair of cycles may appear outside Γ_b, so that we are again in the situation of Fig.16a, and the sequence of bifurcations described in Fig.16 may repeat.

We finally remark that the sequence of bifurcations here described, that cause the transition of a pair of cycles from outside to inside a closed invariant curve, may occur through different mechanisms when the map is noninvertible. In fact, in noninvertible maps the invariant curve may intersect the critical set LC_{-1}, and when this occurs the periodic points of a cycle may be part inside and part outside the closed invariant curve (see [36], [17]).

1.9 From an Invariant Closed Curve to Two Closed Curves (Double Homoclinic Loop)

The last case we consider in this chapter is an example of double homoclinic loop that involves a repelling closed curve Γ_u and a saddle point S. Two attracting sets, $A_i, i = 1, 2$, are also coexisting, or cyclical ones. The repelling closed invariant curve Γ_u surrounds the two attracting sets A_i and the saddle S. The stable set of S, $W^S(S)$, formed by the union of the preimages of any rank of the local stable set, turns around infinitely many times approaching the repelling curve Γ_u, as qualitatively shown in Fig.17a. $W^S(S)$ constitutes the boundary that separates the basins of A_1 and A_2. As the parameters are varied along the bifurcation path, the repelling closed invariant curve Γ_u shrinks in the proximity of the saddle S, and consequently the stable and unstable sets of the saddle approach each other, until Γ_u disappears or, more precisely, *becomes a chaotic repellor at the homoclinic tangency* (see Fig.17b) at which the unstable set of S, $W^U(S)$, has a contact with the sta-

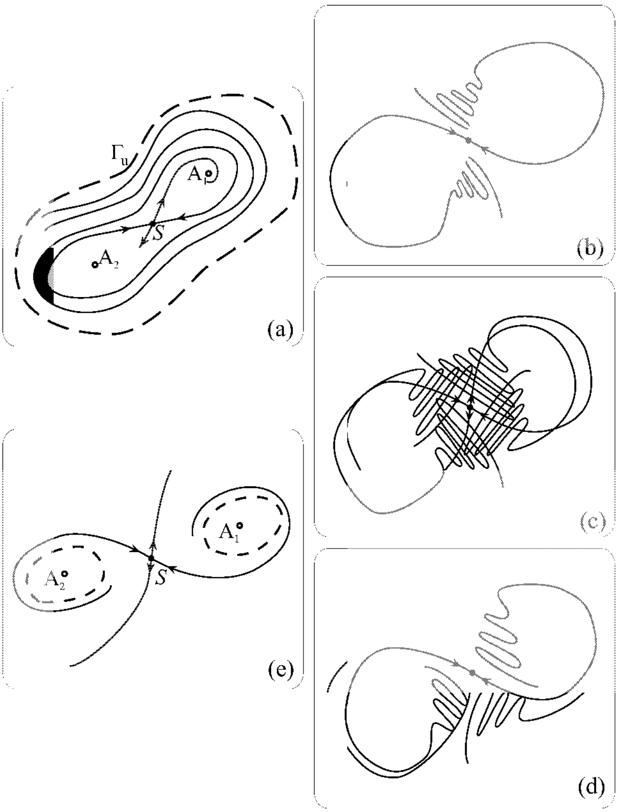

Figure 17: *Qualitative representation of a mechanism causing the transition from an invariant closed curve to two closed curves.*

ble one. This homoclinic tangency is followed by a transverse intersections of the two manifolds, $W^U(S)$ and $W^S(S)$, and a dynamic scenario like the one shown in Fig.17c is obtained, which is followed by another homoclinic

tangency (see Fig.17d) leading to the disappearance of all the homoclinic orbits of S and of the chaotic repellor. After this second tangency, $W^U(S)$ is completely outside of the stable set, so that the stable and unstable sets are again disjoint, $W^U(S) \cap W^S(S) = \emptyset$, and the preimages of the local stable manifolds reach *two disjoint closed invariant curves which have been created around the two attracting sets* A_i, see Fig.17e.
If the map is symmetric with respect to the saddle S then the homoclinic tangencies of the manifolds occur at the same time (an example of business cycle model leading to such a bifurcation can be found in Chapter 8). In the case of a map without symmetry properties, we still may have a transition from the situation of Fig.17(a) to that of Fig.17(e), but the two homoclinic loops may occur separately, that is, first the manifolds $W_1^U(S)$ and $W_1^S(S)$ are involved and then $W_2^U(S)$ and $W_2^S(S)$, or vice-versa (an example of business cycle model leading to such a bifurcation can be seen in Chapter 11).

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