10 'Floors' and/or 'Ceilings' and the Persistence of Business Cycles

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10.1 Introduction

In chapters 3, 6, 7 and 12 of this book and in a number of other recent contributions (e.g. Gallegati *et al.* 2003) the dynamics of Hicks' (1950) discrete-time multiplier-accelerator model has been analysed in depth and the role played by 'floors' and/or 'ceilings' clarified. In this chapter we intend to tackle the same problem with reference to models formulated in continuos-time, in particular Goodwin's (1951) model of the interaction between the dynamic multiplier and the nonlinear accelerator. Of course, the choice of this model is not casual; rather, it is dictated by a number of factors. First, although Goodwin's article was published in January 1951, in the first issue of volume 19 of *Econometrica*, it was certainly already in 'incubation' and its main idea 'in the air' some years earlier. This is testified to by the fact that the paper (with the provisional title "The business cycle as a self-sustaining oscillation") had already been presented by Goodwin at the American Winter Meeting of the Econometric Society held in Cleveland, Ohio on 27-30 December 1948 (for a summary, see Goodwin 1949). Moreover, from what we read in a footnote contained in the second page of the published version of the paper, we can infer that Goodwin became aware of Hicks' contribution only at the end of his research on the topic, when he was making the final revision of his paper. In short, we can consider it as contemporaneous, if not antecedent, to Hicks' contribution. The second reason for our choice is that, at the same time as Duesenberry (1950), Goodwin (1950) also promptly wrote an important and well known review of Hicks' book. In it, as in Duesenberry's review, it is clearly stated and explained that "either the 'ceiling' or the 'floor' will suffice" (Goodwin 1950, p. 319) in order to maintain and perpetuate the cycle. The final reason for our choice is the fact that in the lit-

erature (e.g., Le Corbeiller 1958, 1960, de Figueiredo 1958, Sasakura 1996, Velupillai 1990, 1991, 1998, 2004), when reference is made to the 'Goodwin oscillator' (or 'oscillator with a Goodwin characteristic' or 'two-stroke oscillator' or 'one-sided oscillator' or 'two-straight-line oscillator') what is meant is the discovery by Goodwin of an oscillator capable of generating persistent fluctuations with only one barrier.

The analysis that follows is an attempt to clarify some of the issues raised in this literature. In order to prepare the ground for this, the next section is devoted to a concise presentation of the model.

10.2 The Multiplier-Nonlinear Accelerator Interaction

The final equation of Goodwin's (1951) model is the following second order differential equation with a forcing term:

$$\varepsilon\theta\ddot{y} + \{[\varepsilon + (1-\alpha)\,\theta]\,\dot{y} - \phi\,(\dot{y})\} + (1-\alpha)\,y = O_A\,(t+\theta) \qquad (1)$$

where y is income, $\phi\,(\dot{y})$ induced investment, $O_A\,(\cdot)$ the sum of the autonomous components of consumption $\beta\,(\cdot)$ and investment $l\,(\cdot)$, $0 < \alpha < 1$ the marginal propensity to consume, ε, $\theta > 0$ the time-lag of the dynamic multiplier and the time-lag between investment decisions and the resulting outlays respectively and where $\dot{y} = dy/dt$, $\ddot{y} = d^2y/dt^2$.

Goodwin arrives at equation (1) by means of a 'step by step' procedure that has the purpose of removing one by one the unrealistic aspects of the simple multiplier and accelerator principle. Leaving out the first step, which leads to a rather crude model in which there is either investment at the maximum rate allowed by the existing productive capacity ($\dot{k}^* > 0$) or disinvestment at the maximum rate allowed by not replacing capital goods which are being scrapped for depreciation ($\dot{k}^{**} < 0$),[1] such a procedure can be described as follows.

First, the instantaneous multiplier is replaced by the dynamic multiplier (Goodwin 1951, p. 9):[2]

$$y = c + \dot{k} - \varepsilon\dot{y} = \beta\,(t) + \alpha y + \dot{k} - \varepsilon\dot{y} \qquad (2)$$

where $c = \beta\,(t) + \alpha y$ is consumption, k the capital stock and $\dot{k} = dk/dt$, net investment.

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[1]This is the only version of the model that is usually considered in textbooks. See, for example, Gandolfo (1997, pp. 464-465) and Gabisch and Lorenz (1989, pp. 118-122).

[2]See also Goodwin (1948), where this continuous-time formulation of the dynamic multiplier was first introduced.

Second, net investment is assumed to consist of an autonomous component $l(t)$ and an induced component $\phi(\dot{y})$:

$$\dot{k} = l(t) + \phi(\dot{y}) \tag{3}$$

The latter, in its turn, is assumed to be determined by the nonlinear accelerator, such that the simple acceleration principle (with an acceleration coefficient equal to $v > 0$) holds only over some middle range but passes to complete inflexibility at either extremity $\dot{k}^*$ and $\dot{k}^{**}$.

Fig.1 illustrates the case considered by Goodwin (1951, p. 7), namely, the case of an asymmetric nonlinear accelerator with $|\dot{k}^*| > |\dot{k}^{**}|$.
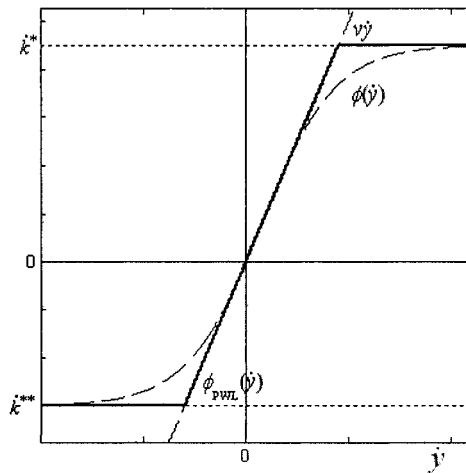

Figure 1: *The nonlinear asymmetric accelerator with $|\dot{k}^*| > |\dot{k}^{**}|$.*

As shown in the figure, $\phi(\dot{y})$ is well approximated by the following piecewise linear investment function:

$$\phi_{PWL}(\dot{y}) = \begin{cases} \dot{k}^* & \text{if } \dot{y} > \dot{k}^*/v \\ v\dot{y} & \text{if } \dot{k}^{**}/v \leq \dot{y} \leq \dot{k}^*/v \\ \dot{k}^{**} & \text{if } \dot{y} < \dot{k}^{**}/v \end{cases} \tag{4}$$

that, for simplicity, is used in all the simulations that follow.

Finally, the time-lag θ between decisions to invest and the corresponding outlays is taken into account (Goodwin 1951, pp. 11-12), such that (3) becomes:

$$\dot{k}(t) = l(t) + \phi(\dot{y}(t - \theta)) \tag{5}$$

Introducing (5) into (2) and rearranging, we obtain:

$$\varepsilon\dot{y}(t + \theta) + (1 - \alpha) y(t + \theta) - \phi(\dot{y}(t)) = O_A(t + \theta)$$

from which, expanding the two leading terms $\dot{y}(t + \theta)$ and $y(t + \theta)$ in a Taylor series and dropping all but the first two terms in each, (1) is readily obtained.

It is not too difficult to understand that the whole 'step by step' procedure, including the final approximation, is simply tantamount to assuming that (2) holds and that induced investment adjusts to its desired level, given by the nonlinear accelerator, with a time-lag of length equal to θ:

$$\dot{k} - l(t) = \phi(\dot{y}) - \theta\frac{d}{dt}\left[\dot{k} - l(t)\right] \tag{6}$$

Combining (6) in (2), we get:

$$\varepsilon\theta\ddot{y} + \{[\varepsilon + (1 - \alpha)\theta]\dot{y} - \phi(\dot{y})\} + (1 - \alpha) y = \left(\theta\frac{d}{dt} + 1\right) O_A(t) \tag{7}$$

that, as is easy to check, is nothing other than the original final equation (1) of Goodwin's model in the case in which the expansion in Taylor series and approximation is applied also to the leading term $O_A(t + \theta)$.

The subsequent analysis in Goodwin's article is based on the simplifying assumption that all autonomous expenditures are constant, so that[3]

$$O_A(t) = O_A^*, \forall t$$

Thus, equation (7), using the new variable $z = y - O_A^*/(1 - \alpha)$, can be written in terms of deviations from equilibrium as:

$$\varepsilon\theta\ddot{z} + \{[\varepsilon + (1 - \alpha)\theta]\dot{z} - \phi(\dot{z})\} + (1 - \alpha) z = 0$$

or, choosing the time-unit so as to have $\theta = 1$:

$$\ddot{z} + \frac{1}{\varepsilon}[(\varepsilon + s)\dot{z} - \phi(\dot{z})] + \frac{s}{\varepsilon}z = 0 \tag{8}$$

where $0 < s = 1 - \alpha < 1$ is the marginal propensity to save.

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[3]Clearly, in this case equations (1) and (7) are identical.

Equation (8) is a generalisation of a well known differential equation which, in physical applications, goes under the name of *Lord Raileigh equation* (see, for example, Le Corbeiller 1936, 1960). Using the terminology introduced by Le Corbeiller (1960), given any variable x, a *Lord Raileigh-Type* (LRT), is a differential equation of the type:

$$\ddot{x} + F(\dot{x}) + x = 0 \tag{9}$$

where the characteristic function (or, 'characteristic') $F(\dot{x})$ is such that (9) has a unique periodic solution (limit cycle). Thus, equation (8) – when $v > \varepsilon + s$ and, as a consequence, the origin is locally unstable – is a LRT equation with characteristic equal to:[4]

$$F(\dot{z}) = -\frac{1}{\varepsilon}[\phi(\dot{z}) - (\varepsilon + s)\dot{z}]$$

We then know (see Appendix 1) that, for given ε and s, (8) is either a so-called *two-stroke* or a *four-stroke* oscillator, depending on the degree of asymmetry of the investment function.

To appreciate fully and clarify the meaning of this, it is useful first to consider the case of a symmetric investment function, such that $|\dot{k}^*| = |\dot{k}^{**}|$. As shown in Fig.2(*ii*), in this case the model generates a symmetric limit cycle. Using the analogy with a physical system (see Le Corbeiller 1936 and Appendix 1 below), we can then describe it by saying that, over a full cycle, the total energy stored in the system increases along the arcs 2-3 and 4-1 and decreases along the arcs 1-2 and 3-4: this is exactly what is meant by the expression '*four*-stroke oscillator'.

This case, however, does not appear to be worth considering any further given that it implies a very unrealistic cycle such that, in an oscillation from peak to peak, the recession and the expansion are specularly identical and exactly of the same length (see Fig.2(*iii*)). Thus, it misses one of the main advantages of nonlinear modelling listed by Goodwin in his path-breaking

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[4]Strictly speaking, (8) becomes a LRT equation only after it has been reduced to a dimensionless form (see Goodwin 1951, pp. 12-13). To avoid this complication, for illustrative purposes only, we assume that $s = \varepsilon$ in all simulations that follow. This assumption is unnecessary for the generation of the limit cycle, as long as $s + \varepsilon < v$. However, as we will see, it has the advantage of making straightforward the interpretation of the condition for a limit cycle in terms of exchanges of energy.

contribution, namely the capability of making "the depression as different from the boom as we wish" (Goodwin 1951, p. 4).[5]

To make sure that the model has this capability, we need an asymmetric accelerator such as, for example, the one drawn by Goodwin in his paper and that (choosing $\dot{k}^*$ and $\dot{k}^{**}$ arbitrarily such that $|\dot{k}^*| > |\dot{k}^{**}|$) we have reproduced in Fig.1 above. Leaving unchanged all the remaining parameters we used to generate the symmetric cycle of Fig.2, it is not too difficult to understand what are the consequences of this change.
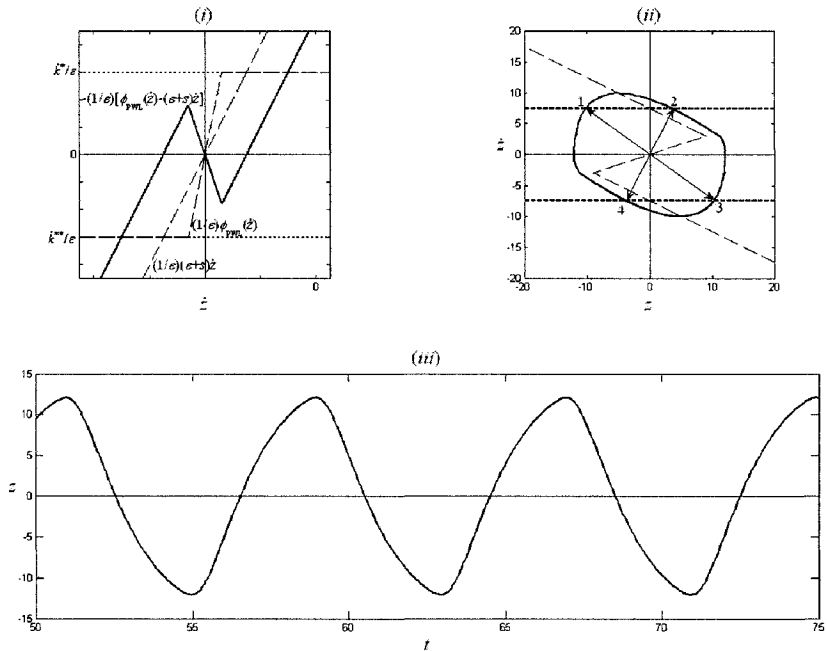

Figure 2: *The Goodwin symmetric (four-stroke) oscillator: (i) the charac-teristic function, (ii) the limit cycle in the phase plane and (iii) the periodic solution for $z(t)$ ($\varepsilon = 0.4$, $s = 0.4$, $v = 2$, $\dot{k}^* = 6$, $\dot{k}^{**} = -6$).*

As shown in Fig.3, the symmetric oscillator of Fig.2 becomes asymmetric and such that, over a cycle from peak to peak, the positive deviations of

[5]If, going on with the quotation, we accept Goodwin's view that such a capability is "one way of assessing the degree of nonlinearity" of a model, we can conclude that the model with a symmetric nonlinear accelerator has the lowest possible degree of nonlinearity.

national income from its equilibrium level are much larger than the negative ones. Moreover, the expansion is a smaller fraction of the period than the recession is.

This is due to the fact that, as shown in Fig.3(i), the 'ceiling' to investment spending is much less restrictive than the 'floor'. It can even happen, as with the values for $\dot{k}^*$ and $\dot{k}^{**}$ we have used in the simulation, that the 'ceiling' never becomes effective over the cycle (see Fig.3(ii)). In this case, the final dynamic equation of the model is said to be a *two-stroke* oscillator, namely, an oscillator such that the total energy stored in the system varies from a maximum to a minimum value (along the arc 1-2) and back again (along the arc 2-1) only once per cycle.[6]
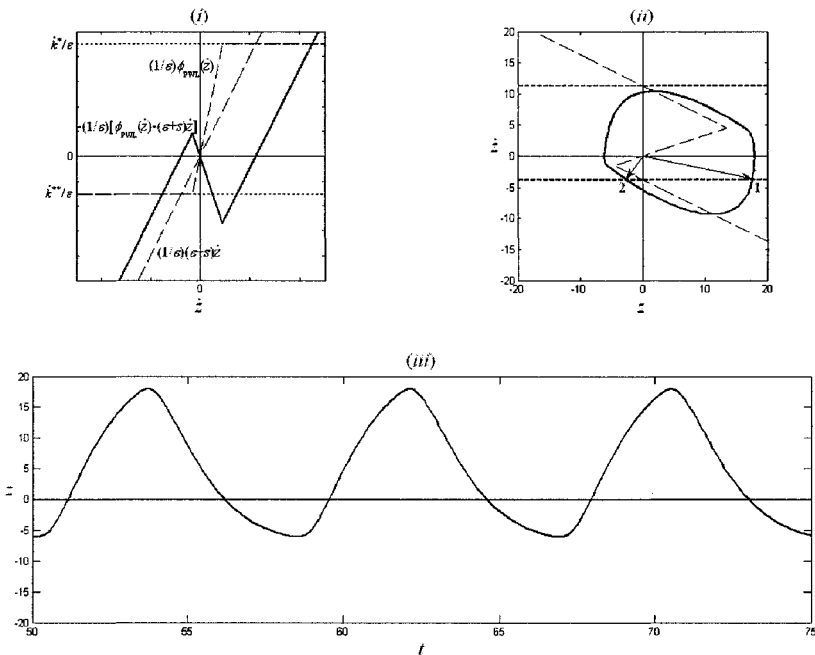

Figure 3: *The Goodwin asymmetric (two-stroke) oscillator with* $|\dot{k}^*| > |\dot{k}^{**}|$ $(\varepsilon = 0.4, s = 0.4, v = 2, \dot{k}^* = 9, \dot{k}^{**} = -3)$.

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[6]The transition from the one to the other type of oscillator as one of the parameters is varied is discussed in Appendix 1 where, for illustrative purposes, following Le Corbeiller (1960) and de Figueiredo (1958), a LRT equation with a cubic characteristic is considered.

For the sake of completeness, it should be noted that the result is exactly the opposite when we take $\dot{k}^* \neq \dot{k}^{**}$ such that $|\dot{k}^*| < |\dot{k}^{**}|$. As shown in Fig.4, in this case the final dynamic equation of the model is still a two-stroke oscillator, but now such that, over the stable limit cycle, the negative deviations from equilibrium dominate the positive ones. Moreover, it can happen, as in the case shown in the figure, that the 'floor' – rather than the 'ceiling' – never becomes effective over a cycle.
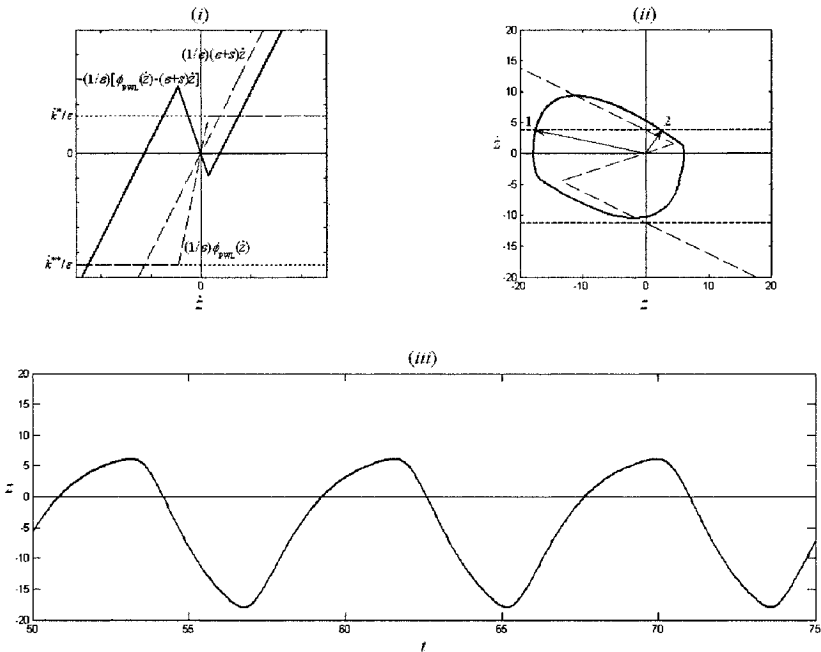

Figure 4: *The Goodwin asymmetric (two-stroke) oscillator with* $|\dot{k}^*| < |\dot{k}^{**}|$ *($\varepsilon = 0.4$, $s = 0.4$, $v = 2$, $\dot{k}^* = 3$, $\dot{k}^{**} = -9$).*

In summary, we have confirmation of the claim made by both Goodwin (1950) and Duesenberry (1950) in their reviews of Hicks' (1950) book on the trade cycle, that either the 'ceiling' or the 'floor' may suffice to perpetuate the cycle.

Under the qualifications which will be given in the next section (see also Appendix 2), the two functions $F(\dot{z})$ pictured in Figs.3(i) and 4(i) are examples of the so-called '*Goodwin characteristic*'.

10.3 The 'Goodwin Characteristic' and the Persistence of the Cycle

The analysis we have developed in the previous section allows us to clarify some issues raised in the recent literature on the topic. To do so, a good starting point is the recent contribution by Sasakura (1996), where the existence of a unique stable limit cycle in Goodwin's model (for the general case of asymmetric nonlinearity of the investment function) is rigorously proved. In particular, it is interesting that, in a footnote at the very end of his paper – as if it were a secondary aspect (but it is not!) – Sasakura (1996, p. 1171) notes that the limit cycle in Goodwin's model is a two-stroke oscillator such that:

> ... it is the floor of investment that is essential to the persistence of business cycles. Goodwin (1982, pp. viii-ix) attaches importance to the ceiling, but his model works as an endogenous business cycle model without it (the Goodwin characteristic!)

Now, as we have seen, it is certainly true that, when $|\dot{k}^*| > |\dot{k}^{**}|$, depending on the relative values of $|\dot{k}^*|$ and $|\dot{k}^{**}|$, it may happen that the model works as an endogenous business cycle model without the 'ceiling'. However, on the basis of the analysis we have developed in the previous section, we know that this, as such, it is not an intrinsic feature of Goodwin's model. As we have shown, when $|\dot{k}^*| < |\dot{k}^{**}|$, exactly the opposite may happen, i.e., the model may endogenously generate a persistent cycle without the 'floor'. Thus, a more appropriate conclusion is to say that, although in his 1951 paper Goodwin considered the case in which $|\dot{k}^*| > |\dot{k}^{**}|$, it is the opposite case, in which $|\dot{k}^*| < |\dot{k}^{**}|$, that is closer to the view expressed by Goodwin (1982) and later fully worked out by him in the first part – on "Macrodynamics" – of Goodwin and Punzo (1987).

The fact is that, if we try to reconstruct the 'genesis' of the 'Goodwin characteristic',[7] we are in a position to arrive at an even more stunning conclusion.

Thanks to Le Corbeiller's (1958) reconstruction, we know that all the debate about the distinction between two- and four-stroke oscillators (and the related concept of the 'Goodwin characteristic') started in December 1950 or thereabouts, i.e., just a couple of weeks before Goodwin's (1951) paper (*with a 'floor' to investment more restrictive than the 'ceiling'*) was actually

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[7]Some useful information in this regard is contained in various contributions by Velupillai (e.g., 1990, 1991, 1998 and 2004).

published. All started when Goodwin, at that time still at Harvard University, went to see Le Corbeiller in his office "in great elation" and showed him that an equation such as (9) with *a characteristic F ($\dot{x}$) made up of two straight lines* could have a periodic solution. That a characteristic of that type could generate a limit cycle was thought to be impossible at that time and this is the reason why Le Corbeiller named this kind of characteristic the 'Goodwin characteristic'. By 1960, thanks to de Figueiredo's Ph.D. thesis, discussed at Harvard in 1958, and to Le Corbeiller's article on "Two-Stroke Oscillators", the theory of this new type of oscillator had been fully developed.

It is not too difficult to understand how the multiplier-nonlinear accelerator interaction we have discussed in the previous section can originate a 'Goodwin characteristic' as rigorously defined in Le Corbeiller (1960). To do so, we must again consider separately the two basically different cases that may arise in the model, according to whether the 'ceiling' is more restrictive of the 'floor' or viceversa.
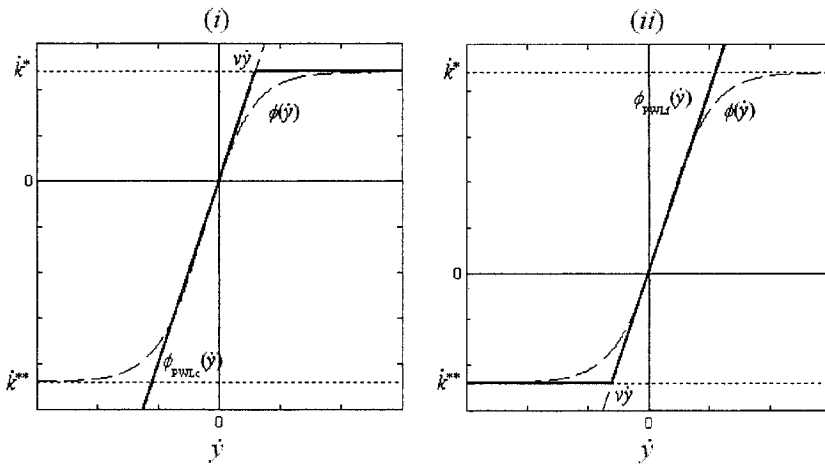

Figure 5: *The piecewise linear accelerator (i) with only the 'ceiling' and (ii) with only the 'floor'.*

As we have seen in the previous section, when the 'floor' is sufficiently less restrictive than the 'ceiling', it may happen that the former does not play any role in the generation of the stable limit cycle. When this happens, we can safely disregarded the 'floor' altogether and assume that the piecewise

linear investment function is such that the linear accelerator always holds until the 'ceiling' is reached, after which investment remains at the maximum rate allowed by the existing productive capacity (see Fig. 5(i)); analytically:

$$\phi_{PWL}(\dot{y}) \approx \phi_{PWLc}(\dot{y}) = \begin{cases} v\dot{y} & \text{for } \dot{y} \leq \dot{k}^*/v \\ \dot{k}^* & \text{for } \dot{y} > \dot{k}^*/v \end{cases} \qquad (10)$$

Introducing (10) into (9), we get a LRT equation with the following characteristic (see Fig.6(i)):

$$F(\dot{z}) = \begin{cases} -(1/\varepsilon)\left[v - (\varepsilon + s)\right]\dot{z} & \text{for } \dot{z} \leq \dot{k}^*/v \\ -(1/\varepsilon)\left[\dot{k}^* - (\varepsilon + s)\dot{z}\right] & \text{for } \dot{z} > \dot{k}^*/v \end{cases} \qquad (11)$$

which is indeed a 'Goodwin characteristic', very similar to the one drawn by hand by Le Corbeiller in his 1958 letter to Goodwin with the intention to reproduce the one drawn by Goodwin at the blackboard during their 1950 meeting.
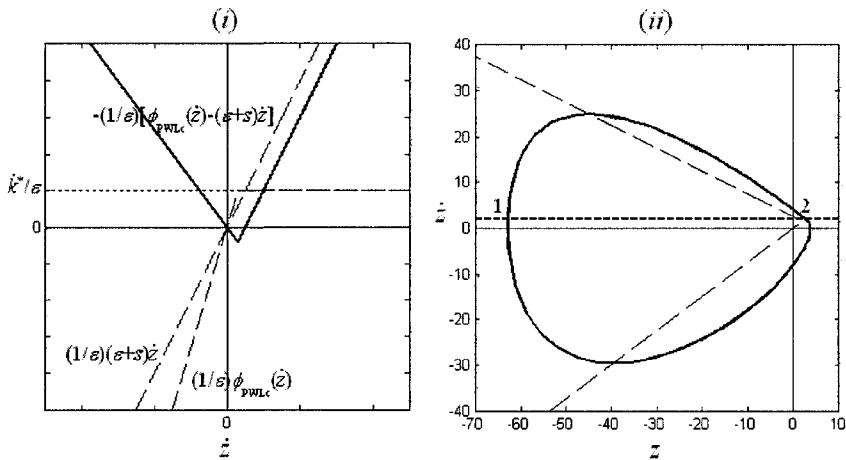

Figure 6: *The Goodwin piecewise linear (two-stroke) oscillator with only the 'ceiling'* $(\varepsilon = 0.6, s = 0.6, v = 2, \dot{k}^* = 3)$.

In order to draw conclusions about the dynamics of the model when (11) is considered, we must study equation (9) separately for each regime (cfr. Appendix 2). In the first regime, where $\dot{z} \leq \dot{k}^*/v$, equation (9) becomes:

$$\ddot{z} - \frac{1}{\varepsilon}\left[v - (\varepsilon + s)\right]\dot{z} + \frac{s}{\varepsilon}z = 0 \tag{12}$$

with characteristic equation given by

$$\lambda^2 - \frac{1}{\varepsilon}\left[v - (\varepsilon + s)\right]\lambda + \frac{s}{\varepsilon}z = 0$$

and eigenvalues

$$\lambda_{1,2} = \frac{1}{2\varepsilon}\left\{v - (\varepsilon + s) \pm \sqrt{\left[v - (\varepsilon + s)\right]^2 - 4s\varepsilon}\right\}$$

where, by assumption, $v > \varepsilon + s$. As a consequence, the unique equilibrium $z = 0$ is either an unstable node when:

$$v > \left(\sqrt{\varepsilon} + \sqrt{s}\right)^2$$

or an unstable focus when:

$$(\varepsilon + s) < v < \left(\sqrt{\varepsilon} + \sqrt{s}\right)^2$$

On the other hand, when the second regime applies, so that $\dot{z} > \dot{k}^*/v$, equation (9) becomes:

$$\ddot{z} + \frac{1}{\varepsilon}\left(\varepsilon + s\right)\dot{z} + \frac{s}{\varepsilon}z = \frac{1}{\varepsilon}\dot{k}^* \tag{13}$$

with singular point $z = \dot{k}^*/s$.

Writing the characteristic equation, we get:

$$\lambda^2 + \frac{1}{\varepsilon}\left(\varepsilon + s\right)\lambda + \frac{s}{\varepsilon}z = 0$$

from which

$$\lambda_{1,2} = \frac{1}{2\varepsilon}\left\{-(\varepsilon + s) \pm (\varepsilon - s)\right\}$$

Thus, since both eigenvalues are real and negative, the equilibrium for this regime is a stable node. Choosing values of the parameters such that in the first regime the equilibrium is an unstable focus, it is not too difficult to show by numerical simulation that the model has a stable limit cycle solution. The results of the simulation for this case are shown in Fig.6(*ii*) where,

as was to be expected (cfr. Fig.4), a stable limit cycle – along which the negative deviations from equilibrium dominate the positive ones – is pictured. Needless to say, much importance in this case is attached to the 'ceiling' to investment spending that, by itself, accounts for the persistence of the cycle. The implication of all this is that Goodwin, in December 1950, at the time of his meeting with Le Corbeiller, already had in mind what he then explicitly theorised years later, namely that "(t)he basic, single, given, short-run non-linearity is full employment, whether of capacity or of labour" (Goodwin 1982, p. *ix*).

The fact is that, in the article that appeared in *Econometrica* in the following month, exactly the opposite case is considered and pictured. To obtain the proper 'Goodwin characteristic' for this case, it is enough to notice that, if the 'ceiling' is so much less restrictive than the 'floor' that it does not play any role in the generation of the limit cycle, the former can be safely disregarded altogether (see Fig. 5(*ii*)) so that the investment function becomes:

$$\phi_{\text{PWL}}\left(\dot{y}\right) \approx \phi_{\text{PWLf}}\left(\dot{y}\right) = \begin{cases} \dot{k}^{**} \text{ for } \dot{y} < \dot{k}^{**}/v \\ v\dot{y} \text{ for } \dot{y} \geq \dot{k}^{**}/v \end{cases} \tag{14}$$

Introducing (14) into equation (9) we get a LRT equation with the following characteristic:

$$F\left(\dot{z}\right) = \begin{cases} -\left(1/\varepsilon\right)\left[\dot{k}^{**} - \left(\varepsilon + s\right)\dot{z}\right] & \text{for } \dot{z} < \dot{k}^{**}/v \\ -\left(1/\varepsilon\right)\left[v - \left(\varepsilon + s\right)\right]\dot{z} & \text{for } \dot{z} \geq \dot{k}^{**}/v \end{cases} \tag{15}$$

Also in this case, on the basis of an analysis similar to the one we have performed for the other case, it is not too difficult to find values of the parameters for which the equation generates a *stable* limit cycle (see Fig. 7). As we see, such a limit cycle is exactly specular with respect to the previous one; in particular, it is now the 'floor' to investment spending that, by itself, accounts for the existence and persistence of the cycle.

This result proves to be very useful in order to understand an aspect discussed in the literature (see, for example, Velupillai 1998, p. 11 and 2004, p. 36), but – we believe – never properly clarified.

Up to now, we have considered oscillators with piecewise-linear characteristics. However, more in general, we can say that a 'Goodwin characteristic' is any characteristic with only one bend and such that equation (9)
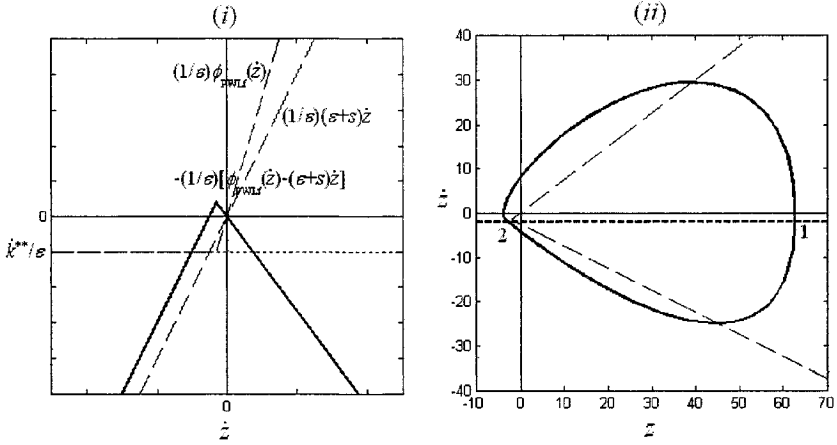
Figure 7: *The Goodwin piecewise linear (two-stroke) oscillator with only the 'floor'* ($\varepsilon = 0.6$, $s = 0.6$, $v = 2$, $\dot{k}^{**} = -3$).
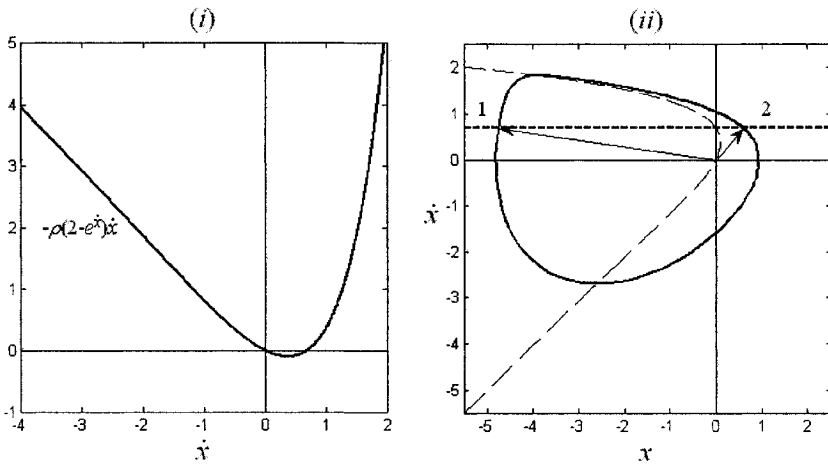

Figure 8: *The two-stroke oscillator with the exponential characteristic* $F\left(\dot{x}\right) = -\rho\left(2 - e^{\dot{x}}\right)\dot{x}$ ($\rho = 0.5$).

admits a unique periodic solution.[8] Some LRT equations with (exponential) characteristic functions which satisfy these conditions are suggested in Le Corbeiller (1960, pp. 390-391). It turns out that all of them, for example

$$\ddot{x} - \rho \left(2 - e^{\dot{x}}\right) \dot{x} + x = 0 \qquad (16)$$

are two-stroke oscillators which generate limit cycles such that the negative amplitude of x is much greater than the positive amplitude (see Fig. 8).

It is possible, however, to imagine cases in which exactly the opposite happens. One of these is suggested by Le Corbeiller in his 1958 letter to Goodwin and is the following:

$$\ddot{x} - \rho \left(2 - e^{-\dot{x}}\right) \dot{x} + x = 0 \qquad (17)$$

The resulting stable limit cycle for this two-stroke oscillator is shown in Fig. 9.
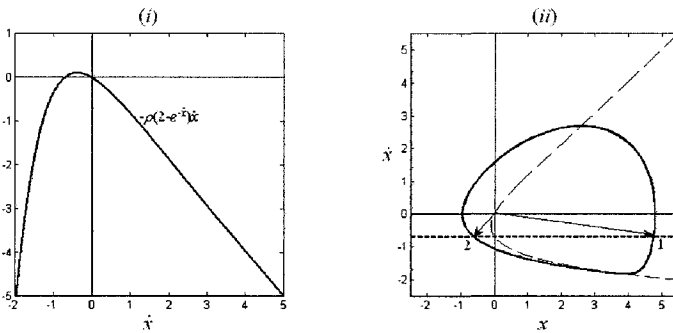

Figure 9: *The two-stroke oscillator with the exponential characteristic* $F(\dot{x}) = -\rho \left(2 - e^{-\dot{x}}\right) \dot{x}$ $(\rho = 0.5)$.

This equation, however, is not included among the examples given in Le Corbeiller (1960), where it is replaced by (16). On the basis of the analysis we have developed in this chapter, we can conclude that, although both (16) and (17) admit a stable limit cycle solution, they are crucially different

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[8]The general condition which must be fulfilled if this is to happen (Le Corbeiller 1960, p. 390) is that, in the equation (9) – written in terms of the 'velocity-controlled damping', $\ddot{x} + R(\dot{x})\dot{x} + x = 0 - R(\dot{x})$ is negative at $\dot{x} = 0$ and there exists a value $\dot{x}_0 > 0$ (< 0), such that $R(\dot{x})$ is negative for $\dot{x} < \dot{x}_0$ ($\dot{x} > \dot{x}_0$) and positive for $\dot{x} > \dot{x}_0$ ($\dot{x} < \dot{x}_0$).

from a qualitative point of view: whether (17) is qualitatively equivalent to
the Goodwin oscillator with only the 'ceiling', (16) is qualitatively equiva-
lent to the Goodwin oscillator with only the 'floor'. It could well be that Le
Corbeiller, after having written in 1958 the letter to Goodwin, realised that
the characteristic of (16) – rather than that of (17) – is what is qualitatively
equivalent to Goodwin's 'blackboard sketch'.

10.4 Conclusions

In this chapter we have used the framework introduced by Le Corbeiller
(1960) to discuss and qualify Sasakura's contention that, although Goodwin
attached importance to the ceiling, the limit cycle in his (1951) model is gen-
erated without it. We have stressed in the introduction that both Duesenberry
(1950) and Goodwin (1950) promptly wrote reviews of Hicks' book, in both
of which it was clearly stated and explained that either the 'ceiling' or the
'floor' will suffice in order to maintain and perpetuate the cycle. Goodwin,
however, not only wrote the review, but, as we have discussed, he actually
discovered such a one-sided oscillator! From the analysis we have devel-
oped, it follows that such a discovery was the outcome of the attempt by
Goodwin to explain the persistence of the cycle with the help of only the
'ceiling', in his opinion the basic, single, given, short-run nonlinearity. It ap-
pears amazing that this result was obtained by Goodwin in December 1950
or thereabouts, namely, just a couple of weeks before his 1951 *Economet-
rica* article – with a final equation qualitatively equivalent to (17), such that
the persistence of the cycle is obtained with no role for the 'ceiling' – was
published.

Appendix 1

As explained by Le Corbeiller (1960, p. 388) and de Figueiredo (1958, p.
xvi), a two-stroke oscillator is an oscillator such that the energy stored in it
varies from a minimum to a maximum value only once when a complete pe-
riod of the variable is traversed, whereas a four-stroke oscillator is the (more
usual) oscillator in which the energy varies from a minimum to a maximum
twice within each period.

To appreciate this distinction and its interpretation in terms of exchanges
of energy, it is useful to consider, equation (9) with the following cubic char-
acteristic (see Le Corbeiller 1960, pp. 392-395 and de Figueiredo 1958, pp.
7.1-7.16):

$$F\left(\dot{x}\right) = -\varepsilon \left[\left(1 - a^2\right)\dot{x} - a\dot{x}^2 - \frac{\dot{x}^3}{3}\right] \quad \varepsilon > 0, \ 0 < a < 1 \quad (18)$$

where x is any given variable. As clearly appears from Fig.10, the parameter a is a measure of the asymmetry of the characteristic function $F(\dot{x})$.
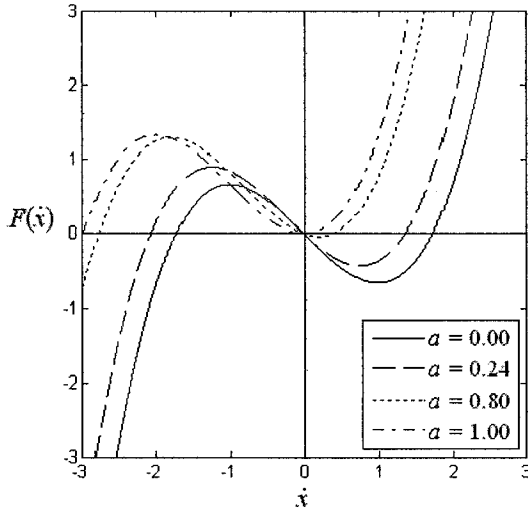

Figure 10: *The characteristic function (18) for different values of the parameter a.*

For $a = 0$, equation (9) reduces to the original Lord Rayleigh equation. This is a (symmetric) four-stroke oscillator such that, as shown in Fig.11(i), over the limit cycle, the total energy stored in the system,

$$\frac{x^2}{2} + \frac{\dot{x}^2}{2}$$

increases along the arcs 2-3 and 4-1 and decreases along the arcs 3-4 and 1-2.

As a increases but remains below a certain critical value, equation (9) with the cubic characteristic (18) is still a four-stroke oscillator, although asymmetric, as in Fig.11(ii), where $a = 0.10$.

The critical case is then shown in Fig.11(iii), where $a = 0.24$ and where the points 3 and 4 of the previous two figures merge.

Finally, the case in Fig.11(iv), where $a = 0.80$, illustrates what happens when the parameter is further increased: the total energy stored in the system

now increases only once over the cycle (along the arc 2-1) and decreases in the remaining part (along the arc 1-2). It goes without saying that, in all four cases considered, these positive and negative energy exchanges are such that they perfectly compensate over the limit cycle.
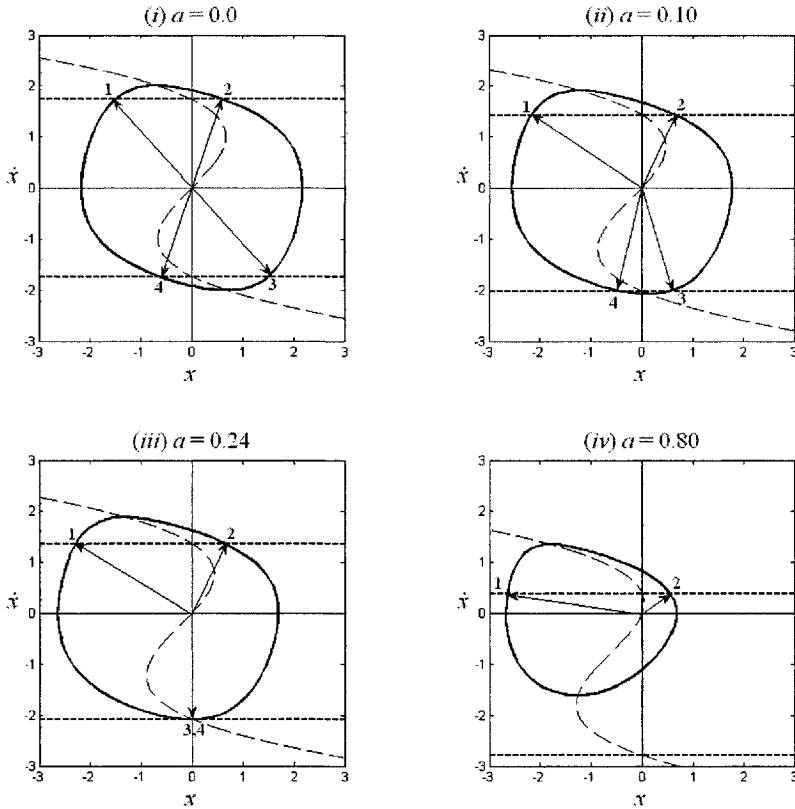

Figure 11: *The limit cycle of the LRT equation (9) with the cubic character- istic (18) for different values of the parameter a.*

To conclude, it is useful to underline two important aspects of the limit cycles of Fig.11 (see Le Corbeiller 1960, p. 302). First, it clearly appears that the distinction between four and two stroke-oscillators does not depend upon the number of points of intersection between the characteristic and the $0y$-axis. Rather, it depends on whether, apart from the origin, there are one or

two such points inside the limit cycle. Second, the existence of a limit cycle is not due to the fact that the characteristic is sigmoid. It appears indeed that the parts of the characteristic outside the limit cycle do not play any role.

This is what opens the way to the possibility of having a limit cycle with a so-called Goodwin characteristic as discussed in the next Appendix and in Section 10.3 above.

Appendix 2

Strictly speaking, the 'Goodwin characteristic', as defined by Le Corbeiller (1958, 1960) and analysed at length in Figueiredo's (1958) Ph.D. thesis, is any characteristic made up of two straight lines, *with or without rounded-off corners*, such that equation (9) is a two-stroke oscillator.

One possibility, analysed in detail by de Figueiredo (1958, Ch. 6), is to assume that in equation (9) we have (see Fig. 12(*i*)):[9]

$$F(\dot{x}) = \begin{cases} -\rho_1\dot{x} \text{ for } \dot{x} \leq \dot{x}_0 \\ \rho_2\dot{x} - (\rho_1 + \rho_2)\dot{x}_0 \text{ for } \dot{x} > \dot{x}_0 \end{cases} \tag{19}$$

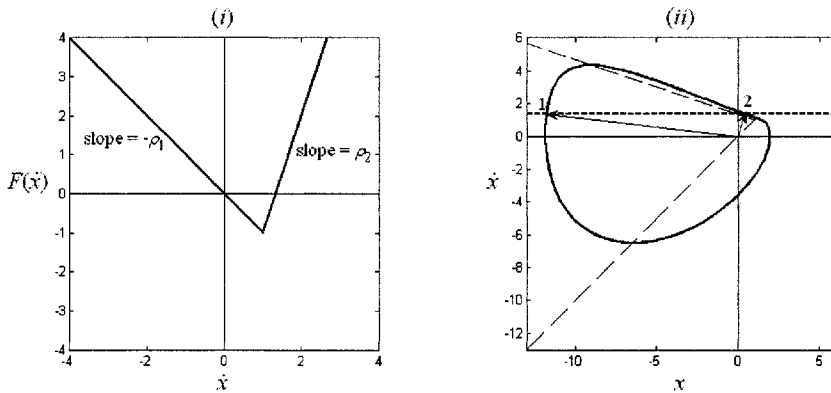where $0 < \rho_1 < 2, \rho_2 > \rho_1 > 0$ and $\dot{x}_0 > 0$.

Figure 12: *The two-stroke oscillator with the piecewise linear characteristic (19)* $(\rho_1 = 1, \rho_2 = 3, \dot{x}_0 = 1)$.

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[9]In de Figueiredo's contribution this is defined as the 'two-straight-line oscillator'. In a footnote, it is then stressed that this oscillator was first observed by R. Goodwin in economics.

With this characteristic (9) is an oscillator with negative damping for $\dot{x} < \dot{x}_0$ and positive damping for $\dot{x} > \dot{x}_0$. Given that it is a piecewise linear equation, we must study its dynamics in the two regimes separately.

In the first regime, such that $\dot{x} \leq \dot{x}_0$, equation (9) becomes:

$$\ddot{x} - \rho_1\dot{x} + x = 0 \tag{20}$$

with $x = 0$ as singular point.

From the characteristic equation:

$$\lambda^2 - \rho_1\lambda + 1 = 0$$

we then find that the eigenvalues:

$$\lambda_{1,2} = \frac{1}{2}\left\{\rho_1 \pm \sqrt{\rho_1^2 - 4}\right\}$$

are always complex with positive real parts. Thus, we can conclude that the equilibrium point is an *unstable focus*.

In the other regime, such that $\dot{x} > \dot{x}_0$, the LRT equation becomes:

$$\ddot{x} + \rho_2\dot{x} + x = (\rho_1 + \rho_2)\dot{x}_0 \tag{21}$$

From the characteristic equation:

$$\lambda^2 + \rho_2\lambda + 1 = 0$$

we then find:

$$\lambda_{1,2} = \frac{1}{2}\left\{-\rho_2 \pm \sqrt{\rho_2^2 - 4}\right\}$$

Thus, the unique equilibrium point of $(21) - x = (\rho_1 + \rho_2)\dot{x}_0 -$ is either a stable focus, when $0 < \rho_2 < 2$, or a stable node when $\rho_2 > 2$.

Both for $0 < \rho_2 < 2$ and $\rho_2 > 2$, it is possible to prove that equation (9) with (19) fulfills the conditions for a unique stable limit cycle given by de Figueiredo in two theorems (see de Figueiredo 1958, Theorem 4.1 and 5.1 respectively). A case with $\rho_2 > 2$ is illustrated in Fig.12.

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