Pierre-François Verhulst's Final Triumph

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The so-called *Logistic function* of Verhulst led a turbulent life: it was first proposed in 1838, it was dismissed initially for being not scientifically sound, it became the foundation of social politics, it fell into oblivion twice and was rediscovered twice, it became the object of contempt, was subsequently applied to many fields for which it was not really intended and it sank to the bottom of scientific philosophy. Today it is cited many times a year. And last but not least, during the past three decades it has been claimed as the prototype of a chaotic oscillation and as a model of a fractal figure.

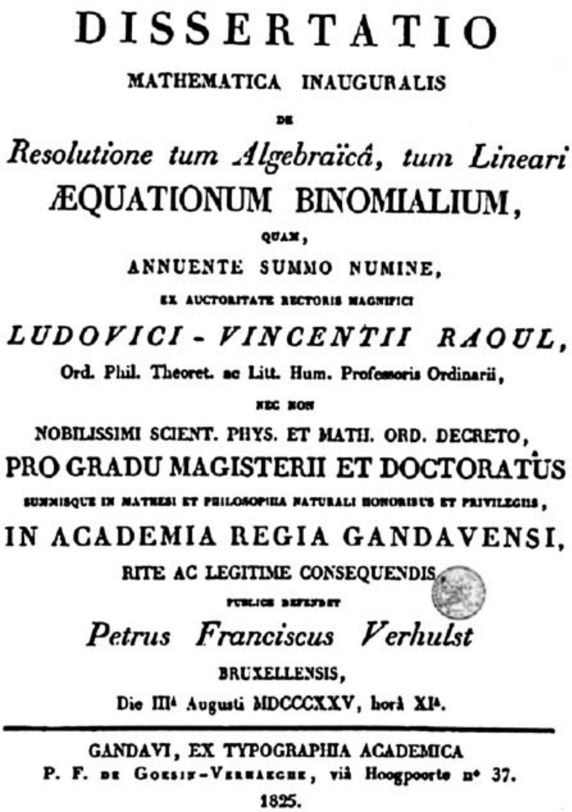
It is only now, 155 years after Verhulst's death, that it becomes clear that his logistic function transcends the importance of pure mathematics and that it plays a fundamental role in many other disciplines. The logistic curve has lived through a long and difficult history before it was finally and generally recognised as a universal milestone marking the road to unexpected fields of research. Only at the end of the 20th century did Verhulst's idea enjoy its definitive triumph. But let us start at the beginning.

On August 3, 1825 the magnificent auditorium of Ghent University was still under construction. It would only be completed early 1826. However, at 11 a.m. of that particular August 3, a small function was held in the provisional hall of the university. In the presence of the then rector of the university, Professor Louis Raoul, a mathematician of scarcely 21 years old defended his doctorate's thesis. Even in those days, twenty-one was very young to take one's PhD. It was clear that, from that moment on, Pierre-François Verhulst would not go through life unnoticed.

1 His Life

He was born in Brussels on October 28, 1804 as the child of wealthy parents. As a pupil at the Brussels Atheneum, where Adolphe Quetelet was his mathematics teacher, he already excelled, and not only because of his knowledge of mathematics. He also had linguistic talents. Twice he won a prize for Latin poetry. However, he had a distinct preference for mathematics. His desire to study exact sciences was so strong that in September 1822, without even having completed his grammar high school, Verhulst enrolled as a student at the University of Ghent. Evidently, his lack of formalism caused some problems when he tried to enrol, although, in those days such matters could easily be resolved with some negotiating and argumentation. It was here that he met Quetelet again, this time as his algebra professor. Just like his studies at the Brussels Atheneum, his academic performance at the University of Ghent was a success. In less than a year, between February 1824 and October 1824, he was honoured with two prizes, one at the University of Leiden for his comments on the theory of maxima, and a second time he won the gold medal of the University of Ghent for a study of variation analysis [1].

In 1825, after only three years of study, Verhulst took his PhD in mathematics with a thesis entitled *De resolutione tum algebraica, tum lineari aequationum binominalium*, in other words, with a thesis in Latin on reducing binomial equations (Fig. 1).



DISSERTATIO

MATHEMATICA INAUGURALIS

DE

Resolutione tum Algebraïcâ, tum Lineari

ÆQUATIONUM BINOMIALIUM,

QUAM,

ANNUENTE SUMMO NUMINE,

EX AUCTORITATE RECTORIS MAGNIFICI

LUDOVICI - VINCENTII RAOUL,

Ord. Phil. Theoret. ac Litt. Hum. Professoris Ordinarii,

NEC NON

NOBILISSIMI SCIENT. PHYS. ET MATH. ORD. DECRETO,

PRO GRADU MAGISTERII ET DOCTORATÛS

SUMMISQUE IN MATHESI ET PHILOSOPHIA NATURALI HONORIBUS ET PRIVILEGIIS,

IN ACADEMIA REGIA GANDAVENSI,

RITE AC LEGITIME CONSEQUENDIS,

PUBLICE DEFENDET

Petrus Franciscus Verhulst

BRUXELLENSIS,

Die IIIâ Augusti MDCCCXXV, horâ XIâ.

GANDAVI, EX TYPOGRAPHIA ACADEMICA

P. F. DE GOESIN-VERHAEGHE, viâ Hoogpoorte n° 37.

1825.

Fig. 1. Doctorate's thesis of Pierre-François Verhulst from 1825

After his studies Verhulst returned to Brussels. He took a keen interest in the calculus of probability and in political economy, an interest which he shared with Quetelet. From then on Quetelet's influence on Verhulst is marked. Indeed, on several occasions Verhulst did some computations to support research carried out by Quetelet.

Moreover, Quetelet's influence was not limited to passing on ideas and stimulating research. It was through his agency that Verhulst was entrusted with a teaching assignment at the "Musée des Sciences et des Lettres" in Brussels in April 1827. A job which he soon had to give up on account of his poor health. Verhulst would be in bad health all of his life as a result of a chronic illness, the nature of which could not be retrieved from the documents that are left from that period. A brief stay in Italy, shortly after his promotion, did not help much to improve his state of health. During his stay in Rome in September 1830, the Belgian Revolution broke out in Brussels. In the mind of Verhulst, who was 26 at that time, a rather peculiar idea began to take shape. An idea only conceivable by young people who in their youthful exuberance and audacity let their imaginations run free. Verhulst always consistently acted upon the consequences of his principles with the self-confidence of a profound conviction. He conceives the rather original idea that the papal state could use a constitution, just like Belgium, his own country which had just become independent. And of course he is not satisfied with the idea alone, but immediately prepares a draft constitution. It seems incredible, yet it is true: the draft constitution was given some consideration by a few cardinals of the papal Curia and was sent to various foreign ministries. However, the matter came to the attention of the Roman bourgeoisie who was not at all pleased with someone from Brussels lecturing the Italians on how to deal with their political matters. The Roman police ordered him to leave the country at once. Verhulst retired to his residence for a couple of days and tried to barricade himself, expecting a siege by the police. But in the end, after having discussed the matter with some friends, he decided to obey the expulsion order and left Italy. Queen Hortense of Holland – at that time living in Rome – made in her memoirs a lively account of the affair. Translated from French: "... A young Belgian savant, Mr Verhulst, had come to Rome for his health. He came very often to my house in the evening; we had frequent discussions together. He asked to speak to me one morning, and brought a plan for a constitution for the Papal States, which he wished to submit to my criticism before giving it to the cardinal-vicar to submit to the pope. I could not help laughing at the singularity of my position. I [the exiled Queen of Holland] to revise a constitution, and for the pope! That seemed to me like a real joke. But my young Belgian friend did not laugh. 'I was talking yesterday evening,' he said to me, 'with several cardinals; their terror is great. I told them of the only way to save the church and the state. They agreed with all my observations. And one of them wishes to submit them to the pope himself. Here is the constitution of which I have sketched the basis ...'" [2]

Back in Brussels, in 1831, he writes a document on behalf of the recently established Congress – the present Belgian parliament – in which he deplores the situation at the university and formulates a way to resolve it [3].

He complained about the political favouritism in the appointment of university professors and the poor standard of the lectures. In spite of his rebellious attitude he is appointed professor at the Royal Military Academy in 1835, and in the same year he is also appointed professor of mathematics at the Université Libre of Brussels, both newly established teaching institutes. However, Verhulst had to give up his professorship at the Université Libre of Brussels in 1840, following a decision of the then Minister of War, which stipulated that professors at the Military School were not allowed to teach in other education institutes. It is not unlikely that Quetelet had a part in the appointments of Verhulst. In 1837 he married a miss Debiefve, who would bear him a daughter about a year later.

Verhulst and Quetelet were closely associated in their life and work [4]. They were both professors at the Military School, they were both members of the Académie royale des Sciences et des Belles Lettres de Bruxelles and they were both interested in mathematical statistics which could be the key to revealing the "natural laws" of human society. Although Verhulst hardly made any general statements regarding the purpose and methodology of these statistics, his practical routine was in line with the theories of Quetelet. The application of mathematics was an essential feature. In both Quetelet's and Verhulst's opinion scientific statistics should be based on a precise mathematical formula to make the accurate incorporation of statistical data possible. However, gradually a significant difference arose in the approach of Verhulst and Quetelet. Verhulst was not in the least interested in what Quetelet called "applied statistics". Verhulst was of the opinion that the calculations were only applicable if there was a direct relation between cause and effect. Quetelet himself did not feel so strongly about such reservations. In contrast he always preferred to find some analogy between physical laws and social phenomena. The debate on this problem, which must have been going on between Verhulst and Quetelet for several years, came to a sudden end with Verhulst's untimely death [4]. It is difficult to determine the precise nature of their relationship from the available documents of that period. Adolphe Quetelet (1796–1876) was eight years older than Verhulst. It is true that Quetelet called Verhulst "successively my pupil, my fellow-worker, my colleague at the Military School, my confrere at the university and the Academy and my friend". However, according to several authors, the relationship between both men was not always as serene as it appeared at first sight. There is one thing we know for sure: they were both interested in mathematical statistics capable of explaining the so-called natural laws of society. Quetelet spoke highly of Verhulst's work, but he had more regard for his compilations than for his original ideas. On one particular occasion, at a public sale, Verhulst managed to get hold of a valuable edition of the complete works of the French mathematician Legendre (1752–1833). The satisfaction of having acquired these works inspired Verhulst to study the "Traité des fonctions elliptiques" and to read the works of the German Abel (1802–1829) and the Norwegian Jacobi (1804–1851), with the intention of making a compilation of all aspects related to elliptic functions. He read and summarized the works of these three famous mathematicians as well as every other document on this subject. Quetelet was full of praise about the result of this study entitled "Traité élémentaire des fonctions elliptiques", which, in fact, was nothing more than a critical résumé of the works of others. However, Quetelet did not approve of what was in fact Verhulst's most original achievement, i.e., the logistic function. After the publication of his "Traité élémentaire des fonctions elliptiques" Verhulst was admitted as a member of the "Académie royale" in 1841. In 1848 Verhulst is appointed director of the scientific department and later, in spite of his deteriorating health, the king appointed him chairman of the Academy. He died a couple of months later on February 15, 1849, at the age of 44.

According to Quetelet, Verhulst was somewhat of an "enfant terrible" [1]. He was self-willed, a man with a social conscience and a man of principle, controversial and often an advocate of extreme ideas, but he also had a strong sense of justice and acted from a deep feeling for his duty. He was straightforward and consistent in his thinking, but on the other hand also conciliatory. As chairman of the Academy he shrank from anything that might have caused dissension. He was never offensive, and the higher his position the more unassuming he became. Although he himself did not have the slightest inclination for losing his temper, he respected the short-temperedness of others. Although he loved taking part in debates, it was more out of a craving for knowledge than in a spirit of contradiction or with the intention of imposing his own views. He was noted for his unperturbed equanimity. It would have been difficult to find a man more conscientious. According to Quetelet's testimony, this sense of duty was marked during the last years of his life, when he still went to work every day. It took him more than an hour to walk the short distance from his house to his office. People saw him trudge along the streets, resting with every step he took, to arrive finally at the academy, panting heavily and completely exhausted.

2 His Work in the Field of Population Growth

Verhulst's first research in the field of population growth dates from shortly after the independence of Belgium. In order to grasp the full import of the research on population growth in the nineteenth century, one must recall the social climate of those days. During the first half of the nineteenth century Flanders went through the worst economic depression in its entire history. Although under the "Ancien régime" in the 18th century it had been one of the most prosperous regions of Europe, it became a backward and shattered region with an impoverished and destitute population in only a few decades' time. In addition to sheer destitution, the pauperization of the population also resulted in demoralization, moral degeneration and social unrest. The same confusion was also seen in other European countries. The correlation between poverty and population was first demonstrated by Thomas Robert Malthus, in his famous Essay on the Principle of Population, which was published in 1798. Malthus stated that poverty is only the inevitable result of overpopulation. In turn, overpopulation was the natural result of the fundamental laws of human society. The ideas of Malthus were the subject of heated debates in the nineteenth century. The necessity of conducting a social policy to curb the pauperization of the population turned the study of the laws of population growth into a scientifically respectable subject. A new discipline, political economics, found enthusiastic adherents everywhere. A demographic study of the population was initially impeded by a lack of statistical material or, even worse, by the unreliability of the available material. It was only in 1820 that progress was made in the methods of compiling and processing statistical data on which demographic conclusions could be based. In Belgium it was again Adolphe Quetelet who organized the collection of data with regard to population figures. He was the initiator of the first census carried out in 1829, the results of which were published in 1832. As chairman of the "Commission centrale de statistique" Quetelet was in charge of the general censuses of 1846, 1856, and 1866. Quetelet also laid the foundations of the international conferences of statistics, the first of which took place in Brussels in 1853.

It was against this background that Verhulst started his research on population growth. His research was based on the ideas of Malthus. In his opinion it could not be denied that the population grew according to a geometric sequence. On the other hand it was incontestable that a number of inhibiting factors also increase in strength as the population grows. Verhulst argued that, as a consequence, the growth of the population was bound by an absolute limit, if only because of the limited availability of habitable land and food supplies. This was an original interpretation, but also a deviation from the original concept of Malthus. Malthus' hypothesis can be formulated by means of a differential equation (with p for the population figure)

$$\frac{\mathrm{d}p}{\mathrm{d}t} = mp \, .$$

Integration of this equation produces the well known exponential growth curve, on which economic Malthusianism is founded. Verhulst did not accept this and considered an alternative. In order to implement the check on population growth, Verhulst had to subtract a still unknown factor from the right-hand side of the equation; a factor which, according to Verhulst, is dependent on the population figure itself. He started from the most obvious hypothesis, namely that the growth coefficient m is not constant but in proportion to the distance of the population size from its saturation point.

In other words Verhulst introduced an inhibitory term, proportional to the square of the population size. Consequently, Verhulst stated that

$$\frac{\mathrm{d}p}{\mathrm{d}t} = mp - np^2 .$$

The solution of this differential equation gave rise to a function which was to project the population growth

$$p = \frac{mp_0 \mathrm{e}^{mt}}{np_0 \mathrm{e}^{mt} + m - np_0} ,$$

where p_0 represents the population figure at a given time $t = 0$. Verhulst verified this formula by comparing the real population figures of France, Belgium, Essex and Russia with the result of his calculations. The correspondence was striking, although the available figures related to a period of only twenty years. Verhulst created a new term for his equation and called it the logistic function.

Verhulst never explained why he chose the term "logistique". Yet, in the nineteenth century this French term was used to designate the art of computation, as opposed to a branch of theoretical mathematics such as the theory of proportions and relations. The term was also frequently used in connection with logarithms in astronomic calculations.

As a matter of fact the military meaning of the word "logistic" also found its origin around that period. The third supplement to the sixth edition of the etymological dictionary of the Académie Française first mentions the term in 1835. The military meaning of the word also comprises the calculation of the provisionment of an army or of a population. The "logistic problem" par excellence is the provisioning of the population. Through his contacts at the Military School, Verhulst must have been familiar with military terminology. Verhulst probably used this term to launch the idea of an arithmetical strategy that could be used to calculate the saturation point of a population as well as the time at which that point would be reached within a given percentage.

Verhulst's results were published in 1838 [5] as a modest "Notice sur la loi que la population suit dans son accroissement" in the "Correspondance Mathématique et Physique", a journal of which Quetelet was editor-in-chief. Verhulst regarded his work as a first step towards a much more elaborate study which would be published in 1845 and 1847 in the form of a "Mémoire de l'Académie royale des Sciences et Belles-Lettres de Bruxelles" [6, 7]. For more details on the life of Verhulst, see [8] and its references.

3 The Logistic Function After 1849

From then on this logistic principle of Verhulst led a most peculiar life. It may be said that after Verhulst's death his principle was completely forgotten. One can only guess why this was the case. But Quetelet's rather ambiguous eulogy [1] on Verhulst at the Academy a few months after his death had something to do with it. In a condescending, almost contemptuous tone Quetelet expresses his reservations with regard to Verhulst's principle and even with regard to his former "friend" himself. Quetelet had previously considered another principle regarding population growth, founded on the analogy with a falling stone in a viscous medium which encounters more resistance as its speed of fall increases. Verhulst considered this concept too dogmatic and had always rejected it strongly. For in Verhulst's mind there was only one thing that mattered: to find a correspondence between his calculations and the real population figures, whereas Quetelet attached greater importance to a formal analogy between the laws of physics and the behavioural pattern of a population: much more than Verhulst, Quetelet was obsessed with the notion – which was popular in the nineteenth century – to presuppose exact causal mechanisms without which the world would not be able to function. The title of his magnum opus "La Physique sociale" already outlines Quetelet's tendency to compare human social behaviour to the laws of physics. However, to state that Quetelet's attitude was the decisive factor in the scarce dissemination of Verhulst's ideas in the nineteenth century, would be a limited representation of the facts. At least as important was the fact that Verhulst's work never developed into a practicable theory that could be tested by demographers. John Miner of Johns Hopkins University translated Quetelet's French eulogy on Verhulst into English and published it in 1933 [9].

Whatever the reason may be, it is a fact that Verhulst's work was completely ignored during the whole nineteenth century. The logistic curve was rediscovered only in 1920. In that year two renowned American demographers, Raymond Pearl and Lowell Reed [10], who were not acquainted with Verhulst's publications, formulated the sigmoid growth curve a second time. It was only when their manuscript was already at the printer's that they were informed of Verhulst's work which had been published 75 years earlier. In later publications they recognise their omission and they adopt the term "logistic" from Verhulst [11].

The data of the United States census available to Pearl and Reed only made up half of a logistic curve, and the population level was far from reaching its saturation point. Nevertheless, they endeavoured to make an extrapolation and stated that the American population – at that time only 80 million people – would grow to a saturation point of 198 million people and that this saturation point would only be reached by the end of the twentieth century. Unlike Verhulst, Pearl and Reed did not deduce the curve's equation from any preliminary thinking. On the contrary, reflexions on the inhibitive effect of diminishing ambient factors as a result of the population growth only appear towards the end of the article, and only to support the application of the sigmoid curve. In other words, Pearl and Reed start from the idea that population growth follows a sigmoid curve. In addition they regard the sigmoid curve of population growth as a genuine principle of population growth. On the one hand this was based on the fact that the logistic curve supported the data fairly well, and on the other hand on the fact that, based on reasonable assumptions, it provided a fairly accurate picture of the future evolution of the population. In 1924, Pearl [12] compared his curve "in a modest way" with Kepler's law of planetary motion and with Boyle's law of gases... For many years, the emphasis which Pearl and Reed put on the systematic nature of the logistic curve led to many heated and bitter discussions which would only come to an end with Pearl's death in 1940. In spite of, or maybe thanks to, these fierce discussions, the logistic curve is sometimes also called the Verhulst–Pearl curve.

A first sign of real recognition of Verhulst's merits came in 1925 [13], when the English statistician Udny Yule recognised that Verhulst was far ahead of his time: "...Probably owing to the fact that Verhulst was greatly in advance of his time, and that the then existing data were quite inadequate to form any effective test to his views, his memoirs fell into oblivion; but they are classics on their subject..." But even that was not sufficient to make Verhulst's reputation and his name was lost again. Verhulst's formula got its final victory only after 1965. From then on scientists from various countries and domains start to refer to Verhulst's publications (Fig. 2). There are at least five reasons for this.

First of all there is the major breakthrough of ecology as a new scientific discipline: on account of the scope of their research ecologists are particularly interested in the growth and the evolution of populations. Verhulst's formula appeared to be an excellent basis for calculating ecological growth problems. A second aspect of Verhulst's formula was that it required a considerable

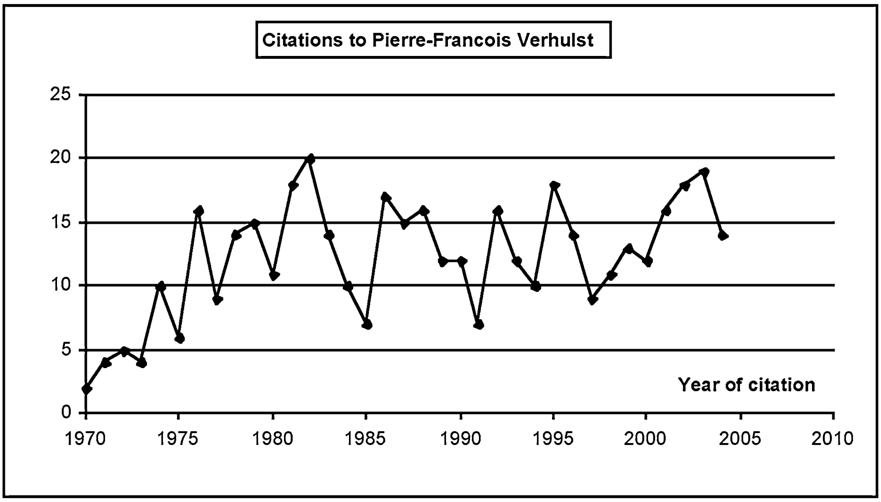


Fig. 2. Citations to the publications of Pierre-François Verhulst

degree of computation. It was only with the advent of the electronic calculator and later the computer that the laborious job of making endless calculations could be carried out with a minimum of effort.

A third factor was the discovery that the S-shaped logistic function could also be applied to a wide variety of other fields, such as chemical autocatalysis, Michaelis–Menten kinetics, cancer chemotherapy, the Hill equation, the Langmuir isotherm, velocity equations of the first and second order of magnitude, oxidation-reduction potentials, erythrocyte haemolysis, the flow of streaming gases, etc. Verhulst's principle was even applied to economics and sociology. It seemed as if everything could be defined using the same sigmoidal curve. Many scientists carried it beyond the limit and applied Verhulst's formula, whether it was relevant or not. This led to a situation in which over the past thirty years Verhulst's work was cited in just about every country of the world, from Brazil to the People's Republic of China, from the Soviet Union to the United States of America. His publications are now cited about 15 times a year, which is quite remarkable considering that his work goes back more than one hundred and sixty years. It is quite amusing in this context to see that each year several authors mention 1938 and 1945 as the year of publication of his works, thinking that 1838 or 1845 must have been a printing error. The journal "Correspondance Mathématique et Physique" ended its publications in 1841. It was in fact published and edited by Quetelet himself on behalf of the Belgian mathematicians. It would reappear only at the end of Quetelet's life from 1874 to 1880 under the name of "Nouvelle Correspondance Mathématique et Physique" and from 1881 to 1961 as "Mathesis".

4 Verhulst's Principle and Chaos Theory

But there is a fourth reason why the work by Verhulst received so much attention all of a sudden: its implication in chaos theory. Already in 1963 Edward Lorenz used a one-dimensional mapping equivalent to the Verhulst mapping to explain certain aspects of his by now famous simplified weather forecast model. In 1976 the biologist Robert May [14] stated explicitly that the logistic model should be studied as early as possible in one's scientific education in order to start understanding nonlinear phenomena. Since the work of May, Feigenbaum [15], and others the Verhulst model has become the paradigm for the period-doubling route to chaos, as is for example nicely illustrated in "The Beauty of Fractals" by H.O. Peitgen and P.H. Richter [16] (one of the first mathematical "coffee table books").

Meanwhile several authors have adopted this idea and it seems to be generally acknowledged now that Verhulst's logistic function is the basis of modern chaos theory, although Verhulst himself had absolutely no idea that something like that lay hidden in his formula.

To obtain deterministic chaos from Verhulst's formula one has to replace the continuous logistic differential equation by its discrete form

$$p_{n+1} - p_n = rp_n(1 - p_n)$$

or equivalently

$$p_{n+1} = p_n + rp_n(1 - p_n).$$

In this difference equation p_n denotes the population size at time n, and $r > 0$ is still the growth coefficient; the carrying capacity has been normalized to 1. Using this prototype of a nonlinear iterative process one calculates the evolution of a population by starting with some initial population p_0 (between 0 and 1) and by applying the formula again and again, thus obtaining successively p_1, p_2, p_3, and so on.

When carrying out this iteration scheme one finds that the resulting evolution of the population depends strongly on the value of the growth parameter r (Fig. 3):

1. For $r < 2$ the population sequence tends to the limit value 1. For $r < 1$ this happens in a monotone way, similar to the behaviour in the differential equation (Fig. 3(a)), but for $1 < r < 2$ in an oscillatory way (Fig. 3(b)). As r increases to 2 these oscillations also increase, both in amplitude and length: for $r = 1.95$ the limit is reached only after more than 2000 steps!
2. For values of r between 2 and 2.5699... the sequence displays, after some initial steps, a periodic behaviour with a period which depends on r. When r increases one first observes an oscillation between a maximum and a minimum (period 2, Fig. 3(c)), then an oscillation between 4 different local extremes (period 4, Fig. 3(d)), and subsequently oscillations with period 8 (Fig. 3(e)), period 16, and so on. Such a period-doubling cascade has been identified as one of the typical ways in which a system can go from orderly to chaotic behaviour.
3. For most values of r larger than 2.5699... (and less than 3) the sequence shows no regularity (periodicity) any more (Fig. 3(f)). For such values of r the system is "chaotic", a regime which is mainly characterized by a few hallmarks as described in the next paragraph.

The main characteristic of a chaotic system is its extreme susceptibility to a change in the initial condition (illustrated for the Verhulst model in Fig. 4). Two sequences with almost identical values for p_0 will at first behave in a virtually identical manner, but then suddenly diverge so that from then on there is no correlation between the two oscillations. A similar sensitivity is also observed with respect to a change in the growth parameter r. Another phenomenon is that a chaotic system sometimes seems to behave regularly for a number of steps in the iteration. For example, for $r = 2.7$ and $p_0 = 0.05$ there is an apparent regularity (a fixed point) between step 590 and step 670 (Fig. 5(a)); with $r = 2.7001$ and $p_0 = 0.05$ there is an apparent period-two behaviour between step 298 and step 316 (Fig. 5(b)). Under further iteration these apparent regularities disappear again. Predictability and chaos alternate with each other, but in a basically unpredictable manner.

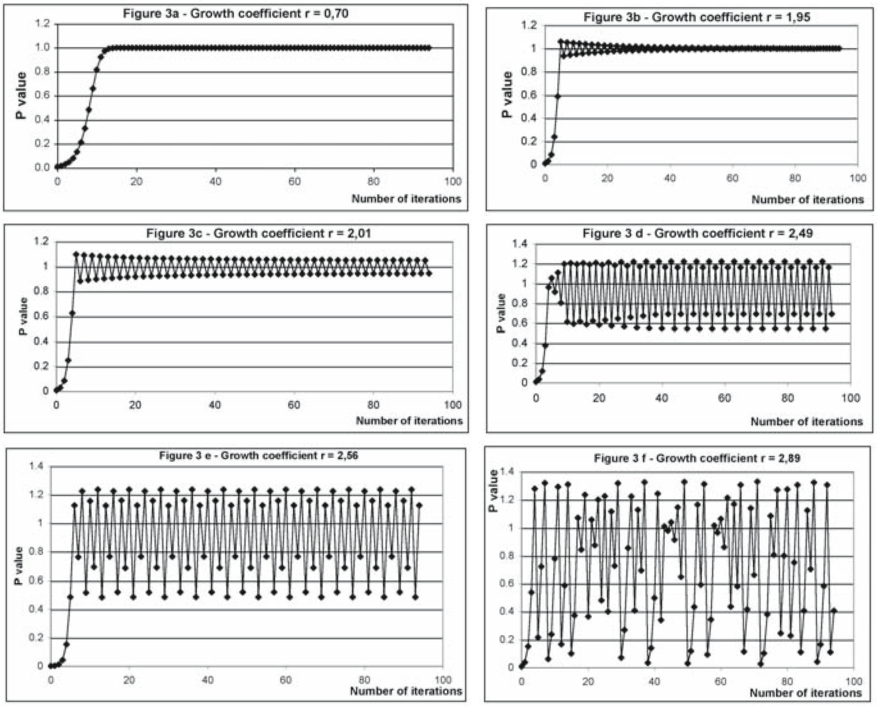


Fig. 3. Deterministic chaos obtained from Verhulst's formula

At the moment when the system becomes chaotic, the size of the population at each step in the iteration will be different from its value at any of the previous steps. There is no stability or regularity any more. Moreover, the long-term evolution of the population will strongly depend on the chosen initial value p_0. Even the smallest deviation – say in the hundredth or thousandth decimal – from the initial value will have a significant effect and in the end, result in a totally different evolution. It is important to notice that also our computers which work with a fixed number of decimals, are subject to this type of unpredictability, however powerful they may be.

5 Logistic Fractal of Verhulst

And finally, a fifth factor can be identified which contributes to the late triumph of Verhulst's logistic function. Indeed, using the logistic formula, one can produce fractal figures comparable to the well-known Mandelbrot fractal. For that purpose we consider again the discrete Verhulst iteration,

$$p_{n+1} = p_n + rp_n(1 - p_n),$$

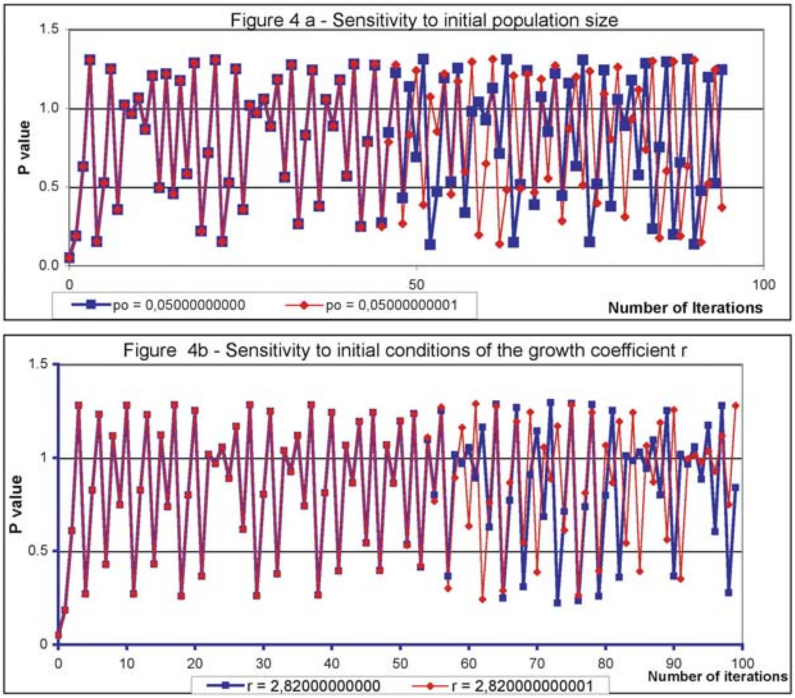


Fig. 4. Example of extreme susceptibility to the initial condition in Verhulst's formula

but this time we allow p and r to be complex, and therefore related to points in the plane. More precisely, p and r take values of the form $a + bi$, and are then identified with the point (a, b) in the plane. The iteration is started by fixing a nonzero value for p_0, for instance $0.01 + 0.01i$. For each value of r one can then calculate the resulting iteration sequence. One finds that there are two possible results: either the sequence stays bounded, or it diverges to infinity. The r-values for which the sequence stays bounded form a set which we call a Verhulst fractal; observe that this Verhulst fractal depends on the choice of the initial value p_0. In a similar way as for the Mandelbrot set, such Verhulst fractals are easily generated on a computer: points not belonging to the fractal evolve towards infinity at different speeds, and by assigning different colors to different speeds one obtains patterns such as in Fig. 6. In this figure the black points form the Verhulst fractal; each picture in the sequence is an enlargement of part of the preceding picture. What we learn from these pictures is that the boundary of the Verhulst set has a fractal structure, in the sense that however much we enlarge this boundary, it will never become a simple line or curve. At each scale new details appear, and the figure never reaches a limit.

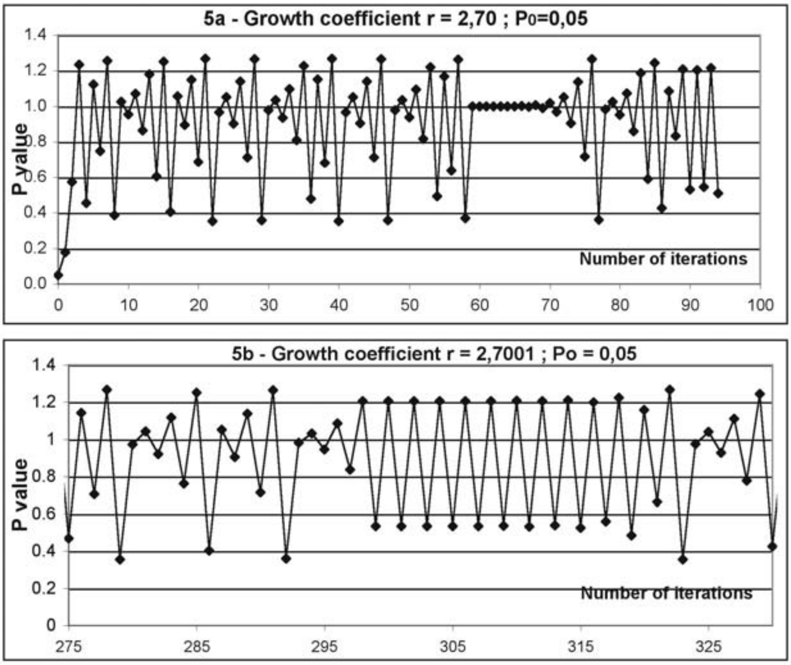


Fig. 5. Predictability and chaos alternating in Verhulst's formula

6 Conclusion

Hence, twice in the past three decades Verhulst's logistic function obtained a new, additional meaning. The first time as a model of a chaotic oscillation and the second time as an example of a fractal figure. The realization that complex phenomena can be represented by means of a simple algebraic equation has radically changed our way of thinking in the past years. Robert May was one of the first people to understand its broader social significance: "Not only in research, but also in the everyday world of politics and economics, we would all be better off if more people realized that simple non-linear systems do not necessarily possess simple dynamical properties."

Verhulst's function is but one of the many examples of a non-linear, chaotic system, although it clearly illustrates the essence of deterministic chaos. It also illustrates how a discovery can go through a real evolution of its own and how the underlying significance of a discovery can change radically as a result of the evolution of its scientific context. Some scientific ideas have to wait for a long period before they come to their final triumph. Verhulst's logistic function is certainly one among them.

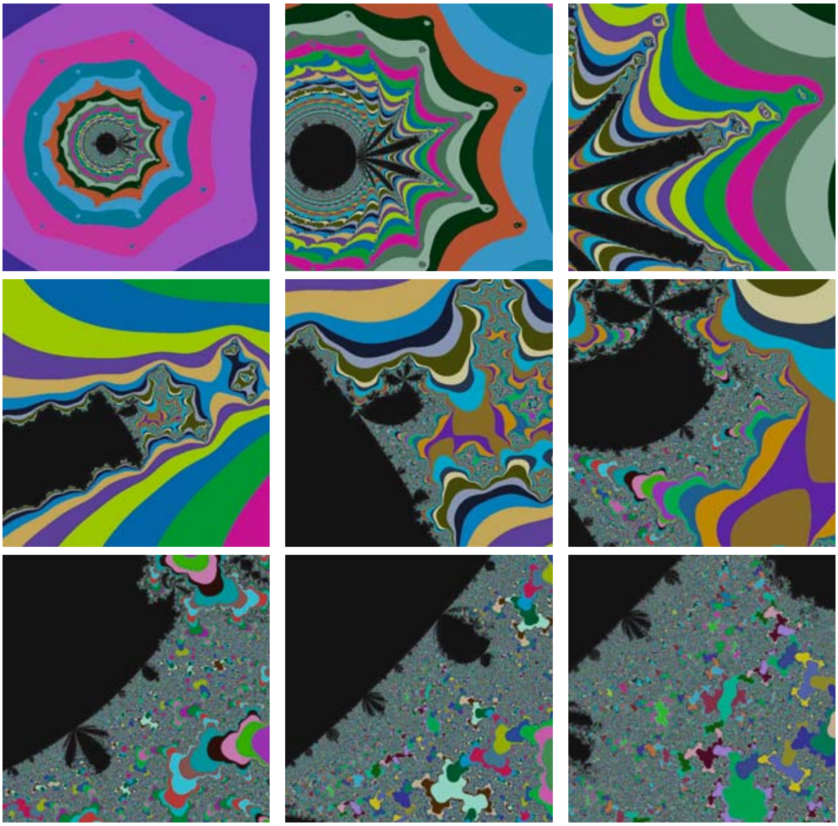


Fig. 6. The Logistic fractal of Verhulst for the value $p_0 = -10^{-7}$; each figure to the right and downwards is an enlargement of the preceding figure

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