6

Embedding and Local Confluence

In this chapter, we continue to present important results for adhesive HLR systems which have been introduced in Section 3.4 of Part I already. The Embedding Theorem is one of the classical results for the graph case presented in [Ehr79]. For the categorical presentation of most of the results in this chapter, we introduce in Section 6.1 the concept of initial pushouts, which is a universal characterization of the boundary and the context, discussed in Section 3.2 for the graph case. This allows us to present the Embedding and Extension Theorems in Section 6.2, which characterize under what conditions a transformation sequence can be embedded into a larger context. The main ideas of the Embedding and Extension Theorems and of the other results in this chapter have been explained already in Section 3.4.

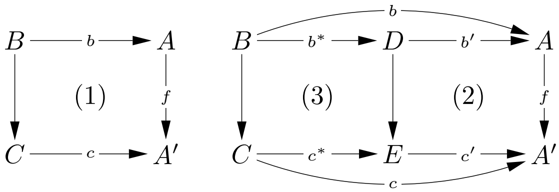
The concepts of critical pairs and local confluence were motivated originally by term rewriting systems, and were studied for hypergraph rewriting systems in [Plu93] and for typed attributed graph transformation systems in [HKT02]. The general theory of critical pairs and local confluence for adhesive HLR systems according to [EHPP04] is presented in Sections 6.3 and 6.4.

We start this chapter with the concept of initial pushouts in Section 6.1, because they are needed in the Extension Theorem. Initial pushouts and the Extension Theorem are both needed in the proof of the Local Confluence Theorem, which is the most important result in this chapter, because it has a large number of applications in various domains.

6.1 Initial Pushouts and the Gluing Condition

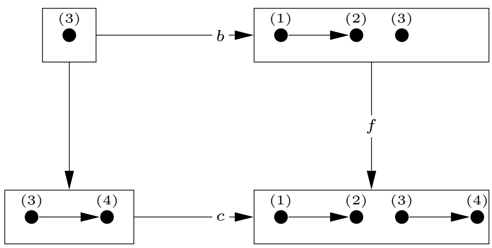
An initial pushout formalizes the construction of the boundary and the context which were mentioned earlier in Subsection 3.4.2. For a morphism $f : A \to A'$, we want to construct a boundary $b : B \to A$, a boundary object B, and a context object C, leading to a pushout. Roughly speaking, A' is the gluing of A and the context object C along the boundary object B.

Definition 6.1 (initial pushout). *Given a morphism $f : A \to A'$ in a (weak) adhesive HLR category, a morphism $b : B \to A$ with $b \in \mathcal{M}$ is called the* boundary *over f if there is a pushout complement of f and b such that (1) is a pushout which is* initial *over f. Initiality of (1) over f means, that for every pushout (2) with $b' \in \mathcal{M}$ there exist unique morphisms $b^* : B \to D$ and $c^* : C \to E$ with b^*, $c^* \in \mathcal{M}$ such that $b' \circ b^* = b$, $c' \circ c^* = c$ and (3) is a pushout. B is then called the* boundary object *and C the* context *with respect to f.*



Example 6.2 (initial pushouts in Graphs). The boundary object B of an injective graph morphism $f : A \to A'$ consists of all nodes $a \in A$ such that $f(a)$ is adjacent to an edge in $A' \backslash f(A)$. These nodes are needed to glue A to the context graph $C = A' \backslash f(A) \cup f(b(B))$ in order to obtain A' as the gluing of A and C via B in the initial pushout.

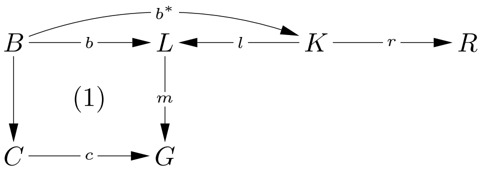
Consider the following morphism $f : A \to A'$ induced by the node labels. Node (3) is the only node adjacent to an edge in $A' \backslash f(A)$ and therefore has to be in the boundary object B. The context object C contains the nodes (3) and (4) and the edge between them. All morphisms are inclusions.



In **Graphs**, initial pushouts over arbitrary morphisms exist. If the given graph morphism $f : A \to A'$ is not injective, we have to add to the boundary object B all nodes and edges $x, y \in A$ with $f(x) = f(y)$ and those nodes that are the source or target of two edges that are equally mapped by f. □

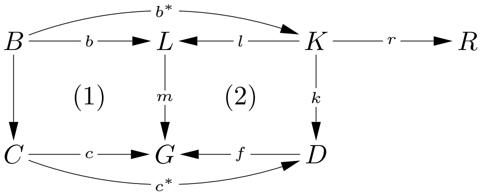
The concept of initial pushouts allows us to formulate a gluing condition analogous to that in the graph case (see Definition 3.9), leading to the existence and uniqueness of contexts in Theorem 6.4, which generalizes the graph case considered in Fact 3.11.

Definition 6.3 (gluing condition in adhesive HLR systems). *Given an adhesive HLR system AHS over a (weak) adhesive HLR category with initial pushouts, then a match $m : L \to G$ satisfies the gluing condition with respect* *to a production* $p = (L \overset{l}{\leftarrow} K \overset{r}{\rightarrow} R)$ *if, for the inital pushout (1) over* m*, there is a morphism* $b^* : B \rightarrow K$ *such that* $l \circ b^* = b$*:*



In this case $b, l \in \mathcal{M}$ *implies* $b^* \in \mathcal{M}$ *by the decomposition property of* $\mathcal{M}$*.*

Theorem 6.4 (existence and uniqueness of contexts). *Given an adhesive HLR system AHS over a (weak) adhesive HLR category with initial pushouts,a match* $m : L \rightarrow G$ *satisfies the gluing condition with respect to a production* $p = (L \overset{l}{\leftarrow} K \overset{r}{\rightarrow} R)$ *if and only if the context object* D *exists, i.e. there is a pushout complement (2) of* l *and* m*:*



If it exists, the context object D *is unique up to isomorphism.*

Proof. If the gluing condition is fulfilled, then we can construct from $b^* \in \mathcal{M}$ and $B \rightarrow C$ a pushout (3) with the pushout object D and the morphisms k and c^*, where (3) is hidden behind (1) and (2). This new pushout (3), together with the morphisms c and $m \circ l$, implies a unique morphism f with $f \circ c^* = c$ and $m \circ l = f \circ k$, and by pushout decomposition of (3), (2) is also a pushout, leading to the context object D.

If the context object D with the pushout (2) exists, the initiality of pushout (1) implies the existence of b^* with $l \circ b^* = b$.

The uniqueness of D follows from the uniqueness of pushout complements shown in Theorem 4.26. □

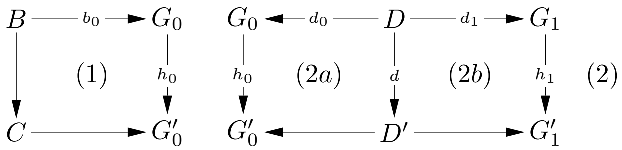
We shall now show an interesting closure property of initial pushouts, which we need for technical reasons (see the proof of Theorem 6.16). The closure property shows that initial pushouts over $\mathcal{M}'$-morphisms are closed under composition with double pushouts along $\mathcal{M}$-morphisms. In the (typed) graph case, we can take as $\mathcal{M}'$ the class of all (typed) graph morphisms or the class of all injective (typed) graph morphisms.

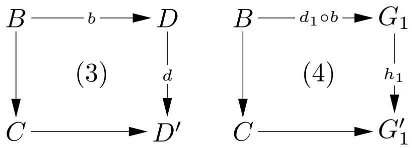
Lemma 6.5 (closure property of initial POs). *Let* $\mathcal{M}'$ *be a class of morphisms closed under pushouts and pullbacks along* $\mathcal{M}$*-morphisms (see Remark*

6.6), with initial pushouts over $\mathcal{M}'$-morphisms. Then initial pushouts over $\mathcal{M}'$-morphisms are closed under double pushouts along $\mathcal{M}$-morphisms.

This means that, given an initial pushout (1) over $h_0 \in \mathcal{M}'$ and a double-pushout diagram (2) with pushouts (2a) and (2b) and d_0, $d_1 \in \mathcal{M}$, we have the following:

1. *The composition of (1) with (2a), defined as pushout (3) by the initiality of (1), is an initial pushout over $d \in \mathcal{M}'$.*
2. *The composition of the initial pushout (3) with pushout (2b), leading to pushout (4), is an initial pushout over $h_1 \in \mathcal{M}'$.*



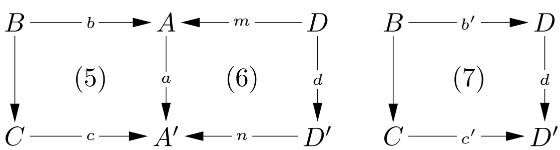


Remark 6.6. The statement that $\mathcal{M}'$ is closed under pushouts along $\mathcal{M}$-morphisms means that, for a pushout $C \overset{n}{\rightarrow} D \overset{g}{\leftarrow} B$ over $C \overset{f}{\leftarrow} A \overset{m}{\rightarrow} B$ with $m, n \in \mathcal{M}$ and $f \in \mathcal{M}'$, it holds also that $g \in \mathcal{M}'$. There is an analogous definition for pullbacks.

Proof. We prove this lemma in three steps.

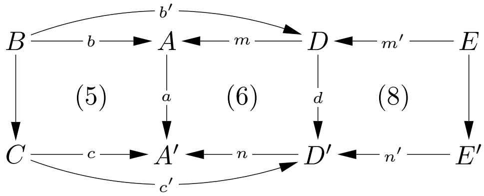
Step I. Initial pushouts are closed under pushouts (in the opposite direction) in the following sense.

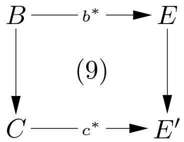
Given an initial pushout (5) over $a \in \mathcal{M}'$ and a pushout (6) with $m \in \mathcal{M}$, then there is an initial pushout (7) over $d \in \mathcal{M}'$ with $m \circ b' = b$ and $n \circ c' = c$:



Since (5) is an initial pushout, there are unique morphisms b' and c' with $b', c' \in \mathcal{M}$ such that (7) is a pushout. It remains to show the initiality and that $d \in \mathcal{M}'$.

For any pushout (8) with $m' \in \mathcal{M}$, we have the result that the composition (8) + (6) is a pushout, with $m \circ m' \in \mathcal{M}$. Since (5) is an initial pushout, there are morphisms $b^* : B \rightarrow E$ and $c^* : C \rightarrow E' \in \mathcal{M}$ with $m \circ m' \circ b^* = b = m \circ b'$ and $n \circ n' \circ c^* = c = n \circ c'$, and (9) is a pushout:

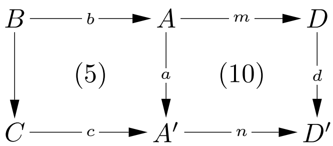




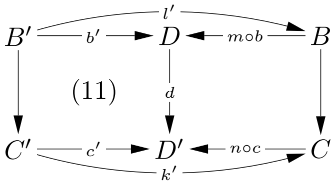
Since m and n are monomorphisms, it holds that $b' = m' \circ b^*$ and $c' = n' \circ c^*$. Therefore (7) is an initial pushout. Finally, pushout (6) is also a pullback by Theorem 4.26, part 1, with $a \in \mathcal{M}'$ such that the closure property of $\mathcal{M}'$ implies $d \in \mathcal{M}'$.

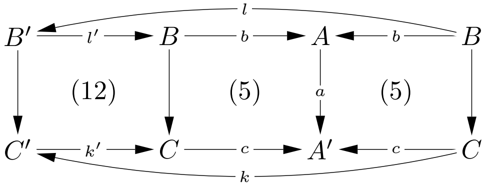
Step II. Initial pushouts are closed under pushouts (in the same direction) in the following sense.

Given an initial pushout (5) over $a \in \mathcal{M}'$ and a pushout (10) with $m \in \mathcal{M}$, then the composition (5) + (10) is an initial pushout over $d \in \mathcal{M}'$:



Since $\mathcal{M}'$-morphisms are closed under pushouts along $\mathcal{M}$-morphisms, we have $d \in \mathcal{M}'$. The initial pushout (11) over d then exists. Comparing (5) + (10) with (11), we obtain unique morphisms $l' : B' \to B$ and $k' : C' \to C \in \mathcal{M}$ with $m \circ b \circ l' = b'$and $n \circ c \circ k' = c'$, and (12) is a pushout:





(12) + (5) is then also a pushout and, from the initial pushout (5), we obtain unique morphisms $l : B \to B'$ and $k : C \to C' \in \mathcal{M}$ with $b \circ l' \circ l = b$ and $c \circ k' \circ k = c$. Since b and c are monomorphisms, we obtain $l' \circ l = id_B$ and $k' \circ k = id_C$, and since l' and k' are monomorphisms they are also isomorphisms. This means that (5) + (10) and (11) are isomorphic, and (5) + (10) is an initial pushout over $d \in \mathcal{M}'$.then also

Step III. Initial pushouts are closed under double pushouts.

Square (3) is an initial pushout over $d \in \mathcal{M}'$, which follows directly from Step I.

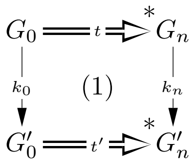
(1) is a pushout along the $\mathcal{M}$-morphism b_0 and therefore a pullback by Theorem 4.26, part 1, and since $\mathcal{M}'$ is closed under pullbacks, we have $B \to C \in \mathcal{M}'$. We then have $d \in \mathcal{M}'$ and $h_1 \in \mathcal{M}'$ with $\mathcal{M}'$ closed under pushouts, and by applying Step II we also have the result that (4) is an initial pushout over $h_1 \in \mathcal{M}'$. □

6.2 Embedding and Extension Theorems

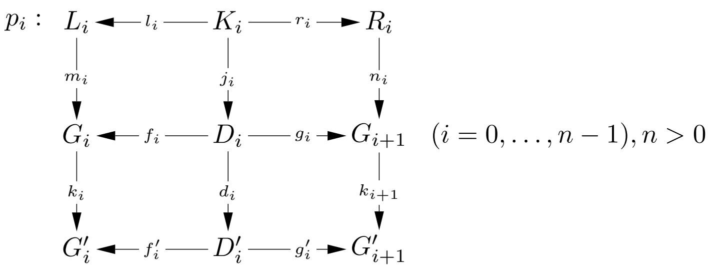
We now present the Embedding and Extension Theorems, which allow us to extend a transformation to a larger context. The ideas behind these theorems were given in Section 3.4 in Part I.

An extension diagram describes how a transformation $t : G_0 \overset{*}{\Rightarrow} G_n$ can be extended to a transformation $t' : G'_0 \overset{*}{\Rightarrow} G'_n$ via an extension morphism $k_0 : G_0 \to G'_0$ that maps G_0 to G'_0.

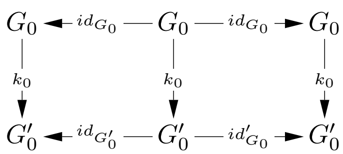
Definition 6.7 (extension diagram). *An* extension diagram *is a diagram (1), as shown below,*



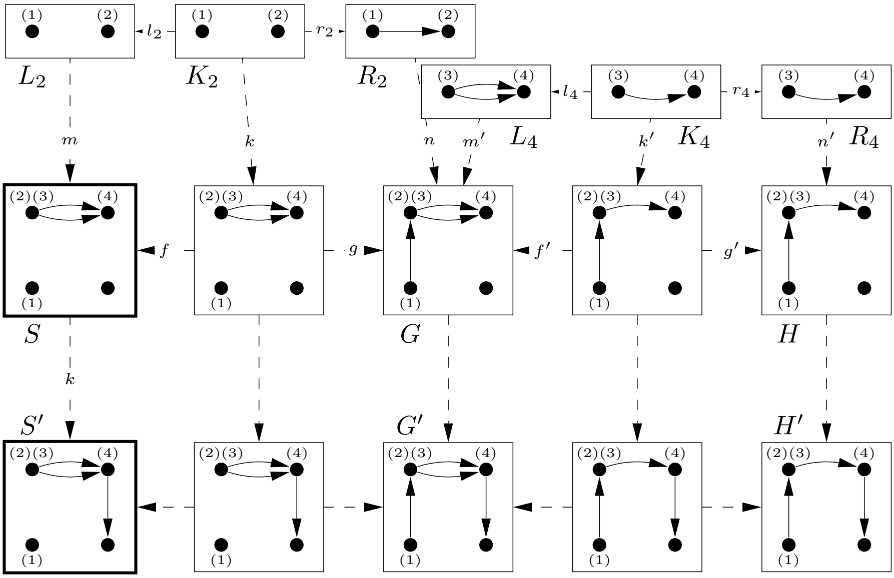
where $k_0 : G_0 \to G'_0$ is a morphism, called an extension morphism, and $t : G_0 \overset{*}{\Rightarrow} G_n$ and $t' : G'_0 \overset{*}{\Rightarrow} G'_n$ are transformations via the same productions $(p_0, \dots, p_{n-1})$ and matches $(m_0, \dots, m_{n-1})$ and $(k_0 \circ m_0, \dots, k_{n-1} \circ m_{n-1})$ respectively, defined by the following DPO diagrams:



For $n = 0$, the extension diagram is given up to isomorphism by



Example 6.8 (extension diagram). Consider the transformation sequence $t : S \Rightarrow G \Rightarrow H$ from Example 5.5 and the extension morphism $k : S \to S'$ as shown in the following diagram. The complete diagram is an extension diagram over t and k.



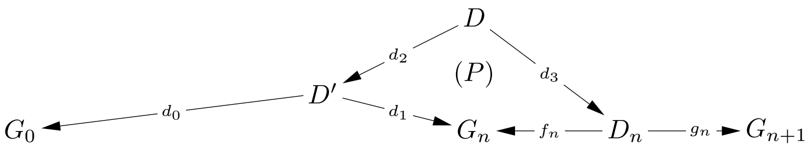
□

The consistency condition given in Definition 6.12 for a transformation $t : G_0 \overset{*}{\Rightarrow} G_n$ and an extension morphism $k_0 : G_0 \to G'_0$ means intuitively that the boundary object B of k_0 is preserved by t. In order to formulate this property, we use the notion of a derived span $der(t) = (G_0 \leftarrow D \rightarrow G_n)$ of the transformation t, which connects the first and the last object.

Definition 6.9 (derived span). *The* derived span *of an identical transformation $t : G \overset{id}{\Rightarrow} G$ is defined by $der(t) = (G \leftarrow G \rightarrow G)$ with identical morphisms.*

The derived span of a direct transformation $G \overset{p,m}{\Longrightarrow} H$ is the span $(G \leftarrow D \rightarrow H)$ (see Def. 5.2).

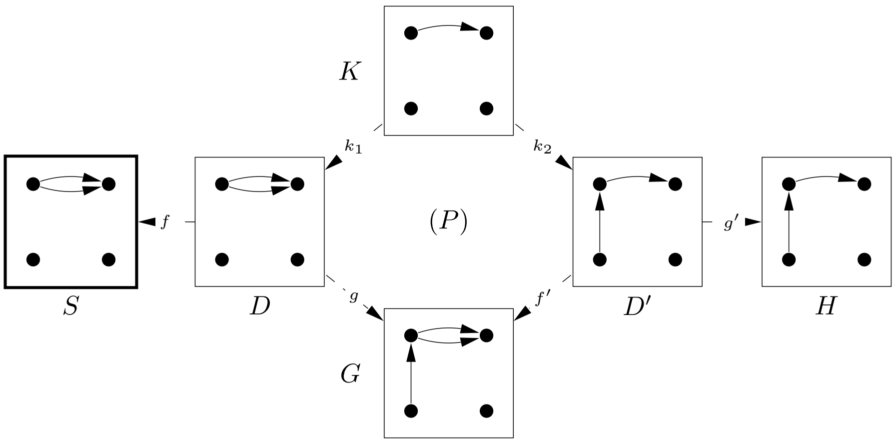
For a transformation $t : G_0 \overset{*}{\Rightarrow} G_n \Rightarrow G_{n+1}$, the derived span is the composition via the pullback (P) of the derived spans $der(G_0 \overset{*}{\Rightarrow} G_n) = (G_0 \overset{d_0}{\leftarrow} D' \overset{d_1}{\rightarrow} G_n)$ and $der(G_n \Rightarrow G_{n+1}) = (G_n \overset{f_n}{\leftarrow} D_n \overset{g_n}{\rightarrow} G_{n+1})$. This construction leads to the derived span $der(t) = (G_0 \overset{d_0 \circ d_2}{\longleftarrow} D \overset{g_n \circ d_3}{\longrightarrow} G_{n+1})$:



In the case $t : G_0 \Rightarrow^* G_n$ where $n = 0$, we have either $G_0 = G_n$ and $t : G_0 \overset{id}{\Rightarrow} G_0$ (see above) or $G_0 \cong G'_0$ with $der(t) = (G_0 \overset{id}{\leftarrow} G_0 \rightarrow G'_0)$.

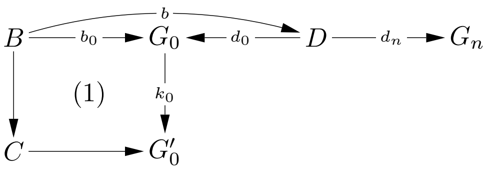
Remark 6.10. The derived span of a transformation is unique up to isomorphism and does not depend on the order of the pullback constructions.

Example 6.11 (derived span). Consider the direct transformation sequence $t : S \Rightarrow G \Rightarrow H$ from Example 5.5. The following diagram shows the construction of the derived span $der(t) = (S \overset{f \circ k_1}{\longleftarrow} K \overset{g' \circ k_2}{\longrightarrow} H)$ with the pullback (P):



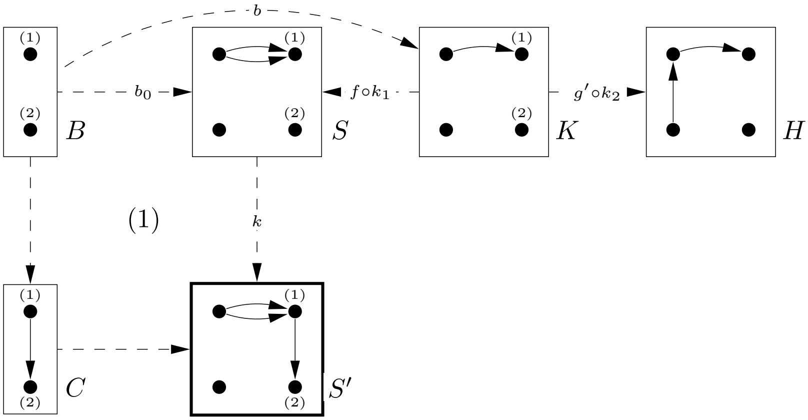
□

Definition 6.12 (consistency). *Given a transformation $t : G_0 \overset{*}{\Rightarrow} G_n$ with a derived span $der(t) = (G_0 \overset{d_0}{\leftarrow} D \overset{d_n}{\rightarrow} G_n)$, a morphism $k_0 : G_0 \rightarrow G'_0$ is called* consistent *with respect to t if there exist an initial pushout (1) over k_0 and a morphism $b \in \mathcal{M}$ with $d_0 \circ b = b_0$:*



Example 6.13 (consistency). Consider the direct transformation sequence $t : S \Rightarrow G \Rightarrow H$ from Example 5.5 with the derived span $der(t) = (S \overset{f \circ k_1}{\longleftarrow} K \overset{g' \circ k_2}{\longrightarrow} H)$ as constructed in Example 6.11. The extension morphism $k : S \rightarrow S'$ given in Example 6.8 is then consistent with respect to t.

We can construct the initial pushout (1) over k as shown in the following diagram. For the morphism b depicted, it holds that $f \circ k_1 \circ b = b_0$:



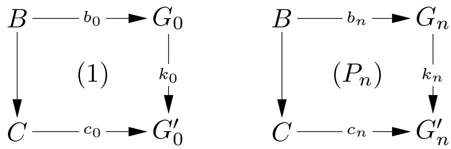
□

Using the following Embedding and Extension Theorems, we can show that consistency is both sufficient and necessary for the construction of extension diagrams. Both theorems are abstractions of the corresponding Theorems 3.28 and 3.29 for the graph case.

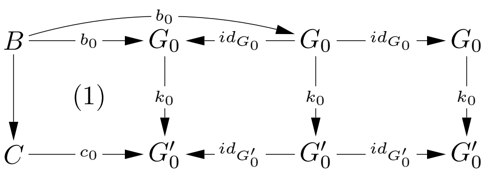
Theorem 6.14 (Embedding Theorem). *Given a transformation* $t : G_0 \overset{*}{\Rightarrow} G_n$ *and a morphism* $k_0 : G_0 \to G'_0$ *which is consistent with respect to* t*, then there is an extension diagram over* t *and* k_0*.*

Proof. We prove this theorem by induction over the number of direct transformation steps n.

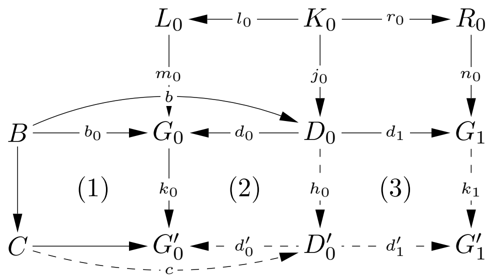
Consider a transformation $t : G_0 \overset{n}{\Rightarrow} G_n$ with a derived span $(G_0 \overset{d_0}{\leftarrow} D_n \overset{d_n}{\rightarrow} G_n)$, the initial pushout (1) over $k_0 : G_0 \to G'_0$, and a morphism $b : B \to D_n$ with $d_0 \circ b = b_0$. We show that there is a suitable extension diagram and suitable morphisms $b_n = d_n \circ b : B \to G_n$ and $c_n : C \to G'_n$, such that (P_n) is a pushout:



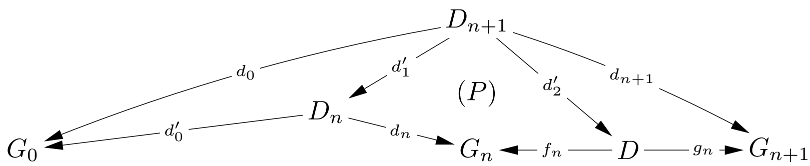
Basis. $n = 0$. Consider the transformation $t : G_0 \overset{id}{\Rightarrow} G_0$ with the derived span $(G_0 \leftarrow G_0 \rightarrow G_0)$ and the morphism $k_0 : G_0 \to G'_0$, consistent with respect to t. There is then the initial pushout (1) over k_0 and a morphism $b = b_0 : B \to G_0$, and we have the following extension diagram:



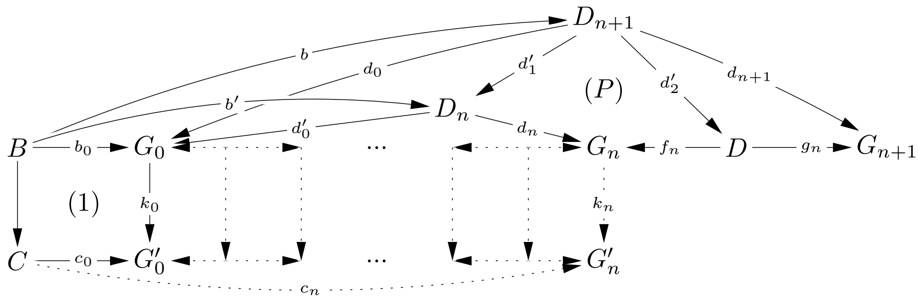
$n = 1$. Given the solid arrows in the following diagram, we can construct the pushout object D'_0 over $C \leftarrow B \stackrel{b}{\rightarrow} D_0$, derive the induced morphism d'_0 from this constructed pushout and, by pushout decomposition, conclude that (2) is also a pushout. Finally, we construct the pushout (3) over $D'_0 \stackrel{h_0}{\leftarrow} D_0 \stackrel{d_1}{\rightarrow} G_1$ and obtain the required extension diagram, and the morphisms $b_1 = d_1 \circ b : B \rightarrow G_1$ and $c_1 = d'_1 \circ c : C \rightarrow G'_1$. By pushout composition, (P_1) is a pushout.



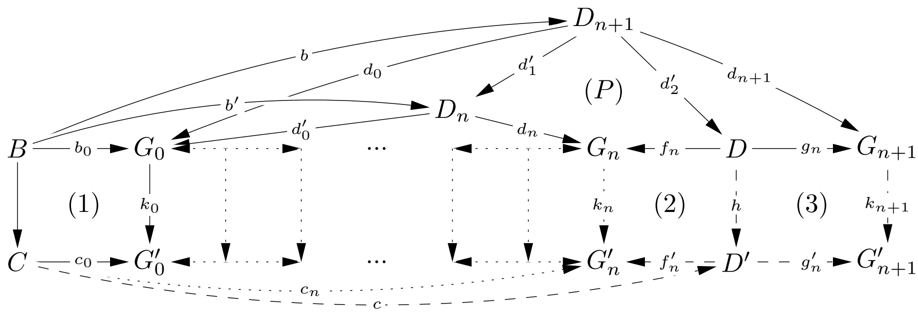
Induction step. Consider the transformation $t : G_0 \stackrel{n}{\Rightarrow} G_n \stackrel{p_n,m_n}{\Longrightarrow} G_{n+1}$ with a derived span $der(t) = (G_0 \stackrel{d_0}{\longleftarrow} D_{n+1} \stackrel{d_{n+1}}{\longrightarrow} G_{n+1})$. There is then a transformation $t' : G_0 \stackrel{n}{\Rightarrow} G_n$ with $der(t') = (G_0 \stackrel{d'_0}{\leftarrow} D_n \stackrel{d_n}{\rightarrow} G_n)$ such that (P) is the pullback obtained from the construction of the derived span, and we have the result that $d'_0 \circ d'_1 = d_0$ and $g_n \circ d'_2 = d_{n+1}$:



Since $k_0 : G_0 \rightarrow G'_0$ is consistent with respect to t, we have an initial pushout (1) over k_0 and a morphism $b : B \rightarrow D_{n+1}$ with $b_0 = d_0 \circ b = d'_0 \circ d'_1 \circ b$. This means that k_0 is also consistent with respect to t', using the morphism $b' = d'_1 \circ b$. We can apply the induction assumption, obtaining an extension diagram for t' and k_0 and morphisms $b_n = d_n \circ b' : B \rightarrow G_n$ and $c_n : C \rightarrow G'_n$ such that (P_n) is a pushout. This is denoted by the dotted arrows in the following diagram:



Now we construct the pushout object D' over $C \leftarrow B \xrightarrow{d'_2 \circ b} D$ and derive the induced morphism f'_n by applying $k_n \circ f_n$ and c_n to this constructed pushout. Since (P_n) is a pushout and it holds that $f_n \circ d'_2 \circ b = d_n \circ d'_1 \circ b = d_n \circ b' = b_n$, it follows by pushout decomposition that (2) is also a pushout. Finally, we construct the pushout (3) over $D' \overset{h}{\leftarrow} D \overset{g_n}{\to} G_{n+1}$ and obtain the required extension diagram and the morphisms $b_{n+1} = d_{n+1} \circ b : B \to G_{n+1}$ and $c_{n+1} = g'_n \circ c : C \to G'_{n+1}$. By pushout composition, (P_{n+1}) is a pushout.



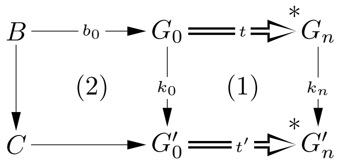
□

Example 6.15 (Embedding Theorem in *ExAHS*). Consider the transformation sequence $t : S \Rightarrow G \Rightarrow H$ in Example 5.5 and the extension morphism $k : S \to S'$ given in Example 6.8. In Example 6.13, we have verified that k is consistent with respect to t. We can conclude, from the Embedding Theorem, that there is an extension diagram over k and t. Indeed, this is the diagram presented in Example 6.8. □

Similarly to the graph case considered in Section 3.4, the next step is to show, in the following Extension Theorem, that the consistency condition is also necessary for the construction of extension diagrams, provided that we have initial pushouts over $\mathcal{M}'$-morphisms. Moreover, we are able to give a direct construction of G'_n in the extension diagram (1) below. This avoids the need to give an explicit construction of $t' : G'_0 \overset{*}{\Rightarrow} G'_n$.

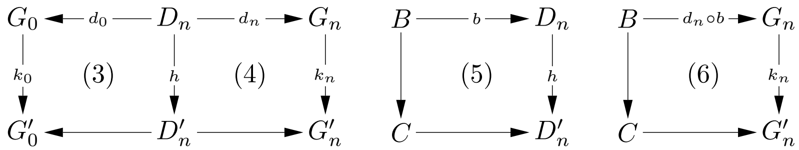
For technical reasons, we consider again, in addition to the class $\mathcal{M}$ of the (weak) adhesive HLR category, a class $\mathcal{M}'$ with suitable properties; such a class has already been used in Lemma 6.5. In the (typed) graph case, we can take $\mathcal{M}'$ as the class of all (typed) graph morphisms or as the class of all injective (typed) graph morphisms.

Theorem 6.16 (Extension Theorem). *Given a transformation* $t : G_0 \overset{*}{\Rightarrow} G_n$ *with a derived span* $der(t) = (G_0 \overset{d_0}{\leftarrow} D_n \overset{d_n}{\rightarrow} G_n)$ *and an extension diagram (1),*



with an initial pushout (2) over $k_0 \in \mathcal{M}'$ *for some class* $\mathcal{M}'$*, closed under pushouts and pullbacks along* $\mathcal{M}$*-morphisms and with initial pushouts over* $\mathcal{M}'$*-morphisms, then we have the following, shown in the diagram below:*

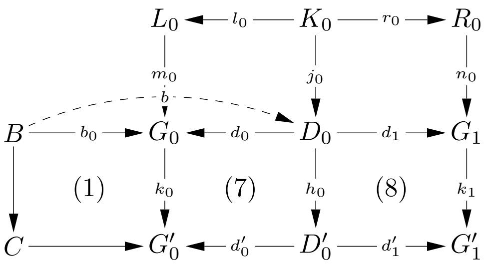
1. k_0 *is consistent with respect to* $t : G_0 \overset{*}{\Rightarrow} G_n$*, with the morphism* $b : B \to D_n$*.*
2. *There is a direct transformation* $G'_0 \Rightarrow G'_n$ *via* $der(t)$ *and* k_0 *given by the pushouts (3) and (4) with* $h, k_n \in \mathcal{M}'$*.*
3. *There are initial pushouts (5) and (6) over* $h \in \mathcal{M}'$ *and* $k_n \in \mathcal{M}'$*, respectively, with the same boundary–context morphism* $B \to C$*.*



Proof. We prove this theorem by induction over the number of direct transformation steps n.

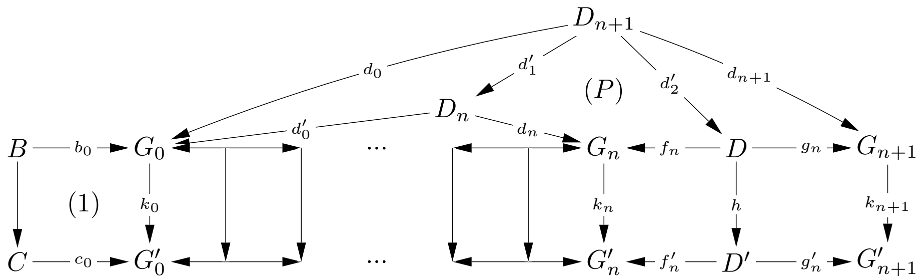
Basis. $n = 0$, $n = 1$. Given the solid arrows in the following diagram, for $n = 1$ and $t : G_0 \overset{p_0,m_0}{\Longrightarrow} G_1$ with $der(t) = (G_0 \overset{d_0}{\leftarrow} D_0 \overset{d_1}{\rightarrow} G_1)$, we conclude that:

1. k_0 is consistent with respect to t, since (1) is an initial pushout over k_0, and, since (7) is a pushout, we have $b : B \to D_0$ with $d_0 \circ b = b_0$.
2. (7) and (8) correspond to the required pushouts (3) and (4). In fact, (7) is a pushout along the $\mathcal{M}$-morphism d_0 and therefore a pullback. Since $\mathcal{M}'$ is closed under pullbacks along $\mathcal{M}$-morphisms, with $k_0 \in \mathcal{M}'$, it follows that $h_0 \in \mathcal{M}'$ also. (8) is a pushout along the $\mathcal{M}$-morphism d_1, and since $\mathcal{M}'$ is closed under pushouts along $\mathcal{M}$-morphisms, $k_1 \in \mathcal{M}'$ follows.
3. The initial pushouts corresponding to (5) and (6) follow directly from Lemma 6.5, where $d_0 \circ b = b_0$ has already been shown in item 1.

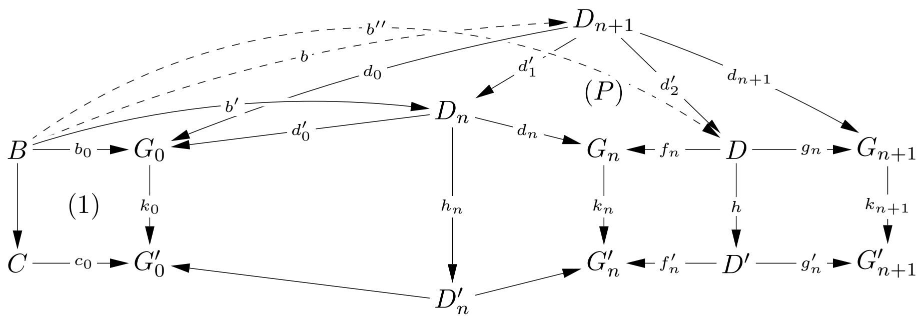


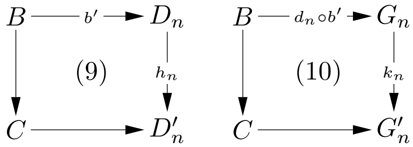
The case $n = 0$ can be dealt with analogously by substituting D_0 and G_1 by G_0, and d_0 and d_1 by id_{G_0}.

Induction step. Consider the transformation $t : G_0 \overset{n}{\Rightarrow} G_n \overset{p_n,m_n}{\Longrightarrow} G_{n+1}$ with a derived span $der(t) = (G_0 \overset{d_0}{\longleftarrow} D_{n+1} \overset{d_{n+1}}{\longrightarrow} G_{n+1})$, the following extension diagram, and the initial pushout (1) over $k_0 : G_0 \to G'_0$. There is then a transformation $t' : G_0 \overset{n}{\Rightarrow} G_n$ with $der(t') = (G_0 \overset{d'_0}{\leftarrow} D_n \overset{d_n}{\to} G_n)$ such that (P) is the pullback obtained from the construction of the derived span, and we have the result that $d'_0 \circ d'_1 = d_0$ and $g_n \circ d'_2 = d_{n+1}$:



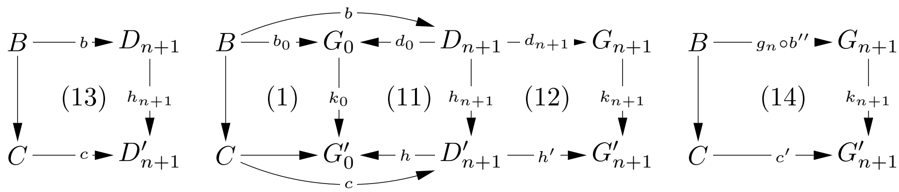
By the induction assumption, k_0 is consistent with respect to t', with a morphism $b' : B \to D_n$ such that $d'_0 \circ b' = b_0$, and there exists a transformation $G'_0 \Rightarrow G'_n$ via $der(t')$ with initial pushouts (9) over $h_n \in \mathcal{M}'$ and (10) over $k_n \in \mathcal{M}'$:





We then have the following:

1. The initiality of (10) implies a morphism $b'' : B \to D$ with $f_n \circ b'' = d_n \circ b'$. Since (P) is a pullback, there is an induced morphism $b : B \to D_{n+1}$ with $d'_2 \circ b = b''$ and $d'_1 \circ b = b'$. We then have the result that $d'_0 \circ d'_1 \circ b = d'_0 \circ b' = b_0$. Thus k_0 is consistent with respect to t.
2. Since k_0 is consistent with respect to t we can easily construct the transformation $G'_0 \Rightarrow G'_{n+1}$ via $der(t)$. First we construct the pushout (13) and obtain the induced morphism $h : D'_{n+1} \to G'_0$. By pushout decomposition, (11) is also a pushout. Lemma 6.5, with the initial pushout (10) over $k_n \in \mathcal{M}'$, implies that there is an initial pushout (14). Since $g_n \circ b'' = g_n \circ d'_2 \circ b = d_{n+1} \circ b$ (see item 1), we obtain from the pushout (13), in comparison with the object G'_{n+1}, a unique morphism $h' : D'_{n+1} \to G'_{n+1}$, with $h' \circ c = c'$ and $h' \circ h_{n+1} = k_{n+1} \circ d_{n+1}$. By pushout decomposition, it follows that (12) is a pushout.

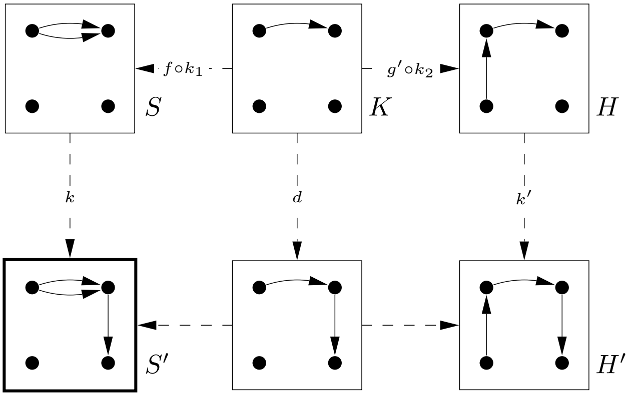


3. Lemma 6.5 states that (13) is an initial pushout over h_{n+1} and (14) is an initial pushout over k_{n+1} with $g_n \circ d'_2 \circ b = g_n \circ b''$ (as shown in items 1 and 2).

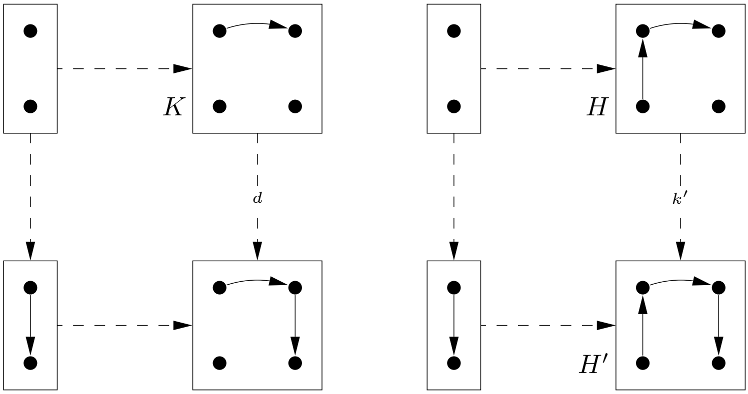
□

Example 6.17 (Extension Theorem in *ExAHS*). For the extension diagram in Example 6.8 with the derived span $der(t) = (S \overset{f \circ k_1}{\longleftarrow} K \overset{g' \circ k_2}{\longrightarrow} H)$ constructed in Example 6.11, we have shown in Example 6.13 that an initial pushout over k exists and that k is consistent with respect to t. Applying Theorem 6.16, we can conclude further that:

- There is a transformation $S' \Rightarrow H'$ via $der(t)$ and k, with d and k' being injective:



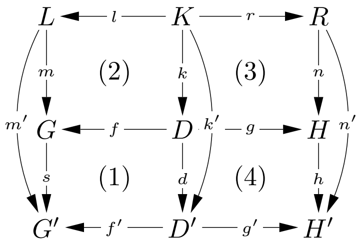
- There are initial pushouts over d and k':



□

In the following, we present a restriction construction which is in some sense inverse to the embedding construction in the Embedding Theorem (Theorem 6.14). The Restriction Theorem, however, is formulated only for direct transformations, in contrast to Theorem 6.14, which is formulated for general transformations. In [Ehr79], it was shown for the graph case that there is a corresponding theorem for the restriction of general graph transformations; however, this requires a consistency condition similar to Definition 6.12. It is most likely that such a general Restriction Theorem can also be formulated for adhesive HLR systems. However, in the following we need only the Restriction Theorem for direct transformations.

Theorem 6.18 (Restriction Theorem). *Given a direct transformation $G' \overset{p,m'}{\Longrightarrow} H'$, a morphism $s : G \to G' \in \mathcal{M}$, and a match $m : L \to G$ such that $s \circ m = m'$, then there is a direct transformation $G \overset{p,m}{\Rightarrow} H$ leading to the following extension diagram:*



Remark 6.19. In fact, it is sufficient to require $s \in \mathcal{M}'$ for a suitable morphism class $\mathcal{M}'$, where the $\mathcal{M}$–$\mathcal{M}'$ pushout–pullback decomposition property holds (see Definition 5.27).

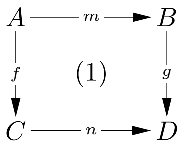
Proof. First we construct the pullback (1) over s and f' and obtain the induced morphism k from (1) in comparison with $m \circ l$ and k'. From the PO–PB decomposition, both (1) and (2) are pushouts using $l, s \in \mathcal{M}$. Now we construct the pushout (3) over k and r and obtain the induced morphism h; by pushout decomposition, (4) is also a pushout. □

6.3 Critical Pairs

We now present the concept of critical pairs, which leads in the next section to the Local Confluence Theorem. The ideas behind this have already been given in Section 3.4 of Part I.

Throughout this section, let $\mathcal{M}'$ be a morphism class closed under pushouts and pullbacks along $\mathcal{M}$-morphisms. This means that, given (1) with $m, n \in \mathcal{M}$, we have the results that:

- if (1) is a pushout and $f \in \mathcal{M}'$, then $g \in \mathcal{M}'$ also and
- if (1) is a pullback and $g \in \mathcal{M}'$, then $f \in \mathcal{M}'$ also:



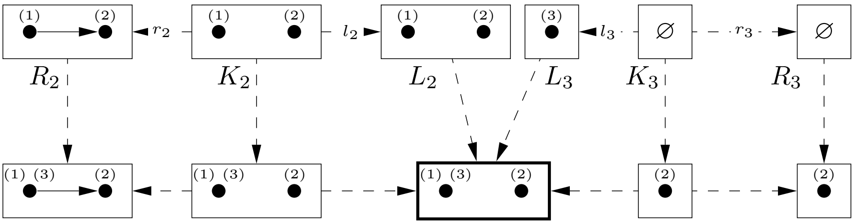
For the completeness of critical pairs considered in Lemma 6.22 and the Local Confluence Theorem given in Theorem 6.28, we need in addition the $\mathcal{M}$–$\mathcal{M}'$ pushout–pullback decomposition property (see Definition 5.27). In the (typed) graph case we take $\mathcal{M}' = \mathcal{M}$ as the class of all injective (typed) graph morphisms, but in Part III, for typed attributed graphs, we shall consider different morphism classes $\mathcal{M}$ and $\mathcal{M}'$.

On the basis of the $\mathcal{E}'$–$\mathcal{M}'$ pair factorization in Definition 5.25, we can define a critical pair as a pair of parallel dependent direct transformations, where both matches are a pair in $\mathcal{E}'$.

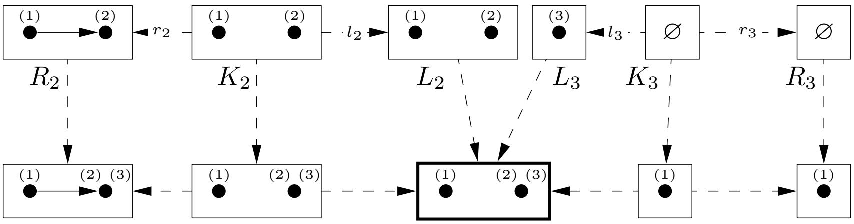
Definition 6.20 (critical pair). *Given an $\mathcal{E}'$–$\mathcal{M}'$ pair factorization, a* critical pair *is a pair of parallel dependent direct transformations $P_1 \overset{p_1,o_1}{\Longleftarrow} K \overset{p_2,o_2}{\Longrightarrow} P_2$ such that $(o_1, o_2) \in \mathcal{E}'$ for the corresponding matches o_1 and o_2.*

Example 6.21 (critical pairs in $ExAHS$). Consider the adhesive HLR system $ExAHS$ introduced in Example 5.5. We use an $\mathcal{E}'$–$\mathcal{M}'$ pair factorization, where there are pairs of jointly epimorphic morphisms in $\mathcal{E}'$, and $\mathcal{M}'$ is the class of all monomorphisms. We then have the following five critical pairs (up to isomorphism).

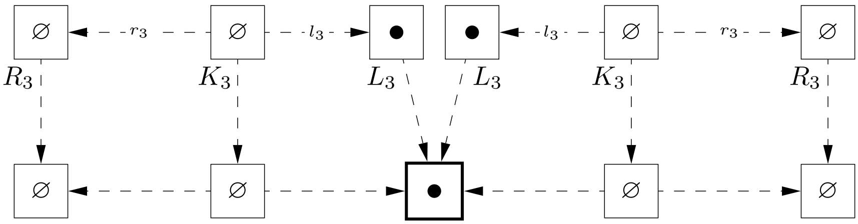
The first critical pair consists of the productions *addEdge* and *deleteVertex*, where *deleteVertex* deletes the souce node of the edge inserted by *addEdge*. Therefore these transformations are parallel dependent. The choice of the matches and their codomain object makes sure that they are jointly surjective:



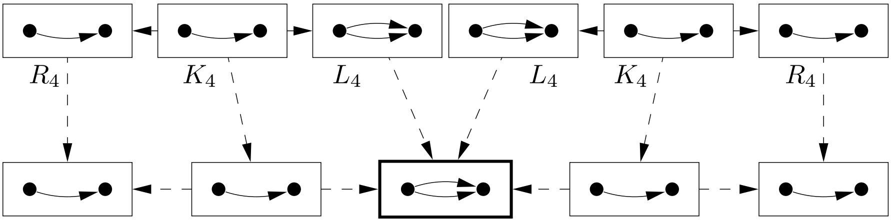
The second critical pair has the same productions *addEdge* and *deleteVertex*, but *deleteVertex* deletes the target node of the edge inserted by *addEdge*:



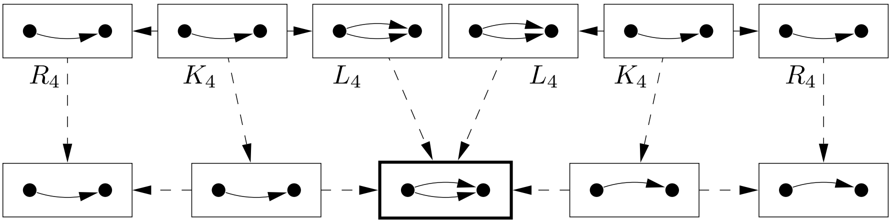
The third critical pair contains the production *deleteVertex* twice: the same vertex is deleted by both transformations:



The fourth critical pair contains the production *del1of2edges* twice. The same edge is deleted by both transformations:



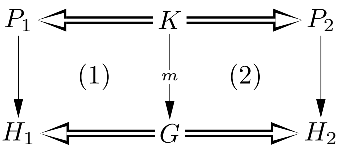
The last critical pair consists also of the production $del1of2edges$ twice. In this case, different edges are deleted by the transformations. However, to apply $del1of2edges$, both edges are necessary:



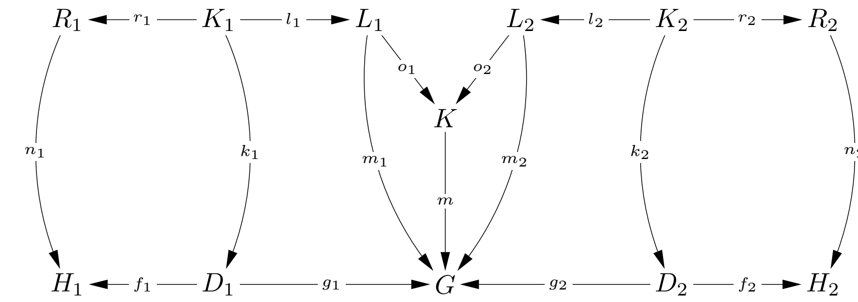
□

The following lemma shows that every pair of parallel dependent direct transformations is an extension of a critical pair. It generalizes Lemma 3.33 from graphs to high-level structures.

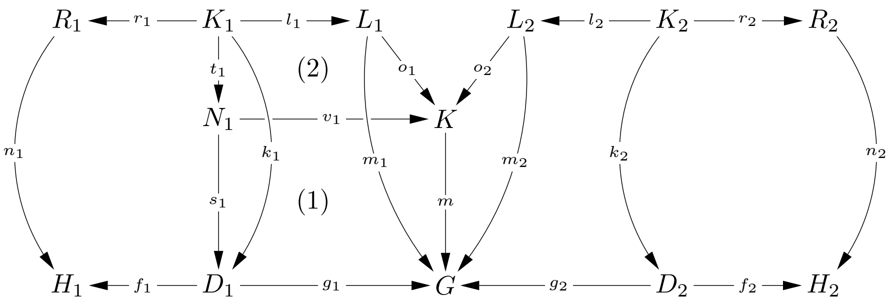
Lemma 6.22 (completeness of critical pairs). *Consider an adhesive HLR system with an $\mathcal{E}'$–$\mathcal{M}'$ pair factorization, where the $\mathcal{M}$–$\mathcal{M}'$ pushout–pullback decomposition property holds (see Definition 5.27). The critical pairs are then complete. This means that for each pair of parallel dependent direct transformations $H_1 \overset{p_1,m_1}{\Longleftarrow} G \overset{p_2,m_2}{\Longrightarrow} H_2$, there is a critical pair $P_1 \overset{p_1,o_1}{\Longleftarrow} K \overset{p_2,o_2}{\Longrightarrow} P_2$ with extension diagrams (1) and (2) and $m \in \mathcal{M}'$:*



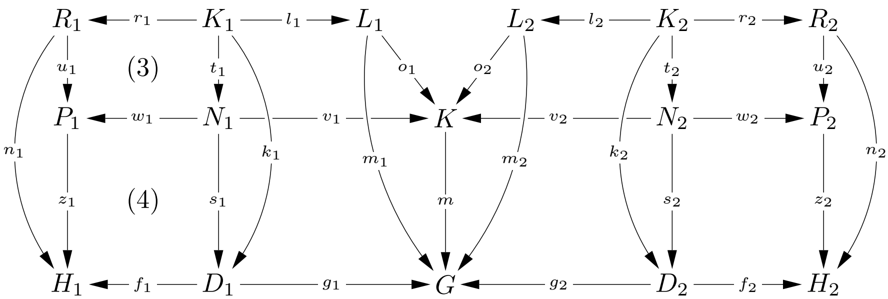
Proof. From the $\mathcal{E}'$–$\mathcal{M}'$ pair factorization, for m_1 and m_2 there exists an object K and morphisms $m : K \to G \in \mathcal{M}'$, $o_1 : L_1 \to K$, and $o_2 : L_2 \to K$, with $(o_1, o_2) \in \mathcal{E}'$ such that $m_1 = m \circ o_1$ and $m_2 = m \circ o_2$:



We can construct the required extension diagram. First we construct the pullback (1) over g_1 and m and derive the induced morphism t_1. By applying the $\mathcal{M}$–$\mathcal{M}'$ pushout–pullback decomposition property, we find that both squares (1) and (2) are pushouts, because $l_1 \in \mathcal{M}$ and $m \in \mathcal{M}'$:



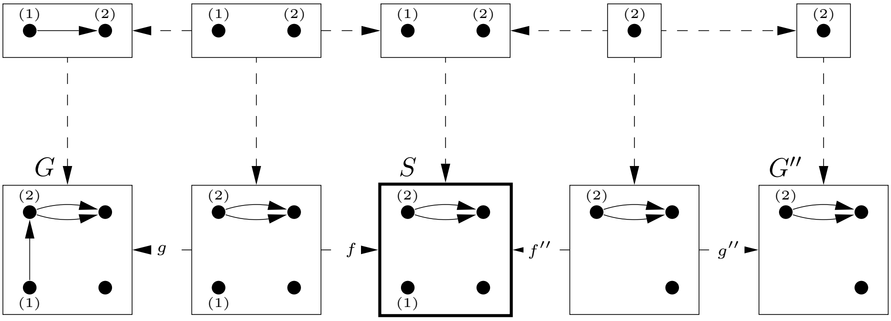
We then construct the pushout (3) over r_1 and t_1 and derive the induced morphism z_1. By pushout decomposition, the square (4) is a pushout. The same construction is applied to the second transformation. This results in the following extension diagrams, where the lower part corresponds to the required extension diagrams (1) and (2) with $m \in \mathcal{M}'$:



Now we show that $P_1 \Leftarrow K \Rightarrow P_2$ is a critical pair. We know that $(o_1, o_2) \in \mathcal{E}'$, by construction. It remains to show that the pair $P_1 \overset{p_1,o_1}{\Longleftarrow} K \overset{p_2,o_2}{\Longrightarrow} P_2$ is parallel dependent. Otherwise, there are morphisms $i : L_1 \to N_2$ and $j : L_2 \to N_1$ with $v_2 \circ i = o_1$ and $v_1 \circ j = o_2$. Then $g_2 \circ s_2 \circ i = m \circ v_2 \circ i = m \circ o_1 = m_1$ and $g_1 \circ s_1 \circ j = m \circ v_1 \circ j = m \circ o_2 = m_2$, which means that $H_1 \overset{p_1,m_1}{\Longleftarrow} G \overset{p_2,m_2}{\Longrightarrow} H_2$

are parallel independent, which is a contradiction. Thus, $P_1 \overset{p_1,o_1}{\Longleftarrow} K \overset{p_2,o_2}{\Longrightarrow} P_2$ is a critical pair. □

Example 6.23 (completeness of critical pairs). The pair of parallel dependent direct transformations $G \Leftarrow S \Rightarrow G''$ in Example 5.11 leads, by the construction in the proof, to the first critical pair from Example 6.21 and the following extension diagrams. We have not shown the productions, but only the actual extensions.

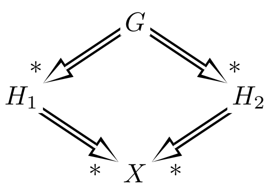


□

6.4 Local Confluence Theorem

We now present the Local Confluence Theorem for adhesive HLR systems. This theorem has been considered in Section 3.4 in Part I for graph transformation systems. As shown in Section 3.4 for graphs and in the following for adhesive HLR systems, local confluence and termination imply confluence, which is the main property of interest. Termination is discussed for the case of graphs in Section 3.4 and analyzed in more detail for the case of typed attributed graph transformation systems in Chapter 12 in Part III.

Definition 6.24 (confluence). *A pair of transformations* $H_1 \overset{*}{\Leftarrow} G \overset{*}{\Rightarrow} H_2$ *is* confluent *if there are transformations* $H_1 \overset{*}{\Rightarrow} X$ *and* $H_2 \overset{*}{\Rightarrow} X$*:*



An adhesive HLR system is locally confluent *if this property holds for each pair of direct transformations. The system is* confluent *if this holds for all pairs of transformations.*

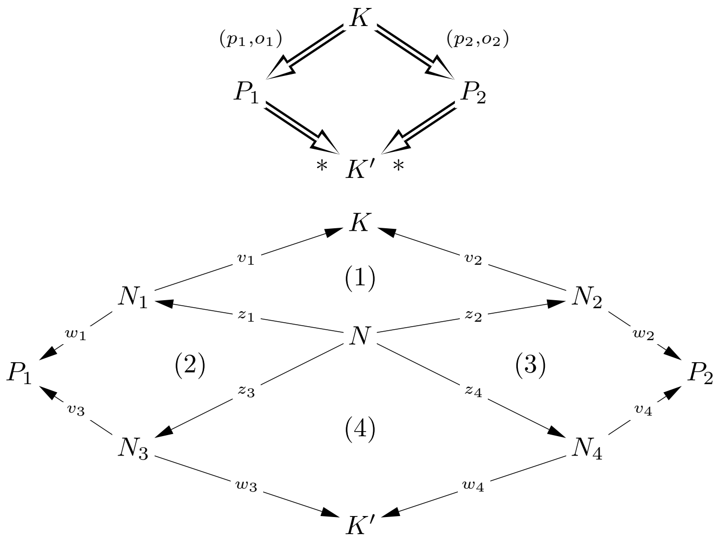
Lemma 6.25 (termination and local confluence imply confluence). *Every terminating and locally confluent adhesive HLR system is confluent.*

Proof. See Section C.2. □

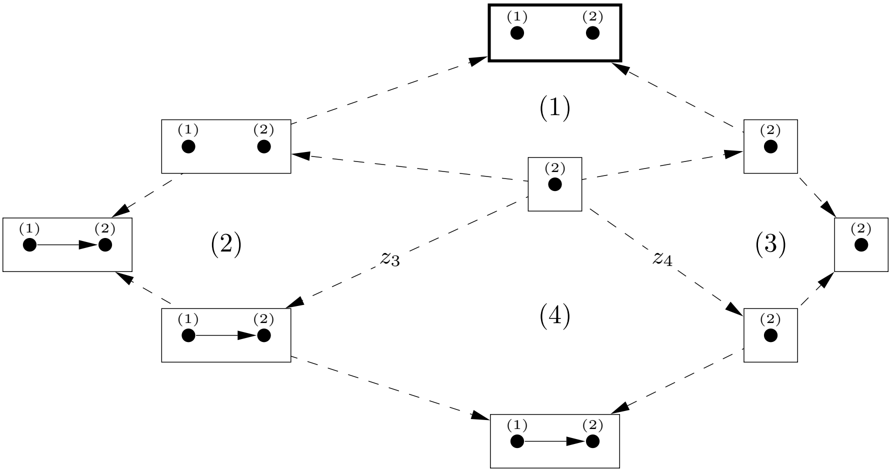
It remains to show local confluence. Roughly speaking, we have to require that all critical pairs are confluent. Unfortunately, however, confluence of critical pairs is not sufficient to show local confluence. As discussed in Subsection 3.4.3, we need strict confluence of critical pairs, which is defined in the following.

Definition 6.26 (strict confluence of critical pairs). *A critical pair* $K \overset{p_1,o_1}{\Longrightarrow} P_1$, $K \overset{p_2,o_2}{\Longrightarrow} P_2$ *is called* strictly confluent, *if we have the following:*

1. Confluence.*: the critical pair is confluent, i.e. there are transformations* $P_1 \overset{*}{\Rightarrow} K'$, $P_2 \overset{*}{\Rightarrow} K'$ *with derived spans* $der(P_i \overset{*}{\Rightarrow} K') = (P_i \overset{v_{i+2}}{\leftarrow} N_{i+2} \overset{w_{i+2}}{\rightarrow} K')$ *for* $i = 1, 2$.
2. Strictness. *Let* $der(K \overset{p_i,o_i}{\Longrightarrow} P_i) = (K \overset{v_i}{\leftarrow} N_i \overset{w_i}{\rightarrow} P_i)$ *for* $i = 1, 2$, *and let* N *be the pullback object of the pullback (1). There are then morphisms* z_3 *and* z_4 *such that (2), (3), and (4) commute:*

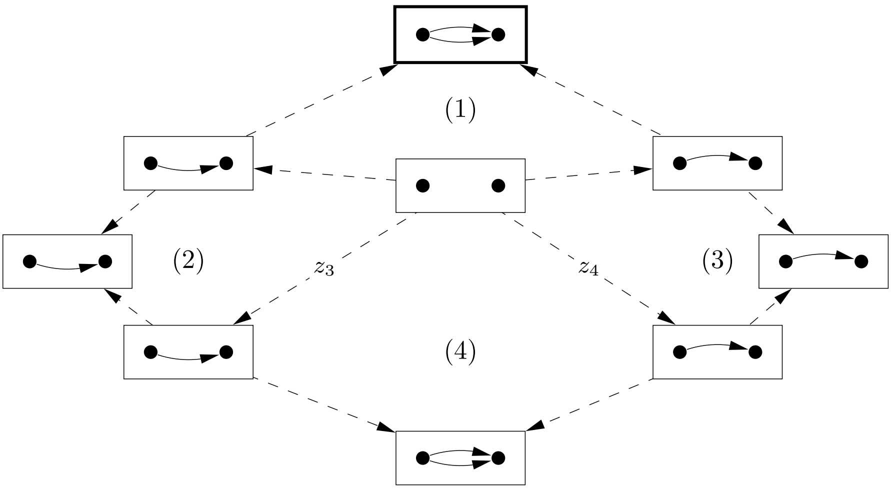


Example 6.27 (strict confluence in $ExAHS$**).** In our adhesive HLR system $ExAHS$, all critical pairs defined in Example 6.21 are strictly confluent. The confluence of the first and the second critical pair is established by applying no further transformation to the first graph and applying $addVertex$ and $addEdge$ to the second graph. This is shown in the following diagram for the first critical pair, and works analogously for the second pair. The strictness condition holds for the morphisms z_3 and z_4 shown:



The third critical pair is also confluent, since both transformations result in the empty graph. In the strictness diagram, all graphs except for K are empty, and therefore the strictness condition is fulfilled. Similarly, for the fourth critical pair, both transformations result in the same graph, with two nodes and one edge between them. This is the graph for all objects in the strictness diagram except K, which has two edges between the two nodes.

For the last critical pair, we can reverse the deletion of the edges by applying the production *addEdge* to both graphs. The following diagram shows that the strictness condition holds, since all morphisms are inclusions:



□

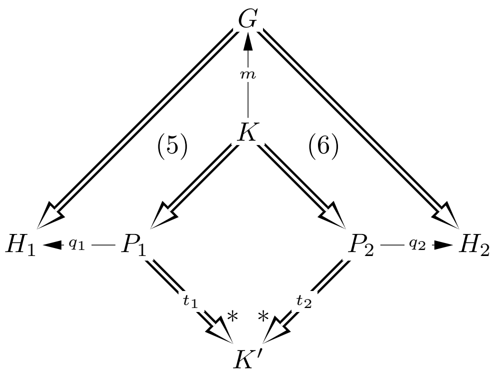
Now we are able to prove the following Local Confluence Theorem for adhesive HLR systems, which generalizes Theorem 3.34 for the graph case. In the special case of graphs, $\mathcal{E}'$ is the class of pairs of jointly surjective (typed) graph morphisms and $\mathcal{M}' = \mathcal{M}$ is the class of all injective (typed) graph morphisms. In the case of typed attributed graphs considered in Part III, we shall consider different choices for $\mathcal{E}'$, $\mathcal{M}'$, and $\mathcal{M}$.

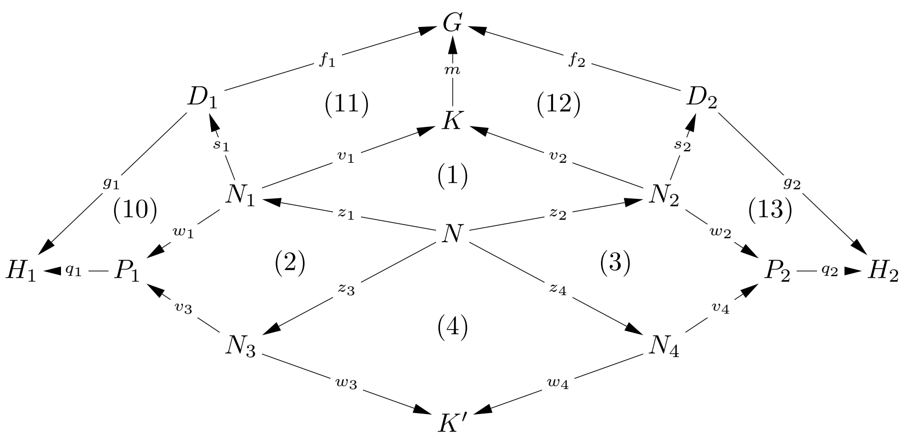
Theorem 6.28 (Local Confluence Theorem and Critical Pair Lemma). *Given an adhesive HLR system AHS with an $\mathcal{E}'$–$\mathcal{M}'$ pair factorization, let $\mathcal{M}'$ be a morphism class closed under pushouts and pullbacks along $\mathcal{M}$-morphisms, with initial pushouts over $\mathcal{M}'$-morphisms and where the $\mathcal{M}$–$\mathcal{M}'$ pushout–pullback decomposition property is fulfilled. AHS is then locally confluent if all its critical pairs are strictly confluent.*

Proof. For a given pair of direct transformations $H_1 \overset{p_1,m_2}{\Leftarrow} G \overset{p_2,m_2}{\Rightarrow} H_2$, we have to show the existence of transformations $t_1' : H_1 \overset{*}{\Rightarrow} G'$ and $t_2' : H_2 \overset{*}{\Rightarrow} G'$.

If the given pair is parallel independent, this follows from Theorem 5.12.

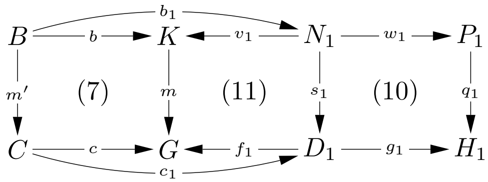
If the given pair is parallel dependent, Lemma 6.22 implies the existence of a critical pair $P_1 \overset{p_1,o_1}{\Longleftarrow} K \overset{p_2,o_2}{\Longrightarrow} P_2$ with the extension diagrams (5) and (6) below, and $m \in \mathcal{M}'$. By assumption, this critical pair is strictly confluent, leading to transformations $t_1 : P_1 \overset{*}{\Rightarrow} K'$, $t_2 : P_2 \overset{*}{\Rightarrow} K'$ and the following diagrams:



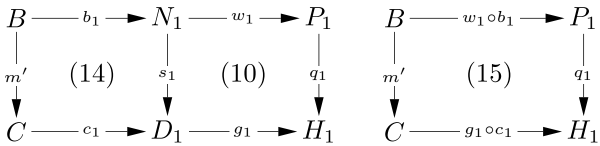


Since $v_1, v_2 \in \mathcal{M}$, (1) is a pullback, and $\mathcal{M}$ is closed under pullbacks, we have the result that $z_1, z_2 \in \mathcal{M}$. The fact that $w_1, w_2, v_3, v_4 \in \mathcal{M}$, (2) and (3) are commutative, and $\mathcal{M}$ is closed under decomposition gives us $z_3, z_4 \in \mathcal{M}$.

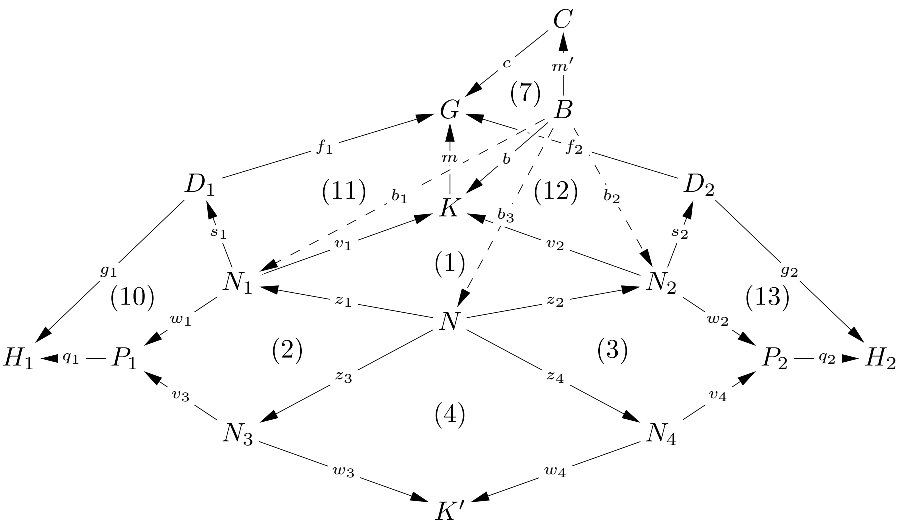
Now let (7) be an initial pushout over $m \in \mathcal{M}'$, and consider the double pushouts (10) and (11) corresponding to the extension diagram (5):



The initiality of (7), applied to the pushout (11), leads to unique morphisms b_1, $c_1 \in \mathcal{M}$ such that $v_1 \circ b_1 = b$, $f_1 \circ c_1 = c$, and (14) is a pushout. By Lemma 6.5, (14) is an initial pushout over s_1 and (15) is an initial pushout over q_1:



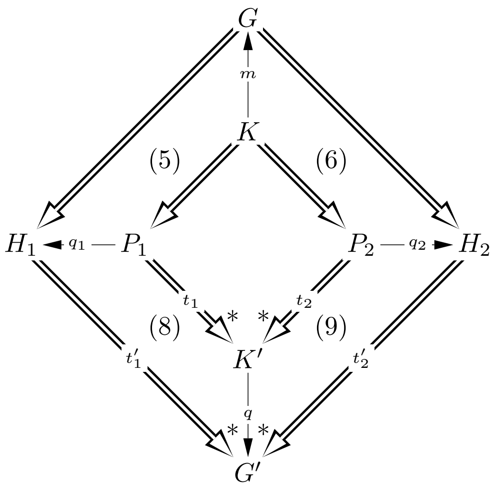
Dually, we obtain morphisms b_2, $c_2 \in \mathcal{M}$ with $v_2 \circ b_2 = b$ from (12) and (13). Using the pullback property of (1) with $v_1 \circ b_1 = b = v_2 \circ b_2$, we obtain a unique $b_3 : B \to N$ with $z_1 \circ b_3 = b_1$ and $z_2 \circ b_3 = b_2$. Moreover, b_1, $z_1 \in \mathcal{M}$ implies $b_3 \in \mathcal{M}$ by the decomposition property of $\mathcal{M}$:



In order to show the consistency of q_1 with respect to t_1, with the initial pushout (15) over q_1, we have to construct $b'_3 : B \to N_3 \in \mathcal{M}$ such that

$v_3 \circ b_3' = w_1 \circ b_1$. This holds for $b_3' = z_3 \circ b_3$, since then $v_3 \circ b_3' = v_3 \circ z_3 \circ b_3 \overset{(2)}{=} w_1 \circ z_1 \circ b_3 = w_1 \circ b_1$. It holds that $b_3' \in \mathcal{M}$, by the composition of $\mathcal{M}$-morphisms.

Dually, q_2 is consistent with respect to t_2, using $b_4' = z_4 \circ b_3 \in \mathcal{M}$ and the commutativity of (3). By Theorem 6.14, we obtain extension the diagrams (8) and (9), where the morphism $q : K' \to G'$ is the same in both cases:



This equality can be shown using part 3 of Theorem 6.16, where q is determined by an initial pushout of $m' : B \to C$ and $w_3 \circ b_3' : B \to K'$ in the first case and $w_4 \circ b_4' : B \to K'$ in the second case, and we have $w_3 \circ b_3' = w_4 \circ b_4'$ given by the commutativity of (4). □

Example 6.29 (local confluence of ***ExAHS*****).** In *ExAHS*, we have $\mathcal{M}' = \mathcal{M}$ and initial pushouts over injective graph morphisms. Therefore all preconditions for the Local Confluence Theorem are fulfilled. Since all critical pairs in *ExAHS* are strictly confluent, as shown in Example 6.27, *ExAHS* is locally confluent. □