Sensitivity Analysis for Fuzzy Shortest Path Problem

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Abstract. The shortest path problem is an optimization problem in which the best path between two considered objects is searched for in accordance with an optimization criterion, which has to be minimized. In this paper this problem is investigated in the case when the distances between the nodes are fuzzy numbers. The problem is formulated as a linear optimization problem with fuzzy coefficients in the objective function. This problem is solved using crisp parametric two-criterial linear optimization. Special emphasis is given to the sensitivity of the solution with respect to the fuzzy objective function coefficients.

1 Introduction

Consider a directed graph $G = (V, E)$, where $V = \{v_1, v_2, ..., v_n\}$ is a set of nodes and $E = \{(v_1, w_1), \ldots, (v_m, w_m)\}$ is a set of directed edges. Each edge $(v, w) \in E$ connects two nodes $v, w \in V$ of the graph G. There is a positive number (or a weight) $c(v, w)$ associated with each edge $(v, w) \in E$ that can represent the length of this edge, the time needed to cover it, etc. Given two nodes $q \in V$ and $s \in V$ a path from q to s is a sequence of edges $\{(u_0, u_1), (u_1, u_2), \ldots, (u_{t-2}, u_{t-1}), (u_{t-1}, u_t)\} \subseteq E$ with $u_0 = q$ and $u_t = s$, where $\{q, u_1, \ldots, u_t, s\} \subseteq V$ are all distinct. In the shortest path problem a path from q to s is searched with the minimal length $\sum_{i=0}^{t-1} c(u_i, u_{i+1})$.

Denote a set of arrows beginning in node $v \in V$ by $\Gamma_+(v) := \{(\overline{v}, \overline{w}) \in E : \overline{v} = v\}$ and $\deg^+(v) := |\Gamma_+(v)|$ is the *outdegree* of $v \in V$. Similarly, the set $\Gamma_-(w) := \{(\overline{v}, \overline{w}) \in E : \overline{w} = w\}$ describes a set of arrows which are ended in the node $w \in V$ and $\deg^-(w) := |\Gamma_-(w)|$ is the *indegree* of node $w \in V$. Then, the shortest path problem can be modelled as a linear optimization problem [6, 8] as:

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$$\begin{aligned} &\sum_{(v,w)\in E} c(v,w)x(v,w) \to \min \\ &\sum_{w\in\Gamma_+(v)} x(v,w) - \sum_{u\in\Gamma_-(v)} x(u,v) = g(v),\ \forall v \in V \\ &x(v,w) \geq 0,\ \forall (v,w) \in E, \end{aligned} \tag{1}$$

where

$$g(v) = \begin{cases} 0, & \text{if } v \notin \{q,s\} \\ 1, & \text{for } v = q \\ -1, & \text{for } v = s. \end{cases} \tag{2}$$

Denote the feasible set of this problem by M. Walks in graph G correspond to integer feasible solutions of this problem. $M \neq \emptyset$ whenever there is a path from q to s. Since the coefficient matrix of the constraints of this problem is totally unimodular [7], the vertices of M have integer components and, hence, are the incidence vectors of the walks in G, i.e. sequences of edges starting in q and ending in s but possibly crossing one node multiply. In this case it is said that the walk contains cycles. If all the distances between the nodes of the graph G are positive, optimal vertex solutions of problem (1) are paths, they do not contain cycles. Hence, in what follows, $c(v,w) > 0$ for all $v, w \in V$ is assumed.

Usually it is supposed that the parameters $c(v,w)$ in the objective function of this model are exactly known. However, in many real situations these data can not be given exactly because of the influence of various factors of environment. Then the problem can be appropriately modelled using a graph with fuzzy parameters. This corresponds to a model (1) with fuzzy coefficients in the objective function.

Focus in this paper is on the situation when the membership functions of the distances are not precisely known in advance. This could be considered as a realistic situation. Then, both the dependency of the optimal (fuzzy) solution of the problem on the fuzzy distances as well as a path from q to s in the graph being more or less equally "good" for all possible distances are of special interest. For related investigations of a fuzzy linear optimization problem the interested reader is referred to the paper [4].

2 The two-criterial optimization approach

Assume now that in model (1) the weights $c(v,w)$ are fuzzy numbers of the type $L-L$ [5]:

$$\tilde{c}(v,w) = (\underline{c}(v,w); \overline{c}(v,w); \alpha(v,w); \beta(v,w))_{L-L},\ \forall\ (v,w) \in E \tag{3}$$

where $\underline{c}(v,w), \overline{c}(v,w)$ - are the left and right borders of the fuzzy number $\tilde{c}(v,w)$ corresponding to the maximal reliability level ($\lambda = 1$) and $\alpha(v,w)$ and $\beta(v,w)$ are non-negative real numbers (cf. Figure 1). To guarantee that $\tilde{c}(v,w)$ is positive it is assumed that $\underline{c}(v,w) - \alpha(v,w) > 0$ for all edges $(v,w) \in E$.

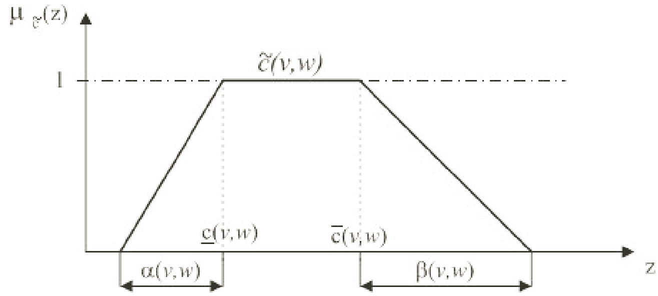

Fig. 1. Used type of fuzzy numbers

A fuzzy number $\tilde{c}(v,w)$ is defined as a fuzzy set in the space of real numbers with the following membership function [5]:

$$\mu_{\tilde{c}}(z) = \begin{cases} 1 & \text{if } \underline{c} \leq z \leq \overline{c}, \\ L\left(\dfrac{\underline{c}-z}{\alpha}\right) & \text{if } z \leq \underline{c}, \\ L\left(\dfrac{z-\overline{c}}{\beta}\right) & \text{if } z \geq \overline{c}, \end{cases} \tag{4}$$

where L is a shape function, which satisfies to following conditions:
- L is a continuous non-increasing function on $[0,\infty)$ with $L(0)=1$;
- L is strictly decreasing on that part of $[0,\infty)$ on which it is positive.

The shortest path problem in a directed graph with fuzzy weights (or "fuzzy shortest path problem" for short) is problem (1) with $c(v,w)$ being replaced with the fuzzy weights $\tilde{c}(v,w)$ in the objective function. In analogy with [3, 2] this problem can be associated with a set of the following problems, which depend on a parameter $\theta \in (0,1)$:

$$\begin{aligned} & f_1(x,\theta) = \sum_{(v,w)\in E} (\underline{c}(v,w) - \alpha(v,w)\theta)x(v,w) \rightarrow \min \\ & f_2(x,\theta) = \sum_{(v,w)\in E} (\overline{c}(v,w) + \beta(v,w)\theta)x(v,w) \rightarrow \min \\ & \sum_{w\in\Gamma_+(v)} x(v,w) - \sum_{u\in\Gamma_-(v)} x(u,v) = g(v),\ \forall v \in V, \\ & x(v,w) \geq 0,\ \forall (v,w) \in E. \end{aligned} \tag{5}$$

This model is based on the preference relation

$$a \leq b \Longleftrightarrow \underline{a} \leq \underline{b} \wedge \overline{a} \leq \overline{b},$$
$$a < b \Longleftrightarrow a \leq b \wedge a \neq b,$$

between intervals $a = [\underline{a}, \overline{a}]$ and $b = [\underline{b}, \overline{b}]$ [2].

Thus, to find a shortest path between the nodes q and s problem (5) has to be solved for all $\theta \in (0,1)$.

Problem (5) is a two-criterial optimization problem. One solution concept for such problems is to find one (or better all) Pareto-optimal solution(s).

Definition 1. *A point $x^* \in X$ is a Pareto-optimal solution of a two-criterial optimization problem*

$$\begin{aligned} & f_1(x,\theta) \to \min \\ & f_2(x,\theta) \to \min \\ & x \in X \end{aligned}$$

at $\theta = \theta^*$ if there does not exist another point $\overline{x} \in X$ with $f_1(\overline{x},\theta^*) \leq f_1(x^*,\theta^*)$ and $f_2(\overline{x},\theta^*) \leq f_2(x^*,\theta^*)$ with at least one strict inequality.

Hence, the sets of Pareto-optimal solutions $\Psi(\theta)$ of problem (5) are searched for all $\theta \in [\underline{\theta},\overline{\theta}]$.

As result a number of different paths in the graph G are computed and each such path is Pareto-optimal for problem (5). All these paths can now be used to compose the fuzzy optimal solution $\tilde{x}$ of the initial fuzzy shortest path problem. Let $\Psi(\theta)$ denote the set of Pareto-optimal solutions of problem (5) for fixed θ. Then the frequency of $x \in \Psi(\theta)$ for $\theta \in [0,1]$ can be used to determine a membership function for $\tilde{x}$ [3]:

$$\mu_{FS}(x) = \left| \left\{ \begin{array}{cc} \lambda \in [0,1] : & x \text{ is a Pareto-optimal vertex} \\ & \text{of the problem (5) for } \theta = L^{-1}(\lambda) \end{array} \right\} \right| .$$

Here, $|Q|$ means the geometric measure of the set Q. Since the set of Pareto-optimal points can be computed using parametric linear programming, $\Psi(\theta)$ equals the union of faces of M. By parametric linear programming, too, Pareto-optimal solutions for one parameter value θ^0 remain Pareto-optimal for all parameter values within some interval $[\underline{\theta},\overline{\theta}]$. This implies that

$$\mu_{FS}(x) = \sum_{i=1}^{l} (L(\theta_{2i-1}) - L(\theta_{2i})) \tag{6}$$

where $\{\theta_i\}_{i=1}^{2l}$ is such that x is Pareto-optimal for problem (5) for all $\theta \in [\theta_{2i-1},\theta_{2i}],\ i = 1,\ldots,l$.

3 Sensitivity analysis

Usually the fuzzy numbers $\tilde{c}(v,w)$ have been determined by a group of experts. Asking other experts other fuzzy numbers will result. Hence, it is an interesting question to consider the dependency of the solutions obtained from the parameters of the fuzzy numbers $\tilde{c}(v,w)$. In the following only the special case of the question is considered in which these numbers are perturbed by an additive number $\delta(v,w), \forall\ (v,w) \in E$ (cf. Figure 2). Applying such perturbations to the problem (5) the following model arises:

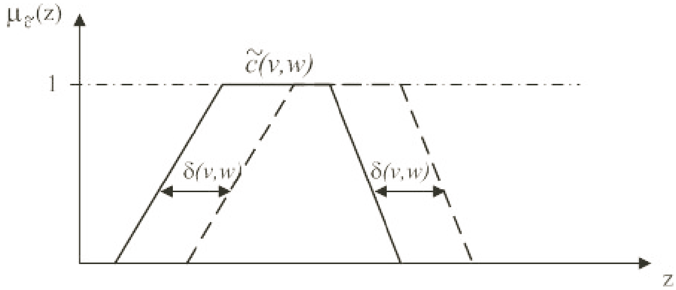

Fig. 2. Perturbed fuzzy numbers

$$\begin{aligned}
&f_1(x,\theta) = \sum_{(v,w)\in E} [\underline{c}(v,w) - \alpha(v,w)\theta + \delta(v,w)]\, x(v,w) \to \min \\
&f_2(x,\theta) = \sum_{(v,w)\in E} [\overline{c}(v,w) + \beta(v,w)\theta + \delta(v,w)]\, x(v,w) \to \min \\
&\sum_{w\in\Gamma_+(v)} x(v,w) - \sum_{u\in\Gamma_-(v)} x(u,v) = g(v),\ \forall v \in V, \\
&x(v,w) \geq 0,\ \forall (v,w) \in E.
\end{aligned} \tag{7}$$

The interesting point here is the determination of the range in which the $\delta(v,w)$ may vary without violating Pareto-optimality of some path in G. Let

$$\mathcal{R}(x,\theta) := \{\delta : x \in \Psi^{\delta}(\theta)\}$$

denote this set and call it *region of stability* of the path in G with incidence vector x. Here, $\Psi^{\delta}(\theta)$ denotes the set of Pareto-optimal vertices of problem (7).

Theorem 1. *For fixed θ and each feasible point x the set $cl\,\mathcal{R}(x,\theta)$ is a convex polyhedron.*

Proof. Abbreviate the coefficient matrix of the constraints in M by A such that

$$M = \{x : Ax = g,\ x \geq 0\}.$$

The matrix A is the incidence matrix of G, having exactly two nonzero entries in each column. The columns correspond to the edges $(v,w) \in E$ of G with a 1 in the row v and a -1 in row w. In this notation, the vector x is determined by $x_{(v,w)} := x(v,w)$.

Then, the normal cone to M at some incidence vector x to a path in G is

$$N_M(x) = \{z : \exists\ y,\ \exists t \geq 0 \text{ with } z = A^{\top}y + It, x^{\top}t = 0\},$$

where I denotes the unit matrix. An incidence vector x is Pareto-optimal for problem (7) iff there exists $\gamma \in (0,1)$ such that x is an optimal solution of the problem

$$\left.\begin{array}{l}\gamma f_1(x,\theta) + (1-\gamma) f_2(x,\theta) \to \min \\ Ax = g \\ x \geq 0\end{array}\right\} \quad (8)$$

cf.e.g. [10]. A necessary and sufficient optimality condition for this problem is

$$-\gamma \nabla f_1(x,\theta) - (1-\gamma)\nabla f_2(x,\theta) \in N_M(x) \quad (9)$$

by linear programming. Here,

$$\nabla f_1(x,\theta) = \Big(\underline{c}(v,w) - \alpha(v,w)\theta + \delta(v,w)\Big)_{(v,w)\in E}$$

and

$$\nabla f_2(x,\theta) = \Big(\overline{c}(v,w) + \beta(v,w)\theta + \delta(v,w)\Big)_{(v,w)\in E}$$

are independent of x. This implies that (9) is a system of linear equalities and inequalities in δ, y, t, γ. Hence, the projection of the solution set of this system onto the δ–space is a convex polyhedron.

Formula (9) can be used both to compute the bounds θ_i in (6) by setting $\delta \equiv 0$ and the dependency of θ_i from δ in a neighborhood of $\delta \equiv 0$. The θ_i are the bounds of θ for which x enters the set of Pareto-optimal solutions respectively leaves this set. Note, that this system is no longer linear if θ is not constant. This results in nonconvex regions of stability which is also reflected by the results in [3].

4 Robust optimization

In contrast to sensitivity analysis where the dependency of shortest paths on variations of the membership functions of the distances is investigated, robust optimization intends to find paths in G which are at the same time "equally good" with respect to all membership functions in a certain set [1]. For that, let P denote a set of all possible realizations of membership functions for the distances between the nodes of the graph G and assume that the membership functions in P are composed by the elements in a convex bounded polyhedron for simplicity. Hence, this polyhedron is given by

$$Q = \text{ conv } \left\{ \Big(\underline{c}^k(v,w), \alpha^k(v,w), \overline{c}^k(v,w), \beta^k(v,w)\Big)_{(v,w)\in E} : k = 1,\ldots,K \right\},$$

the convex hull of its K vertices

$$\Big(\underline{c}^k(v,w), \alpha^k(v,w), \overline{c}^k(v,w), \beta^k(v,w)\Big)_{(v,w)\in E}.$$

This results in the two-criterial optimization problem

$$\left.\begin{array}{l} z_1 \to \min \\ z_2 \to \min \\ \sum\limits_{(v,w)\in E} (\underline{c}(v,w) - \alpha(v,w)\theta)\, x(v,w) \le z_1 \;\forall\, (\underline{c},\alpha,\overline{c},\beta) \in Q \\ \sum\limits_{(v,w)\in E} (\overline{c}(v,w) + \beta(v,w)\theta)\, x(v,w) \le z_2 \;\forall\, (\underline{c},\alpha,\overline{c},\beta) \in Q \\ Ax = g \\ x \ge 0, \end{array}\right\} \qquad (10)$$

where $(\underline{c},\alpha,\overline{c},\beta)$ is an abbreviation of the matrix

$$\Big(\underline{c}(v,w), \alpha(v,w)\overline{c}(v,w), \beta(v,w)\Big)_{(v,w)\in E}.$$

It is easy to see that the first and second group of inequalities in (10) are satisfied if and only if

$$\tilde{f}_1(x,\theta) := \max_{(\underline{c},\alpha,\overline{c},\beta)\in Q} \sum_{(v,w)\in E} (\underline{c}(v,w) - \alpha(v,w)\theta)\, x(v,w) \le z_1$$

and

$$\tilde{f}_2(x,\theta) := \max_{(\underline{c},\alpha,\overline{c},\beta)\in Q} \sum_{(v,w)\in E} (\overline{c}(v,w) + \beta(v,w)\theta)\, x(v,w) \le z_2.$$

For fixed θ linear functions are maximized over a convex bounded polyhedron which implies that the maximum is attained at a vertex of Q. Hence,

$$\tilde{f}_1(x,\theta) := \max_{k=1,\ldots,K} \sum_{(v,w)\in E} \left(\underline{c}^k(v,w) - \alpha^k(v,w)\theta\right) x(v,w) \le z_1$$

and

$$\tilde{f}_2(x,\theta) := \max_{k=1,\ldots,K} \sum_{(v,w)\in E} \left(\overline{c}^k(v,w) + \beta^k(v,w)\theta\right) x(v,w) \le z_2$$

which are convex, piecewise linear functions. Summing up, for computing a robust solution of the fuzzy linear optimization problem, the set of Pareto-optimal solutions of the following problem has to be determined:

$$\left.\begin{array}{l} \tilde{f}_1(x,\theta) = \max\limits_{k=1,\ldots,K} \sum\limits_{(v,w)\in E} \left(\underline{c}^k(v,w) - \alpha^k(v,w)\theta\right) x(v,w) \to \min \\ \tilde{f}_2(x,\theta) = \max\limits_{k=1,\ldots,K} \sum\limits_{(v,w)\in E} \left(\overline{c}^k(v,w) + \beta^k(v,w)\theta\right) x(v,w) \to \min \\ Ax = g \\ x \ge 0, \end{array}\right\} \qquad (11)$$

To compute Pareto-optimal solutions for (11) the problem

$$\left.\begin{array}{l}\gamma\tilde{f}_1(x,\theta)+(1-\gamma)\tilde{f}_2(x,\theta)\to\min\\ Ax=g\\ x\geq 0,\end{array}\right\}\qquad(12)$$

is solved for $\gamma \in [0,1]$. The following result is a consequence of convex (multicriterial) optimization [9, 10].

Theorem 2. *Let problem (12) has unique optimal solutions for $\gamma = 0$ and $\gamma = 1$. Then, an incidence vector x^0 of a path in G can have a positive membership function value for the robust fuzzy shortest path problem only if there is $\theta \in [0,1]$ such that*

$$0 \in \gamma\partial f_1(x^0,\theta) + (1-\gamma)\partial f_2(x^0,\theta) + N_M(x^0).$$

Here, $\partial f_i(x^0,\theta)$ equals the subdifferential of the function $f_i(x^0,\theta)$.

Analogously to the proof of Theorem 1 this makes the computation of bounds $\{\theta_i\}_{i=1}^{2l}$ possible such that an incidence vector x^0 is Pareto-optimal for problem (11) for all $\theta \in \cup_{i=1}^{l}[\theta_{2i-1},\theta_{2i}]$. This implies that the membership function of such a point can be computed in a similar manner to (6).

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