VIII

Shear Force

VIII.1 General Considerations

Pure bending is a very rare loading condition. In fact, slender members are very often under the action of shear forces caused by transversal loading or by end moments. The presence of the shear force V implies that the bending moment cannot be constant, since $V = \frac{\mathrm{d}M}{\mathrm{d}z}$ (non-uniform bending: $M \neq 0$ and $V \neq 0$). The shear force is balanced by shearing stresses τ_{zx} and τ_{zy}, acting on the cross-section of the bar. Denoting by V_x and V_y the components of the shear force in the reference axes x and y, the shearing stress distribution in the cross-section must obey the conditions

$$\int_\Omega \tau_{zx}\,\mathrm{d}\Omega = V_x \qquad \text{and} \qquad \int_\Omega \tau_{zy}\,\mathrm{d}\Omega = V_y\ . \tag{184}$$

A supplementary condition is furnished by the reciprocity of shearing stresses in perpendicular facets, which is also an equilibrium condition (see Subsect. II.3.a). According to this condition, if there are no shear forces with a component in the longitudinal direction, applied in the lateral surface of the bar, the shearing stress will be zero in that direction and, as a consequence, in the points of the cross-section which are close to the boundary, the component of the shearing stress which is perpendicular to it will also be zero (Fig. 101). Thus, in the points of the cross-section at an infinitesimal distance to its boundary, *the shearing stress will be tangent to the border line.*

It is obvious that there are infinite stress distributions which obey this condition and also satisfy (184). We have, therefore, a problem with an infinite degree of indeterminacy. The law of conservation of plane sections cannot be used to solve the problem, since, as explained in Sects. V.10 and VII.1, the shear force is not a symmetrical internal force. Besides, the superposition principle cannot be used to analyse the effects of the bending moment and of the shear force separately. In fact, this principle refers to distinct sets of external loads and it is not possible to find a system of transversal forces

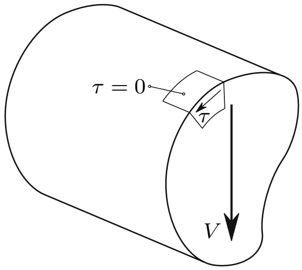


Fig. 101. Shearing stress at the boundary of the cross-section

which causes shear force without introducing a bending moment, since $M = \int V\,\mathrm{d}z + C$, although the opposite is possible, as seen in the analysis of the bending moment.

For these reasons, the analysis of the effect of the shear force expounded here is limited to prismatic bars made of materials with linear elastic behaviour. Furthermore, the following starting hypothesis must be considered (Saint-Venant's hypothesis): *the normal stresses caused by the bending moment in the case of non-uniform bending may be computed by the expressions developed for circular bending.* The validity of this hypothesis will be discussed later. First, it is used to develop the basic tool for the analysis of the effect of the shear force acting on the cross-section: the expression for the computation of the *longitudinal shear force*, i.e., the shear force acting on longitudinal cylindrical surfaces which are parallel to the bar's axis.

VIII.2 The Longitudinal Shear Force

In a prismatic bar under non-uniform bending let us consider the piece defined by two cross-sections at an infinitesimal distance $\mathrm{d}z$ from each other. In this piece let us consider a longitudinal cylindrical surface, defined by the fibres contained in a straight or curved line of the cross-section (Sect. VII.2), as represented in Fig. 102 (squared surface). That line divides the cross-section into two distinct parts, which means that the longitudinal surface divides the piece of bar into two distinct bodies. In order to simplify the development, we first analyse only the case of plane bending.

The equilibrium conditions of the piece of bar as a whole yield the well-known relations between the transversal load P, the shear force in the cross-section V and the bending moment M. Using the sign conventions represented by considering as positive the directions depicted in Fig. 102, we get

$$\begin{cases} \sum F_y = 0 \Rightarrow P = -\frac{\mathrm{d}V}{\mathrm{d}z} \\ \sum M_x = 0 \Rightarrow V = \frac{\mathrm{d}M}{\mathrm{d}z}\ . \end{cases} \qquad (185)$$

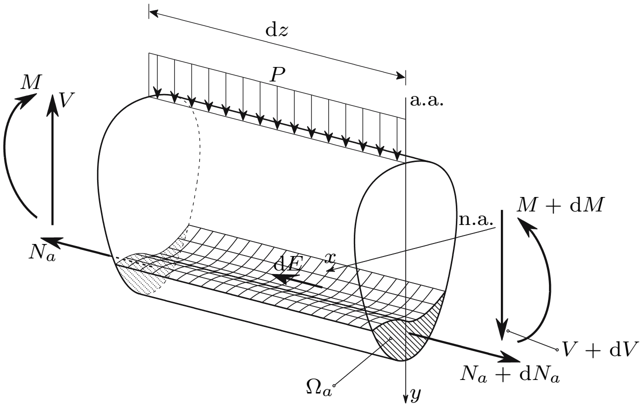


Fig. 102. Longitudinal shear force in a prismatic bar under non-uniform bending

Let us now consider the equilibrium condition of the longitudinal forces acting on the part of the bar defined by the hatched area Ω_a of the left and right cross-sections, which is separated from the remaining bar by the squared longitudinal surface. In the areas Ω_a of the left and right cross-sections, normal stresses caused by the bending moment are acting. According to the Saint-Venant's hypothesis, the forces resulting from these stresses in the left and right cross-sections are given by the expressions (Fig. 102)

$$\sigma = \frac{My}{I} \Rightarrow \begin{cases} N_a = \int_{\Omega_a} \sigma \, \mathrm{d}\Omega_a = \frac{M}{I} \int_{\Omega_a} y \, \mathrm{d}\Omega_a = \frac{MS}{I} \\ N_a + \mathrm{d}N_a = \frac{M+\mathrm{d}M}{I} \int_{\Omega_a} y \, \mathrm{d}\Omega_a = \frac{MS}{I} + \frac{S\,\mathrm{d}M}{I} \ . \end{cases} \tag{186}$$

In these expressions $S = \int_{\Omega_a} y \, \mathrm{d}\Omega_a$ represents the first area moment of the area Ω_a with respect to the neutral axis. The resultant of these two opposite forces – $\mathrm{d}N_a$ – must be balanced by the *longitudinal shear force* $\mathrm{d}E$, acting on the contact surface between the two bodies (the squared surface). Thus, this force takes the value ($\mathrm{d}M = V \mathrm{d}z$, (185))

$$\mathrm{d}E = N_a + \mathrm{d}N_a - N_a = \frac{S\,\mathrm{d}M}{I} = \frac{VS}{I}\mathrm{d}z \ . \tag{187}$$

If the equilibrium of the upper part were to be considered instead, an equal force with opposite direction would be obtained, since the unbalanced force $\mathrm{d}N_a$ would have the opposite direction. The first area moment would be $-S$, since the area moment of the whole cross-section in relation to the neutral axis is zero. From (187) we can see that, of all possible longitudinal surfaces, the neutral surface has the maximum longitudinal shear force, because the maximum absolute value of the first area moment S corresponds the whole tensioned area (or to the whole compressed area) of the cross-section.[1]

The longitudinal shear force per unit length is called the *longitudinal shear flow* and is given by the expression

$$f = \frac{\mathrm{d}E}{\mathrm{d}z} = \frac{VS}{I} \; . \tag{188}$$

In the case of inclined bending, the longitudinal shear force may be computed by superposing the forces corresponding to the decomposition of the bending moment and the shear force in the principal axes of inertia, which leads to the expression (cf.(150), $\mathrm{d}M_x = V_y\,\mathrm{d}z$ and $\mathrm{d}M_y = -V_x\,\mathrm{d}z$)

$$\mathrm{d}E = \left(\frac{V_y S_x}{I_x} + \frac{V_x S_y}{I_y}\right)\mathrm{d}z \; ,$$

where $S_x = \int_{\Omega_a} y\,\mathrm{d}\Omega_a$ and $S_y = \int_{\Omega_a} x\,\mathrm{d}\Omega_a$ are the first area moments of Ω_a with respect to the principal axes x and y, respectively. An alternative expression for inclined bending is presented in Subsect. VIII.3.f.

In order to illustrate the importance of this internal force caused by the shear force V, let us consider the cantilever beam depicted in Fig. 103, which is made of two bars with square cross-section $b \times b$.

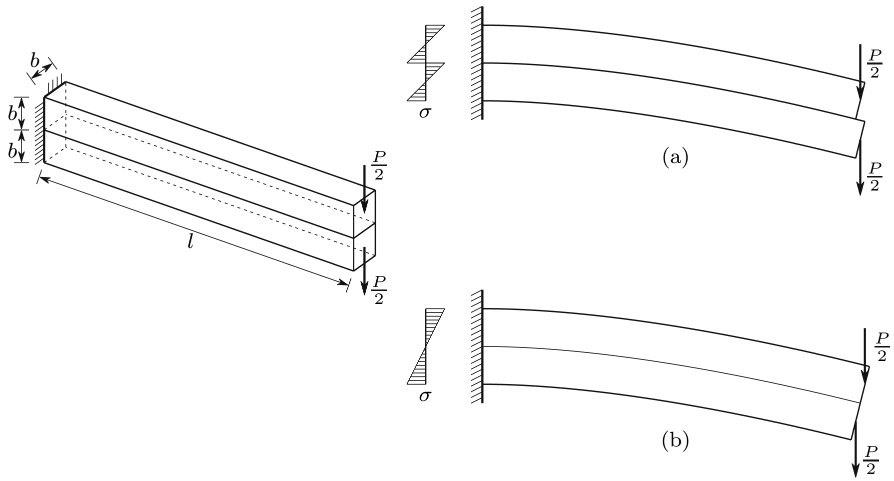


Fig. 103. Non-uniform bending of a built-up beam: (**a**) without friction in the contact surface; (**b**) bars perfectly connected together

If the contact surface between the two bars is lubricated, so that the friction force between the bars is eliminated, each bar will bend independently and a

[1] The same holds in the case of inclined bending, since the maximum value of $\mathrm{d}E$ corresponds to the difference between the resultants of the normal stresses acting on the whole tensioned area (or on the whole compressed area) of the cross-section.

relative sliding in the contact surface of the bars takes place, leading to the deformation and stress distribution represented in Fig. 103-a. The maximum stress caused by the bending moment, which occurs in the left end cross-section, may be computed considering the force $\frac{P}{2}$ acting on one beam with square cross-section $b \times b$, yielding

$$M = M_{\max} = \frac{Pl}{2} \Rightarrow \sigma^a_{\max} = \frac{M_{\max}}{\frac{I}{v}} = \frac{\frac{Pl}{2}}{\frac{b^3}{6}} = 3\frac{Pl}{b^3} \ .$$

In the same cross-section the curvature takes the value

$$\frac{1}{\rho_a} = \frac{M_{\max}}{EI} = \frac{\frac{Pl}{2}}{E\frac{b^4}{12}} = 6\frac{Pl}{Eb^4} \ .$$

If the two bars are perfectly connected together, so that the above-mentioned sliding is prevented, the two bars behave as a single unit with a cross-section $b \times 2b$. Thus, the deformation and the stress distribution take the forms represented in Fig. 103-b. The maximum stress and curvature are then given by

$$M = M_{\max} = Pl \Rightarrow \begin{cases} \sigma^b_{\max} = \dfrac{M_{\max}}{\frac{I}{v}} = \dfrac{Pl}{\frac{b(2b)^2}{6}} = \dfrac{3}{2}\dfrac{Pl}{b^3} = \dfrac{1}{2}\sigma^a_{\max} \\ \dfrac{1}{\rho_b} = \dfrac{M_{\max}}{EI} = \dfrac{Pl}{E\frac{b(2b)^3}{12}} = \dfrac{6}{4}\dfrac{Pl}{Eb^4} = \dfrac{1}{4}\dfrac{1}{\rho_a} \ . \end{cases}$$

We conclude that, by preventing the sliding in the contact surface, the bending stiffness is multiplied by four and the loading capacity of the beam duplicates, since the maximum stress caused by a given load P is divided by two, i.e., twice the load may be applied for the same maximum stress. In this case, the connection between the two bars must resist the shear flow (188)

$$f = \frac{\mathrm{d}E}{\mathrm{d}z} = \frac{VS}{I} = \frac{Pb^2\frac{b}{2}}{\frac{b(2b)^3}{12}} = \frac{3}{4}\frac{P}{b} \ .$$

In order to see how the cross-section deforms in the presence of a shear force, let us consider a piece with infinitesimal length $\mathrm{d}z$, of a bar with a rectangular cross-section. The bar is under non-uniform plane bending with the action axis parallel to height h, as represented in Fig. 104. The width b of the cross-section is very small, compared with the height h, so the shearing stresses in the cross-section may be considered as constant and parallel to the sides of the cross-section in the whole width.

In the horizontal surface *abcd* the same shearing stress τ as in the cross-section is acting, as a consequence of the reciprocity of the shearing stresses. In this surface, the stress distribution may be admitted as uniform, since the dimension $\mathrm{d}z$ is infinitesimal. The resultant of this shearing stress is the

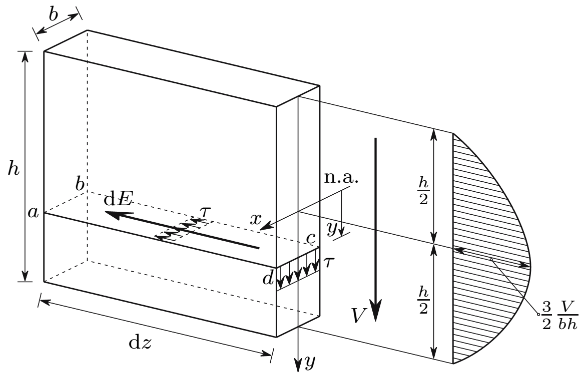


Fig. 104. Shearing stresses caused by the shear force V in a rectangular cross-section with small width

longitudinal shear force given by (187). Thus, the shearing stress takes the value

$$\tau b \mathrm{d}z = \mathrm{d}E = \frac{VS}{I}\mathrm{d}z \Rightarrow \tau(y) = \frac{VS(y)}{Ib} \Rightarrow \tau(y) = \frac{V}{I}\frac{1}{2}\left(\frac{h^2}{4} - y^2\right) . \quad (189)$$

This expression defines a parabolic stress distribution, as represented in Fig. 104. The maximum value of the shearing stress occurs on the neutral axis ($y = 0$) and takes the value $\tau_{\max} = \frac{Vh^2}{8I} = \frac{3}{2}\frac{V}{bh}$.

Since the shearing strain is proportional to the sharing stress $\left(\gamma = \frac{\tau}{G}\right)$, the cross-section must deform in such a way, that the shearing stress vanishes in the fibres farthest from the neutral axis ($y = \frac{h}{2} \Rightarrow \tau = 0$) and attains a maximum value on the neutral axis ($y = 0 \Rightarrow \tau = \tau_{\max}$). If the cross-section were to remain plane, the shearing strain would be constant in the cross-section (Fig. 105-a) and the distribution of shearing stresses would not be as represented by (189). Thus, we conclude that, either the starting hypothesis for the analysis of the effect of the shear force is wrong (the Saint-Venant hypothesis), or the cross-section must deform as represented in Fig. 105-b.

However, by considering all pieces of infinitesimal length dz separately, we verify that, provided that the shear force is constant, the same warping in all cross-sections takes place. This means that the deformations of the different pieces are *compatible*, i.e., that the deformed infinitesimal pieces fit perfectly together. Thus, no additional normal stresses are needed to make deformations compatible, which means that the strain distribution resulting from Saint-Venant's hypothesis obeys all conditions of compatibility.

This example shows that the cross-section may warp without the need to change the length of the fibres ($\overline{aa} = \overline{a'a'}$, Fig. 105), provided that the shear force does not vary along the axis of the bar. Since the deformation caused by the shear force does not require changes in the fibres' length, this force may be resisted without altering the distribution of the normal stresses corresponding

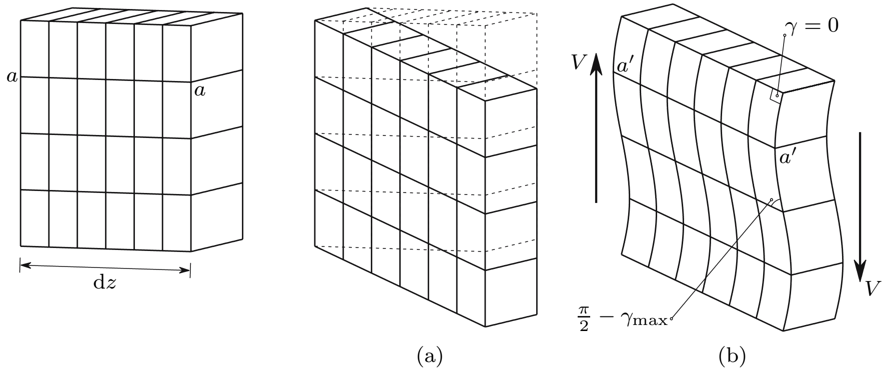


Fig. 105. Warping of a rectangular cross-section caused by the shear force V

to circular bending, i.e., there is no objection to the validity of the Saint-Venant hypothesis. This conclusion may be generalized to a cross-section of any shape, since the shearing strains corresponding to any distribution of shearing stresses may occur without the need to change the length of the fibres, provided that the warping is the same in all cross-sections.

These considerations are not a complete proof of the validity of the Saint-Venant hypothesis in the case of constant shear force. However, they do show that this possibility exists and the solutions of the Theory of Elasticity for particular problems confirm that, *if the shear force is constant, the distribution of normal stresses caused by the bending moment is the same as in circular bending, i.e., it is the same as when the cross-sections remain plane and perpendicular to the bar's axis.* This means that the law of conservation of plane sections is a sufficient condition for a linear distribution of the longitudinal strains in the cross-section, although it may not be necessary, as we conclude from the above considerations.

In the case of a non-constant shear force, this is no longer valid. However, as discussed in Sect. VII.7, the error affecting the computation of the normal stresses and, as a consequence, the computation of the longitudinal shear force by means of (187), is very small and may even vanish (see Sect. VIII.6).

From a practical point of view, (187) may thus be considered exact. However, the computation of the shearing stress from the longitudinal shear force always requires simplifying hypotheses, which introduce errors, whose importance depends on the shape of the cross-section. Thus, good approximations for the shearing stress distribution are obtained for symmetrical cross-sections, if the action axis of the shear force coincides with the symmetry axis and in the cases of *thin-walled* cross-sections. In other cases it is generally not possible to compute the shearing stresses by means of the elementary theory presented in this book. These cases, as well as the errors introduced by the simplifying hypotheses used are discussed below.

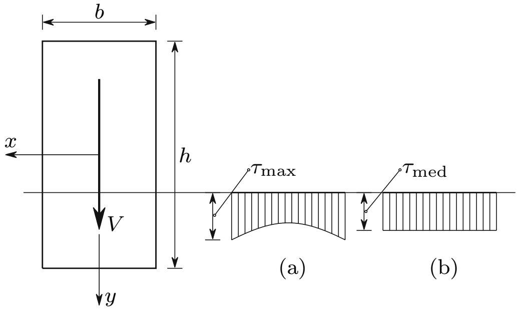


Fig. 106. Shearing stress τ_{zy} in a rectangular cross-section: (**a**) real distribution; (**b**) admitted distribution

VIII.3 Shearing Stresses Caused by the Shear Force

VIII.3.a Rectangular Cross-Sections

In rectangular cross-sections under plane bending the simplifying hypothesis which consists of considering the shearing strain as constant in the width of the cross-section is usually considered: that is, the stress varies only in the direction parallel to the action axis of the shear force. This corresponds to the generalization to rectangular sections with any width/height ratio of the assumptions used in previous section for the small width case. In the case of inclined non-uniform bending, the shear force is decomposed in the symmetry axes. Thus, in a point defined by its coordinates x and y, the two components of the shearing stress are ((189) and Fig. 104)

$$\begin{cases} \tau_{zy} = \frac{V_y}{I_x}\frac{1}{2}\left(\frac{h^2}{4} - y^2\right) = \frac{V_y}{bh}\left[\frac{3}{2} - 6\left(\frac{y}{h}\right)^2\right] \\ \tau_{zx} = \frac{V_x}{I_y}\frac{1}{2}\left(\frac{b^2}{4} - x^2\right) = \frac{V_x}{bh}\left[\frac{3}{2} - 6\left(\frac{x}{b}\right)^2\right] \; . \end{cases} \tag{190}$$

The Theory of Elasticity provides a solution for this problem, which is obtained without the simplifying hypothesis above. This solution indicates that the shearing stress is not constant in the direction perpendicular to the action axis of the shear force unless the Poisson's coefficient vanishes, but it has a maximum in the points close to the lateral sides, as represented in Fig. 106-a.

The maximum value of the shearing stress, which occurs for $x = \pm\frac{b}{2}$ and $y = 0$, may be computed by the expression (cf. e.g. [4])

$$\tau_{\max} = \alpha \frac{3}{2}\frac{T}{\Omega}$$

$$\text{with} \qquad \alpha = 1 + \frac{\nu}{1+\nu}\left(\frac{h}{b}\right)^2\left[\frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1,2,3,\ldots}^{\infty}\frac{1}{n^2\cosh\left(n\pi\frac{h}{b}\right)}\right] \; . \tag{191}$$

The coefficient α represents the correction to be applied to the maximum stress obtained from (190), in the case of plane bending, $\tau_{\max} = \frac{3}{2}\frac{V}{\Omega}$. This coefficient depends on the height/width ratio (h/b) and on the Poisson coefficient of the material, ν. The following table gives values of α, computed from (191), for some cases.

α	$\nu = 0$	0.05	0.1	0.15	0.2	0.25	0.3	0.4	0.5
$h/b = 0.25$	1.0000	1.2352	1.4491	1.6443	1.8233	1.9879	2.1399	2.4113	2.6466
0.50	1.0000	1.0944	1.1802	1.2585	1.3303	1.3964	1.4574	1.5663	1.6606
0.75	1.0000	1.0498	1.0951	1.1365	1.1744	1.2093	1.2415	1.2990	1.3488
1.00	1.0000	1.0301	1.0574	1.0823	1.1052	1.1263	1.1457	1.1804	1.2104
1.25	1.0000	1.0198	1.0379	1.0543	1.0694	1.0833	1.0961	1.1190	1.1388
1.50	1.0000	1.0140	1.0266	1.0382	1.0488	1.0586	1.0676	1.0837	1.0977
2.00	1.0000	1.0079	1.0151	1.0217	1.0277	1.0333	1.0384	1.0475	1.0554
4.00	1.0000	1.0020	1.0038	1.0054	1.0069	1.0083	1.0096	1.0119	1.0139

This table shows that the error of the solution furnished by (190) increases with the value of the Poisson coefficient and decreases as the height/width ratio increases. The dependence of the error on the relation $\frac{h}{b}$ has greater practical relevance, since structural materials with a Poisson coefficient smaller than 0.05 are not common, while rectangular cross-sections with height/width ratios superior to 2 are widely used.

VIII.3.b Symmetrical Cross-Sections

In practical applications cross-sections that are symmetrical with respect to the action axis of the shear force are common. In these cases, the computation of the shearing stresses may be carried out by considering two simplifying hypotheses: the vertical component of the shearing stress τ_{zy} is constant in the direction perpendicular to the symmetry axis; the total stress vectors τ in a line perpendicular to the symmetry axis have directions converging to the point defined by the two tangents to the cross-section's contour on that line, as represented in Fig. 107.

The vertical component of the shearing stress may then be computed in the same way as in the rectangular cross-section, taking the value

$$\tau_{zy} = \frac{VS(y)}{Ib(y)} \,. \tag{192}$$

The horizontal component and the resultant stress may then be obtained from this value and angle ψ, yielding

$$\tau_{zx} = \tau_{zy}\tan\psi \quad \Leftrightarrow \quad \tau = \sqrt{\tau_{zx}^2 + \tau_{zy}^2} = \frac{\tau_{zy}}{\cos\psi} = \frac{VS}{Ib\cos\psi} \,. \tag{193}$$

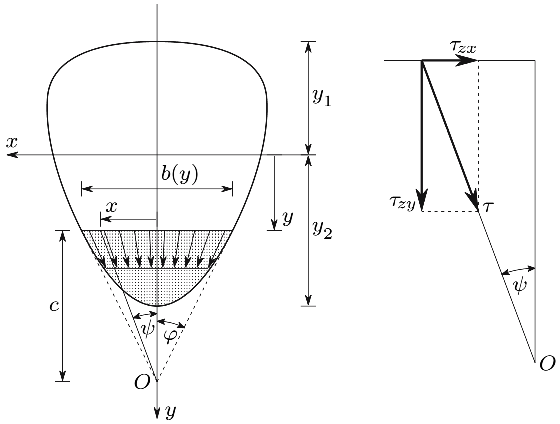


Fig. 107. Simplifying hypotheses for the computation of the shear force in a symmetrical cross-section

The maximum stress for a given value of y occurs clearly on the contour of the cross-section, taking the value $\tau_{\text{max}} = \frac{VS}{Ib\cos\varphi}$.

As an applied example let us consider a circular cross-section (Fig. 108). The first area moment of the surface element defined by the central angle β is given by the expression (Fig. 108)

$$\mathrm{d}S = \overbrace{r\sin\beta}^{x}\ \overbrace{r\,\mathrm{d}\beta\,\sin\beta}^{\mathrm{d}y'}\ \overbrace{r\cos\beta}^{y'} = r^3\sin^2\beta\cos\beta\,\mathrm{d}\beta\ .$$

Integrating to the whole area defined by angle α (Fig. 108), we get

$$S = \int_{-\alpha}^{\alpha} r^3\sin^2\beta\cos\beta\,\mathrm{d}\beta = \frac{2}{3}r^3\sin^3\alpha\ . \tag{194}$$

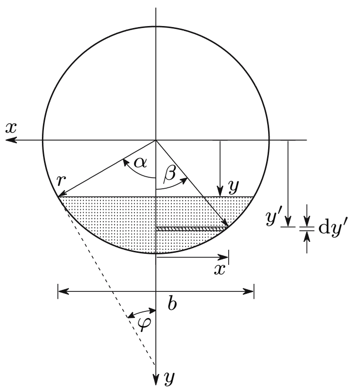


Fig. 108. Computation of the shearing stress in a circular cross-section

The shearing stress τ_{zy} corresponding to the area moment S (194) is then ($b = 2r\sin\alpha$)

$$\tau_{zy}(\alpha) = \frac{VS}{Ib} = \frac{V\frac{2}{3}r^3\sin^3\alpha}{\frac{\pi r^4}{4}2r\sin\alpha} = \frac{4}{3}\frac{V}{\pi r^2}\sin^2\alpha = \frac{4}{3}\frac{V}{\Omega}\sin^2\alpha\ .$$

For a given value α, the maximum stress occurs at the boundary. From (193) we get

$$\tau = \frac{\tau_{zy}}{\cos\varphi} = \frac{\tau_{zy}}{\sin\alpha} = \frac{4}{3}\frac{V}{\Omega}\sin\alpha\ .$$

This expression attains a maximum for $\alpha = \frac{\pi}{2}$ (neutral axis), which means that the maximum shear stress in the cross-section takes the value

$$\alpha = \frac{\pi}{2} \ \Rightarrow\ \tau = \tau_{\max} = \frac{4}{3}\frac{V}{\Omega}\ .$$

The solution given by the Theory of Elasticity for this problem indicates that, unless the Poisson coefficient takes the value $\nu = 0.5$ (incompressible material), the stress distribution is not uniform in the neutral axis. The maximum value occurs in the centre of the circle and takes the value [4]

$$\tau_{\max} = \gamma\frac{4}{3}\frac{V}{\Omega} \quad \text{with} \quad \gamma = \frac{9+6\nu}{8(1+\nu)}\ .$$

The error for the approximate solution vanishes for $\nu = 0.5$ ($\gamma = 1$) and takes the maximum value for a vanishing Poisson's coefficient ($\gamma = 1.125$). For the mean value $\nu = 0.25$, we get $\gamma = 1.05$. In the case of steel ($\nu = 0.3$) the error is 3.8% ($\gamma = 1.038$). We conclude that the error introduced by the simplifying hypotheses is relatively small.

VIII.3.c Open Thin-Walled Cross-Sections

Many of the slender members currently used in structural engineering, especially in metallic constructions, have thin-walled cross-sections, i.e., cross-sections made of straight or curved elements with small thickness, in comparison with the cross-section dimensions. Usual *profile sections*, such as I-beams, channel beams, angle sections, Z-sections, T-beams, circular or rectangular tubes, etc., are examples of this kind of member. In this Sub-section, we will deal with open thin-walled cross-sections, i.e., simply-connected thin-walled cross-sections.

As seen in the study of the shearing stresses in rectangular cross-sections, if the width is small compared with the height, the simplifying hypothesis of considering constant stresses in the thickness b is very close to the actual distribution. The same happens in thin-walled cross-sections, like that represented in Fig. 109. Thus, by considering the longitudinal surface which is perpendicular to the centre line of the cross-section wall and contains the point where the shearing stress is to be computed, the shearing stress may be obtained from the longitudinal shear force dE. From (187) we get

$$\mathrm{d}E = \tau e \,\mathrm{d}z = \frac{VS}{I} \Rightarrow \tau = \frac{\mathrm{d}E}{e\,\mathrm{d}z} = \frac{VS}{Ie} , \qquad (195)$$

where e represents the wall thickness in the point where τ is computed. The computation of the area moment S of thin walls may be simplified if the area is considered as concentrated on the centre line. Denoting by s a coordinate which follows that line (Fig. 109), we get for the first area moment needed to compute the shearing stress in the point defined by s[2]

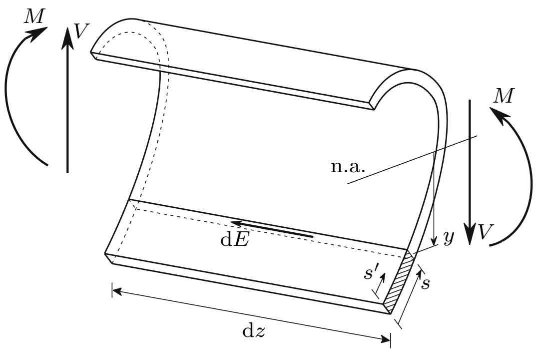


Fig. 109. Longitudinal shear force in a thin-walled cross-section

$$S(s) = \int_0^s e(s')y(s')\,\mathrm{d}s' .$$

In order to illustrate these considerations, the shearing stress distribution in the cross-section represented in Fig. 110, caused by a vertical shear force V is analysed.

In the flange element $\overline{AB}$ the area moment corresponding to the point of the centre line defined by the coordinate s_1 may be expressed by

$$S(s_1) = s_1 e \left(\frac{h}{4} + \frac{s_1}{2}\right) .$$

The shearing stress in this point is then

$$\tau(s_1) = \frac{VS(s_1)}{Ie} = \frac{V}{I}\left(\frac{hs_1}{4} + \frac{s_1^2}{2}\right) .$$

[2]If the same approximation is made for the moment of inertia, a completely consistent theory for thin-walled cross-sections with infinitesimal wall thickness is obtained, in the sense that the computed resultant of the shearing stress exactly balances the applied shear force. Otherwise, a discrepancy will appear, which is introduced by the wall curvature or by angle points in the centre line.

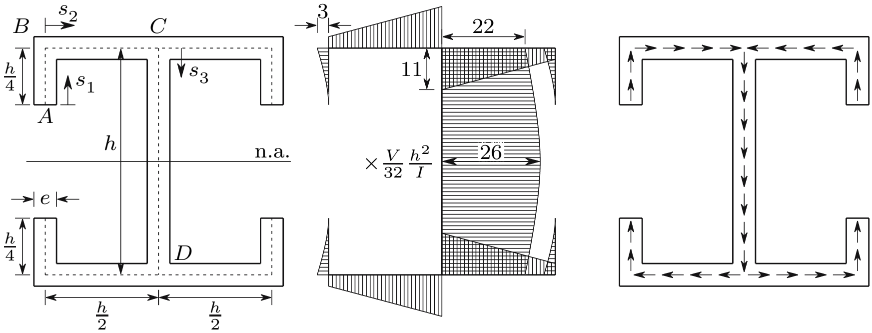


Fig. 110. Shearing stresses caused by a vertical positive shear force in a symmetrical open thin-walled cross-section

The maximum stress occurs for the maximum value of s_1 (point B), taking the value

$$s_1 = \frac{h}{4} \Rightarrow \tau = \tau_{\max}^{AB} = \frac{3}{32} h^2 \frac{V}{I} \ .$$

In the flange element $\overline{BC}$ the area moment and the shearing stress may be expressed in terms of coordinate s_2, yielding

$$S\left(s_2\right) = \frac{3h^2 e}{32} + s_2 e \frac{h}{2} \Rightarrow \tau = \frac{V}{I}\left(\frac{h}{2} s_2 + \frac{3h^2}{32}\right) \ .$$

In this wall segment the stress is a linear function of s_2 and takes the maximum value in point C

$$s_2 = \frac{h}{2} \Rightarrow \tau = \tau_{\max}^{BC} = \frac{11}{32} h^2 \frac{V}{I} \ .$$

Finally, in the web (wall segment $\overline{CD}$) the area moment may be expressed as a function of coordinate s_3, yielding

$$S(s_3) = \frac{22}{32} h^2 e + s_3 e \left(\frac{h}{2} - \frac{s_3}{2}\right) \Rightarrow \tau = \frac{V}{I}\left(\frac{22}{32} h^2 + \frac{s_3 h}{2} - \frac{s_3^2}{2}\right) \ .$$

This expression represents a parabolic stress distribution. The maximum value occurs on the neutral axis and takes the value

$$s_3 = \frac{h}{2} \Rightarrow \tau = \tau_{\max}^{CD} = \frac{26}{32} h^2 \frac{V}{I} \ .$$

The direction of the shearing stresses may be obtained from the direction of the longitudinal shear force. For example, in order to get the stress direction in the flange element $\overline{AB}$, let us consider the balance of the longitudinal forces acting on a piece of this flange element, as represented in Fig. 111.

Let us assume a positive shear force (downward direction). As the flange element $\overline{AB}$ is above the neutral axis, it is in the compressed zone, if the

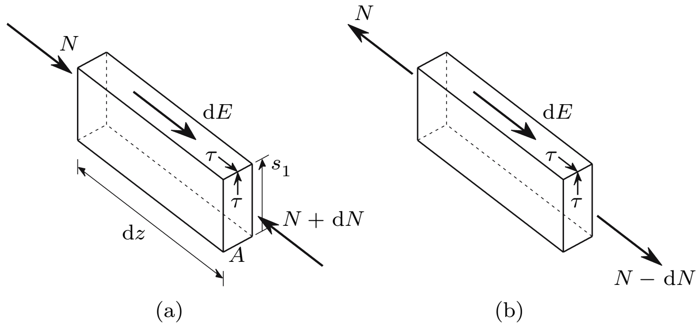


Fig. 111. Determination of the direction of the shearing stresses in the flange element $\overline{AB}$ (Fig. 110): (**a**) positive bending moment; (**b**) negative bending moment

bending moment is positive. A positive shear force will cause an increase in the bending moment, as $\mathrm{d}M = V\,\mathrm{d}z$, which will cause an increase $\mathrm{d}N$ in the compressive stress resultant N (Fig. 111-a). In the case of a negative bending moment, the flange element $\overline{AB}$ will be in the tensioned zone. However, a positive shear force will cause a decrease in the absolute value of the bending moment ($\mathrm{d}M > 0$ and $M < 0$) and, therefore, a decrease in the tensile stress resultant N, as represented in Fig. 111-b. In both cases, the same direction is obtained for the shearing stress τ, as expected, since this stress is caused by the shear force, which is the same in the two cases.

The direction of the shearing stresses in the segments $\overline{BC}$ and $\overline{CD}$ could be obtained in the same way. The symmetry of the cross-section leads to the directions of the shearing stresses represented in Fig. 110.

An additional tool to obtain the direction of the shearing stresses is furnished by the condition of constant shear flow in a point of convergence of two or more centre lines of the cross-section walls, as points B and C (Fig. 110). This condition may be obtained from the balance equation of the longitudinal

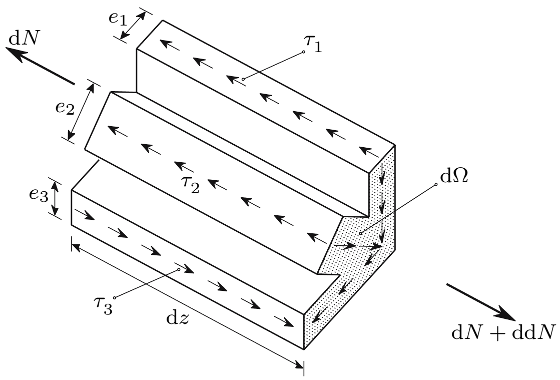


Fig. 112. Shear flow in a nodal point of a thin-walled cross-section

forces acting on an infinitesimal neighbourhood of one of these points (nodal points). In the case represented in Fig. 112, this equation takes the form

$$\overbrace{(-\tau_1 e_1 - \tau_2 e_2 + \tau_3 e_3)\,\mathrm{d}z}^{\text{infinitesimal quantity of second order}} + \underbrace{\mathrm{d}\sigma\,\mathrm{d}\Omega}_{\text{infinite simal quantity of third order (dd}N\text{)}} = 0\ .$$

The product $e\mathrm{d}z$ is an infinitesimal quantity of second order, since the thickness e is infinitesimal (cf. Footnote 55). Because $\mathrm{d}\Omega$ is also a second order infinitesimal quantity, $\mathrm{d}\sigma\mathrm{d}\Omega$ will be an infinitesimal quantity of third order. Thus, $\mathrm{d}\sigma\mathrm{d}\Omega$ is an infinitesimal quantity of higher order, which may be neglected, yielding

$$\overbrace{\tau_1 e_1 + \tau_2 e_2}^{\text{ingoing shear flow}} = \underbrace{\tau_3 e_3}_{\text{outgoing shear flow}}\ . \tag{196}$$

Generalizing (196) to a number n of centre lines converging to a nodal point, we get

$$\sum_{i=1}^{n} \tau_i e_i = 0\ .$$

Taking the reciprocity of shearing stresses into consideration, this expression means that the sum of the products τe heading into the nodal point is equal to the sum of the products τe heading out. In other words, *the shear flow entering the node is equal to the shear flow leaving the node.* For example, in point C (Fig. 110) the shear flow entering the node is $2 \times \frac{11}{32}\frac{Vh^2 e}{I}$ and the outgoing flow is $\frac{22}{32}\frac{Vh^2 e}{I}$.

VIII.3.d Closed Thin-Walled Cross-Sections

If the cross-section is doubly-connected, i.e., if the centre line of the wall is a closed line, a longitudinal cut, like the one represented in Fig. 109, is not enough to separate the cross-section into two distinct parts. This means that two cuts must be made and that the longitudinal shear force $\mathrm{d}E$, given by (187), is the sum of the resultants of two different longitudinal shearing stresses, τ_1 and τ_2. The value of the shearing stress cannot be computed, therefore, unless an additional relation between τ_1 and τ_2 is found. However, in the case of a symmetrical cross-section, with respect to the action axis of the shear force, these stresses will be equal, provided that the two cuts are made in symmetrical points of the centre line, as represented in Fig. 118. In this case, the shearing stress may be computed by the expression

$$2\tau e\,\mathrm{d}z = \mathrm{d}E = \frac{VS}{I}\mathrm{d}z \Rightarrow \tau = \frac{VS}{2Ie}\ , \tag{197}$$

where S is the first area moment of the shaded area in Fig. 113.

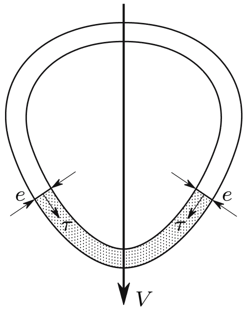


Fig. 113. Computation of the shearing stress in a closed symmetrical thin-walled cross section

If the cross-section is not symmetrical with respect to the action axis of the shear force, the problem becomes a statically indeterminate one, whose solution may be computed by means of the force method. As seen in Sect. VI.4, this method consists of releasing a sufficient number of connections to get a statically determinate problem, followed by the computation of the forces needed to avoid displacements in the released connections. In the present problem, the longitudinal connection in a point of the cross-section wall is released, so that an open cross-section is obtained. Under the action of the shear force, the two sides of the cut suffer a longitudinal relative displacement, as represented in Fig. 114-a. This displacement must then be eliminated, by applying a pair of shear forces dE to both sides of the cut (Fig. 114-b). The resulting stress in any point of the cross-section may be obtained by the superposition principle, by adding the stresses corresponding to the two situations (Fig. 114-c).

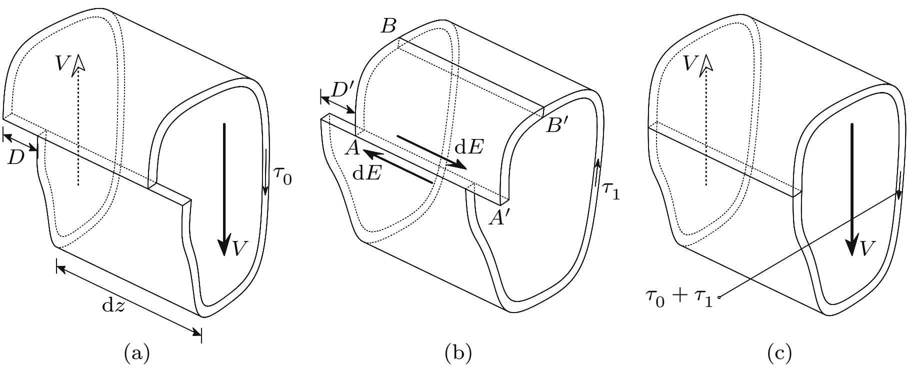


Fig. 114. Computation of the shear stresses in a non-symmetrical closed thin-walled cross-section

The relative displacement in direction z of two points of the centre line, located at an infinitesimal distance ds of each other, is $\mathrm{d}D = \gamma_0\,\mathrm{d}s$.[3] Thus, in the open cross-section, the relative displacement D of both sides of the cut, caused by the shear force V (Fig. 114-a), may be computed by integrating the shear strain γ_0 along the complete centre line of the wall, which yields ($\tau_0 = G\gamma_0$)

$$D = \oint \gamma_0\,\mathrm{d}s = \frac{1}{G}\oint \tau_0\,\mathrm{d}s = \frac{V}{IG}\oint \frac{S}{e}\,\mathrm{d}s\ . \tag{198}$$

In the situation depicted in Fig. 114-b, the shear flow $f = \tau_1(s)e(s)$ is constant along the whole centre line of the wall,[4] since there are no other forces applied to the bar apart from the pair of forces dE. This conclusion is easily drawn by establishing the balance condition of the longitudinal forces acting on the piece defined by the longitudinal cut $\overline{AA'}$ and by any other longitudinal surface $\overline{BB'}$ (Fig. 114-b). This condition immediately means that the product $\tau_1 e = \frac{\mathrm{d}E}{\mathrm{d}z} = f$ is constant, even if e varies along the centre line. The longitudinal relative displacement D' caused by the pair of forces dE, is then ($\tau_1 = \frac{f}{e}$)

$$D' = \oint \gamma_1\,\mathrm{d}s = \oint \frac{\tau_1}{G}\,\mathrm{d}s = \frac{f}{G}\oint \frac{\mathrm{d}s}{e}\ . \tag{199}$$

The condition of compatibility requires that the displacement D' eliminates displacement D, which allows the computation of the shear flow f

$$D + D' = 0 \Rightarrow f = -\frac{V}{I}\frac{\oint \frac{S}{e}\mathrm{d}s}{\oint \frac{\mathrm{d}s}{e}} \Rightarrow \tau_1(s) = \frac{f}{e(s)}\ . \tag{200}$$

The shearing stress in the closed cross-section (Fig. 114-c) may then be computed by adding the stresses τ_0 and τ_1.

The closed line integrals appearing in the expressions above obviously refer to the line limiting the closed part of the cross-section, that is, they do not include simply-connected walls, as in the cross-section represented in Fig. 115.

The expressions above are valid for doubly-connected cross-sections, i.e., closed cross-sections with only one channel. In cross-sections with higher degrees of connection a number of longitudinal cuts equal to the degree of connection minus one is necessary to get a statically determinate problem, i.e., an open cross-section. As a consequence, the conditions of compatibility of the deformations in all the longitudinal cuts yield, instead of (200), a system

[3]This simple relation requires that the fibres remain parallel to each other in the deformation caused by the shear force. This condition is satisfied if there is no rotation of the cross-sections around a longitudinal axis of the prismatic bar, i.e., if torsion does not take place (see example XII.8).

[4]This shear flow defines a torsional moment (twisting moment or torque, see Chap. X.3). This moment corresponds to the translation of the shear force, from the shear centre of the open cross-section to the shear centre of the closed cross-section (see Sect. VIII.4 and example VIII.12).

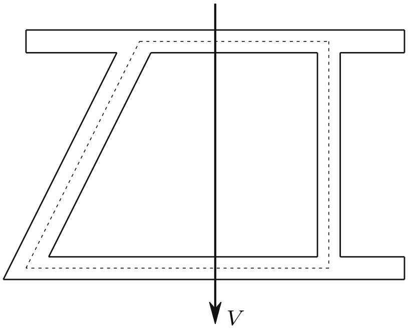


Fig. 115. Line, to which the closed line integrals in (198), (199) and (200) are referred (*dashed line*)

with a number of equations equal to the degree of connection minus one (see example VIII.7).

VIII.3.e Composite Members

In composite members the longitudinal shear force may be determined in the same way as in the case of homogeneous bars (187). The normal stress is in this case given by (169). Assuming, for simplicity, plane bending, as in the case represented in Fig. 116, we get the following expression for the longitudinal shear force (Fig. 116-b)

$$\left\{\begin{aligned} \mathrm{d}\sigma_a &= \frac{\mathrm{d}M\,E_a}{J_n} y \\ \mathrm{d}\sigma_b &= \frac{\mathrm{d}M\,E_b}{J_n} y \end{aligned}\right. \Rightarrow \left|\begin{aligned} \mathrm{d}E &= \int_{\Omega_{a1}} \mathrm{d}\sigma_a \,\mathrm{d}\Omega_a + \int_{\Omega_{b1}} \mathrm{d}\sigma_b \,\mathrm{d}\Omega_b \\ &= \tfrac{V}{J_n}\left(E_a \int_{\Omega_{a1}} y \,\mathrm{d}\Omega_a + E_b \int_{\Omega_{b1}} y \,\mathrm{d}\Omega_b\right) \mathrm{d}z\ . \end{aligned}\right. \tag{201}$$

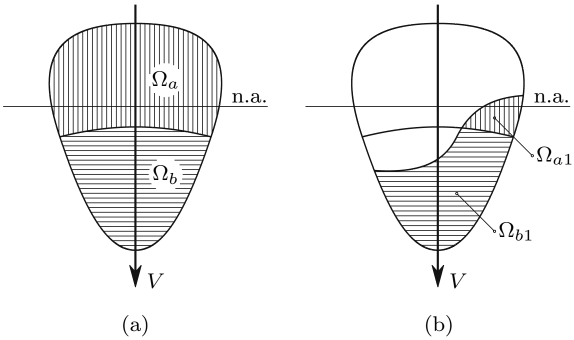


Fig. 116. Determination of the longitudinal shear force in composite members

In composite members, the longitudinal shear force in the contact surface between the two materials must usually be computed. In this particular case (201) takes a simpler form and the longitudinal shear force may be computed by any of the following expressions

$$\begin{cases} \Omega_{a1} = \Omega_a \\ \qquad\qquad \Rightarrow \dfrac{\mathrm{d}E}{\mathrm{d}z} = \dfrac{VE_aS_a}{J_n} = \dfrac{VS_a}{I_{ha}} \quad \text{with} \quad S_a = \displaystyle\int_{\Omega_a} y\,\mathrm{d}\Omega_a \\ \Omega_{b1} = 0 \\ \Omega_{a1} = 0 \\ \qquad\qquad \Rightarrow \dfrac{\mathrm{d}E}{\mathrm{d}z} = \dfrac{VE_bS_b}{J_n} = \dfrac{VS_b}{I_{hb}} \quad \text{with} \quad S_b = \displaystyle\int_{\Omega_b} y\,\mathrm{d}\Omega_b \ . \\ \Omega_{b1} = \Omega_b \end{cases} \tag{202}$$

VIII.3.f Non-Principal Reference Axes

In some cross-sections it is easy to compute the moments and product of inertia with respect to non-principal central axes, as well as distances and area moments. In Fig. 117 two examples of this kind of cross-section are represented.

In these cases it may be useful to compute the normal and shearing stresses directly from these axes, especially if one of them is parallel to the action axis.

The normal stresses may by computed by means of (140). From this equation an expression for the computation of the longitudinal shear force may then be developed. If the bending moment has only the M_x component and the axial force vanishes, the normal stress may be computed by the expression

$$\sigma = \frac{I_y y - I_{xy} x}{I_x I_y - I_{xy}^2} M_x \ .$$

The same line of reasoning used to develop (186), leads to the following expression for the longitudinal shear force (cf. Figs. 102 and 117)

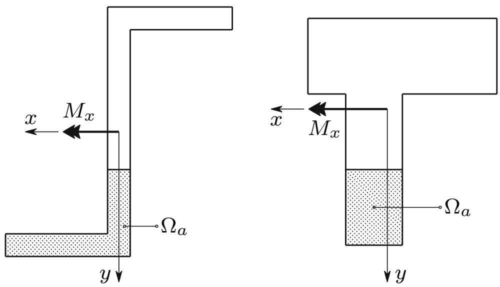


Fig. 117. Computation of the longitudinal shear force with non-principal reference axes

$$\mathrm{d}E = \int_{\Omega_a} \frac{I_y y - I_{xy} x}{I_x I_y - I_{xy}^2} \mathrm{d}M_x \,\mathrm{d}\Omega_a = V_y \frac{I_y S_x - I_{xy} S_y}{I_x I_y - I_{xy}^2} \mathrm{d}z$$
$$\text{with} \quad V_y = \frac{\mathrm{d}M_x}{\mathrm{d}z}, \quad S_x = \int_{\Omega_a} y \,\mathrm{d}\Omega_a \quad \text{and} \quad S_y = \int_{\Omega_a} x \,\mathrm{d}\Omega_a \ . \tag{203}$$

The shearing stresses may be computed from this expression, in the same way as was done on the basis of (187) (see example VIII.10).

VIII.4 The Shear Centre

When inclined circular bending was analysed (Sect. VII.4), we showed that a parallel displacement of the action axis does not change the normal stresses induced by the bending moment in the cross-section. However, if a shear force is acting (non-uniform bending), the equilibrium condition requires that the action axis of the shear force has a position which coincides with the line of action of the resultant of the shearing stresses. The position of the action axis of the shear force is therefore not arbitrary. There are two internal forces introducing shearing stresses in the cross-section: the shear force and the torsional moment. The expressions presented for the shearing stresses in this Chapter only take the shear force into consideration, since they are all based on the relation $\mathrm{d}M = V\,\mathrm{d}z$ (185). It is therefore assumed that the torsional moment is zero. If it is not, additional shearing stresses will appear. These stresses will be analysed in Chap. X.

Thus, to avoid torsion, the action axis of the shear stress must coincide with the line of action of the resultant of the shearing stresses computed by means of the expressions which were developed from (187) (longitudinal shear force caused by the cross-sectional shear force). By considering two shear forces with the directions of the principal central axes of inertia, and computing the position of the line of action of the resultant of the shearing stresses in each case, a point is defined by the intersection of these two lines, which has the following property: *if the line of action of the shear force passes through this point, it will not induce torsion of the bar*. This point is the *shear centre* of the cross-section.

The shear centre plays the same role in relation to the transversal forces, as the centroid in relation to the longitudinal (axial) forces: if the resultant axial force passes through the centroid of the cross-section, it will not induce bending; otherwise, composed bending will take place, with a bending moment given by the product of the axial force and the distance of its line of action to the centroid. In the same way, if the resultant of the forces acting on the cross-section plane (the shear force) does not pass through the shear centre, it will introduce a torsional moment, with a value given by the product of the shear force and the distance of its line of action to the shear centre.

The computation of the torsional moment must thus be made in relation to the shear centre, while the bending moment is computed with respect to the centroid. In the case of a cross-section with a symmetry axis, the shear centre is on this axis, since, for an action axis of the shear force coinciding with the symmetry axis, the shearing stress distribution will also be symmetric, which means that the line of action of its resultant coincides with the symmetry axis. Thus, if the cross-section has two symmetry axes the centroid and the shear centre will coincide. In other cases, these two points usually occupy different positions in the cross-section's plane.

We will demonstrate later (Chap. XII) that in prismatic bars made of materials with linear elastic behaviour, the shear centre coincides with the *torsion centre*, which is defined as the point around which the cross-section rotates in the twisting deformation induced by the torsional moment. For this reason, these two designations are sometimes indistinctly used.

While it is very easy to compute the position of the line of action of the resultant of the normal stresses in the case of pure axial force, since these stresses are constant in the cross-section, the computation of the line of action of the resultant of the shearing stresses is often complex, since the distribution of the stresses caused by the shear force is required. As seen in the previous sections, good approximations for these stresses are obtained only in the cases of symmetrical cross-sections with respect to the action axis of the shear force and in thin-walled cross-sections. In the first case, the position of the resultant is known. In the case of non-symmetrical cross-sections which cannot be considered as thin-walled, the problem of computing the shear centre's position cannot be solved by the approach used in the Strength of Materials. But the knowledge of the position of the shear centre is most important in the case of open thin walled cross-sections, since this kind of member is very weak in torsion, as will be seen in Chap. X. Fortunately, the stresses caused by the shear force in these cross-sections are easily computed with good approximation, as seen in Subsect. VIII.3.c.

In order to illustrate these considerations, the position of the shear centre of the channel cross-section represented in Fig. 118 is computed. As this cross-section has a symmetry axis, the shear centre will be located on this axis.

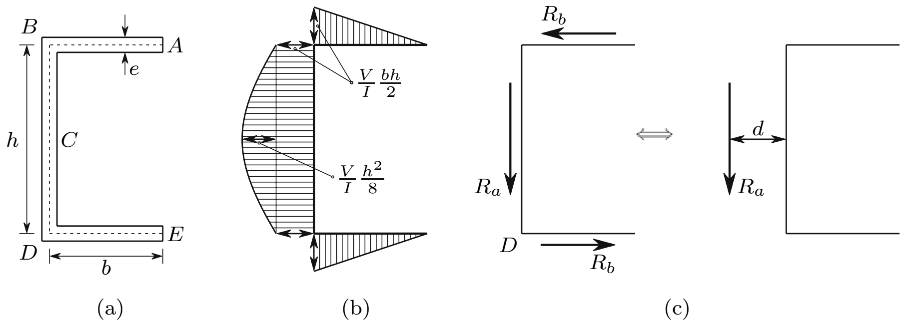


Fig. 118. Computation of the position of the shear centre in a thin-walled cross-section

Thus, in order to determine its position, it is enough to compute the distance d from the line of action of the resultant of the shearing stresses, introduced by a shear force perpendicular to the symmetry axis, to the centre line of the web (Fig. 118-c).

As the example of (Fig. 110) shows, the shearing stress has a linear distribution in the wall segments which are parallel to the neutral axis, and a parabolic distribution in the others. Besides, we know that the maximum stress occurs on the neutral axis. For these reasons, in example of (Fig. 118) the stress distribution is completely defined by the values in points B and C. For point B we get from (195)

$$S = be\frac{h}{2} \Rightarrow \tau = \frac{V}{I}\frac{bh}{2} \ .$$

For point C the same expression yields the value

$$S = \frac{beh}{2} + \frac{h}{2}e\frac{h}{4} \Rightarrow \tau = \frac{V}{I}\left(\frac{bh}{2} + \frac{h^2}{8}\right) \ .$$

The resultants of the shearing stress in the web (R_a) and in the flanges (R_b) may be computed from the diagram areas in Fig. 118-b, multiplied by the thickness e, yielding

$$\begin{aligned} R_a &= \frac{V}{I}\left(\frac{bh}{2}he + \frac{2}{3}\frac{h^2}{8}he\right) = \frac{V}{I}\Bigg[\underbrace{\frac{eh^3}{12}}_{I_a} + \underbrace{2 \times be\left(\frac{h}{2}\right)^2}_{I_b - \frac{be^3}{12}}\Bigg] \approx V \\ R_b &\ \frac{1}{2}\frac{V}{I}\frac{bh}{2}be = \frac{V}{I}\frac{b^2he}{4} \approx \frac{3}{\left(\frac{h}{b}\right)^2 + 6\frac{h}{b}}V \ . \end{aligned} \tag{204}$$

It must be remarked here that, as mentioned in Footnote 55, an exact balance between the shear force and the resultant of the shearing stresses is only achieved if the moment of inertia of the flange, with respect to is centre line ($\frac{be^3}{12}$), is neglected.[5] The condition of equivalence of moments with respect to point D (Fig. 118-c) allows the computation of the distance d, which defines the position of the shear centre

$$R_b h = R_a d \Rightarrow d = \frac{3}{\left(\frac{h}{b}\right)^2 + 6\frac{h}{b}}h \ . \tag{205}$$

[5] From a mathematical point of view, the theory expounded for thin-walled cross-sections is only valid if the thickness of the walls is infinitesimal, in comparison with the cross-section dimensions. In this case, the moment of inertia of the flange with respect to its centre line, $\frac{be^3}{12}$, is an infinitesimal quantity of third order, which may be neglected in presence of the infinitesimal quantity of first order resulting from the parallel-axis theorem, $\frac{beh^2}{4}$.

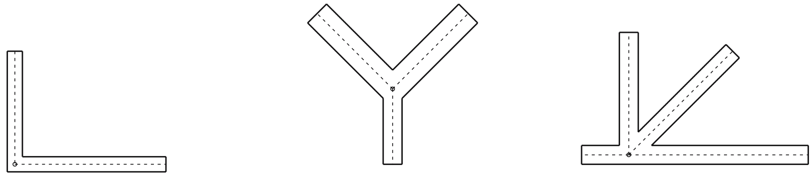


Fig. 119. Shear centre in thin-walled cross-sections with concurrent and straight wall elements

The thin-walled cross-sections with concurrent and straight wall elements, like those represented in Fig. 119, are a particularly simple case of determination of the shear centre. In fact, as the resultants of the shearing stresses in the different wall elements pass through the intersection of the centre lines, the moment of the shearing stress in relation to this point vanishes, which means that it is the shear centre.

VIII.5 Non-Prismatic Members

VIII.5.a Introduction

The basic equation for the analysis of the effect of the shear force (187) has been deduced for prismatic bars. So when the above expressions for the computation of shearing stresses are applied to non-prismatic members, errors are introduced. In order to get an idea of the importance of these errors, two examples of non-prismatic members, which are simple enough for an exact solution to be given by the Theory of Elasticity, are analysed.

VIII.5.b Slender Members with Curved Axis

As explained in Sect. VIII.2, the expression obtained for the shearing stress in a rectangular cross-section with a small thickness (189) coincides with the exact solution of the Theory of Elasticity. Thus, in a bar with the same cross-section, but with a curved axis, the discrepancies between the exact solution and the results obtained using (187) may be attributed to the fact that the bar's axis is not a straight line.

The bar represented in Fig. 120 has a circular axis and a rectangular cross-section with the dimensions $b \times h$ ($b \ll h$). The shear force in the cross-section B defined by the angle θ takes the value $V = -P\cos\theta$.

The shearing stress in that cross-section may be expressed as a function of the dimensionless coordinate η, which, multiplied by the height of the cross-section h, defines the distance to the centre line ($-\frac{1}{2} \leq \eta \leq \frac{1}{2}$, Fig. 120). The exact solution obtained by the Theory of Elasticity for the shearing stress on the cross-section defined by the angle θ may be defined by the expression [4]

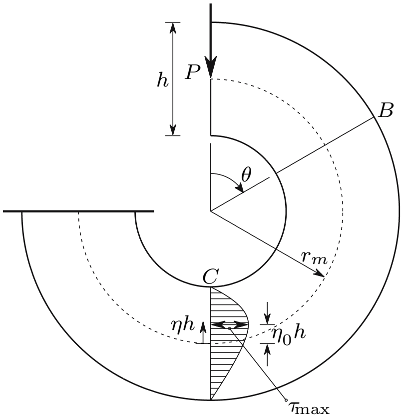


Fig. 120. Shearing stresses induced by the shear force in a bar with a curved axis

$$\tau = -\frac{3}{2}\frac{P\cos\theta}{bh}\underbrace{\frac{1+\eta\alpha+\frac{\left(1-\frac{\alpha^2}{4}\right)^2}{(1+\eta\alpha)^3}-\frac{2\left(1+\frac{\alpha^2}{4}\right)}{1+\eta\alpha}}{3-\frac{3}{\alpha}\left(1+\frac{\alpha^2}{4}\right)\ln\frac{2+\alpha}{2-\alpha}}}_{\gamma} \quad \text{with} \quad \alpha=\frac{h}{r_m}\,. \qquad (206)$$

In the limit case of a prismatic bar ($\alpha = 0$, $\theta = \pi$) this solution yields the same value as (190) $\left(\tau_{\max} = \frac{3}{2}\frac{V}{bh}\right)$.

When the relation α between the height of the cross-section and the curvature radius of the centre line r_m increases, the difference between the distributions of shearing stresses given by (206) and by the expression developed for prismatic bars increases also. This difference remains small, however, even for larger curvatures, as may be easily confirmed by computing the values of η and γ corresponding to the maximum shearing stress ($\eta = \eta_0 \Rightarrow \gamma = \gamma_{\max}$) for some values of α

α	0.0000	0.1000	0.2500	0.5000	0.7500	1.0000	1.5000
η_0	0.0000	0.0250	0.0626	0.1259	0.1905	0.2565	0.3885
$\gamma_{\max}$	1.0000	1.0009	1.0056	1.0233	1.0573	1.1166	1.4402

VIII.5.c Slender Members with Variable Cross-Section

In bars with variable cross-section the expressions developed on the basis of (187) may lead to completely erroneous results, at least in relation to the location of the maximum stress in the cross-section. For example, in the problem represented in Fig. 86, the exact solution shows that the shearing stress vanishes in the neutral axis and attains the maximum value in the farthest points from the neutral axis, as may be easily ascertained by a two-dimensional analysis of the stress state in those points, which totally contradicts the solution developed for prismatic bars.

Regarding the value of the maximum shearing stress in the cross-section, significant errors may also be introduced by the theory of prismatic bars, as may be easily verified by computing the maximum shearing stress in cross-section $\overline{AA'}$ (Fig. 86). From (164) we find that the maximum radial stress occurs in point A and takes the value

$$\varphi = \frac{\alpha}{2} \Rightarrow \sigma_r = \sigma_{r-\max} = \frac{2}{\alpha - \sin\alpha}\frac{P}{br}\sin\frac{\alpha}{2} \, .$$

A two-dimensional analysis of the stress state shows that the shearing stress in a vertical facet takes the value

$$\tau_{\max} = \frac{1}{2}\sigma_{r-\max}\sin\alpha = \sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sigma_{r-\max} = \frac{2\sin^2\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\alpha - \sin\alpha}\frac{P}{br} \, .$$

The theory of prismatic bars yields the following value for the maximum shearing stress in the same cross-section, $\tau_{\max-\mathrm{p}}$

$$\begin{cases} h = 2r\sin\frac{\alpha}{2} \\ \\ V = P \end{cases} \Rightarrow \tau_{\max-\mathrm{p}} = \frac{3}{2}\frac{V}{bh} = \frac{3}{4}\frac{1}{\sin\frac{\alpha}{2}}\frac{P}{br} \, .$$

The relation between the exact value $\tau_{\max}$ and the value yield by the theory of prismatic bars, $\tau_{\max-\mathrm{p}}$, depends only on angle α and may expressed by parameter β

$$\beta = \frac{\tau_{\max}}{\tau_{\max-\mathrm{p}}} = \frac{8}{3}\frac{\sin^3\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\alpha - \sin\alpha} \, .$$

The following Table gives the values of β corresponding to some values of angle α.

α	1°	10°	20°	30°	45°	60°
β	1.999	1.988	1.952	1.892	1.764	1.593

This example shows that the actual value of the maximum shearing stress in a slender member with a variable cross-section may be substantially higher than the value given by the theory of prismatic bars.

VIII.6 Influence of a Non-Constant Shear Force

The solution of the Theory of Elasticity for the shearing stresses in the example depicted in Fig. 85 (162) shows that (189) is exact ($V = p\left(\frac{l}{2} - z\right)$), although the expression defining the normal stresses σ_z (162) is different from the expression developed for the case of pure bending, on the basis of the law of conservation of plane sections. The variation of the shear force thus affects the distribution of normal stresses, but does not change the distribution of shearing stresses. This is due to the fact that the second element of the expression of σ_z (which represents the correction to be added to (146), to take the variation of the shearing stress into account) is independent of z, i.e., it is constant in all cross-sections, so it does not introduce a longitudinal shear force.

Also in the case of the example depicted in Fig. 120 the shear force is not constant. However, the distribution of shearing stresses in the cross-section is not altered by the variation of the shear force, since the exact solution (206) shows that the shearing stress is proportional to the shear force $V = -P\cos\theta$.

Considering these examples and the fact that the normal stress computed by means of the expressions developed on the basis of the Saint Venant hypothesis are very close to the exact solution (Sect. VII.7), we may conclude that the variation of the shear force does not affect the validity of the fundamental expression for studying the effect of the shear force (187).

VIII.7 Stress State in Slender Members

Generally, in slender members, the stresses that act on perpendicular facets to the cross-section plane and are parallel to it – σ_x, σ_y and τ_{xy} – either vanish, as happens if there are no forces applied on the bar element under consideration and the Poisson coefficient is constant, or are sufficiently small to be neglected (see example VIII.16). We thus have a plane stress state. Obviously, this does not apply to the regions in the vicinity of sudden changes in the cross-section dimensions, angle points of the bar's axis or strongly concentrated loads. However, in these cases the theory of prismatic bars is not valid.

According to these considerations, the stress state in a slender member under non-uniform bending may be analysed in the plane perpendicular to the cross-section which contains the shear force vector in the point under consideration.[6] In thin-walled cross-sections this is the longitudinal plane containing the wall centre line. In this kind of cross-section and also in rectangular sections under plane bending, the plane stress state may be visualized by means of the *principal stress trajectories.* These lines represent, in each point, the principal directions of the stress state. As the stress tensor only has a normal and a shearing component, the maximum principal stress is always a tensile one and the minimum principal stress is always compressive, as may be easily verified by drawing the Mohr circles corresponding to tensile and compressive

[6]This conclusion remains valid if a torsional moment is also acting, since this internal force only causes shearing stresses in the cross-section and not σ_x, σ_y or τ_{xy}, as will be seen in Chap. X.

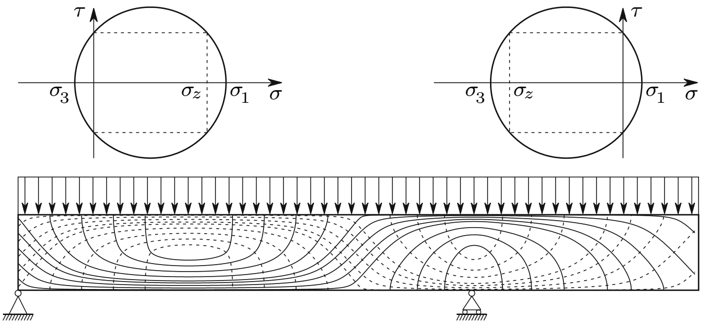


Fig. 121. Principal stress trajectories in a simply supported beam: —— tensile trajectories; ------ compressive trajectories

normal stresses σ_z, as represented in Fig. 121. The same Figure also shows the principal stress trajectories in a simply supported beam under a uniformly distributed load.

In the points on the neutral surface a purely deviatoric stress state is installed, since $\sigma = 0$. The principal direction are thus at 45° angles with the cross-section plane. If there are no shearing loads applied on the surface of the bar, one of the principal directions is perpendicular and the other is tangent to the surface. However, in the right end cross-section the principal directions are indeterminate, since there are no stresses in theses points. The principal stress trajectories, with an inclination of 45°, appearing in the left end cross-section result from the fact that the principal directions were computed by means of the theory of prismatic bars, assuming that the left reaction force is applied as a shear force acting on that cross-section. In the same way, the perturbation introduced by the concentrated load corresponding to the reaction force on the right support was not considered. Actually, in this region the compressive trajectories converge to the support. Furthermore, this reaction force was considered to be distributed over a small length, in order to avoid discontinuities in the shearing stress distribution, which would introduce corners into the principal stress trajectories.

The safety evaluation in bars under non-uniform bending usually includes three points:

– verification of the maximum normal stress in the fibres farthest from the neutral axis;
– verification of the maximum shearing stress, which usually occurs on the neutral axis;
– verification of the two-dimensional stress state in the points where the shearing and normal stresses simultaneously reach higher values. In these points, a yielding or a rupture criterion must be used. In ductile materials the von Mises criterion is generally used (see example VIII.17).

The third verification is especially important in I-beams and channels, in the points where web and flange connect, in the case of cross-sections where both the bending moment and the shear force attain higher values, as in the cross-sections which are close to the right support in the beam represented in Fig. 121. In these points, the normal stress is not much lower than the maximum value, since they are close to the fibres farthest from the neutral axis. The shearing stress is also close to the maximum value appearing on the neutral axis, because the area moment of the flanges is not substantially smaller than the area moment of half section. These considerations are summarized in Fig. 122.

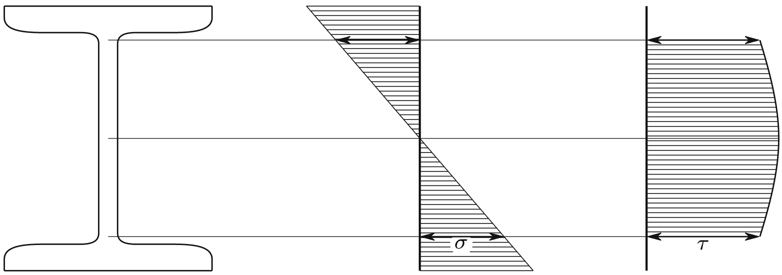


Fig. 122. Stress state (σ, τ) in connection points between web and flange in a I-beam

VIII.8 Examples and Exercises

VIII.1. Figure VIII.1 shows the cross-section of a simply supported beam with a span $100a$, under a uniformly distributed load p. The beam is made by connecting a bar with rectangular cross-section $a \times 3a$ and four bars with square cross-section $a \times a$. Determine the longitudinal shear force acting in each connection.

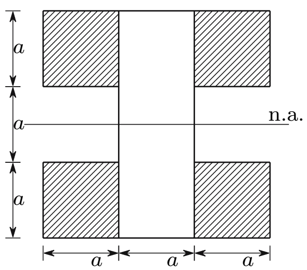


Fig. VIII.1

Resolution

The problem may be solved by directly applying (187). To this end, it is necessary to compute the moment of inertia of the cross-section and the first area moment S of one of the hatched areas in Fig. VIII.1 with respect to the neutral axis. These quantities take the values

$$I = \frac{(3a)^4}{12} - \frac{2a \times a^3}{12} = \frac{79}{12}a^4 \quad \text{and} \quad S = a^3 \ ,$$

respectively. The longitudinal shear force per unit length in each connection is then

$$\frac{\mathrm{d}E}{\mathrm{d}z} = \frac{VS}{I} = V\frac{a^3}{\frac{79}{12}a^4} = \frac{12}{79}\frac{V}{a} \ .$$

Since the longitudinal shear force is proportional to the cross-sectional shear force V, we conclude that it varies linearly between $-\frac{12}{79}\frac{50pa}{a} = -\frac{600}{79}p$ and $\frac{600}{79}p$ ($V_{\max} = 50pa$).

VIII.2. Determine the maximum shearing stress in a cross-section with the shape of an isosceles triangle of base b and height h, caused by a shear force acting on the symmetry axis (Fig. VIII.2-a).

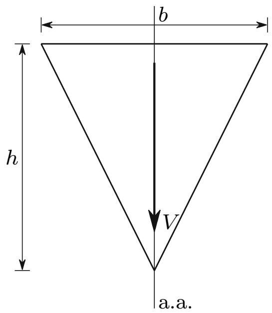


Fig. VIII.2-a

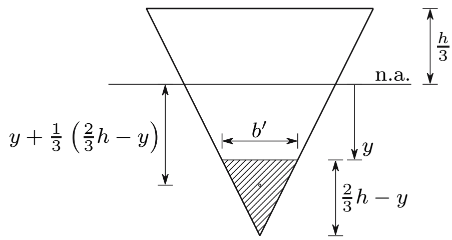


Fig. VIII.2-b

Resolution

Since the cross-section is symmetrical with respect to the action axis of the shear force, the problem may be solved by means of the theory expounded in Subsect. VIII.3.b. The moment of inertia of the cross-section takes the value $I = \frac{bh^3}{36}$ (Fig. VIII.2-a). The first area moment of the area defined by the distance y (shaded area in Fig. VIII.2-b) is given by the expression

$$S = \frac{1}{2}b'\left(\frac{2}{3}h - y\right)\left[y + \frac{1}{3}\left(\frac{2}{3}h - y\right)\right] \quad \text{with} \quad b' = \frac{b}{h}\left(\frac{2}{3}h - y\right) \ .$$

The vertical component of the shearing stress may obtained from (192), yielding

$$\tau_{xy} = \frac{VS}{Ib'} = \frac{V}{I}\left(\frac{1}{9}hy + \frac{2}{27}h^2 - \frac{1}{3}y^2\right) .$$

Differentiating this expression in relation to y and equating to zero, the value of y corresponding to the maximum shearing stress is obtained

$$\frac{\mathrm{d}\tau_{xy}}{\mathrm{d}y} = 0 \Rightarrow \frac{1}{9}h - \frac{2}{3}y = 0 \Rightarrow y = \frac{h}{6} .$$

The maximum value of the vertical component of the shearing stress is then

$$y = \frac{h}{6} \Rightarrow \tau_{xy} = \tau_{xy}^{\max} = \frac{1}{12}\frac{Vh^2}{I} = \frac{1}{12}\frac{36}{bh^3}Vh^2 = 3\frac{V}{bh} = \frac{3}{2}\frac{V}{\Omega} .$$

Since angle φ (Fig. 107) is constant, we conclude that the maximum shearing stress occurs at the sides of the cross-section at the distance $y = \frac{h}{6}$ from the neutral axis and takes the value (cf. (193))

$$\tau_{\max} = \frac{\tau_{xy}^{\max}}{\cos\varphi} = \frac{3}{\cos\left(\arctan\frac{b}{2h}\right)}\frac{V}{bh} .$$

As a rule, the maximum shearing stress occurs on the neutral axis. In this case it does not take place, which is because the cross-section width is not constant in the region around the neutral axis.

VIII.3. Determine the distribution of shearing stresses induced by a vertical shear force V in the open thin-walled cross-section depicted in Fig. VIII.3-a.

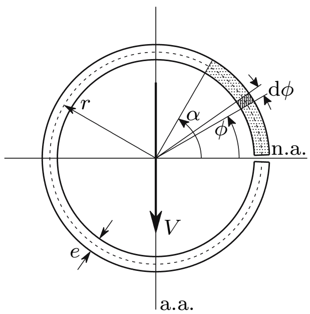


Fig. VIII.3-a

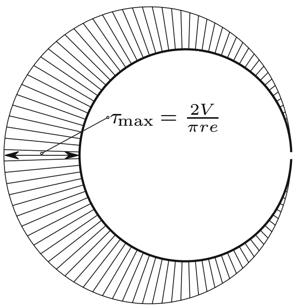


Fig. VIII.3-b

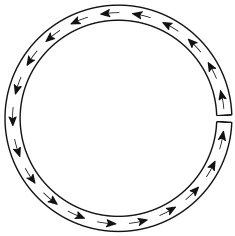


Fig. VIII.3-c

Resolution

Using polar coordinates, we may define the position of a point on the centre line by means of angle α (Fig. VIII.3-a). Denoting the integration variable by ϕ, the first area moment of the shaded area defined by angle α takes the value

$$\mathrm{d}S = er\,\mathrm{d}\phi \times r\sin\phi \Rightarrow S(\alpha) = er^2 \int_0^{\alpha} \sin\phi\,\mathrm{d}\phi = (1-\cos\alpha)\,er^2 \ .$$

The moment of inertia of the cross-section with respect to the neutral axis may also be computed by integration along the centre line, yielding

$$I = \int_0^{2\pi} (r\sin\phi)^2 \times er\,\mathrm{d}\phi = er^3 \int_0^{2\pi} \sin^2\phi\,\mathrm{d}\phi = \pi er^3 \ .$$

The shearing stress is then defined by the expression

$$\tau = \frac{VS}{Ie} = \frac{V}{\pi er}(1-\cos\alpha) \ .$$

This stress distribution defines the diagram represented in Fig. VIII.3-b. The direction of the shearing strain may be found, as described in Subsect. VIII.3.c (Fig. 111), which leads to the directions represented in Fig. VIII.3-c.

VIII.4. Determine the distribution of shearing stresses in a thin-walled circular tube with a wall-thickness e and a radius of the center line r.

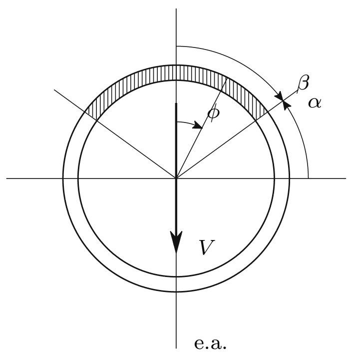


Fig. VIII.4

Resolution

Since we have a symmetrical cross-section, (197) may be used to compute the shearing stresses. Defining the position of the two symmetrical longitudinal cuts by angle β (Fig. VIII.4), we get, for the first area moment of the hatched area,

$$S(\beta) = 2\int_0^{\beta} r\cos\phi \times er\,\mathrm{d}\phi = 2er^2\sin\beta \ .$$

The shearing stress then takes the value

$$\tau = \frac{VS}{2Ie} = \frac{V}{I} r^2 \sin\beta \ .$$

VIII.5. Determine the distribution of shearing stresses in the cross-section considered in example VIII.4 without using symmetry considerations.

Resolution

The open cross-section considered in example VIII.3 may be used as the statically determinate base problem. Denoting the moment of inertia of the cross-section by I, the shearing stress in the open cross-section is given by the expression

$$\tau_0 = \frac{V}{I} r^2 (1 - \cos\alpha) \ .$$

The closed line integrals contained in (200) are given by the expressions

$$\oint \frac{S}{e} \mathrm{d}s = \int_0^{2\pi} \underbrace{r^2 (1 - \cos\alpha)}_{\frac{S}{e}} \underbrace{r\,\mathrm{d}\alpha}_{\mathrm{d}s} = 2\pi r^3 \qquad \text{and} \qquad \oint \frac{\mathrm{d}s}{e} = \frac{2\pi r}{e} \ .$$

Substituting these expressions into (200), we get

$$\tau_1 = -\frac{V}{Ie} \frac{\oint \frac{S}{e} \mathrm{d}s}{\oint \frac{\mathrm{d}s}{e}} = -\frac{V}{Ie} \frac{2\pi r^3}{\frac{2\pi r}{e}} = -\frac{V}{I} r^2 \ .$$

The shearing stress in the closed cross-section is then

$$\tau = \tau_0 + \tau_1 = -\frac{V}{I} r^2 \cos\alpha \ .$$

This value coincides with the solution obtained in example VIII.4, since $\sin\beta = \cos\alpha$. The difference in the sign results from the fact that in example VIII.4 the shearing stress is considered positive when it has the direction of progression of angle β, while in example VIII.3 the direction of progression of angle α is adopted as positive.

VIII.6. Determine the distribution of shearing stresses in the thin-walled cross-section represented in Fig. VIII.6-a. The cross-section wall has a constant thickness e.

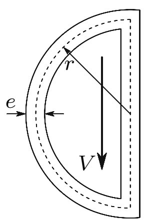


Fig. VIII.6-a

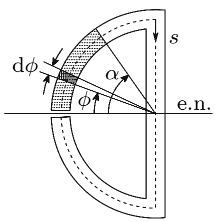


Fig. VIII.6-b

Resolution

Since the cross-section is doubly-connected and the action axis of shear force is not coincident with, or parallel to a symmetry axis, the shearing stress must be computed by means of (198)–(200). As a statically determinate base problem the open thin-walled cross-section represented in Fig. VIII.6-b may be used.

As in example VIII.3, the coordinate α may be considered to define the position of a point in the centre line of the curved part of the wall (Fig. VIII.6-b). Thus, the first area moment of the shaded area defined by angle α may be computed by means of the expression

$$S(\alpha) = \int_0^{\alpha} r \sin\phi \times er \,\mathrm{d}\phi = er^2 \int_0^{\alpha} \sin\phi \,\mathrm{d}\phi = (1-\cos\alpha)\, er^2 \,.$$

The shearing stress corresponding to this area moment is, then,

$$\tau_0(\alpha) = \frac{VS}{Ie} = \frac{V\,(1-\cos\alpha)\, er^2}{Ie} = \frac{Vr^2}{I}\,(1-\cos\alpha) \;.$$

In the straight wall, the position of a point on the centre line may be defined by coordinate s (Fig. VIII.6-b). The first area moment and the corresponding shearing stresses are then

$$S(s) = S\left(\alpha = \frac{\pi}{2}\right) + se\left(r - \frac{s}{2}\right) = er^2 + ers - e\frac{s^2}{2} \quad \tau_0(s) = \frac{V}{I}\left(r^2 + rs - \frac{s^2}{2}\right) \;.$$

Evaluating the integrals contained in (200), we get

$$\oint \frac{\mathrm{d}s}{e} = \frac{1}{e}\,(\pi r + 2r)$$

$$\oint \frac{S}{e}\,\mathrm{d}s = 2\int_0^{\frac{\pi}{2}} \frac{(1-\cos\alpha)\, er^2}{e} \underbrace{r\,\mathrm{d}\alpha}_{\mathrm{d}s} + \int_0^{2r} \left(r^2 + rs - \frac{s^2}{2}\right) \mathrm{d}s = \left(\pi + \frac{2}{3}\right) r^3 \,.$$

From (200) we get the shear flow

$$f = -\frac{V}{I}\frac{\oint \frac{S}{e}\,\mathrm{d}s}{\oint \frac{\mathrm{d}s}{e}} = -\frac{V}{I}\frac{\left(\pi + \frac{2}{3}\right) r^3}{\frac{1}{e}\,(\pi r + 2r)} = -\frac{Ver^2}{I}\frac{3\pi + 2}{3\pi + 6} \,.$$

Since the wall thickness is constant, the shearing stress corresponding to this shear flow is also constant and takes the value

$$\tau_1 = \frac{f}{e} = -\frac{Vr^2}{I}\frac{3\pi + 2}{3\pi + 6} \,.$$

The total shearing stress in the close thin-walled cross-section may be obtained by adding the stresses in the statically determinate base problem (τ_0) to the stress τ_1, which yields, respectively for the curved and straight walls, the expressions

$$\tau(\alpha) = \tau_0(\alpha) + \tau_1 = \frac{Vr^2}{I}(1-\cos\alpha) - \frac{Vr^2}{I}\frac{3\pi+2}{3\pi+6} = \frac{Vr^2}{I}\left(\frac{4}{3\pi+6} - \cos\alpha\right)$$

$$\tau(s) = \tau_0(s) + \tau_1 = \frac{V}{I}\left(r^2 + rs - \frac{s^2}{2} - \frac{3\pi+2}{3\pi+6}\right) .$$

In Figs. VIII.6-c a diagram showing the distribution of the shearing stress values in the cross-section is presented. Figure VIII.6-d shows the direction of the shearing stresses.

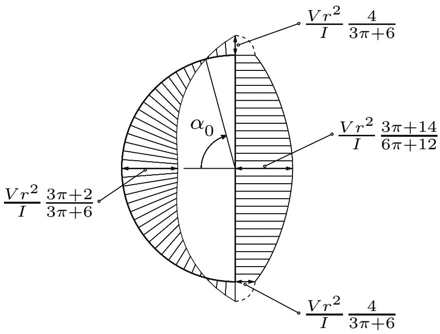


Fig. VIII.6-c

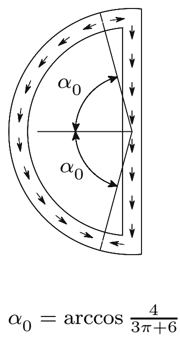


Fig. VIII.6-d

VIII.7. Develop expressions allowing the computation of the shearing stress caused by the shear force in a triply-connected thin-walled cross-section.

Resolution

The degree of static indeterminacy is two, since it is necessary to cut the cross-section wall in two points to get an open thin-walled cross-section. Let us assume that the two cuts are made in the points c_1 and c_2 (Fig. VIII.7). Denoting the coordinates along the centre line in the three walls by s_1, s_2 and s_3, the relative displacement D_1 in cut c_1 may be obtained by applying (198) to the channel defined by points a, b and c_1, which yields, considering as positive the coordinates which define a clockwise rotation around the channel

$$D_1 = \frac{V}{IG}\oint\frac{S}{e}\,\mathrm{d}s = \frac{V}{IG}\left(\int_{c_1}^{b}\frac{S}{e}\,\mathrm{d}s_1 - \int_a^b\frac{S}{e}\,\mathrm{d}s_3 + \int_a^{c_1}\frac{S}{e}\,\mathrm{d}s_1\right) .$$

In the same way, we get for the relative displacement D_2 in cut c_2

$$D_2 = \frac{V}{IG}\left(\int_{c_2}^{a}\frac{S}{e}\,\mathrm{d}s_2 + \int_a^b\frac{S}{e}\,\mathrm{d}s_3 + \int_b^{c_2}\frac{S}{e}\,\mathrm{d}s_2\right) .$$

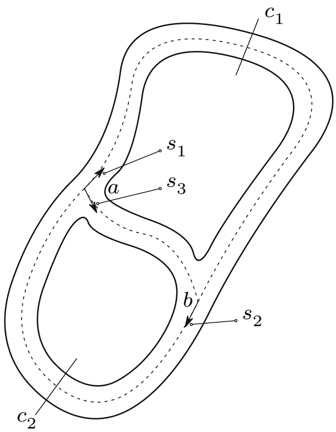


Fig. VIII.7

The displacements D_1 and D_2 are eliminated by the shear flows f_1, f_2 and f_3, corresponding to coordinates s_1, s_2 and s_3, respectively (Fig. VIII.7). Applying (199) to the two channels of the cross-section, we get

$$D_1' = \frac{f}{G}\oint \frac{\mathrm{d}s}{e} = \frac{f_1}{G}\int_{c_1}^{b} \frac{\mathrm{d}s_1}{e_1} - \frac{f_3}{G}\int_{a}^{b} \frac{\mathrm{d}s_3}{e_3} + \frac{f_1}{G}\int_{a}^{c_1} \frac{\mathrm{d}s_1}{e_1}$$

$$D_2' = \frac{f}{G}\oint \frac{\mathrm{d}s}{e} = \frac{f_2}{G}\int_{c_2}^{a} \frac{\mathrm{d}s_2}{e_2} + \frac{f_3}{G}\int_{a}^{b} \frac{\mathrm{d}s_3}{e_3} + \frac{f_2}{G}\int_{b}^{c_2} \frac{\mathrm{d}s_2}{e_2} \ .$$

The shear flows f_1, f_2 and f_3 are the unknowns of the problem, which may be computed by solving the system of equations represented by the compatibility conditions $D_1 + D_1' = 0$ and $D_2 + D_2' = 0$ and the condition of equilibrium of the flows in node a or in node b, $f_1 + f_3 = f_2$ (Fig. 112).

VIII.8. The beam represented in Fig. VIII.8 is made of concrete and reinforced with two steel plates, as shown. Assuming that the concrete does not crack in the tensioned zone, determine the longitudinal shear force in each steel-concrete connection. Consider $E_{\text{steel}} = 10E_{\text{concrete}}$.

Resolution

The weighted moment of inertia of the cross-section takes the value ($E_s = E_{\text{steel}}$ and $E_c = E_{\text{concrete}}$)

$$J_n = 2\left[\frac{b\left(\frac{b}{10}\right)^3}{12} + \frac{b^2}{10}\left(b + \frac{b}{20}\right)^2\right]E_s + E_c\frac{b\,(2b)^3}{12}$$

$$= b^4 E_s\left[0.2207 + \frac{E_c}{E_s}\frac{2}{3}\right] = 0.2873 b^4 E_s \ .$$

The first moment of the area occupied by a steel plate in the cross-section is

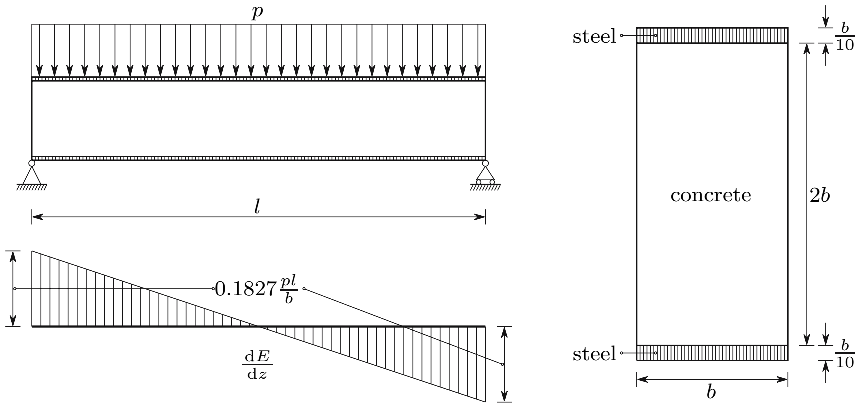


Fig. VIII.8

$$S_a = b\frac{b}{10}\left(b + \frac{b}{20}\right) = 0.105b^3 \ .$$

Substituting the values of J_n and S_a in the first of (202), we get the longitudinal shear force per unit length

$$\frac{\mathrm{d}E}{\mathrm{d}z} = \frac{VE_sS_a}{J_n} = \frac{VE_s 0.105b^3}{0.2873b^4E_s} = 0.3654\frac{V}{b} \ .$$

This force attains the maximum value in the cross-sections over the supports, taking the value (Fig. VIII.8)

$$V_{\max} = \frac{pl}{2} \Rightarrow \left(\frac{\mathrm{d}E}{\mathrm{d}z}\right)_{\max} = 0.1827\frac{pl}{b} \ .$$

VIII.9. In the cantilever beam considered in example VII.11 (Fig. VII.11-a), determine the distribution of the longitudinal shear force per unit length in the connection between the two materials.

Resolution

The horizontal shear force V_x does not cause a longitudinal shear force in the surface between the two materials, since axis y (Fig. VII.11-b) is a symmetry axis and that surface is perpendicular to this axis. The longitudinal shear force introduced by the cross-sectional shear force V_y may be computed by means of (202).

The weighted moment of inertia of the cross-section takes the value $J_x = 1915.34Ea^4$ (cf. example VII.11). The moment of the area occupied by material a, with respect to the neutral axis, is

$$S_a = 40a^2 \times 2.91176a \approx 116.470\,a^3 \ .$$

Considering a coordinate z with origin in the free end and pointing leftwards, the shear force V_y is defined by the expression

$$V_y(z) = 10paz\ .$$

The longitudinal shear force per unit length, as function of coordinate z, takes then the value (202)

$$E'(z) = \frac{\mathrm{d}E}{\mathrm{d}z} = \frac{V_y S_a E_a}{J_x} = \frac{10paz \times 116.470\,a^3 \times 2E}{1915.34Ea^4} \approx 1.21618\,pz\ .$$

This force attains the maximum value in the built-in end, yielding

$$z = z_{\max} = 200a \Rightarrow E' = E'_{\max} = 243.236\,pa\ .$$

VIII.10. Determine the shearing stress in the point of connection between the web and a flange in the cross-section represented in Fig. VIII.10.

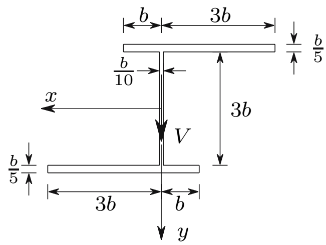


Fig. VIII.10

Resolution

The problem may be solved by means of (203). To this end, it is necessary to compute the moments and the product of inertia in relation to axes x and y. These quantities take the values

$$I_x = \frac{\frac{b}{10}(3b)^3}{12} + 2\left[\frac{4b\left(\frac{b}{5}\right)^3}{12} + 4b\frac{b}{5}\left(1.5b + \frac{b}{10}\right)^2\right] = 4.32633b^4$$

$$I_y = 2\left(\frac{\frac{b}{5}(3b)^3}{3} + \frac{\frac{b}{5}b^3}{3}\right) + \frac{3b\left(\frac{b}{10}\right)^3}{12} = 3.73358b^4$$

$$I_{xy} = 4b \times \frac{b}{5} \times b \times \left(1.5b + \frac{b}{10}\right) \times 2 = 2.56b^4\ .$$

In order to get the shearing stress in the connection point between the web and a flange, the first area moments of a flange, with respect to axes x and y

must be computed. Considering the bottom flange, these quantities are given by the expressions

$$S_x = 4b \times \frac{b}{5} \times \left(\frac{3}{2}b + \frac{b}{10}\right) = 1.28b^3 \qquad \text{and} \qquad S_y = 4b \times \frac{b}{5} \times b = 0.8b^3 \ .$$

The longitudinal shear force per unit length in this point is, then, (203)

$$\frac{\mathrm{d}E}{\mathrm{d}z} = \frac{3.73358b^4 \times 1.28b^3 - 2.56b^4 \times 0.8b^3}{4.32633b^4 \times 3.73358b^4 - (2.56b^4)^2} V = 0.28450 \frac{V}{b} \ .$$

Thus, the shearing stress in that point takes the value

$$\tau = \frac{\mathrm{d}E}{\frac{b}{10}\,\mathrm{d}z} = \frac{0.2845}{0.1b}\frac{V}{b} = 2.845 \frac{V}{b^2} \ .$$

As an alternative, this stress could be determined by decomposing the shear force in the two principal directions of inertia. However the volume of computation would be substantially larger, since it would be necessary to compute the principal moments and directions of inertia and two shearing stresses (one for each plane of bending).

VIII.11. Determine the position of the shear centre in the cross-section represented in Fig. VIII.3-a.

Resolution

As seen in example VIII.3, the stresses caused by the shear force in the cross-section are given by the expression

$$\tau = \frac{V}{\pi e r} (1 - \cos \alpha) \ .$$

The position of the shear centre on the symmetry axis may be obtained by the equivalence condition of the moments of the shearing stresses and of the shear force with respect to any point of the cross-section's plane. The point which leads to the simplest expressions is the centre of the cross-section. In this case, we have

$$M = \int_0^{2\pi} r \times \tau e r \,\mathrm{d}\alpha = \frac{Vr}{\pi} \int_0^{2\pi} (1 - \cos \alpha)\,\mathrm{d}\alpha = 2Vr \ .$$

The shearing stress resultant obviously takes the value V (this may easily be confirmed by evaluating the integral $\int_0^{2\pi} -\tau e r \cos \alpha \,\mathrm{d}\alpha$). The moment of the stresses will be equal to the moment of the shear force if the action axis of V is at a distance d from the cross-section centre given by the expression

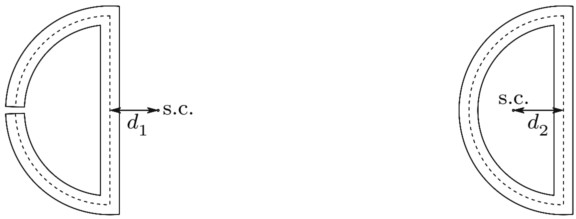
$$V \times d = 2Vr \ \Rightarrow \ d = 2r \ .$$

Since the cross-section is symmetrical in relation to the horizontal axis passing through the centre, the shear centre will be on this axis, at a distance $2r$ from the centroid, to the left.

VIII.12. Considering the cross-section defined in example VIII.6, determine the position of the shear centre:

(a) in the statically determinate base problem (bar with the longitudinal cut, Fig. VIII.6-b);

(b) in the closed cross-section.



$d_1 = \frac{6\pi-12}{3\pi+4} r \approx 0.51022\, r$ $d_2 = \frac{4\pi+24}{3\pi^2+10\pi+8} r \approx 0.52976\, r$

Fig. VIII.12

Resolution

In order to determine the position of the shear centre, the moment of inertia must be expressed as a function of the given geometrical data. Considering that the cross-section is thin-walled, this quantity may be given by the expression (Fig. VIII.6-b)

$$I = 2\int_0^{\frac{\pi}{2}} (r\sin\phi)^2\, re\,\mathrm{d}\phi + \frac{e\,(2r)^3}{12} = \left(\frac{\pi}{2} + \frac{2}{3}\right) er^3 \, . \qquad \text{(VIII.12-a)}$$

(a) The simplest expression for the moment of the shearing stresses is found if the centre of the half-circumference defining the centre line of the curved wall is used as reference point. This moment has a clockwise direction and takes the value

$$\tau_0 = \frac{Vr^2}{I}(1-\cos\alpha) \Rightarrow \left| \begin{array}{l} M = 2\int_0^{\frac{\pi}{2}} r \times \tau er\,\mathrm{d}\alpha \\ \quad = \frac{2Ver^4}{I}\int_0^{\frac{\pi}{2}} (1-\cos\alpha)\,\mathrm{d}\alpha = \frac{Ver^4}{I}(\pi-2) \, . \end{array} \right.$$

This moment will be equivalent to the moment of the shear force with respect to the same point if its line of action is at a distance d to the right of this point, so the following condition is satisfied

$$Vd = M \Rightarrow Vd = \frac{Ver^4}{I}(\pi-2) \Rightarrow d = \frac{er^4}{I}(\pi-2) = \frac{6\pi-12}{3\pi+4} r \, .$$

where I has been substituted by (VIII.12-a).

(b) In the case of the closed cross-section, the procedure is similar, the only difference being, that the shearing stress in the curved wall is given by the expression

$$\tau(\alpha) = \frac{Vr^2}{I}\left(\frac{4}{3\pi + 6} - \cos\alpha\right) .$$

The moment of the stresses with respect to the centre of the half-circumference, considered as positive in the clockwise direction, is then given by the expression

$$M = \frac{2Ver^4}{I}\int_0^{\frac{\pi}{2}}\left(\frac{4}{3\pi + 6} - \cos\alpha\right) = -\frac{Ver^4}{I}\frac{2\pi + 12}{3\pi + 6} .$$

The equivalence condition of the moments of the stresses and the shear force yields the position of the shear centre

$$Vd = M \;\Rightarrow\; Vd = -\frac{Ver^4}{I}\frac{2\pi + 12}{3\pi + 6} \;\Rightarrow\; d = -\frac{4\pi + 24}{3\pi^2 + 10\pi + 8}r .$$

The minus sign means that the shear centre is located to the left of the reference point.

Figure VIII.12 shows these two cross-sections with their corresponding shear centres. As mentioned in Footnote 57, the shear flow f computed in example VIII.6 corresponds to a torsional moment. The relation between the shear flow and the torsional moment will be studied in Sect. X.3. This relation is given in (240) and may be written in the form $T = 2Af$, where A represents the area limited by the wall's centre line in the closed cross-section. In the present case, we have

$$\begin{cases} f = \frac{Ver^3}{I}\frac{3\pi+2}{3\pi+6} \\ A = \frac{\pi r^2}{2} \\ I = \left(\frac{\pi}{2} + \frac{2}{3}\right)er^3 \end{cases} \Rightarrow T = 2 \times \frac{\pi r^2}{2} \times \frac{Ver^3}{\left(\frac{\pi}{2}+\frac{2}{3}\right)er^3}\frac{3\pi+2}{3\pi+6} \approx 1.03398\,Vr .$$

This torsional moment is equal to $V(d_1 + d_2)$, that is, it corresponds to the translation of the shear force from the shear centre of the open cross-section to the shear center of the closed section.

VIII.13. The cantilever beam represented in Fig. VIII.13-a has a closed cross-section from A to B and an open cross-section from B to C. The wall has constant thickness e. The plane of application of the distributed load p contains the centre line of the web in segment BC of the beam. Draw the diagram representing the torsional moment of the beam.

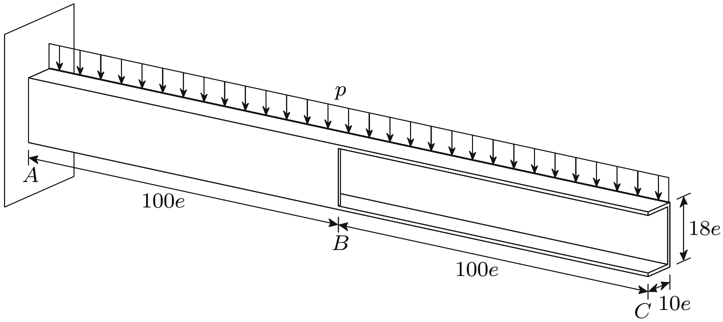


Fig. VIII.13-a

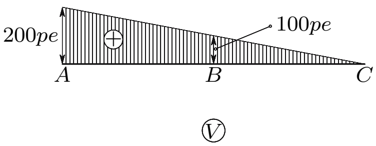


Fig. VIII.13-b

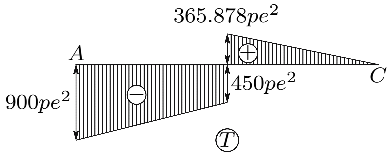


Fig. VIII.13-c

Resolution

The shear centre of the cross-section in segment AB coincides with the centroid, since the cross-section is doubly symmetric. The distance from the action axis of the shear force to the shear centre is $4.5\,e$. In the case of segment BC, the position of the shear centre may be computed from (205). In this case, we have

$$\begin{cases} h = 17e \\ b = 9e \end{cases} \Rightarrow d = \frac{3}{\left(\frac{h}{b}\right)^2 + 6\frac{h}{b}} h = \frac{3}{\left(\frac{17e}{9.5e}\right)^2 + 6\frac{17e}{9.5e}} 17e \approx 3.65878\,e\ .$$

The diagram representing the distribution of the shear force is given in Fig. VIII.13-b. By multiplying the shear force by the distance of its line of action to the shear centre, a negative torsional moment is obtained for segment AB, while in part BC it takes a positive value, as represented in Fig. VIII.13-c.

VIII.14. Draw a diagram representing the distribution of torsional moment in the cantilever beam considered in example VII.11 and represented in Fig. VII.11-a.

Resolution

The beam has a thin-walled cross-section with straight centre lines converging in a point, so that this point – the centroid of the rectangle made of material a – coincides with the shear centre of the cross-section. In fact, the non-homogeneity of the cross-section does not affect the validity of the line of reasoning developed for this kind of cross-sections (Fig. 119).

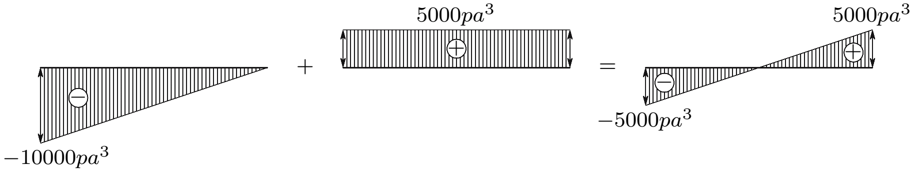


Fig. VIII.14-a, VIII.14-b VIII.14-c

The vertical load p causes a negative torsional moment with the value $V_y \times 5a$, as represented in Fig. VIII.14-a. The horizontal load $P = 500pa^2$, induces a constant positive torsional moment with the value $P \times 10a$, which is shown in Fig. VIII.14-b. By superposing these two diagrams, we get the total moment given in Fig. VIII.14-c.

VIII.15 Figure VIII.15 represents a thin-walled cross-section made of two materials a (hatched areas) and b, with the moduli of elasticity $E_a = 9E$ and $E_b = 3E$. Determine the position of the shear centre of this cross-section.

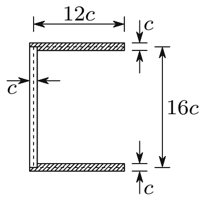


Fig. VIII.15

Resolution

The longitudinal shear force in composite bars under plane bending may be computed by means of (201). Representing by S_a and S_b the first area moments contained in (201), we may develop the following expression for the computation of the shearing stress

$$\tau = \frac{\mathrm{d}E}{e\,\mathrm{d}z} = \frac{V\,(E_a S_a + E_b S_b)}{J_n e} \ .$$

Since the cross-section has a symmetry axis, the shear centre is on this axis and its position may be obtained by considering a shear force with the direction of the other principal axis of inertia (a vertical axis in this case). The weighted moment of inertia, with respect to the horizontal axis takes the value

$$J_n = \frac{c\,(16c)^3}{12} 3E + 2 \times 12c^2 \times (8c)^2 \times 9E = 14848Ec^2 \ .$$

The procedure leading to the computation of the position of the shearing stress resultant from the above indicated expressions for the shearing stress τ and for the moment of inertia I is exactly the same as that used in the example described in Sect. VIII.4. The distribution of shearing stress takes the same form, as indicated in Fig. 118-b. The stress in the angle point between the web and a flange takes the value ($S_a = 12c^2 \times 8c$ and $S_b = 0$)

$$\tau_1 = \frac{V \times 9E \times 12c^2 \times 8c}{14848Ec^4 \times c} = \frac{27}{464}\frac{V}{c^2} \approx 58.190 \times 10^{-3}\frac{V}{c^2} \, .$$

As in the case of Fig. 118-b, we compute the difference between the maximum stress in the web and τ_1. This difference is defined by the first area moment of the half web with respect to the neutral axis, taking the value

$$\tau_2 = \tau_{\max} - \tau_1 = \frac{V}{14848Ec^4 \times c} 8c^2 \times 4c \times 3E = \frac{3}{464}\frac{V}{c^2} \, .$$

The resultant of the shearing stresses in a flange is equal to the area of the stress diagram (triangular diagram, Fig. 118-b), multiplied by the thickness $e = c$

$$R_b = \frac{1}{2}\tau_1 \times 12c \times c = \frac{81}{232} V \, .$$

The moment of the shearing stresses with respect to one of the angle points of the wall's centre line may be computed directly from R_b, yielding

$$M = R_b \times 16c = \frac{162}{29} Vc \, .$$

The resultant of the shearing stresses is equal to the shear force. This fact may easily be confirmed by computing the stress resultant in the web, R_a, which is simultaneously the total stress resultant, since the stresses in the flanges are perpendicular to the shear force

$$R_a = \tau_1 \times 16c^2 + \frac{2}{3}\tau_2 \times 16c^2 = \left(\frac{27}{464} \times 16 + \frac{2}{3}\frac{3}{464} \times 16\right) V = V \, .$$

By equating the moment of the stresses to the moment of the shear force, we get the distance d of the action axis of this force to the centre line of the web, such that it does not induce torsion. This distance defines the position of the shear centre (Fig. 118-c)

$$M = V \times d \Rightarrow \frac{162}{29} Vc = Vd \Rightarrow d = \frac{162}{29} c \approx 5.5862\, c \, .$$

VIII.16. In the beam represented in Fig. 85, find the relation between the maximum values of the stresses σ_y and σ_z (162), as a function of the relation between the height h of the cross-section and the span l.

Resolution

From the second of (162) we can see that the maximum value of σ_y occurs in the points belonging to the upper fibres and that it takes the value $-p$. Dividing this value by $\sigma_{z-\max}$ (162), we get the relation

$$\frac{\sigma_{y-\max}}{\sigma_{z-\max}} = \frac{-p}{\frac{ph^3}{\frac{h^3}{12}}\left[\frac{1}{16}\left(\frac{l}{h}\right)^2 + \frac{1}{60}\right]} = -\frac{5}{\frac{15}{4}\left(\frac{l}{h}\right)^2 + 1} \, .$$

In slender members we generally have $l \geq 10h$. For $l = 10h$, $\sigma_{y-\max}$ is only about 1% of $\sigma_{z-\max}$.

VIII.17. Develop expressions for the direct application of the Tresca's and von Mises' yielding criteria to the safety evaluation of slender members.

Resolution

The principal stresses of the stress state in a point of a slender member are given by the expressions (cf. (38), with $\sigma_x = \sigma$, $\sigma_y = 0$ and $\tau_{xy} = \tau$)

$$\sigma_1 = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} \qquad \text{and} \qquad \sigma_3 = \frac{\sigma}{2} - \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} \, ,$$

where σ and τ are the normal and shearing stresses in a point of the cross-section. Substituting these values in the expression of Tresca's yielding criterion (104), we get

$$\sigma_1 - \sigma_3 \leq \sigma_{\text{all}} \Rightarrow \sqrt{\sigma^2 + 4\tau^2} \leq \sigma_{\text{all}} \, .$$

Using the same procedure in relation to von Mises' yielding criterion (105), we get ($\sigma_2 = 0$)

$$\sqrt{\sigma_1^2 - \sigma_1\sigma_3 + \sigma_3^2} \leq \sigma_{\text{all}} \Rightarrow \sqrt{\sigma^2 + 3\tau^2} \leq \sigma_{\text{all}} \, .$$

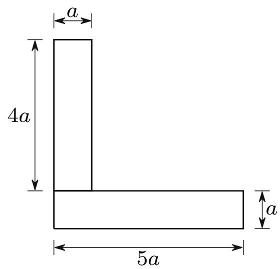


Fig. VIII.19

VIII.18. A cantilever beam of length l, carrying a uniformly distributed load p, is made of two materials, a and b, as depicted in Fig. VII.24. The two materials have linear elastic behaviour, with elasticity moduli $E_a = 4E$ and $E_b = E$. Determine the longitudinal shear force in the connection between one rectangle of material a and the material b around it.

VIII.19. Figure VIII.19 represents the cross-section of a cantilever beam of length $100a$. The beam is composed of two bars with rectangular cross-section, connected as shown in the figure. The beam supports a vertical, uniformly distributed load p.

(a) What is the longitudinal shear force in the connection between the two bars?

(b) What should be the position of the plane containing the loading, so that no torsion takes place?