VII

Bending Moment

VII.1 Introduction

As mentioned in Subsect. V.10.c, a prismatic bar under a constant bending moment is a symmetric problem in relation to any plane containing a cross-section and, as a consequence, the law of conservation of plane sections is valid. Therefore, although the axis of a bar under a bending moment does not remain a straight line – it acquires *curvature* – the cross-sections remain plane and perpendicular to the bar axis, provided that the bending moment is constant.

In the case of a varying bending moment the symmetry is lost not only because the moments acting in both ends of a piece of bar are different, but also due to the appearance of a shear force, which is equal to the derivative of the function describing the bending moment, in relation to a coordinate with the direction of the bar axis.

However, the stresses induced in a slender member by a constant bending moment, are not changed if a constant shear force is applied, and are a very close approximation to the actual stress distribution caused by the bending moment in the case of a non-constant shear force, as will be seen later (Sect. VII.7 and Chap. VIII). For these reasons, the stresses induced by bending are studied by considering a zero shear force, which means a constant bending moment.

It is usual to distinguish between the following types of bending:

– *Pure or circular bending.* This type of bending occurs when the only internal force in the bar is a constant bending moment, i.e., the axial (N) and shear (V) forces and the torsional moment (T) are zero. The designation of circular bending is suggested by the fact that the deformed axis of the initially prismatic bar is an arc of circumference, when the bending moment is constant (a constant moment implies a constant curvature).

– *Non-uniform bending.* This designation is normally used for a loading causing bending moment and shear force, that is, for a non-constant bending moment. The axial force and the torsional moment are zero.
– *Composed bending.* This designation is used in this book for a loading causing bending moment and axial force. The bending moment may be constant (circular composed bending: $M \neq 0$, $N \neq 0$, $V = 0$, $T = 0$) or variable (non-uniform composed bending: $M \neq 0$, $N \neq 0$, $V \neq 0$, $T = 0$).

Each of these three types of bending may be sub-divided into *plane* and *inclined* bending. In the first case the plane containing the deformed bar is parallel to the plane containing the couple of forces which defines the bending moment. In the second case, the first plane is inclined in relation to the second one.

VII.2 General Considerations

When a bar is under the action of symmetrical internal forces (constant axial force and bending moment) the symmetry of the problem leads to the conclusion that the cross-sections remain plane and perpendicular to the bar's axis, as seen before. The same symmetry conditions allow the conclusion that the shearing stress in the cross-section vanishes.[1] We may also easily demonstrate that, *if Poisson's coefficient is constant*, the normal and shearing stresses acting in facets which are perpendicular to the cross-section's plane vanish (cf. Sect. VII.6). Here we are considering the geometry of the member defined in relation to a rectangular Cartesian reference frame x, y, z, with axis z parallel to the bar's axis. In accordance with these considerations, we will have a one-dimensional stress state, i.e., a stress tensor with the components $\sigma_x = \sigma_y = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0$ and $\sigma_z \neq 0$.

The analysis of the normal stresses in the cross-section ($\sigma_z \neq 0$) may be carried out directly from the law of conservation of plane sections and equilibrium considerations. To this end, let us consider two cross-sections of a prismatic bar at an infinitesimal distance l_0 from each other. Considering a reference system with its origin in the centroid of the left cross-section, the position of the points pertaining to the right section in the deformed configuration may be defined by the equation of an inclined plane (Fig. 72)

$$z\,(x, y) = l_0 + a_1 x + b_1 y + c_1 \ .$$

In fact, it is easily demonstrated that, as l_0 is an infinitesimal distance, the displacements in the plane (x, y) are infinitesimal quantities of higher order, as compared with the displacements in direction z, so they may be neglected.

[1] If there were any shearing stresses, they would be represented by vectors with opposite directions in the end sections of a small piece of bar, which is not compatible with the complete symmetry of the problem in relation to the middle cross-section of any piece of the bar.

The strain ε_z at the point of the cross-section defined by the coordinates x and y is then

$$\varepsilon_z(x,y) = \frac{l - l_0}{l_0} = \frac{z - l_0}{l_0} = \frac{a_1}{l_0}x + \frac{b_1}{l_0}y + \frac{c_1}{l_0}\,. \tag{138}$$

The validity of this expression does not depend on the rheological behaviour of the material of which the bar is made, since it depends directly on the law of conservation of plane sections. The application of the law of conservation of plane sections reduces to three (a_1, b_1 and c_1) the number of parameters needed to define completely the relative motion of two cross-sections at an infinitesimal distance from each other. As this is exactly the number of equilibrium conditions which may be established for a system of parallel forces in a three-dimensional space (the stresses σ_z) the problem of computation of the stresses induced by the bending moment becomes statically determinate. If the bar is homogeneous and is made of a material with

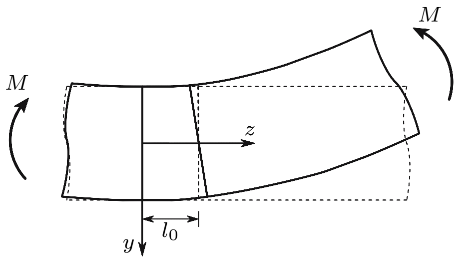

Fig. 72. Relative motion of two cross-sections in the bending deformation: ------ original configuration; —— deformed configuration

linear elastic behaviour, we have $\sigma_x = \sigma_y = 0$, as mentioned above. Therefore, the normal stress in the cross-section, σ_z, may be obtained by means of the one-dimensional Hooke's law, $\sigma = E\varepsilon$[2]

$$\sigma = \sigma_z = E\varepsilon_z = ax + by + c \quad \text{with} \quad \begin{cases} a = \dfrac{a_1}{l_0}E \\ b = \dfrac{b_1}{l_0}E \\ c = \dfrac{c_1}{l_0}E\,. \end{cases}$$

The constants a, b and c may be obtained by means of the aforementioned equilibrium conditions. Considering the components M_x and M_y of the bending moment as positive when they take the direction defined by a positive (tensile) stress acting in a point with positive x and y coordinates, we get

[2]In this expression and in the following account the index z is omitted ($\sigma = \sigma_z$), since σ_z is the only normal stress which needs to be considered in the theory of bending.

$$\begin{cases} N \;= \int_\Omega \sigma \,\mathrm{d}\Omega = \int_\Omega (ax + by + c)\,\mathrm{d}\Omega = c\Omega \\ M_x = \int_\Omega \sigma y \,\mathrm{d}\Omega = \int_\Omega \left(axy + by^2 + cy\right)\mathrm{d}\Omega = aI_{xy} + bI_x \\ M_y = \int_\Omega \sigma x \,\mathrm{d}\Omega = \int_\Omega \left(ax^2 + bxy + cx\right)\mathrm{d}\Omega = aI_y + bI_{xy} \;. \end{cases} \tag{139}$$

The first area moments $\int_\Omega x\,\mathrm{d}\Omega$ and $\int_\Omega y\,\mathrm{d}\Omega$ vanish, since the axes x and y pass through the centroid of the cross-section. The quantities $I_x = \int_\Omega y^2\,\mathrm{d}\Omega$, $I_y = \int_\Omega x^2\,\mathrm{d}\Omega$ and $I_{xy} = \int_\Omega xy\,\mathrm{d}\Omega$ are the moments and the product of inertia with respect to the central x and y axes. Solving this system of equations, we get

$$\begin{cases} a = \dfrac{M_y I_x - M_x I_{xy}}{I_x I_y - I_{xy}^2} \\ b = \dfrac{M_x I_y - M_y I_{xy}}{I_x I_y - I_{xy}^2} \\ c = \dfrac{N}{\Omega} \end{cases} \Rightarrow \sigma = \frac{M_y I_x - M_x I_{xy}}{I_x I_y - I_{xy}^2}x + \frac{M_x I_y - M_y I_{xy}}{I_x I_y - I_{xy}^2}y + \frac{N}{\Omega}\;. \tag{140}$$

This expression furnishes the stresses induced in the cross-section of a homogeneous prismatic member, made of a material with a linear elastic constitutive law, under the action of a bending moment and an axial force. It is important to note that the validity of this expression is not restricted to infinitesimal relative rotations of the cross-sections (i.e., to an infinitesimal curvature of the deformed member), since it was not necessary to use this approximation to deduce (140).

The analysis of the different types of bending referred to in Sect. VII.1 could be performed from (140), by particularizing it to the different cases, namely pure or composed and plane or inclined bending. However, in order to make the physical understanding easier, we expound the bending theory in the opposite sequence, i.e., we start with the most simple case (pure plane bending) and progressively generalize the conclusions to more complex cases. Equation (140) will however still be used for some particular problems. The last part of this chapter contains some problems where (140) is not valid: prismatic members made of two materials with linear elastic behaviour, and members made of materials with nonlinear behaviour in particularly simple cases.

In order to systematize the exposition of the theory of bending, some frequently used concepts are first defined:

- *action axis of the bending moment*: axis defined in the plane of the cross-section which is perpendicular to the vector representation of the bending moment (moments are usually represented by double-headed arrows); an equivalent definition is the intersection of the cross-section plane with the plane containing the couple of forces which defines the bending moment; this axis is simply called *action axis*, if it is not necessary to distinguish it from the action axis of the shear force;

- *action axis of the shear force*: line of action of the shear force acting on the cross-section;[3]
- *fibre*: prism with an infinitesimal cross-section area ($d\Omega$) with its axis parallel to the axis of the slender member;
- *neutral axis*: axis of rotation of a cross-section in relation to another, infinitesimally close, cross-section, in the deformation caused by the bending moment (and by the axial force in the case of composed bending); this name (neutral) comes from the fact, that the normal stress vanishes in the points of the cross-section belonging to this axis, since there is no elongation of the fibres passing through it; if the axial force is zero, or takes a sufficiently small value, the neutral axis divides the cross-section in compression and tension zones;
- *neutral surface*: surface defined by the points contained in the neutral axes of the cross-sections in the deformed member; it is also the surface defined by the fibres which do not suffer elongation or shortening (neutral fibres);
- *deflection curve*: line defining the shape of the bar's axis after the deformation; it may or may not be contained in a plane;
- *deflection plane*: plane containing the deflection curve; if this curve is not contained in a plane, the deflection plane varies along the deflection curve and is defined in an infinitesimal axis' length; this plane is perpendicular to the neutral axis.

VII.3 Pure Plane Bending

A prismatic bar is said to be under pure plane bending if the bending moment is constant and there is no axial force (pure bending) and the deflection plane is perpendicular to the vector representing the bending moment (plane bending). In this case, the cross-section rotates around an axis parallel to this vector, which means that the action axis of the bending moment is perpendicular to the neutral axis. This kind of bending takes place, for example, in a bar whose cross-section has a symmetry axis, if the action axis coincides with that axis, as represented in Fig. 73.

In the deformation of the bar, the neutral fibre AB did not change its length l. As the bending moment is constant, the curvature of the deflection curve is also constant, which means that this curve has the shape of a circumference arc. The angle φ, defining the relative rotation of the end cross-sections, may be related to the curvature radius ρ by the expression

$$\rho\varphi = l \Rightarrow \varphi = \frac{l}{\rho} \,. \tag{141}$$

[3] If the forces causing the non-uniform bending are all in the same plane and are perpendicular to the axis of the bar, this plane may be called *plane of actions*. In this case, the action axes of the bending moment and the shear force coincide with the intersection of the plane of actions and the cross-section plane.

Fibre CD, located at a distance y from the neutral axis, suffers a strain, which may be related to the curvature $\frac{1}{\rho}$ by the expression (Fig. 73)

$$\Delta l = \varphi\,(\rho + y) - \varphi\rho = \varphi y = \frac{l}{\rho} y \;\Rightarrow\; \varepsilon = \frac{\Delta l}{l} = \frac{1}{\rho} y \,. \tag{142}$$

This relation between the curvature $\frac{1}{\rho}$ and the strain at the point defined by coordinate y has been obtained directly by means of geometrical considerations based of the law of conservation of plane sections. It is therefore valid independently of the rheological properties of the material of the bar. Nor is its validity limited by the size of the deformations.

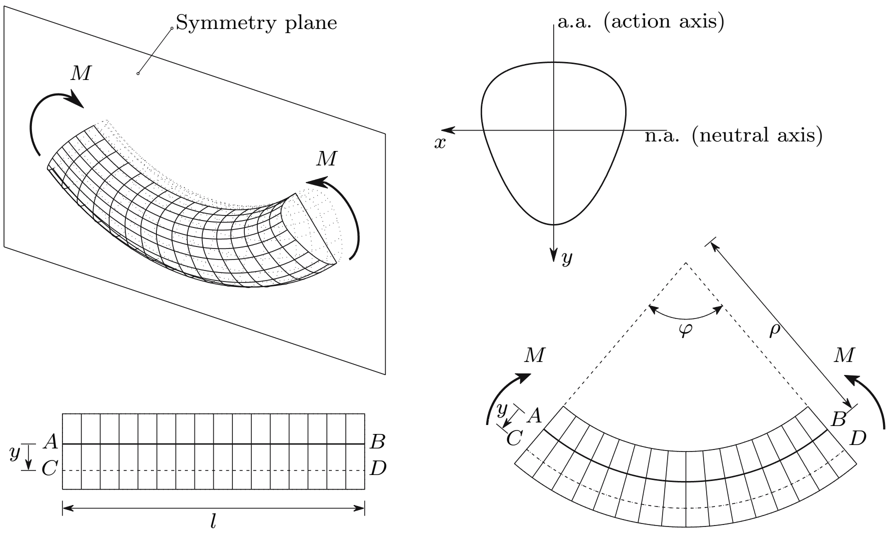

Fig. 73. Plane circular bending of a bar with a symmetric cross-section

If the bar is homogeneous and is made of a material with linear elastic behaviour, the stress may be found using the one-dimensional Hooke's law

$$\sigma = E\varepsilon = \frac{Ey}{\rho} \,. \tag{143}$$

The position of the neutral axis is obtained by the condition of equilibrium of the normal forces to the cross-section. As the axial force is zero, the resultant of the normal stresses must vanish. This condition yields

$$\int_\Omega \sigma \,\mathrm{d}\Omega = 0 \;\Rightarrow\; \int_\Omega \frac{Ey}{\rho}\,\mathrm{d}\Omega = 0 \;\Rightarrow\; \int_\Omega y \,\mathrm{d}\Omega = 0 \,. \tag{144}$$

This integral represents the first area moment of the cross-section with respect to the neutral axis (the distance y is defined in relation to this axis).

As this moment is zero, the neutral axis must pass through the centroid of the cross-section.

The resulting moment of the stresses acting in the cross-section must be equal to the bending moment M. From this condition, a relation between the curvature and the bending moment may be obtained

$$M = \int_\Omega \sigma y \,\mathrm{d}\Omega = \frac{E}{\rho}\int_\Omega y^2 \,\mathrm{d}\Omega \;\Rightarrow\; \frac{1}{\rho} = \frac{M}{EI} \quad \text{with} \quad I = \int_\Omega y^2 \,\mathrm{d}\Omega \; . \tag{145}$$

In this expression I represents the moment of inertia of the cross-section in relation to the neutral axis. The quantity $EI = \frac{\mathrm{d}M}{\mathrm{d}\left(\frac{1}{\rho}\right)}$ is called the *bending stiffness*, since it relates the amount of bending deformation (the curvature) to the internal force causing it (the bending moment).

By substituting (145) in (143), we get a relation between the stress and the bending moment

$$\sigma = \frac{My}{I} \; . \tag{146}$$

It is obvious from (143) or (146), that the stress attains its maximum value at the most distant point from the neutral axis. Denoting this distance by v ($v = |y|_{\max}$), we get, for the maximum absolute value of the stress in the cross-section,

$$|\sigma|_{\max} = \frac{M}{\left(\frac{I}{v}\right)} \; . \tag{147}$$

The quantity $\left(\frac{I}{v}\right)$, which directly relates the bending moment to the maximum stress is called the *section modulus*. It depends only on the geometry of the cross-section and allows the direct design of it from a given bending moment and a given nominal value for strength of the used material.[4] If the maximum allowable stress is the nominal strength of the material, σ_{all}, the section modulus must obey the design condition

$$\sigma_{\max} \le \sigma_{\text{all}} \;\Rightarrow\; \left(\frac{I}{v}\right) \ge \frac{M}{\sigma_{\text{all}}} \; .$$

For two reasons the part of the cross-section which is close to the neutral axis plays a very small role in the resistance to the bending moment:

– the stress at a given point of the cross-section is proportional to the distance to the neutral axis (146); if the highest allowable stress is installed in the farthest fibres, in the points which are close to the neutral axis the stress will be proportionally lower, which means that the material is only used in a small part of its loading capacity;

[4] Usually the design problem is indeterminate, i.e., there is an infinite number of solutions which obey the design condition represented by the maximum acting stress. In the present case (as in the case of the axial force) the geometrical component of the cross-section's strength may be defined by only one parameter, which means that it may be directly computed from this design condition.

– the contribution of a given stress to the resistance to the bending moment depends on the distance to the neutral axis, since the moment of the stress increases with the distance to that axis; in fact, if the distance between the lines of action of the tensile and compressive stress resultants increases, the moment of the couple formed by these two forces also increases.[5]

For these reasons, from the point of view of economizing material, it is better if the cross-section has a shape with as little material as possible placed in the neighbourhood of the neutral axis. This may be achieved by increasing the height of the cross-section (taken as the dimension perpendicular to the neutral axis) or by giving it an appropriate form, such as an I-shape. In the first case the height increase may be limited by a maximum allowable size (for architectural reasons for example, in the case of Civil Engineering constructions), or by the possibility of structural instability caused by the compressive stresses (lateral buckling), if the width/height ratio is too small. In the second case, the minimum thickness of the vertical element of the I-shaped section (the web) may be imposed for stability reasons in the compressed zone as well, or by the shearing stresses caused by a shear force, as it will be seen later (Chap. VIII). Considering, for example, a rectangular cross-section and an I-beam, the section moduli, expressed in terms of the section's height h and the cross-section's area Ω are, respectively, (see example VII.5)

$$\left(\frac{I}{v}\right) = \frac{bh^2}{6} \approx 0.167\Omega h \qquad \text{and} \qquad \left(\frac{I}{v}\right) \approx 0.33\Omega h \; .$$

We conclude that, for the same area and the same height, the bending strength of the beam with the I-shaped cross-section is approximately twice the strength of the beam with rectangular cross-section.

VII.4 Pure Inclined Bending

In the general case of an action axis which is not a symmetry axis of the cross-section the angle between the neutral axis and the action axis is not known a priori. In order to analyse this more general case, let us consider the cross-section represented in Fig. 74, under the action of a bending moment M with a vertical action axis.

The law of conservation of plane sections leads to the conclusion that, in exactly the same way, as in the case of plane bending, the stress is proportional to the distance to the neutral axis, as defined by (143). The condition of equilibrium of the forces acting in the axial direction (z), used in the plane

[5] This conclusion becomes more obvious, if the stress does not vary with the distance to the neutral axis, as happens in the case of materials with elastic perfectly plastic behaviour, when the yielding strain is exceeded. This kind of problems is introduced in Sect. VII.10.c (see example VII.26).

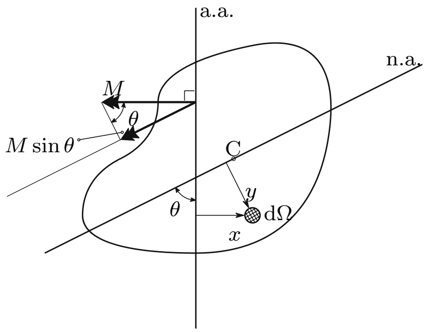

Fig. 74. Inclined bending of a bar without symmetry axes

bending case, is valid in exactly the same way for inclined bending, leading to the conclusion that the neutral axis passes through the centroid C of the cross-section (Fig. 74), as expressed by (144).

As in plane bending, we may establish a relation between the curvature of the deformed bar and the bending moment by means of the equilibrium condition of the moments around the neutral axis. This condition and (143) yield

$$M \sin\theta = \int_\Omega \sigma y \,\mathrm{d}\Omega = \frac{EI_n}{\rho} \Rightarrow \frac{1}{\rho} = \frac{M}{EI_\theta} \quad \text{with} \quad I_\theta = \frac{I_n}{\sin\theta} \,, \tag{148}$$

where I_n represents the moment of inertia of the cross-section in relation to the neutral axis. The curvature may be eliminated from this expression by substituting (148) in (143), yielding

$$\sigma = \frac{My}{I_\theta} \,. \tag{149}$$

The orientation of the neutral axis i.e., the angle θ between the action and neutral axes, is still unknown. In order to define it, the condition of equilibrium of moments around the action axis may be used. Since the moment of the stresses in relation to this axis must vanish, using (143) we get

$$\int_\Omega \sigma x \,\mathrm{d}\Omega = 0 \Rightarrow \frac{E}{\rho}\int_\Omega xy \,\mathrm{d}\Omega = 0 \Rightarrow I_{xy} = 0 \,,$$

where $I_{xy} = \int_\Omega xy \,\mathrm{d}\Omega$ is the product of inertia with respect to the action and neutral axes. The condition $I_{xy} = 0$ means that those axes are conjugate in relation to the ellipse of inertia of the point defined by its intersection.

If the action axis is displaced, remaining parallel to the original position, the product of inertia does not change. In fact, denoting the translation of the axis by a, the product of inertia considering the new position of the action axis is given by

$$\int_{\Omega}(x+a)\,y\,\mathrm{d}\Omega = I_{xy} + aS_n = 0\ ,$$

where $S_n = \int_{\Omega} y\,\mathrm{d}\Omega$. This integral represents the first area moment of the cross-section in relation to the neutral axis. This quantity vanishes, since the neutral axis passes through the section's centroid. As the product of inertia with respect to the neutral axis and any axis parallel to the action axis is zero, we may conclude that the action and neutral axis are conjugate in relation to the centroidal ellipse of inertia.

The conjugate of a principal axis of inertia is the other principal axis.[6] Thus, we may conclude that *bending will be plane, if the action axis is parallel to a principal centroidal axis.* If the two principal moments of inertia have the same value, any axis passing through the centroid of the cross-section is a principal axis, which means that in bars with such cross-sections the bending is always plane. A square cross-section is an example of this kind of section.[7]

The angle θ could be obtained from the equation defined by $I_{xy} = 0$, by expressing I_{xy} as a function of θ. However, in most cases the principal moments of inertia and the corresponding principal directions are easily computed or may be obtained from tables with the geometrical characteristics of current cross-sections. For this reason, the inclined bending is usually analysed by decomposing the vector representing the bending moment (the moment vector) in the centroidal principal directions of inertia, as represented in Fig. 75. In this way, the inclined bending may be analysed as the superposition of two cases of plane bending. In fact, as the moment vector is perpendicular to the action axis, the bending will be plane, if this vector has the direction of a centroidal principal direction of inertia. Another possibility for the computation of the stresses in inclined bending is the use of non-principal reference axes (140).

Considering the separate action of the principal bending moments M_x and M_y, the stress acting in the point defined by the coordinate pair (x, y) is given by the expression

$$\sigma = \frac{M_x}{I_x}y - \frac{M_y}{I_y}x = M\left(\frac{\cos\alpha}{I_x}y - \frac{\sin\alpha}{I_y}x\right)\ . \tag{150}$$

The minus sign appearing in the stress caused by M_y results from the fact, that a bending moment M_y, positive when it has the same direction as

[6]Principal axes are two perpendicular axes, in respect of which the product of inertia is zero. Principal axes and principal directions of the inertial tensor are computed in the same way as the principal stresses and directions of the stress tensor, in the two-dimensional case. If the cross-section has a symmetry axis, it is one of the principal centroidal axes.

[7]These considerations are only valid if the material has a linear elastic behaviour. However, if the cross-section is symmetric in relation to the action axis, the bending will be plane, regardless of the linearity or nonlinearity of the material behaviour.

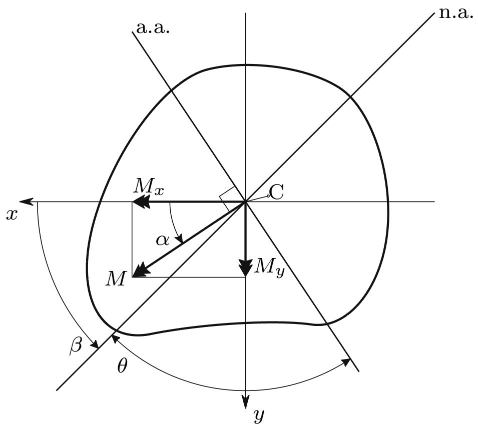

Fig. 75. Decomposition of inclined bending in two cases of plane bending

axis y, causes compression (negative stresses) in the points with a positive x-coordinate.[8] Since in the points belonging to the neutral axis the stress vanishes, the condition $\sigma = 0$ may be used to obtain its position, yielding

$$\sigma = 0 \Rightarrow y = x\frac{I_x}{I_y}\tan\alpha = x\tan\beta \quad \text{with} \quad \tan\beta = \frac{I_x}{I_y}\tan\alpha \, . \tag{151}$$

From this expression we see that, if $I_x > I_y$ then $\beta > \alpha$, i.e., in the inclined bending the neutral axis deviates from the perpendicular direction to the action axis, rotating in the direction of the principal axis with the smallest moment of inertia. In other words, the neutral axis is between the moment vector M and the principal axis with the smallest moment of inertia.

The maximum stress obviously occurs in the farthest point from the neutral axis. In many usual cross-sections, such as I-beams, C-channels, rectangular sections, etc., it is possible to identify those points without having to compute the orientation of the neutral axis. These cross-sections are particular cases of shapes with a rectangular convex contour and a symmetry axis. In this kind of sections, the farthest point from the neutral axis is one of the corners, which is also one of the farthest points in the two possible cases of plane bending ($M = M_x$ and $M = M_y$). In these cases, the maximum absolute value of the normal stress may be obtained directly from the two section moduli in plane bending, i.e., by adding the maximum stresses caused by M_x and M_y

$$|\sigma|_{\max} = \frac{|M_x|}{\left(\frac{I}{v}\right)_x} + \frac{|M_y|}{\left(\frac{I}{v}\right)_y} \, . \tag{152}$$

[8]This expression may also be obtained be particularizing (140) for pure bending ($N = 0$) and the principal axes of inertia ($I_{xy} = 0$). In this case, the minus sign does not appear, since, in the sign convention used in Sect. VII.2, the positive direction of M_y coincides with the negative direction of axis y.

In other types of cross-section, such as a curved contour or polygonal sections with a larger number of sides, the position of the farthest point is often not obvious, which means that the orientation of the neutral axis must be computed, in order to identify the farthest point from the neutral axis (151).

The total deformation may also be computed by superposing the curvatures around the principal axes x and y. Considering two sections located at an infinitesimal distance $\mathrm{d}l$ from each other, their relative rotations around axes x and y are (cf. (141) and (145))

$$\mathrm{d}\varphi_x = \frac{M_x\,\mathrm{d}l}{EI_x} \qquad \text{and} \qquad \mathrm{d}\varphi_y = \frac{M_y\,\mathrm{d}l}{EI_y}\ . \tag{153}$$

These rotations are infinitesimal (even for a large curvature), which mean that they may be treated as vectors with the directions of axes x and y, respectively. The resultant vector takes the direction of the neutral axis, as may be easily confirmed (cf. Fig. 75 and (151))

$$\frac{\mathrm{d}\varphi_y}{\mathrm{d}\varphi_x} = \frac{I_x}{I_y}\frac{M_y}{M_x} = \frac{I_x}{I_y}\tan\alpha = \tan\beta\ .$$

The curvature of the bar is then given by the expression

$$\mathrm{d}\varphi = \sqrt{\mathrm{d}\varphi_x^2 + \mathrm{d}\varphi_y^2} \Rightarrow \frac{1}{\rho} = \frac{\mathrm{d}\varphi}{\mathrm{d}l} = \frac{M}{E}\sqrt{\frac{\cos^2\alpha}{I_x^2} + \frac{\sin^2\alpha}{I_y^2}} = \frac{M}{EI_\theta}\ . \tag{154}$$

The last equality of this expression results from the two obtained expressions for the curvature, (148) and (154). The analytical proof of this equality is relatively lengthy, and so it is not presented here (see example VII.9).

As the quantities M_x, M_y, E, I_x and I_y, contained in (153) do not vary along the axis of the bar, the integration of $\mathrm{d}\varphi$ in a length l is straightforward, yielding the relative rotation of two sections located at a distance l from each other

$$\varphi = \int_l \mathrm{d}\varphi = \frac{Ml}{E}\sqrt{\frac{\cos^2\alpha}{I_x^2} + \frac{\sin^2\alpha}{I_y^2}} = \frac{Ml}{EI_\theta}\ .$$

It should be stressed that the validity of this expression is not restricted to infinitesimal rotations, i.e., it is valid for any value of the rotation φ.

VII.5 Composed Circular Bending

Let us consider a prismatic bar under the action of forces whose resultant is parallel to the axis of the bar. As seen in Sect. VI.1, if the line of action of this resultant passes through the centroid of the cross-section, the axis of the bar remains a straight line, which means that the bar is under pure axial

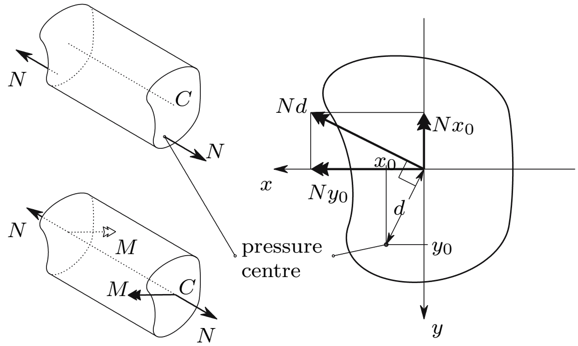

Fig. 76. Bending moments caused by an eccentric tensile axial force

loading. If that does not happen, additional bending takes place. The bending moment is given by the product of the axial force and the distance of its line of action to the centroid of the cross-section. The action axis is in this case the line joining the centroid to the point where the line of action of the axial force intersects the cross-section's plane. In the following description this point is called the *pressure centre*. We therefore have circular composed bending, since $M \neq 0$, $N \neq 0$, $V = 0$ and $T = 0$. This problem is usually analysed by superposition of the effects of the bending moment and the axial force. In the case of inclined bending the moment is decomposed in the principal axes of inertia of the cross-section, as shown in Fig. 76.

The stress acting in a point with coordinates (x, y) may be obtained by adding the stresses caused by the axial force N and by the bending moments $M_x = Ny_0$ and $M_y = -Nx_0$ (Fig. 76). Taking (150) into consideration, we get[9]

$$\sigma = \frac{N}{\Omega} + \frac{Ny_0 y}{I_x} + \frac{Nx_0 x}{I_y} = \frac{N}{\Omega}\left(1 + \frac{y_0 y}{i_x^2} + \frac{x_0 x}{i_y^2}\right) , \tag{155}$$

where i_x and i_y represent the radii of gyration with respect to the principal axis x and y ($I_x = \Omega\, i_x^2$ and $I_y = \Omega\, i_y^2$). The position of the neutral axis may be obtained by the condition of zero stresses, yielding

$$\sigma = 0 \quad \Rightarrow \quad 1 + \frac{y_0 y}{i_x^2} + \frac{x_0 x}{i_y^2} = 0 \quad \Rightarrow \quad \begin{cases} x = 0 \Rightarrow y = y_1 = -\frac{i_x^2}{y_0} \\ y = 0 \Rightarrow x = x_1 = -\frac{i_y^2}{x_0} \, . \end{cases} \tag{156}$$

The position of the neutral axis may be defined by the coordinates of the points where it intersects the principal axes of inertia $(0, y_1)$ and $(x_1, 0)$.

[9]This expression may also be determined by particularizing (140) to principal axes of inertia ($I_{xy} = 0$). The sign convention used for the bending moments does not play any role, since the eccentric axial force F is directly related to the stress.

VII.5.a The Core of a Cross-Section

In composed bending the neutral axis does not pass through the centroid, since there the stress corresponding to the axial force N is installed, as may easily be confirmed by making $x = y = 0$ in (155). From (156) we conclude that the distance of the neutral axis to the centroid increases when the coordinates of the pressure centre, x_0 and y_0 decrease, and vice versa. In the limit case, $x_0 = 0$ and $y_0 = 0$, which corresponds to $M = 0$, that distance is infinite. These considerations are illustrated in the example in Fig. 77.

When the pressure centre is sufficiently close to the centroid, the neutral axis does not intersect the cross-section, which means the the stresses are all tensile, or all compressive. This region around the centroid, where the pressure centre is located, when the neutral axis does not intersect the cross-section, is called the *core* of the cross-section. If the pressure centre is on the core's border, the neutral axis is tangent to the cross-section, as in example 3 of Fig. 77.

The determination of the cross-section core is important in the cases where a single linear elastic constitutive law for tensile and compressive stresses is not acceptable. This happens frequently with brittle materials, as for example, concrete, stone, soil, etc., and also in contact interfaces which are only resistant to compressive stresses. In these cases, (155) is only valid if the pressure centre is in the core of the cross-section, since otherwise tensile and compressive stresses will appear. This also applies to the material, whose constitutive law is shown in Fig. 78.

The core's limits may be directly computed from (156), by determining the position of the pressure centre which corresponds to a neutral axis that is tangent to the cross-section, and repeating this procedure for a sufficiently

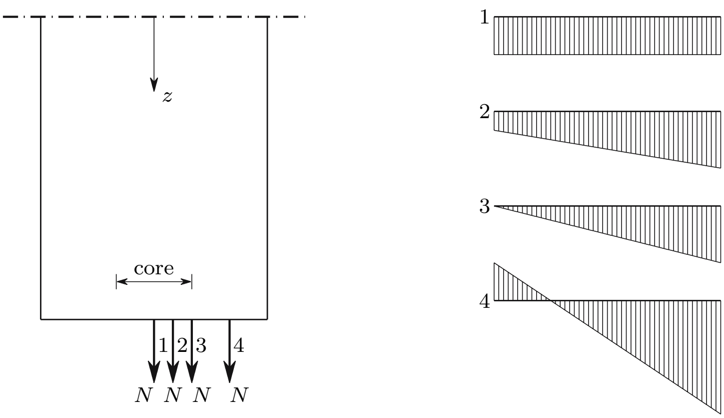

Fig. 77. Stress distribution in the cross-section of a prismatic bar, for different values of the eccentricity of the axial force N

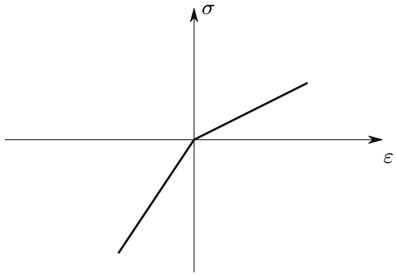

Fig. 78. Material with different linear elastic behaviour for tensile and compressive stresses

high number of pressure centres to define the core with acceptable precision. This technique is useful in the case of a cross-section with a curved boundary.

In the case of a polygonal border, it is more convenient to use another technique, which is based on the reciprocity of (155) in relation to the coordinates of the pressure centre (x_0, y_0) and of a generic point (x, y). In fact, we see immediately from (155) that applying an axial force N on the point with the coordinates (x_0, y_0), we get in the point with coordinates (x, y) the same stress that would be caused on point (x_0, y_0) by the same axial force applied on point (x, y), since the interchange of the roles of x with x_0 and of y with y_0 does not change the result.[10] Therefore, if the axial force is at first applied on (x_0, y_0) and the position of the corresponding neutral axis is computed and, subsequently, the load is applied on different points of this axis, neutral axes will be obtained, which will contain point (x_0, y_0). Thus, we may conclude that the displacement of the pressure centre along a straight line causes a rotation of the corresponding neutral axis around a point. This point corresponds to the pressure centre, whose neutral axis is that straight line.

The core of a polygonal cross-section may be determined by considering a pressure centre on a corner of the cross-section contour and computing the position of the corresponding neutral axis. Pressure centres located on a segment of this axis correspond to neutral axes passing through the corner of the cross-section contour, which do not intersect the cross-section. That line segment is therefore on the limit of the core. By repeating this procedure for all corners the complete core is obtained (Fig. 79-a). Another possibility consists of using the inverse procedure: given a neutral axis joining two corners of the section's contour, the corresponding pressure centre is a corner of the core.

It is obvious that, in the case of a cross-section whose boundary is not completely convex, the tangents to the boundary at the concavities cannot be used to define the core, since they intersect the cross-section. In this kind of

[10]This is a particular case of the Theorem of Maxwell, which is studied in Chap. XII.

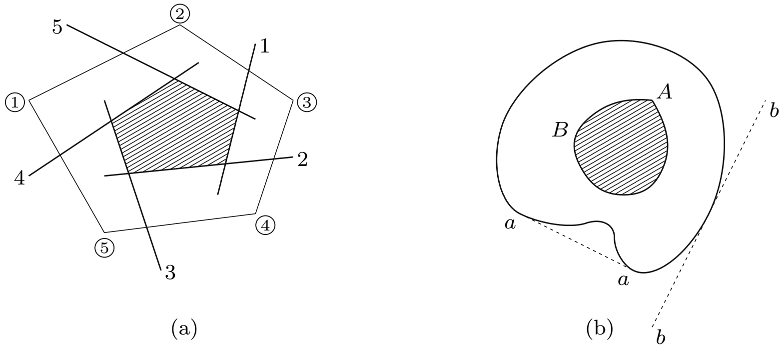

Fig. 79. Determination of the core of a cross-section: (**a**) polygonal cross-section boundary; (**b**) curved cross-section boundary

section the shape of the core is defined by the shape of the convex contour, as represented in Fig. 79-b. The corner A corresponds to the straight part of the convex contour (line $\overline{aa}$). For this reason, the core of an I-beam has the same shape (but not the same dimensions) as a rectangular cross-section (Fig. 80).

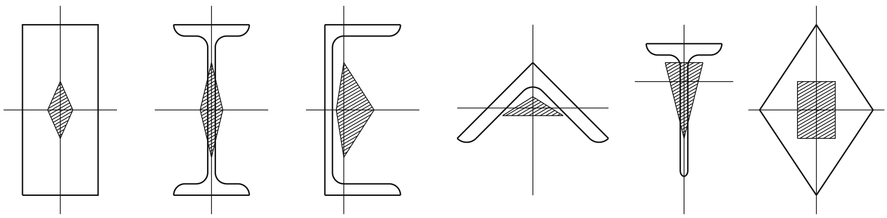

Fig. 80. Core shapes of some usual cross-sections

VII.6 Deformation in the Cross-Section Plane

In order to study the deformation of a cross-section in its plane, let us consider a piece of a prismatic bar whose end sections are sufficiently far from the points of application of the loading to accept the validity of the Saint-Venant principle. If the bar is under circular bending (with or without axial force), the strain distribution in the cross-section is defined by (142), i.e., it is linear, as represented in Fig. 81. Since the validity of what follows is not limited to materials with linear elastic behaviour, we may consider the more general case of a nonlinear stress distribution (Fig. 81).

If we now consider the cross-section divided in narrow strips which are perpendicular to the neutral axis and have an infinitesimal width, these strips may be considered as rectangles under composed circular *plane* bending. We

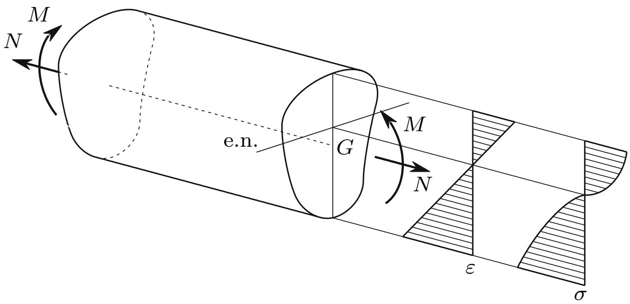

Fig. 81. Piece of a prismatic bar under composed circular bending

should now stress the fact that the division of the cross-section into strips does not change the bending analysis presented above. The only additional consideration is that the stresses σ_x, σ_y and τ_{xy}, which are admitted to vanish, without this being demonstrated (Sect. VII.1), are actually zero in the cross-section divided into strips, since no stresses are applied in its lateral faces, and in the facets parallel to the neutral surface the normal stresses are likewise zero, as there are no lateral forces applied in the piece of the bar under consideration.[11]

The transversal strain of a strip ε_t may be related to the longitudinal strain in the fibre's direction ε by means of the Poisson coefficient ν, yielding

$$\varepsilon_t = -\nu\varepsilon = -\frac{\nu}{\rho} y \; .$$

During the deformation the sides of the strips remain straight, but not parallel, since the transversal strain is proportional to the distance to the neutral axis, *provided that the Poisson's coefficient is constant.* The angle α between the sides of the strip may be related to the curvature of the bar $\frac{1}{\rho}$. The previous expression and the geometrical considerations depicted in Fig. 82 yield the expression

$$\frac{\alpha}{2} = \frac{1}{2}\frac{\mathrm{d}x\,\varepsilon_t}{y} \Rightarrow \alpha = -\frac{\nu}{\rho}\,\mathrm{d}x \; . \tag{157}$$

The displacement δ_y of the points of the strip in the direction perpendicular to the neutral axis may also be obtained from ε_t, and are given by the expression

$$\delta_y = \int_0^y \varepsilon_t \,\mathrm{d}y = -\frac{\nu}{\rho}\frac{y^2}{2} \; .$$

[11]There will be normal stresses in these facets, in the case of a large curvature introduced by bending. These stresses are the radial stresses which equilibrate the longitudinal stresses acting on the fibres when they acquire curvature. These stresses may, however, be neglected for small values of the curvature.

Since the sides of the strip remain straight and since the displacement δ_y only depends on the distance y (this means that δ_y is equal on both sides of the line dividing two contiguous strips), the strips may be assembled after the deformation, reconstructing the bar without any kind of discontinuities, which means that the deformations represented by the previous expressions are *compatible*. We may therefore conclude that the stresses σ_x, σ_y and τ_{xy} also vanish in the undivided bar.

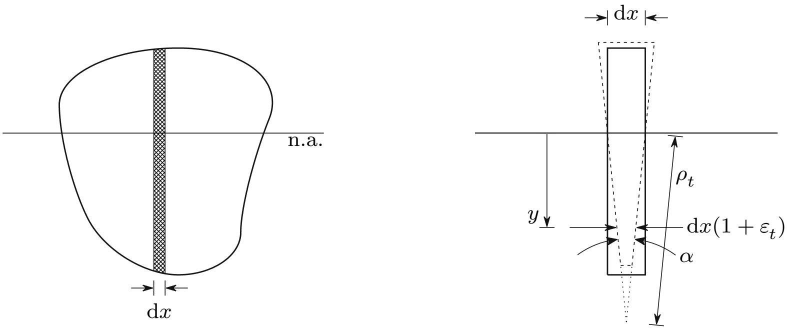

Fig. 82. Transversal deformation of a strip of infinitesimal width and perpendicular to the neutral axis

From Fig. 82 we conclude that, on assembling the deformed strips, the neutral axis is transformed into a circumference arc, whose curvature $\frac{1}{\rho_t}$ may be obtained from (157), yielding

$$\rho_t \alpha = \mathrm{d}x \;\Rightarrow\; \frac{1}{\rho_t} = \frac{\alpha}{\mathrm{d}x} = -\nu \frac{1}{\rho} \, .$$

The neutral surface therefore has an *anticlastic* shape, i.e., it has opposite curvatures in the cross-section plane and in the deflection plane (saddle shape), as represented in Fig. 83 for a bar with rectangular cross-section.

The considerations established in the present section are based only on the law of conservation of plane sections and on a constant value for Poisson's coefficient. Thus, we may consider as demonstrated the statement made in Sect. VII.2 without demonstration that *the normal and shearing stresses acting in facets which are perpendicular to the cross-section plane vanish if the Poisson coefficient is constant.* If this coefficient is not constant, as happens in prismatic bars made of two or more materials, or when plastic deformations take place (cf. Sect. V.3), compatibility conditions for the transversal deformations should be taken into account in the computation of stresses and deformations.

However, these compatibility conditions are usually not considered in the bending analysis of bars with a non-constant Poisson's coefficient. In order to

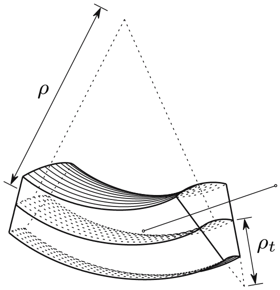

Fig. 83. Deformation caused by a positive bending moment in a bar with rectangular cross-section

get an idea about the importance of the error introduced by this approximation, let us consider the bending of the composite bar represented in Fig. 84. This bar is made of alternate thin layers of two isotropic materials, a and b, which have the same modulus of elasticity E, but different Poisson's coefficients, ν_a and ν_b. The layers all have the same thickness.

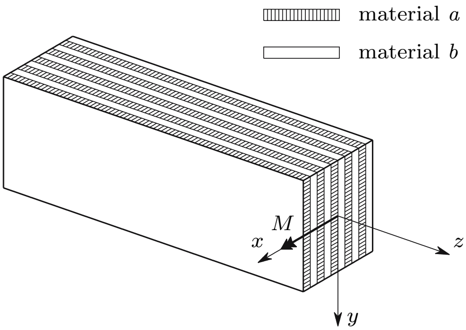

Fig. 84. Composite bar made of two isotropic materials

As the elasticity moduli of the two materials are equal, the solution obtained for the pure plane bending of homogeneous materials is valid, if the transversal compatibility conditions are not considered (143) and (145)

$$\frac{1}{\rho} = \frac{M}{EI} \Rightarrow {}^{a}\sigma_z = {}^{b}\sigma_z = \frac{Ey}{\rho} \, .$$

If the thickness of the layers is very small in relation to the section's dimensions, we may accept that the strain ε_y is the same in both materials. Besides, as the thickness is the same in all layers, the condition of equilibrium in direction y leads to the conclusion that the stresses σ_y are equal and have opposite signs in the two materials (${}^{a}\sigma_y = -\,{}^{b}\sigma_y$). For simplicity, only the extreme case of having the minimum value of the Poisson coefficient in one material ($\nu_a = 0$) and the maximum value in the other ($\nu_b = 0.5$), is analysed here. The stress-strain relations in the two materials are then (cf. (74))

$$\begin{cases} \varepsilon_z = \dfrac{y}{\rho} = \dfrac{{}^a\sigma_z}{E} \\ \varepsilon_y = \dfrac{{}^a\sigma_y}{E} \end{cases} \Rightarrow \begin{cases} {}^a\sigma_z = \dfrac{Ey}{\rho} \\ {}^a\sigma_y = E\varepsilon_y \end{cases} \tag{158}$$

for material a, and

$$\begin{cases} \varepsilon_z = \dfrac{y}{\rho} = \dfrac{1}{E}\left({}^b\sigma_z - \dfrac{1}{2}{}^b\sigma_y\right) \\ \varepsilon_y = \dfrac{1}{E}\left({}^b\sigma_y - \dfrac{1}{2}{}^b\sigma_z\right) \end{cases} \Rightarrow \begin{cases} {}^b\sigma_z = \dfrac{4}{3}\dfrac{Ey}{\rho} + \dfrac{2}{3}E\varepsilon_y \\ {}^b\sigma_y = \dfrac{2}{3}\dfrac{Ey}{\rho} + \dfrac{4}{3}E\varepsilon_y \end{cases}$$

for material b.

The strain ε_y may be obtained from these expressions and the equilibrium condition in direction y, yielding the stress ${}^b\sigma_z$ as function of the curvature

$${}^a\sigma_y = -{}^b\sigma_y \Rightarrow \varepsilon_y = -\frac{2}{7}\frac{y}{\rho} \Rightarrow {}^b\sigma_z = \frac{8}{7}\frac{Ey}{\rho}\,. \tag{159}$$

The moment-curvature relation may be obtained from the condition of equilibrium of moments and from the expressions relating the stresses in the two materials with the curvature (158) and (159), yielding

$$\begin{aligned} M &= \int_{\frac{\Omega}{2}} {}^a\sigma_z y\,\mathrm{d}\Omega + \int_{\frac{\Omega}{2}} {}^b\sigma_z y\,\mathrm{d}\Omega \\ &= \int_{\frac{\Omega}{2}} \frac{Ey^2}{\rho}\,\mathrm{d}\Omega + \int_{\frac{\Omega}{2}} \frac{8}{7}\frac{Ey^2}{\rho}\,\mathrm{d}\Omega \Rightarrow \frac{1}{\rho} = \frac{14}{15}\frac{M}{EI}\,. \end{aligned} \tag{160}$$

Substituting this value in (158) and (159), we get

$${}^a\sigma_z = \frac{14}{15}\frac{My}{I} \approx 0.933\frac{My}{I} \qquad \text{and} \qquad {}^b\sigma_z = \frac{16}{15}\frac{My}{I} \approx 1.067\frac{My}{I}\,. \tag{161}$$

From (160) and (161) we conclude that an error of about 6.7% is introduced into both the computation of the stresses and into the curvature, if the conditions of deformation compatibility in the cross-section plane are not taken into account. However, it should be remembered that this analysis was made for an extreme case ($\nu_a = 0$, $\nu_b = 0.5$). In current materials the error will be smaller. Taking the more usual values $\nu_a = 0.15$ and $\nu_b = 0.30$, for example, a similar analysis shows that the values of the errors decrease to 0.6% in the computation of the curvature and to 1.8% in the computation of the stresses.[12]

[12] In the development of (133) and (134) (stresses induced by the axial force in composite bars) the compatibility conditions of the deformations in the cross-section

VII.7 Influence of a Non-Constant Shear Force

As referred in Sect. VII.1, the stresses computed for the case of pure bending are not changed if a constant shear force is applied. However, in the case of a non-uniform shear force distribution, the stresses computed assuming pure bending are no more an exact description of the actual stress distribution.

In order to get an idea about the importance of the error affecting the computation of the normal stresses, when the expressions developed for circular bending are applied to a bar under a non-constant shear force, we compare the results obtained thereby, with the exact solution of a simple problem, given by the Theory of Elasticity. Let us therefore consider the prismatic bar with a rectangular cross-section, under a uniformly distributed loading p, as represented in Fig. 85. The cross-section has a height h and a width b, which is small compared with h, so that the stress distribution may be assumed as plane. The solution of this problem is given by the functions describing the elements of the stress tensor in plane yz [4]

$$\begin{aligned}
\sigma_z &= \frac{pz\,(l-z)}{2}\frac{y}{I} + \frac{p}{2I}\left(\frac{2}{3}y^3 - \frac{h^2}{10}y\right)\\
\sigma_y &= -\frac{p}{2I}\left(\frac{1}{3}y^3 - \frac{h^2}{4}y + \frac{1}{12}h^3\right)\\
\tau_{yz} &= p\left(\frac{l}{2} - z\right)\frac{1}{2I}\left(\frac{h^2}{4} - y^2\right)\,.
\end{aligned} \tag{162}$$

In the expression of σ_z the first term coincides with the solution given in (146), since $M = \frac{pz(l-z)}{2}$. Therefore, the second term, which depends only on y, represents the error introduced when the stress is computed by means of (146). It may be easily demonstrated, by making $\frac{\partial\sigma_z}{\partial z} = 0$ and $\frac{\partial\sigma_z}{\partial y} = 0$, that the maximum value of σ_z occurs in the extreme fibres of the cross-section ($z = \frac{l}{2}$, $y = \pm\frac{h}{2}$), if $l > \sqrt{0.4}h$. From the first of (162) we get

$$\begin{cases} y = \frac{h}{2} \\ z = \frac{l}{2} \end{cases} \Rightarrow \sigma_z = \sigma_{z-\max} = \frac{ph^3}{I}\left(\frac{\alpha^2}{16} + \frac{1}{60}\right) \quad \text{with} \quad \alpha = \frac{l}{h}\,. \tag{163}$$

The error introduced by (146) ($\sigma_{\max-\text{approx}} = \frac{M_{\max}h}{2I} = \frac{ph^3}{I}\frac{\alpha^2}{16}$), may then be expressed by the relation between $\frac{1}{60}$ and $\frac{\alpha^2}{16}$. This relation depends only on the value of α. In slender members we usually have $l \geq 10h$. For $\alpha = 10$ the second term is only 0.27% of the first one. We conclude therefore that the error introduced by (146) is negligible in this case.

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plane were also not considered, so that an error with the same order of magnitude should be expected (see Footnote 29).

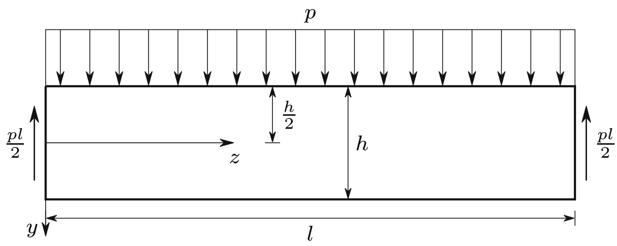

Fig. 85. Beam under a uniformly distributed load

VII.8 Non-Prismatic Members

VII.8.a Introduction

In the same way as in the study of the axial force, the error introduced, when the expressions developed for prismatic members are applied to bars with a non-constant cross-section or with a curved axis, is here investigated by comparing exact solutions given by the Theory of Elasticity with the approximate solutions obtained from the theory of prismatic members in very simple examples.

VII.8.b Slender Members with Variable Cross-Section

As an example of a bar with a non-constant cross-section, the wedge-shaped element with a rectangular cross-section (Fig. 68) is considered again. The force P is now perpendicular to the bar's axis (Fig. 86), so that it causes a bending moment Pr in the cross-section at the distance r of the point of application of the load.

The solution of the Theory of Elasticity is obtained by using polar coordinates (r and θ, Fig. 86) and it shows that in a cylindrical section at the

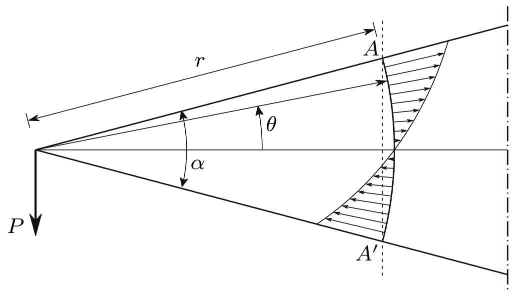

Fig. 86. Stresses caused by bending in a wedge shaped bar

distance r from the vertex the shearing stress vanishes and the radial stress[13] is given by the expression [4]

$$\sigma_r = \frac{2}{\alpha - \sin\alpha}\frac{P\sin\theta}{br} . \tag{164}$$

The maximum stress for a given value of r occurs for the maximum value of θ, $\theta = \frac{\alpha}{2}$. The theory of prismatic members gives, for the same points, the stress $\sigma_{\max-\mathrm{p}}$ (cross-section $AA' \to h = 2r\sin\frac{\alpha}{2}$, $M = Pr\cos\frac{\alpha}{2}$, (147))

$$\frac{I}{v} = \frac{bh^2}{6} = \frac{2}{3}br^2\sin^2\frac{\alpha}{2} \Rightarrow \sigma_{\max-\mathrm{p}} = \frac{M}{\frac{I}{v}} = \frac{P}{br}\frac{3}{2}\frac{\cos\frac{\alpha}{2}}{\sin^2\frac{\alpha}{2}} .$$

The error affecting this approximate expression may be defined by the relation between the maximum stress obtained from (164), $\sigma_{r-\max}$, and $\sigma_{\max-\mathrm{p}}$, which leads to

$$\frac{\sigma_{r-\max}}{\sigma_{\max-\mathrm{p}}} = \frac{4}{3}\frac{\sin^3\frac{\alpha}{2}}{(\alpha - \sin\alpha)\cos\frac{\alpha}{2}} .$$

We verify that the error depends only only α and takes the value

α	$10°$	$20°$	$30°$	$45°$	$60°$
$\sigma_{r-\max}/\sigma_{\max-\mathrm{p}}$	1.0015	1.0062	1.0141	1.0331	1.0622
error	0.15%	0.62%	1.41%	3.31%	6.22%

We conclude that, for low values of angle α, the error is very small.

The Theory of Elasticity yields also a solution, if the force P is substituted by a moment M. In this case, the stress distribution is not purely radial, since the shearing stress $\tau_{r\theta}$ does not vanish. The solution is given by the expressions (M has a counterclockwise direction)

$$\begin{aligned}
\sigma_r &= \frac{2M}{\sin\alpha - \alpha\cos\alpha}\frac{\sin 2\theta}{br^2} \\
\sigma_\theta &= 0 \\
\tau_{r\theta} &= -\frac{M}{\sin\alpha - \alpha\cos\alpha}\frac{\cos 2\theta - \cos\alpha}{br^2} .
\end{aligned}$$

We verify that the shearing stress attains a maximum for $\theta = 0$ and that, for values of $\theta_{\max}$ under $45°$ ($\alpha \leq 90°$), the maximum normal stress occurs at the fibres farthest from the neutral axis and takes the value

$$\theta = \theta_{\max} = \frac{\alpha}{2} \quad \Rightarrow \quad \sigma_r = \sigma_{r-\max} = \frac{1}{br^2}\frac{2M\sin\alpha}{\sin\alpha - \alpha\cos\alpha} = \frac{M}{br^2}\frac{2}{1 - \frac{\alpha}{\tan\alpha}} . \tag{165}$$

[13] It may easily be verified, by evaluating the integral $\int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}} \sigma_r br\sin\theta\,\mathrm{d}\theta$, that the vertical component of the radial stress balances the load P.

This is the maximum principal stress of the stress tensor in point A ($\tau_{r\theta} = 0$ for $\theta = \frac{\alpha}{2}$). The solution of the theory of prismatic members for the same point is

$$\sigma_{\max-\mathrm{p}} = \frac{M}{\left(\frac{I}{v}\right)} = \frac{M}{br^2} \frac{1}{\frac{2}{3}\sin^2\frac{\alpha}{2}} .$$

In the same way as in the previous case, the error introduced by the approximate solution $\sigma_{\max-\mathrm{p}}$ may be expressed by the relation

$$\frac{\sigma_{r-\max}}{\sigma_{\max-\mathrm{p}}} = \frac{4}{3} \frac{\sin^2\frac{\alpha}{2}}{1 - \frac{\alpha}{\tan\alpha}} .$$

As in the previous case, the error depends only on α and takes the values

α	10°	20°	30°	45°	60°
$\sigma_{r-\max}/\sigma_{\max-\mathrm{p}}$	0.9954	0.9818	0.9594	0.9099	0.8430
error	0.46%	1.82%	4.06%	9.01%	15.70%

The error is larger than in the case of the bending moment introduced by the load P but it is also small for low values of α. Besides, as in this case the approximate solution overestimates the actual value of the stress, the error is advantageous for the safety of the structure.

VII.8.c Slender Members with Curved Axis

The errors introduced by the theory of bending for prismatic members, when this is applied to slender members with a curved axis, are studied by comparing the approximate solution given by (147) with the exact solution furnished by the Theory of Elasticity in a curved bar with a circular axis and rectangular cross-section, under a constant bending moment, as represented in Fig. 87.

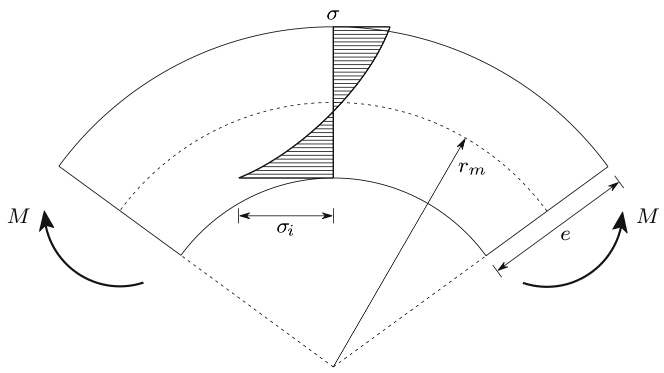

Fig. 87. Stresses introduced by bending in a curved bar

The maximum stress occurs in the extreme fibres of the concave side (σ_i, Fig. 87) and takes the value [4] (b is the cross-section width)

$$\sigma_{\max} = \sigma_i = \frac{6M}{be^2} \frac{4\alpha^3 - 4\alpha^2\beta\left(1+\alpha+\frac{\alpha^2}{4}\right)}{3\beta^2\left(4-2\alpha^2+\frac{\alpha^2}{4}\right)-12\alpha^2} \quad \text{with} \quad \begin{cases} \alpha = \frac{e}{r_m} \\ \beta = \ln\frac{2+\alpha}{2-\alpha}\,. \end{cases}$$

As the approximate solution given by (147) is in this case $\sigma_{i-\text{approx}} = \frac{6M}{be^2}$, the error depends only on the relation α between the dimension e of the bar and the mean radius of curvature r_m and may be expressed by the relation

$$\gamma = \frac{\sigma_i}{\sigma_{i-\text{approx}}} = \frac{4\alpha^3 - 4\alpha^2\beta\left(1+\alpha+\frac{\alpha^2}{4}\right)}{3\beta^2\left(4-2\alpha^2+\frac{\alpha^2}{4}\right)-12\alpha^2}\,.$$

The application of the theory of bending developed for prismatic bars to this curved bar thus leads to the errors

α	0.01	0.05	0.1	0.2	0.3	0.4
γ	1.00335	1.0170	1.0345	1.0717	1.112	1.155
error	0.335%	1.7%	3.45%	7.17%	11.2%	15.5%

Generalizing, we may conclude from these values that, for curved bars where the dimension of the cross-section in the plane containing the axis of the bar exceeds 0.1 times the mean radius of curvature of the bar, the expressions developed for prismatic bars may lead to considerable errors in the computation of bending stresses.

When the parameter α exceeds that value, a bending theory developed by Winkler [8] for curved bars may be used. This theory, although it is based on a simplifying hypothesis – it neglects the effect of the radial stresses (normal stresses acting in facets perpendicular to the curvature radius) – gives results which are very close to the solution furnished by the Theory of Elasticity. The theory of Winkler is also based on the law of conservation of plane sections, but, contrary to the theory of prismatic members, it considers the different initial length of the fibers (the fibres close to the concave face are considerably shorter than the outside fibres). As a consequence of this difference in the initial length of the fibers, the strain and stress distribution is not linear, even though the elongation is proportional to the distance to the neutral axis, but takes a form which is similar to the diagram represented in Fig. 87.

VII.9 Bending of Composite Members

The stresses and deformations induced by a bending moment in a prismatic bar made of more than one material (composite members) are analysed here for the simplest case of a bar made of two materials with linear elastic behaviour. The deformation compatibility conditions in the cross-section plane are not taken into consideration. As a consequence, the theory described here will lead to a small error in the computation of stresses and curvature if the Poisson coefficients of the two materials are different, as seen in Sect. VII.6. This error is not sufficiently high, however, to affect the practical application of the theory.

The law of conservation of plane sections is still valid, since the symmetry conditions used to demonstrate it (cf. Sect. V.10.c) are not affected by the fact that the bar is not homogeneous, provided that the distribution of the two materials in the cross-section is constant. The strain is therefore proportional to the distance to the neutral axis and the strain-curvature relation is still given by (142). Neglecting the stresses acting in facets perpendicular to the cross-section (in accordance with the considerations above), the stresses in the two materials may be related to the curvature of the bar by the expressions

$$\varepsilon = \frac{y}{\rho} \Rightarrow \begin{cases} \sigma_a = \frac{E_a y}{\rho} \\ \\ \sigma_b = \frac{E_b y}{\rho} \ , \end{cases} \tag{166}$$

where E_a and E_b are the elasticity moduli of the two materials and y is the distance of the point under consideration to the neutral axis. We consider here the more general case of inclined bending, as represented in Fig. 88.

As the axial force is zero, the resultant of the normal stresses must vanish. This condition is expressed by the relation

$$\int_\Omega \sigma \, \mathrm{d}\Omega = 0 \Rightarrow E_a \int_{\Omega_a} y \, \mathrm{d}\Omega_a + E_b \int_{\Omega_b} y \, \mathrm{d}\Omega_b = 0 \ . \tag{167}$$

This expression represents the first moment of the areas occupied by each material in the cross-section, with the moment of each area weighted with the modulus of elasticity of the corresponding material, in relation to the neutral axis. Since this moment must vanish, we conclude that the neutral axis passes through the centroid of the cross-section, *computed by weighting the first area moment of each material with the corresponding modulus of elasticity.*

The moment of the stresses in relation to the neutral axis must be equal to the applied bending moment. This condition leads to the expression

$$M \sin\theta = \int_\Omega \sigma y \, \mathrm{d}\Omega = \frac{1}{\rho} J_n \quad \text{with} \quad J_n = E_a \int_{\Omega_a} y^2 \, \mathrm{d}\Omega_a + E_b \int_{\Omega_b} y^2 \, \mathrm{d}\Omega_b \ . \tag{168}$$

J_n is the moment of inertia of the cross-section in relation to the neutral axis, computed by weighting the moment of inertia of the area occupied by each material with the corresponding elasticity modulus. The curvature and the stresses introduced into the bar by the bending moment may be obtained from (168) and (166), yielding

$$\frac{1}{\rho} = \frac{M}{J_\theta} \Rightarrow \begin{cases} \sigma_a = \frac{M E_a}{J_\theta} y \\ \sigma_b = \frac{M E_b}{J_\theta} y \end{cases} \quad \text{with} \quad J_\theta = \frac{J_n}{\sin\theta} \ . \tag{169}$$

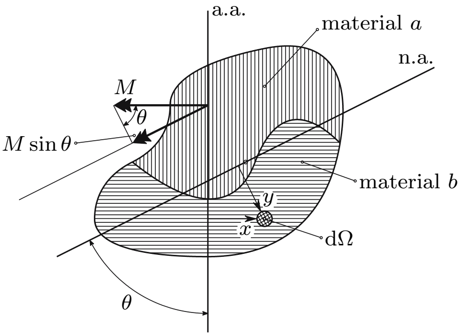

Fig. 88. Circular inclined bending of a prismatic bar made of two materials

The bending stiffness is obviously defined by J_θ. The moment of the stresses in relation to the action axis must vanish. This condition leads to the relation

$$\int_\Omega \sigma x \,\mathrm{d}\Omega = 0 \;\Rightarrow\; E_a \int_{\Omega_a} xy \,\mathrm{d}\Omega_a + E_b \int_{\Omega_b} xy \,\mathrm{d}\Omega_b = 0\,. \tag{170}$$

This expression states that the weighted product of inertia with respect to the action and neutral axes vanishes. From this, we conclude, by establishing the same considerations as in the case of the inclined bending of homogeneous bars (Sect. VII.4), that the action and neutral axes are conjugate in relation to the central ellipse of inertia, if the centroid's position and the moments and product of inertia are computed weighting the areas of each material with the corresponding modulus of elasticity. Thus, plane bending will take place if the action axis is parallel to one of principal directions of inertia, computed with weighting. In the same way as in the homogeneous bars, the moment vector may be decomposed in the principal directions of inertia, allowing the inclined bending to be treated as the superposition of two cases of plane bending (Fig. 75).

In the practical applications we often deal with cross-sections with a symmetry axis and with a plane contact surface between the two materials that is either parallel or perpendicular to the symmetry axis. In these cases the concept of *homogenization* may be useful. This concept allows the composite cross-section to be treated as homogeneous. In order to introduce it, let us divide (167) by E_a and put this quantity in evidence in the expression of J_n (168). For simplicity, let us assume plane bending ($\theta = 90°$). We get

$$\left.\begin{array}{l} \displaystyle\int_{\Omega_a} y \,\mathrm{d}\Omega_a + m_a \int_{\Omega_b} y \,\mathrm{d}\Omega_b = 0 \\[3ex] \displaystyle M = \frac{1}{\rho} E_a I_{ha} \;\Rightarrow\; \frac{1}{\rho} = \frac{M}{E_a I_{ha}} \end{array}\right| \begin{array}{l} \text{with:} \\ \displaystyle m_a = \frac{E_b}{E_a} \qquad \text{and} \\[3ex] \displaystyle I_{ha} = \int_{\Omega_a} y^2 \,\mathrm{d}\Omega_a + m_a \int_{\Omega_b} y^2 \,\mathrm{d}\Omega_b\,, \end{array} \tag{171}$$

where m_a is the homogenizing coefficient of material b in material a. Equations 171 show that, changing the cross-section's shape and dimensions, so that the area of material b is multiplied by a factor m_a, without altering the distances to the neutral axis, i.e., by multiplying the dimensions which are parallel to the neutral axis by m_a, the resulting cross-section may be analysed as homogeneous and made of material a when computing the centroid's position and the curvature caused by the bending moment. The stress in material a may also be computed using the same expression as in the case of a homogeneous bar. Only for the computation of the stress in material b must the homogenizing coefficient be used, as we conclude from (169) ($\theta = 90° \Rightarrow J_\theta = J_n$)

$$\sigma_a = \frac{ME_a}{J_n}y = \frac{ME_a}{E_a I_{ha}}y = \frac{My}{I_{ha}} \qquad \text{and} \quad \sigma_b = \frac{ME_b}{J_n}y = \frac{ME_b}{E_a I_{ha}}y = m_a\frac{My}{I_{ha}} \ .$$

Obviously, the cross-section could also be homogenized in material b, as depicted in Fig. 89. In the case of inclined bending, the geometry of the homogenized section is different for the two principal directions, as exemplified in Fig. 90.

As in the case of homogeneous cross-sections, principal directions of inertia x and y may be defined whose orientation is computed from the weighted moments of inertia and the weighted product of inertia with respect to two orthogonal axes originating in the centroid of the weighted cross-section (see example VII.12). From (169) we easily conclude that the stresses in inclined bending may be computed from the components of the bending moment in the principal directions of inertia, M_x and M_y, by the expressions

$$\begin{cases} \sigma_a = \frac{M_x E_a}{J_x}y - \frac{M_y E_a}{J_y}x \\ \\ \sigma_b = \frac{M_x E_b}{J_x}y - \frac{M_y E_b}{J_y}x \end{cases} \quad \text{with} \quad \begin{cases} J_x = E_a \int_{\Omega_a} y^2 \, \mathrm{d}\Omega_a + E_b \int_{\Omega_b} y^2 \, \mathrm{d}\Omega_b \\ \\ J_y = E_a \int_{\Omega_a} x^2 \, \mathrm{d}\Omega_a + E_b \int_{\Omega_b} x^2 \, \mathrm{d}\Omega_b \ . \end{cases} \tag{172}$$

The curvature of the bar may be computed by superposition of the curvatures around the principal axes, as in the case of the homogeneous members (154)

$$\frac{1}{\rho} = \sqrt{\left(\frac{1}{\rho_x}\right)^2 + \left(\frac{1}{\rho_y}\right)^2} = \sqrt{\frac{M_x^2}{J_x^2} + \frac{M_y^2}{J_y^2}} \ . \tag{173}$$

The equation of the neutral axis may be obtained from any of the conditions $\sigma_a = 0$ or $\sigma_b = 0$, yielding $y = \frac{J_x}{J_y}\frac{M_y}{M_x}x$.

VII.9.a Linear Analysis of Symmetrical Reinforced Concrete Cross-Sections

When analysing reinforced concrete structural elements we generally neglect the tensile strength, since concrete has a much higher resistance to compressive

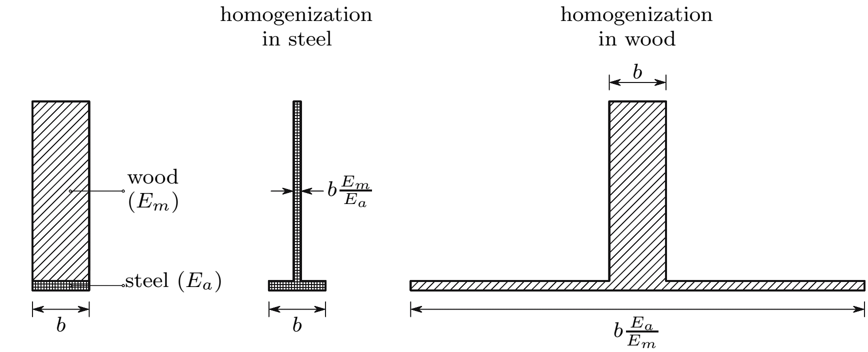

Fig. 89. Homogenization of a composite cross-section

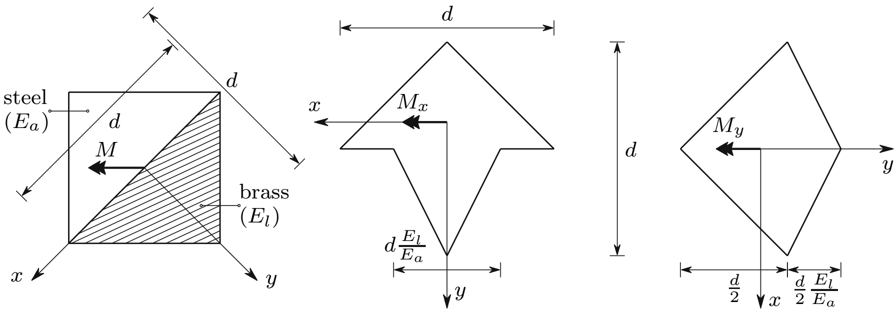

Fig. 90. Homogenization in inclined bending

than to tensile stresses, so that it is usually considered that these stresses are resisted only by the steel bars. In the linear analysis of concrete structures, we admit that the stress is proportional to the strain, as represented in Fig. 91.[14] Looking at this diagram we can immediately see that the above theory for composite bars is not directly applicable, since the stress-strain relation is not defined by the same linear law in tension and compression. However, as the internal tensile forces are supported by the steel bars, this difficulty may be circumvented by considering only the compressed concrete as active. This procedure raises the order of the equation to be solved to compute the position of the section's centroid. In the case of a cross-section that is symmetrical

[14]Practical design and safety evaluation of reinforced concrete slender members are currently done by computing the failure bending moment. Under these conditions, a linear stress-strain relation for the concrete is not admissible (see Subsect. VII.10.d). Therefore, the present analysis is only intended to be an example of the application of the theory of composite prismatic bars. However, in high strength concrete, the stress-strain relation remains very close to a straight line almost until rupture (Fig. 91), and so the better the concrete quality, the better the approximation of this analysis. The curvature of a bar under service loads may also be computed, by considering a linear law.

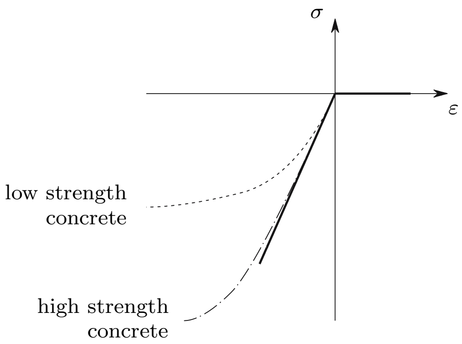

Fig. 91. Admitted stress-strain relation for concrete (*solid line*)

with respect to the action axis, the bending is always plane, regardless of the material behaviour (see footnote 40). This means that the neutral axis is perpendicular to the action axis and that the condition $N = 0$ suffices to compute the position of the neutral axis.

In order to illustrate these considerations, the expressions needed to compute the stresses and curvature caused by plane bending in a rectangular cross-section are developed. The simplest case of having only one layer of steel bars is considered (Fig. 92).

Denoting the areas of steel and concrete by Ω_s and Ω_c respectively, the position of the neutral axis may be obtained from 167, yielding

$$\begin{aligned} E_s\Omega_s(1-k)\,h &= E_c\underbrace{bkh}_{\Omega_c}\frac{kh}{2} \Rightarrow k^2 = 2(1-k)\,\lambda \\ \Rightarrow \quad k &= -\lambda + \sqrt{\lambda^2 + 2\lambda} \quad \text{with} \quad \lambda = \frac{E_s}{E_c}\frac{\Omega_s}{bh}\,. \end{aligned} \tag{174}$$

The weighted moment of inertia of the cross-section in relation to the neutral axis is given by the expression (cf. (168))

$$J_n = E_s\Omega_s(1-k)^2 h^2 + E_c\frac{b\,(kh)^3}{3}\,.$$

Using the first of (174), two other forms may be given to this expression

$$\begin{aligned} J_n &= E_s\Omega_s h^2(1-k)\left(1-\frac{1}{3}k\right) \\ J_n &= \frac{1}{2}E_c b h^3 k^2\left(1-\frac{1}{3}k\right)\,. \end{aligned}$$

The stress in the steel and the maximum stress in the concrete may then be obtained by substituting these values of J_n in (169)

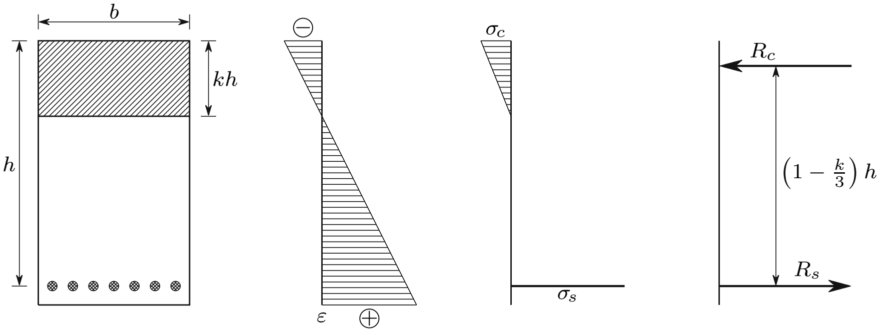

Fig. 92. Reinforced concrete cross-section under a positive bending moment

$$\begin{cases} \sigma_s = \frac{ME_s(1-k)h}{J_n} = \frac{M}{Z\Omega_s} \\ \sigma_{c-\max} = \frac{ME_c kh}{J_n} = \frac{2M}{Zkbh} \end{cases} \quad \text{with} \quad Z = \left(1 - \frac{1}{3}k\right) h \,.$$

Z is the arm of the couple of forces defined by the resultant of the compressive stresses in the concrete, R_c, and by the resultant of the tensile forces in the steel, R_s, (Fig. 92). The curvature of the bar may be computed from any of the above expressions given for J_n, since $\frac{1}{\rho} = \frac{M}{J_n}$.

The superposition principle may be applied to this problem, only if the active cross-section remains the same. Thus, the effects of two positive bending moments with parallel action axes may be superposed, but not the effects of a horizontal and a vertical bending moment. This is because of the lack of complete linearity of the constitutive law considered for the concrete. As a consequence, the analysis of inclined bending is more complicated and cannot be performed on the basis of the bending theory for composite beams described above.

VII.10 Nonlinear bending

VII.10.a Introduction

When the rheological behaviour of the material is not linear, there is no proportionality of stresses and strains and the system of equations obtained from the equilibrium conditions (139) is no longer linear, although the strain distribution remains linear owing to the law of conservation of plane sections (138). In the most general case, the computation of the stresses caused by a bending moment and an axial force in a prismatic bar made of a material, whose one-dimensional constitutive law is described by the function $\sigma = f(\varepsilon) = f\,(ax + by + c)$[15] requires the solution of the system of equations

[15]We still assume that the stresses in facets perpendicular to the cross-section are zero (see SectionVII.6).

$$\begin{cases} N = \displaystyle\int_{\Omega} \sigma \, \mathrm{d}\Omega = \int_{\Omega} f\left(ax + by + c\right) \mathrm{d}\Omega \\ M_x = \displaystyle\int_{\Omega} \sigma y \, \mathrm{d}\Omega = \int_{\Omega} f\left(ax + by + c\right) y \, \mathrm{d}\Omega \\ M_y = \displaystyle\int_{\Omega} \sigma x \, \mathrm{d}\Omega = \int_{\Omega} f\left(ax + by + c\right) x \, \mathrm{d}\Omega \; . \end{cases} \tag{175}$$

The degree of complexity of these equations depends on the shape of the cross-section and on the function $f(\varepsilon)$. The system of equations may admit more than one set of solutions (a, b, c) or be impossible. On the other hand, the computation of the set of internal forces, N, M_x and M_y which corresponds to a given deformation, defined by a set of parameters a, b, and c, is always a determinate problem, as (175) shows.[16]

From these considerations, we can see at once that nonlinear bending is a very wide field. The detailed general analysis of this kind of problems goes beyond the scope of this book, and so only some particularly simple cases are explained.

VII.10.b Nonlinear Elastic Bending

As an example of bending in a nonlinear elastic regime, let us consider a rectangular cross-section under pure plane bending (action axis coinciding with a symmetry axis of the cross-section) made of a material with the constitutive law which is schematically represented in Fig. 93.

Since the compressive and tensile stress-strain relations may be different, the neutral axis generally does not pass through the centroid of the cross-section. Its position may be determined from the condition $N = 0$, as in the linear case, yielding (y_1 and y_2 are defined in absolute value, Fig. 94)

$$\int_{\Omega} \sigma \, \mathrm{d}\Omega = 0 \Rightarrow \begin{cases} \displaystyle\int_{-y_1}^{y_2} \sigma b \, \mathrm{d}y = 0 \\ y = \rho\varepsilon \end{cases} \Rightarrow \int_{\varepsilon_1}^{0} \sigma\left(\varepsilon\right) \mathrm{d}\varepsilon = -\int_{0}^{\varepsilon_2} \sigma\left(\varepsilon\right) \mathrm{d}\varepsilon \; . \tag{176}$$

From (176) we conclude that the neutral axis must take a position which leads to equal compressive and tensile areas defined by the used region of the stress-strain diagram ($A_1 = A_2$, Fig. 94). As the shape of this diagram may not be the same in tension and compression, the position of the neutral axis is not independent of the magnitude of the bending moment, i.e., it may change as the moment increases.

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[16] This conclusion is valid for most of the problems of Solid Mechanics. The computation of the internal forces corresponding to a given set of displacements is always a determinate problem, while the inverse problem is often not determinate. For this reason, the generalization of the displacement method to nonlinear problems is much easier than in the case of the force method.

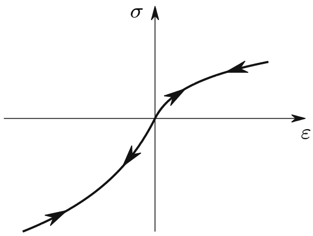

Fig. 93. Nonlinear elastic behaviour

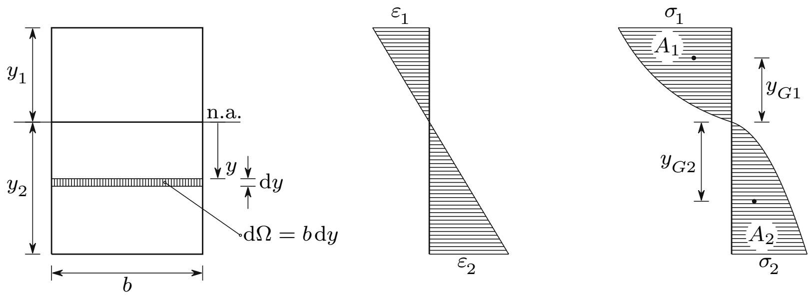

Fig. 94. Bending in nonlinear elastic regime

The bending moment corresponding to the curvature $\frac{1}{\rho} = \frac{\varepsilon}{y}$ (142) may be computed in two steps: first y_1 and y_2 are obtained from the curvature, which can be done by computing the difference between the maximum (tensile) and minimum (compressive) strains

$$\varepsilon_2 - \varepsilon_1 = \frac{y_2}{\rho} + \frac{y_1}{\rho} = \frac{h}{\rho}$$

and computing the values of ε_1 and ε_2 which lead to $A_1 = A_2$; the corresponding bending moment may then by computed by the expression

$$\begin{cases} y_1 = -\rho\varepsilon_1 \\ y_2 = \rho\varepsilon_2 \end{cases}$$

$$\Rightarrow \; M = \int_\Omega \sigma y \, \mathrm{d}\Omega = b \int_{-y_1}^{y_2} \sigma y \, \mathrm{d}y = A_1 b \; y_{G1} + A_2 b \; y_{G2} = A_1 b \left(y_{G1} + y_{G2} \right) \; .$$

In this expression y_{G1} and y_{G2} represent the distances from the neutral axis of the centroids of the areas defined by the stress-strain diagram in compression and in tension, respectively, as represented in Fig. 94.

VII.10.c Bending in Elasto-Plastic Regime

If the yielding strain of a ductile material is exceeded, the stress-strain relation becomes non-linear, which means that the linear bending theory is not valid anymore. Furthermore the material behaviour becomes different for loading and unloading. The analysis of this kind of problem is described in detail here for the example of the elasto-plastic plane bending of a bar with a rectangular cross-section, made of a material with elastic perfectly plastic behaviour (Fig. 61). In the last part of this Sub-section, the plastic analysis of cross-sections with one or two symmetry axes under plane bending is described.

Considering the rectangular cross-section represented in Fig. 95, we conclude that the neutral axis divides the cross-section in two equal parts, since the material behaviour is the same for compressive and tensile stresses.

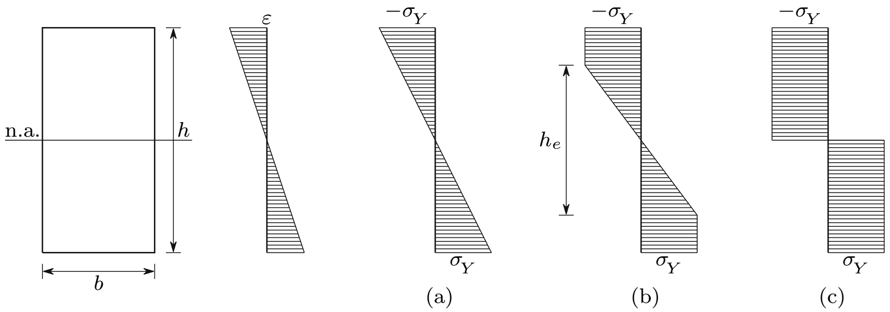

Fig. 95. Plane bending of a rectangular cross-section in the elasto-plastic regime

If the bending moment exceeds the value corresponding to the yielding strain in the fibres farthest from the neutral axis (Fig. 95-a), which is the highest possible bending moment in the elastic phase, the fibres undergoing more strain yield and the bar enters in the elasto-plastic regime (Fig. 95-b). In this phase, the relation between the bending moment and the curvature may be obtained from the strain in the fibres which are still under elastic deformation. With h_e being the height of the part of the section still under elastic deformations (Fig. 95-b), this quantity may be related to the curvature by the expression

$$y = \frac{h_e}{2} \Rightarrow \varepsilon = \varepsilon_Y = \frac{1}{\rho}\frac{h_e}{2} \Rightarrow h_e = 2\rho\varepsilon_Y = \frac{2\rho\sigma_Y}{E} \ . \tag{177}$$

The moment of the stresses with respect to the neutral axis must be equal to the bending moment. From this condition, a relation between the bending moment M and h_e may be obtained

$$M = \frac{b}{4}\sigma_Y \left(h^2 - \frac{h_e^2}{3}\right) \quad \Rightarrow \quad M = M_e \left[\frac{3}{2} - \frac{1}{2}\left(\frac{h_e}{h}\right)^2\right] \quad \text{with} \quad M_e = \frac{bh^2}{6}\sigma_Y \ . \tag{178}$$

Substituting, in (178), h_e with the value given by (177), the relation between the curvature and the bending moment is obtained

$$\frac{1}{\rho} = \frac{2\sigma_Y}{hE} \frac{1}{\sqrt{3 - 2\frac{M}{M_e}}} \qquad \text{with} \quad M \geq M_e \, . \tag{179}$$

From this expression we conclude that the curvature $\frac{1}{\rho}$ goes to infinity, as the bending moment M goes to $\frac{3}{2}M_e$. The maximum bending moment supported by the bar is therefore 1.5 times the maximum bending moment in the elastic regime M_e, i.e., $M_p = 1.5M_e$. M_p represents the yielding bending moment. It corresponds to the limit case represented in Fig. 95-c, as may be easily verified by making $h_e = 0$ in (178). The relation between the yielding bending moment M_p and the maximum bending moment in the elastic regime M_e is called the *shape factor* of the cross-section $\varphi = \frac{M_p}{M_e}$. Thus, in a rectangular cross-section, we have $\varphi = 1.5$.

Let us now consider non-rectangular cross-sections with a symmetry axis. If the action axis is parallel to this axis we will have plane bending. The same happens if the action axis is perpendicular to the symmetry axis. In fact, as the material has the same behaviour in tension and compression, the neutral axis will be the symmetry axis, since, under these conditions, the moment of the stresses in relation to the action axis will vanish, as required by the equilibrium conditions (see Sect. VII.4). A general elasto-plastic analysis of any of these cases is, however, substantially more complex than the rectangular case, since the width of the cross-section is not constant.

Anyway, the most important issue in elasto-plastic analysis is the computation of the yielding bending moment M_p, also called the *plastic moment.* This problem is substantially simpler than the computation of the moment-curvature relation in the elasto-plastic phase, since only the limit case of having a constant tensile or compressive stress σ_Y (yielding stress) on each side of the neutral axis needs to be analysed. Considering the symmetrical cross-section represented in Fig. 96, the condition of equilibrium of the forces acting in the direction of the bar's axis leads to the relation

$$\sigma_Y \Omega_1 - \sigma_Y \Omega_2 = 0 \Rightarrow \Omega_1 = \Omega_2 = \frac{\Omega}{2} \, .$$

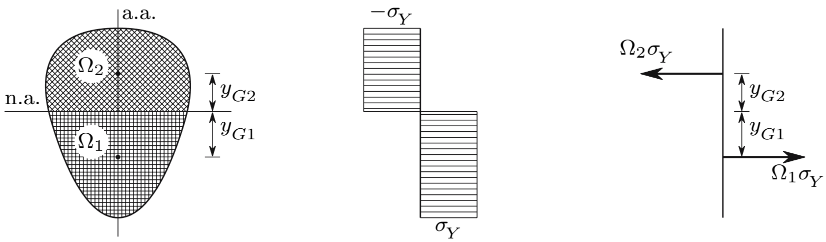

Fig. 96. Fully plastified symmetrical cross-section

Thus, the neutral axis divides the cross-section into two equal areas, which means that it will generally not pass through the centroid, unless it is a symmetry axis. The plastic moment M_p is equal to the moment of the stresses with respect to the neutral axis, taking the value below (y_{G1} and y_{G2} are the distances from the centroids of the tensioned and compressed zones of the cross-section to the neutral axis)

$$M_p = \int_\Omega \sigma y \, \mathrm{d}\Omega = \sigma_Y \int_\Omega |y| \, \mathrm{d}\Omega = \Omega_1 \sigma_Y \; y_{G1} + \Omega_2 \sigma_Y \; y_{G2} = \frac{\Omega}{2} \left(y_{G1} + y_{G2} \right) \sigma_Y \; . \tag{180}$$

The quantity $Z = \frac{M_p}{\sigma_Y} = \frac{\Omega}{2} \left(y_{G1} + y_{G2} \right)$ depends only on the geometry of the cross-section and is called the *plastic section modulus* or simply *plastic modulus*. The shape factor may be obtained from Z and the elastic section modulus $\left(\frac{I}{v}\right)$, $\varphi = \frac{M_p}{M_e} = \frac{Z}{\left(\frac{I}{v}\right)}$. In the following Table, the shape factors of some current symmetrical cross-sections are indicated.

Cross-section	shape factor – φ
rectangle	1.5
isosceles triangle	2.343
rhombus	2
circle	1.7
I-beam	≈1.15
channel	≈1.2

From these examples we conclude that, the less specialized the section for the resistance to the bending moment, the higher its shape factor. This is due to the fact that such sections have more material in the region around the centroid, whose contribution to the bending strength is not exhausted until the cross-section is totally plastified.

Plastification is a gradual process which, from a theoretical point of view, is only finished for an infinite curvature of the bar, as seen above. However, when the height of the elastic zone is small, we may consider the cross-section as totally plastified, since in that case the contribution of the elastic zone to the resistance to the bending moment is very small. The curvature may then be increased practically without any increase in the bending moment, i.e., *yielding of the entire cross-section* takes place. In this case, we say that a *plastic hinge* has been formed. In Fig. 97 the formation process of such a hinge is schematically represented.

If the bar is unloaded after the maximum bending moment in the elastic phase is exceeded, the internal stresses do not disappear totally, since the material behaves elastically in the unloading process (Fig. 61) and some residual deformation is left in the fibres where the yielding strain was exceeded.

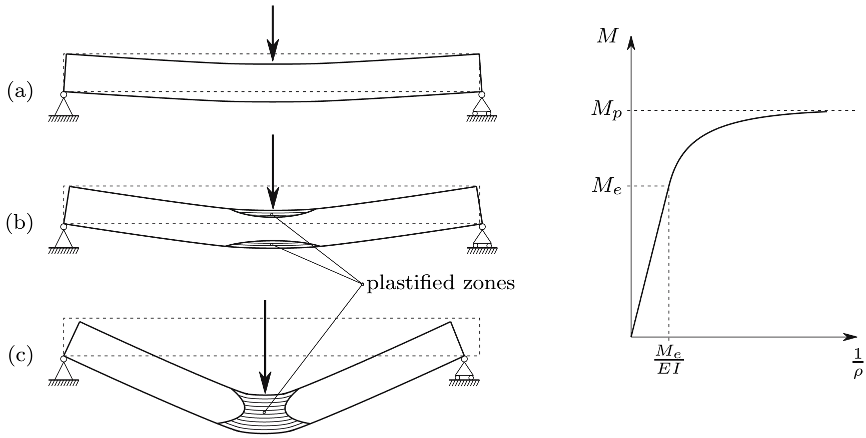

Fig. 97. Formation of a plastic hinge: (**a**) elastic phase; (**b**) elasto-plastic phase; (**c**) plastic phase

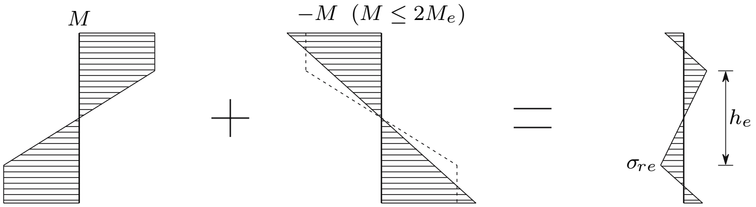

Fig. 98. Residual stresses after unloading in the elasto-plastic regime

The unloading may be understood as the application of a bending moment with the same magnitude and opposite direction. In terms of stresses, the unloading corresponds to the superposition of a linear elastic diagram on the elasto-plastic diagram resulting from the loading, as represented in Fig. 98.

The unloading bending moment will cause yielding of the fibres farthest from the neutral axis, only if the bending moment in the elastic phase exceeds twice the maximum bending moment in the elastic phase. In fact, after yielding under a tensile stress, a stress decrease of $2\sigma_Y$ is needed to cause yielding under compressive stress and vice versa in compression (Fig. 61). Obviously, this is only possible if the cross-section has a shape factor greater than 2, which does not happen in cross-sections used currently to absorb bending moments, as seen above.[17]

[17] This line of reasoning is only accurate if there is no displacement of the neutral axis in the elasto-plastic deformation, i.e., if the neutral axis is a symmetry axis. If it is not, the stress may not decrease in the unloading in part of the fibres located close to the neutral axis. However, in usual cross-sections the displacement of the neutral axis is small and the stresses in the region where it occurs are low, so that the error introduced by this procedure will be small, if any. Anyway, this error would be

The residual curvature of the bar may be computed from the residual stress in the fibres where yielding did not take place in the unloading. Thus, we have

$$\varepsilon = \frac{1}{\rho} y \Rightarrow \frac{1}{\rho} = \frac{\varepsilon}{y} = \frac{2\sigma_{re}}{Eh_e} . \tag{181}$$

As an alternative, if the curvature in the loading phase has been computed, the residual curvature may be computed by superposing the curvature recovered in the unloading on it. As the latter is elastic, we have

$$\left(\frac{1}{\rho}\right)_{\text{residual}} = \left(\frac{1}{\rho}\right)_{\text{loading}} - \frac{M}{EI} . \tag{182}$$

It should be noted that this operation is valid regardless of the size of the displacements and rotations caused by the deformation, since it is the addition of two angles (the curvature is the relative rotation, measured in radians, of two cross-sections at a unit distance from each other). In fact, the results given by (181) and (182) are exactly the same (see example VII.13).

VII.10.d Ultimate Bending Strength of Reinforced Concrete Members

As a final example of the application of the nonlinear bending theory, the ultimate bending strength of a reinforced concrete member, with a rectangular cross-section is computed. The cross-section of the bar is the same as considered in Sect. VII.9. In this example, the rheological behaviour of steel and concrete recommended in the Portuguese concrete norms is used: the steel is considered as elastic perfectly plastic and to have a limit strain of 0.01; the one-dimensional constitutive law for the concrete is described by a parabola for smaller strains ($0 < \varepsilon < 0.002$), followed by a yielding zone ($0.002 < \varepsilon < 0.0035$), as represented in Fig. 99.[18]

The limit bending moment is reached when the steel reaches a strain of 0.01, or when the maximum strain in the concrete reaches 0.0035. In accordance with the above defined constitutive laws, there are four analytical possibilities of ultimate bending strength, which are:[19]

immediately detected, since it would lead to larger stresses than the yielding stress σ_Y.

[18] In the recent European standards (Eurocode 2) different constitutive laws are recommended. Among other differences, the limit strains both for the steel and for the concrete vary with the type of material: for example, high strength concrete has a lower limit strain. For the steel, either an elastic perfectly plastic or a hardening elasto-plastic behaviour may be considered, with no limiting value for the strain in the first case. The constitutive law for concrete is described by a unique curve, which includes a softening zone.

[19] In these expressions and the ones that follow, we consider the compressive stresses and strains in the concrete as positive.

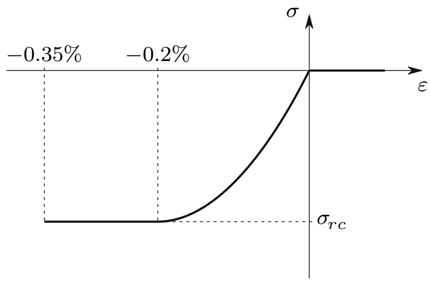

Fig. 99. Stress-strain diagram used in the computation of the ultimate strength of concrete elements

$$\overbrace{\left\{\begin{array}{c}\varepsilon_s = 0.01\\ \varepsilon_c \le 0.002\end{array}\right. \left\{\begin{array}{c}\varepsilon_s = 0.01\\ 0.002 \le \varepsilon_c \le 0.0035\end{array}\right.}^{\text{steel failure}} \overbrace{\left\{\begin{array}{c}\varepsilon_s \le \frac{\sigma_Y}{E_s}\\ \varepsilon_c = 0.0035\end{array}\right. \left\{\begin{array}{c}\frac{\sigma_Y}{E_s} \le \varepsilon_s \le 0.01\\ \varepsilon_c = 0.0035\ .\end{array}\right.}^{\text{concrete failure}}$$

As an example, only the fourth possibility is analysed, i.e., the case of having the ultimate strain in the concrete and the steel in the yielding zone.[20] Under these conditions, the tensile force borne by the steel is $N_s = \Omega_s \sigma_Y$, where σ_Y is the nominal value of the yielding stress and Ω_s is the area of steel in the cross-section. Representing the position of the neutral axis by its distance from the upper side of the cross-section kh (with $0 \le k \le 1$), from the equilibrium condition of the stress resultants we get

$$\underbrace{\frac{2}{3}\frac{2}{3.5}bkh\sigma_{rc}}_{N_{c1}3(\text{parabola})} + \overbrace{\frac{1.5}{3.5}bkh\sigma_{rc}}^{N_{c2}3(\text{rectangle})} = \underbrace{\Omega_s\sigma_Y}_{N_s} \Rightarrow k = \frac{17}{21}\frac{\Omega_s}{bh}\frac{\sigma_Y}{\sigma_{rc}}\ .$$

The strain in the steel may then be computed from this value. The analysis will be valid if

$$\frac{\sigma_Y}{E_s} \le \varepsilon_s = 0.0035\frac{1-k}{k} \le 0.01\ . \qquad (183)$$

The ultimate bending moment may then be obtained by computing the moment of the stresses with respect to the neutral axis

$$M = N_s\,(1 - kh) + N_{c1}\frac{5}{14}kh + N_{c2}\frac{11}{14}kh \quad \text{with} \quad \left\{\begin{array}{ll} N_s &= \Omega_s\sigma_Y\\ N_{c1} &= \frac{8}{21}bkh\sigma_{rc}\\ N_{c2} &= \frac{3}{7}bkh\sigma_{rc}\ .\end{array}\right.$$

[20] In the case of mild reinforcing steel, the yielding zone is about $\frac{9}{10}$ of the total range of strains considered in the stress-strain diagram. For this reason, with the quantities of steel used in current members, the failure takes place under these conditions.

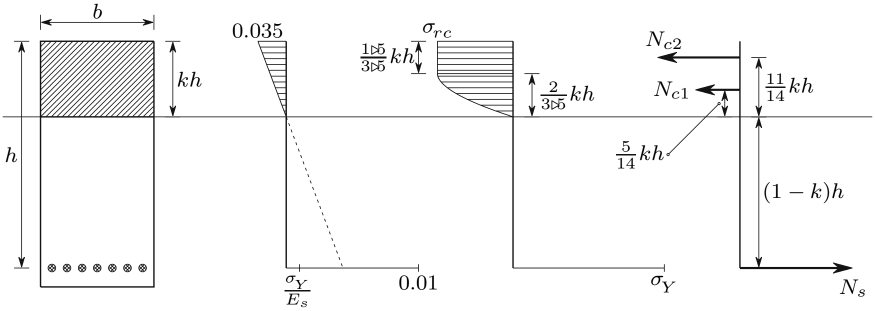

Fig. 100. Limit state of a reinforced concrete rectangular cross-section

The other possibilities of failure may be physically grouped in two. One occurs in members with a very low amount of reinforcing steel, where the steel reaches the maximum strain $\varepsilon_s = 0.01$ before the concrete ($\varepsilon_s > 0.01$ in (183)). The other possibility occurs in members with a very high amount of steel, where the maximum strain in the concrete is attained with the steel in the elastic phase ($\varepsilon_s < \frac{\sigma_Y}{E_s}$). Members with this kind of failure should be avoided, since the collapse is not preceded by plastic deformations, i.e., it is a brittle failure, which is not desirable from the point of view of the safety, as explained in Sect. VI.5. The other failure possibilities could be analysed in the same way as the situation analysed above, by means of the equilibrium conditions of the forces acting in the cross-section.

VII.11 Examples and Exercises

VII.1. Determine the bending strength of a bar with a square cross-section, when the action axis is parallel to:
(a) one of the sides;
(b) one of the diagonals.

Resolution

(a) The moment of inertia of a rectangular cross-section with height h, with respect to the symmetry axis which is parallel to the base b, is given by the expression

$$I = \frac{bh^3}{12} .$$

Since the maximum distance to the neutral axis is in this case $\frac{a}{2}$, the section modulus takes the value ($b = h = a$)

$$\frac{I}{v} = \frac{\frac{a^4}{12}}{\frac{a}{2}} = \frac{a^3}{6} .$$

Denoting by σ_{all} the nominal value of the material's resisting stress, the resisting bending moment is

$$M^{\text{a}}_{\text{all}} = \frac{a^3}{6}\sigma_{\text{all}} \,.$$

(b) If the action axis is parallel to one of the square's diagonals, the neutral axis is the other diagonal. The moment of inertia with respect to this axis is also $I = \frac{a^4}{12}$. In fact, in a square, the moment of inertia is the same with respect to any axis passing through the centroid, since the two principal moments of inertia are equal. The maximum distance to the neutral axis is, in this case, $v = \frac{a}{\sqrt{2}}$. Thus, the resisting bending moment takes the value

$$M^{\text{b}}_{\text{all}} = \frac{I}{v}\sigma_{\text{all}} = \frac{\frac{a^4}{12}}{\frac{a}{\sqrt{2}}}\sigma_{\text{all}} = \frac{a^3}{6\sqrt{2}}\sigma_{\text{all}} = \frac{1}{\sqrt{2}}M^{\text{a}}_{\text{all}} \approx 0.707 M^{\text{a}}_{\text{all}} \,.$$

VII.2. A tree trunk with circular cross-section of diameter d is to be cut to a rectangular cross-section of base b and height h. Determine the dimensions b and h, in order to maximize:

(a) the bending stiffness;
(b) the bending strength.

Resolution

(a) The geometrical parameter which enters into the definition of the bending stiffness is the moment of inertia. Since the diagonal of the rectangle cannot exceed the diameter of the trunk d, the moment of inertia may be expressed as a function of b, yielding $(h^2 + b^2 = d^2)$

$$I = \frac{bh^3}{12} = \frac{b\left(d^2 - b^2\right)^{\frac{3}{2}}}{12} \,.$$

The value of b which maximizes I may be obtained from the condition of a zero derivative in order to b, which gives

$$\frac{\mathrm{d}I}{\mathrm{d}b} = 0 \;\Rightarrow\; d^2 - 4b^2 = 0 \;\Rightarrow\; b = \frac{d}{2} \;\Rightarrow\; h = \frac{\sqrt{3}}{2}d \;\Rightarrow\; \frac{h}{b} = \sqrt{3} \approx 1.732 \,.$$

(b) To obtain the maximum bending strength, the section modulus must be maximized. The same procedure as before yields

$$\frac{I}{v} = \frac{bh^2}{6} = \frac{b\left(d^2 - b^2\right)}{6}; \qquad \frac{\mathrm{d}}{\mathrm{d}b}\left(\frac{I}{v}\right) = 0 \Rightarrow d^2 - 3b^2 = 0$$

$$\Rightarrow \quad b = \frac{d}{\sqrt{3}} \Rightarrow h = \frac{\sqrt{2}}{\sqrt{3}}d \Rightarrow \frac{h}{b} = \sqrt{2} \approx 1.414 \,.$$

VII.3. Compare the section moduli of the three following rectangular cross-sections with the same area $\Omega = bh$. The action axis is parallel to the height h.
(a) Base b and height h.
(b) Base $\frac{b}{2}$ and height $2h$.
(c) Base $2b$ and height $\frac{h}{2}$.

VII.4. Wire made of $S\,235$ steel [10] with a circular cross-section of diameter d is wound around a cylindrical drum for transportation. Determine the minimum diameter D of the winding needed to avoid residual deformation, when the wire is unwound.

Resolution

The wire will not have a residual curvature, if the yielding strain is not exceeded during the winding process, i.e, if the curvature does not exceed the value ((142), $y = \frac{d}{2}$)

$$\varepsilon_{\max} = \frac{1}{\rho}\frac{d}{2} \leq \varepsilon_Y \Rightarrow \frac{1}{\rho} \leq \frac{2\varepsilon_Y}{d} \,.$$

The yielding stress of this steel and its modulus of elasticity are $\sigma_Y = 235MPa$ and $E = 206GPa$ [10], respectively. Thus, the minimum diameter of the winding will be

$$D = 2\rho = \frac{d}{\varepsilon_Y} = \frac{Ed}{\sigma_Y} = \frac{206 \times 10^9}{235 \times 10^6} \approx 877d \,.$$

The exact minimum diameter of the drum would be $D_{\text{drum}} = D - \frac{d}{2}$.

VII.5. Express the section modulus and the moment of inertia as functions of the cross-section area Ω and the height h in the following cross-sections (the action axis is parallel to the height h):
(a) rectangle of base b and height h;
(b) circle of radius r;
(c) isosceles triangle of base b and height h;
(d) rhombus with a horizontal dimension b and height h;
(e) I-beam INP200 [9];
(f) I-beam HE200B [9].

Resolution

(a) Rectangle

$$I = \frac{bh^3}{12} = \frac{1}{12}bhh^2 = \frac{1}{12}\Omega h^2 \approx 0.0833\Omega h^2; \quad \frac{I}{v} = \frac{I}{\frac{h}{2}} = \frac{1}{6}\Omega h \approx 0.1667\Omega h \,.$$

(b) Circle

$$I = \frac{\pi r^4}{4} = \pi r^2 \frac{(2r)^2}{16} = \frac{1}{16}\Omega h^2 = 0.0625\Omega h^2;$$
$$\frac{I}{v} = \frac{I}{\frac{h}{2}} = 0.125\Omega h \ .$$

(c) Triangle

$$I = \frac{bh^3}{36} = \frac{1}{18}\frac{bh}{2}h^2 = \frac{1}{18}\Omega h^2 \approx 0.0556\Omega h^2;$$
$$\frac{I}{v} = \frac{I}{\frac{2}{3}h} = \frac{1}{12}\Omega h \approx 0.0833\Omega h \ .$$

(d) Rhombus:

$$I = \frac{bh^3}{48} = \frac{1}{24}\frac{bh}{2}h^2 = \frac{1}{24}\Omega h^2 \approx 0.0417\Omega h^2;$$
$$\frac{I}{v} = \frac{I}{\frac{h}{2}} = \frac{1}{12}\Omega h \approx 0.0833\Omega h \ .$$

(e) I-beam INP200 ([9], 7.1.1):

$$\begin{cases} I &= 2140cm^4 \\ \Omega &= 33.5cm^3 \\ h &= 20cm \end{cases} \Rightarrow \begin{cases} I &= \frac{2140}{33.5\times 20^2}\Omega h^2 \approx 0.1597\Omega h^2 \\ \frac{I}{v} &= \frac{I}{\frac{h}{2}} \approx 0.3194\Omega h \ . \end{cases}$$

(f) I-beam HE200B ([9], 7.1.3):

$$\begin{cases} I &= 5696cm^4 \\ \Omega &= 78.1cm^3 \\ h &= 20cm \end{cases} \Rightarrow \begin{cases} I &= \frac{5696}{78.1\times 20^2}\Omega h^2 \approx 0.1823\Omega h^2 \\ \frac{I}{v} &= \frac{I}{\frac{h}{2}} \approx 0.3647\Omega h \ . \end{cases}$$

From these examples, we conclude that, by choosing cross-sections with less material in the region around the neutral axis, like the I-beams, both the bending stiffness and the bending strength are substantially increased, without there being any need to increase the amount of material (represented by Ω) or the height of the cross-section.

VII.6. The bar with the square cross-section represented in Fig. VII.6 is made of a brittle material with linear elastic behaviour until rupture. Determine the increase that can be obtained in bending strength by cutting the bar as represented in the Figure.

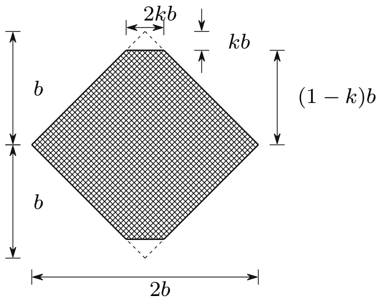

Fig. VII.6

Resolution

Since the material is brittle, rupture takes place for tensile stresses, when the rupture strain is attained, which causes a crack in a transversal direction to the fibres. As a consequence of the stress concentration at the tip of the crack (see Sect. VI.9), it propagates immediately to the whole cross-section causing the failure of the bar. For this reason, the bending strength can only be increased by improving the elastic loading capacity, i.e., by improving the section modulus. The two small symmetrical cuts shown in the Figure reduce the maximum distance to the neutral axis, v without a great reduction of the moment of inertia I, which may increase the section modulus $\frac{I}{v}$.

The moment of inertia of the cross-section with the cuts defined by kb, may be computed from the expressions for the moment of inertia of a rhombus ($I = \frac{bh^3}{48}$) and of a triangle ($I = \frac{bh^3}{36}$) and from the parallel-axis theorem, yielding (Fig. VII.6)

$$
\begin{aligned}
I &= \frac{2b\,(2b)^3}{48} - 2\left[\frac{2kb\,(kb)^3}{36} + \frac{1}{2}2kb\;kb\left(1-\frac{2}{3}k\right)^2 b^2\right] \\
&= \left[\frac{1}{3} - \frac{k^4}{9} - 2k^2\left(1-\frac{2}{3}k\right)^2\right] b^4 \,.
\end{aligned}
$$

The section modulus takes the value

$$
\frac{I}{v} = \frac{I}{(1-k)\,b} = \frac{b^3}{1-k}\left[\frac{1}{3} - \frac{k^4}{9} - 2k^2\left(1-\frac{2}{3}k\right)^2\right] \,.
$$

The value of k which maximizes this quantity may be computed by analytical or numerical means, leading to the conclusion that, for $k = \frac{1}{9}$, the section modulus attains the maximum value $0.35117b^3$. Comparing this value with the section modulus of the original cross-section, we get

$$
\left\{\begin{array}{l}
k = 0 \;\Rightarrow\; \frac{I}{v} = \frac{b^3}{3} = 0.33333b^3 \\
k = \frac{1}{9} \;\Rightarrow\; \frac{I}{v} = \left(\frac{I}{v}\right)_{\max} = 0.35117b^3
\end{array}\right.
\quad \Rightarrow \quad \frac{\left(\frac{I}{v}\right)_{\max}}{\left(\frac{I}{v}\right)} = 1.0535 \,.
$$

We conclude that the cuts can increase the bending strength of the bar by 5.35%.

VII.7. Figure VII.7 represents the cross-section of a bar supporting the indicated bending moment M. Justifying the procedure used, determine the variation of the failure bending moment, when the top and bottom small rectangles (top: $2a \times 4a$; bottom: $2 \times a \times 4a$) are removed, so that an I-shaped cross-section is obtained, for:

(a) a brittle material with linear elastic behaviour and rupture stress σ_r;
(b) a ductile material with elastic perfectly plastic behaviour with yielding stress σ_Y.

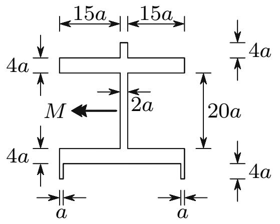

Fig. VII.7

Resolution

(a) In the case of a brittle material with linear elastic behaviour until rupture, the relation between the ultimate bending moments, with and without the small rectangles, coincides with the relation between the section moduli in the two situations. In the case of the original section (Fig. VII.7), the moment of inertia and the section modulus are given by the expressions

$$I_1 = \frac{2a(36a)^3}{12} + \frac{30a(28a)^3}{12} - \frac{30a(20a)^3}{12} = 42656a^4$$
$$\Rightarrow \left(\frac{I}{v}\right)_1 = \frac{42656a^4}{18a} = \frac{21328}{9}a^3 \approx 2369.78a^3 \ .$$

In the case of the cross-section without the small rectangles, the same quantities take the values

$$I_2 = \frac{32a(28a)^3}{12} - \frac{30a(20a)^3}{12} = \frac{115616a^4}{3} \approx 38538.67a^4$$
$$\Rightarrow \left(\frac{I}{v}\right)_2 = \frac{\frac{115616a^4}{3}}{14a} \approx 2752.76a^3 \ .$$

The relation between the section moduli in the two situations is then

$$\frac{\left(\frac{I}{v}\right)_2}{\left(\frac{I}{v}\right)_1} \approx 1.16161 \ .$$

We conclude that the removal of the small rectangles increases the ultimate bending strength by 16%

(b) In the case of a ductile material with elastic perfectly plastic behaviour, the relation between the ultimate bending moments is given by the relation between the plastic section moduli. In the original cross-section (Fig. VII.7) this quantity takes the value

$$Z_1 = 2 \times (18a \times 2a \times 9a + 30a \times 4a \times 12a) = 3528a^3 \ .$$

The removal of the small rectangles causes a fall in the plastic section modulus, which corresponds to the first moment of the removed area

$$Z_2 = Z_1 - 2 \times 2a \times 4a \times 16a = 3272a^3 \ .$$

The relation between the plastic section moduli in the two situations is then

$$\frac{Z_2}{Z_1} = \frac{3272a^3}{3528a^3} \approx 0.92744 \ ,$$

which represents a reduction in the ultimate strength of 7.256%.

VII.8. A bar with a rectangular cross-section with the dimensions $b \times 2b$ supports a bending moment whose action axis is vertical and makes an angle of 45° with the symmetry axis of the rectangle. Determine:

(a) the maximum stress in the cross-section;
(b) the position of the neutral axis;
(c) the curvature of the bar.
(d) Compare the answers to questions a) and c), with the corresponding quantities obtained when the bar is rotated so that the action axis becomes parallel to the larger sides of the rectangle.

Resolution

(a) Since we have a cross-section which has a rectangular convex contour with a symmetry axis, (152) may be used to compute the maximum stress. To this end, it is necessary to compute the following quantities (axis y is parallel to the largest side of the rectangle)

$$\left(\frac{I}{v}\right)_x = \frac{bh^2}{6} = \frac{b\,(2b)^2}{6} = \frac{2}{3}b^3 \qquad \left(\frac{I}{v}\right)_y = \frac{hb^2}{6} = \frac{2bb^2}{6} = \frac{1}{3}b^3$$

$$M_x = M\cos\alpha = \frac{\sqrt{2}}{2}M \qquad M_y = M\sin\alpha = \frac{\sqrt{2}}{2}M \ .$$

Substituting these values in (152), we get

$$\sigma_{\max} = \frac{\frac{\sqrt{2}}{2}M}{\frac{2}{3}b^3} + \frac{\frac{\sqrt{2}}{2}M}{\frac{1}{3}b^3} = \frac{9\sqrt{2}}{4}\frac{M}{b^3} \approx 3.182\frac{M}{b^3} \ .$$

(b) The position of the neutral axis is defined by angle β, which may be obtained by means of (151), yielding

$$\tan\beta = \frac{I_x}{I_y}\tan 45^\circ = \frac{\frac{b(2b)^3}{12}}{\frac{2bb^3}{12}} = 4 \Rightarrow \beta = \arctan(4) \approx 75.96^\circ \ .$$

We conclude that the deflection plane makes an angle of $75.96^\circ - 45^\circ = 30.96^\circ$ with the vertical plane

(c) The curvature may be computed by means of (154), yielding

$$\frac{1}{\rho} = \frac{M}{E}\sqrt{\frac{\frac{1}{2}}{\left[\frac{b(2b)^3}{12}\right]^2} + \frac{\frac{1}{2}}{\left(\frac{2bb^3}{12}\right)^2}} \approx 4.373\frac{M}{b^4E} \ .$$

(d) If the cross-section is rotated so that the larger sides of the rectangle become vertical, the maximum stress and the curvature take the values

$$\sigma_{\max} = \frac{M}{\frac{2}{3}b^3} = 1.5\frac{M}{b^3} \qquad \left(\text{instead of } 3.182\frac{M}{b^3}\right)$$

$$\frac{1}{\rho} = \frac{M}{E\frac{b(2b)^3}{12}} = 1.5\frac{M}{b^4E} \qquad \left(\text{instead of } 4.373\frac{M}{b^4E}\right) \ .$$

VII.9. Demonstrate the last equality of (154).

Resolution

The last equality of (154) is equivalent to the expression

$$\frac{\sin^2\theta}{I_n^2} = \frac{\cos^2\alpha}{I_x^2} + \frac{\sin^2\alpha}{I_y^2} \ , \tag{a}$$

since we have $I_\theta = \frac{I_n}{\sin\theta}$ (148). Equality (a) may be demonstrated on the basis of the following expressions

$$\tan\alpha = \frac{I_y}{I_x}\tan\beta \quad (151) \quad \text{(b)} \qquad \sin^2\alpha = \frac{\tan^2\alpha}{1+\tan^2\alpha} \quad \text{(e)}$$

$$\theta = \frac{\pi}{2} - \beta + \alpha \quad \text{(Fig. 75)} \quad \text{(c)} \qquad \cos^2\alpha = \frac{1}{1+\tan^2\alpha} \quad \text{(f)}$$

$$I_n = I_x\cos^2\beta + I_y\sin^2\beta \quad \text{(d)} \qquad \sin(a+b) = \sin a\cos b + \cos a\sin b \quad \text{(g)} \ .$$

Substituting $\sin^2\alpha$ and $\cos^2\alpha$ in the second term of (a) by (e) and (f), respectively, and using (b), we may establish the relation

$$\frac{\cos^2\alpha}{I_x^2}+\frac{\sin^2\alpha}{I_y^2}=\frac{1}{I_x^2\cos^2\beta+I_y^2\sin^2\beta}\,. \tag{h}$$

By means of (g), we may express $\sin\theta$ as a function of angles α and β, yielding

$$\begin{aligned}\sin\theta&=\sin\left[\left(\frac{\pi}{2}-\beta\right)+\alpha\right]=\cos\alpha\,(\cos\beta+\sin\beta\tan\alpha)\\&=\cos\alpha\left(\cos\beta+\sin\beta\frac{I_y}{I_x}\tan\beta\right)\,.\end{aligned}$$

Using this expression, the first term of (a) may be transformed, so that it is expressed in terms of I_x, I_y and β

$$\begin{aligned}\frac{\sin^2\theta}{I_n^2}&=\cos^2\alpha\frac{\left(\cos\beta+\sin\beta\frac{I_y}{I_x}\tan\beta\right)^2}{I_n^2}\\&=\overbrace{\frac{1}{1+\frac{I_y^2}{I_x^2}\tan^2\beta}}^{\cos^2\alpha}\,\frac{\cos^2\beta+2\frac{I_y}{I_x}\sin^2\beta+\frac{I_y^2}{I_x^2}\frac{\sin^4\beta}{\cos^2\beta}}{I_n^2}\\&=\frac{1}{I_n^2}\frac{\overbrace{I_x^2\cos^4\beta+2I_xI_y\sin^2\beta\cos^2\beta+I_y^2\sin^4\beta}^{I_n^2(\text{ d})}}{I_x^2\cos^2\beta+I_y^2\sin^2\beta}=\frac{1}{I_x^2\cos^2\beta+I_y^2\sin^2\beta}\,.\end{aligned}$$

This result coincides with (h), which shows that (a) is correct.

VII.10. Determine the shape and dimensions of the core of the following cross-sections:
(a) rectangle with base b and height h;
(b) circle with radius r;
(c) rhombus with symmetry axes b and h;
(d) equilateral triangle with side length a;
(e) ellipse with semi-axes lengths a and b.

VII.11. The cantilever beam represented in Fig. VII.11-a is made of two materials, a and b, with elasticity moduli $E_a=2E$ and $E_b=5E$. The beam supports a vertical loading p by surface unit and a horizontal concentrated force $P=500pa^2$, as indicated in the Figure. Determine the maximum normal stress in each of the materials.

Resolution

Both the bending moments caused by the vertical and horizontal forces attain maximum values at the built-in end, so that the maximum values of the normal stress occur in that cross-section.

As the beam is made of two materials with linear elastic behaviour, it is necessary to compute the centroid's position, weighting the first area moments with the elasticity moduli of the two materials, as given by (167). Since the cross-section has a vertical axis of symmetry, the position of the centroid is completely defined by the distance d (Fig. VII.11-b)

$$d = \frac{\Omega_a d_a E_a + \Omega_b d_b E_b}{\Omega_a E_a + \Omega_b E_b} = \frac{241}{34} a \approx 7.08824a$$

$$\text{with} \left\{ \begin{array}{lcl} d_a & = & 10a \\ d_b & = & 4.5a \end{array} \right. \text{and} \left\{ \begin{array}{lcl} \Omega_a & = & 40a^2 \\ \Omega_b & = & 18a^2 \end{array} \right. .$$

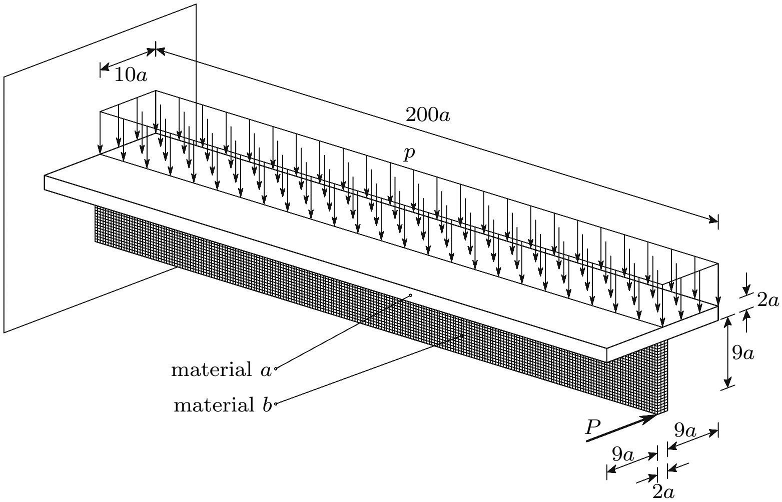

Fig. VII.11-a

The weighted moments of inertia of the cross-section with respect to the principal axes x and y (Fig. VII.11-b) take the values (172)

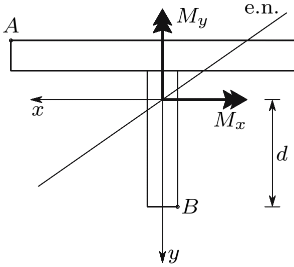

Fig. VII.11-b

$$
\begin{aligned}
J_x &= E_a \int_{\Omega_a} y^2 \,\mathrm{d}\Omega_a + E_b \int_{\Omega_b} y^2 \,\mathrm{d}\Omega_b \\
&= 2E \left[\tfrac{20a(2a)^3}{12} + 40a^2(2.91176a)^2 \right] \\
&+ 5E \left[\tfrac{2a(9a)^3}{12} + 18a^2(2.58824a)^2 \right] \\
&\approx 1915.34Ea^4 \\
J_y &= E_a \int_{\Omega_a} x^2 \,\mathrm{d}\Omega_a + E_b \int_{\Omega_b} x^2 \,\mathrm{d}\Omega_b \\
&= 2E \left[\tfrac{2a(20a)^3}{12} \right] + 5E \left[\tfrac{9a(2a)^3}{12} \right] \approx 2696.67Ea^4 \ .
\end{aligned}
$$

The bending moments at the left end cross-section (built-in end) take the values

$$
M_x = -\frac{10pa\,(200a)^2}{2} = -2\times10^5 pa^3 \qquad M_y = -500pa^2(200a) = -1\times10^5 pa^3 \ .
$$

Since the neutral axis must be in the same quadrant as the resultant bending moment (Fig. VII.11-b), points A and B are the farthest points from the neutral axes in materials a and b, respectively. The stresses in these points may be computed by means of (172), yielding

$$
\begin{aligned}
\sigma_{a-\max} &= \frac{M_x E_a}{J_x} y - \frac{M_y E_a}{J_y} x \\
&= \frac{-2\times10^5 pa^3(2E)}{1915.34a^4E}(-3.91176a) - \frac{-1\times10^5 pa^3(2E)}{2696.67a^4E}(10a) \approx 1558.59\,p \,, \\
\sigma_{b-\max} &= \frac{M_x E_b}{J_x} y - \frac{M_y E_b}{J_y} x \\
&= \frac{-2\times10^5 pa^3(5E)}{1915.34a^4E}(7.08824a) - \frac{-1\times10^5 pa^3(5E)}{2696.67a^4E}(-a) \approx -3886.18\,p \ .
\end{aligned}
$$

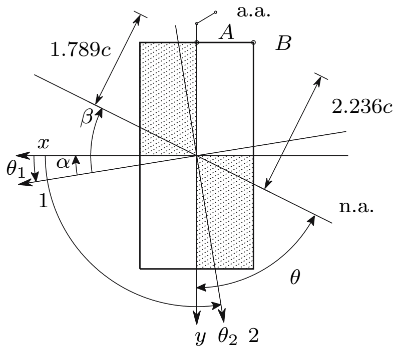

Fig. VII.12

VII.12. Consider the cross-section of a composite bar as depicted in Fig. VI.20. The moduli of elasticity of materials a and b are, respectively, $E_a = 2E$ and $E_b = E$.

(a) Determine the orientation of the principal axes of bending, i.e., the orientation of the action axes which cause plane bending.

(b) Determine the orientation of the neutral axes and the maximum stresses in the two materials, caused by a bending moment M with a vertical action axis.

Resolution

(a) The principal bending axes may be computed from the weighted moments and the product of inertia with respect to the axes x and y represented in Fig. VII.12. These quantities take the values (cf. (168) and (170))

$$
\begin{aligned}
J_x &= 2E\tfrac{c(4c)^3}{12} + E\tfrac{c(4c)^3}{12} = 16Ec^4 \\
J_y &= 2E\tfrac{2c(2c)^3}{12} + E\tfrac{2c(2c)^3}{12} = 4Ec^4 \\
J_{xy} &= -2E\tfrac{c^2(2c)^2}{4} \times 2 + E\tfrac{c^2(2c)^2}{4} \times 2 \\
&= -2Ec^4 \; .
\end{aligned}
$$

The principal directions of bending may be determined by means of the same expression that is used for the computation of the principal directions of inertia in homogeneous cross-sections, yielding

$$
\tan 2\theta = -\frac{2J_{xy}}{J_x - J_y} = \frac{4}{16-4} \Rightarrow \begin{cases} \theta_1 = 9.22^\circ \\ \theta_2 = 99.22^\circ \; . \end{cases}
$$

(b) The orientation of the neutral axis may be found by means of the expression resulting from the condition $\sigma_a = 0$ in (172), which yields $\beta = \arctan\left(\frac{J_1}{J_2}\tan\alpha\right)$, where 1 and 2 are the weighted principal directions of inertia (the principal directions of bending), α is the angle between the positive directions of the moment vector and axis 1 and β is the angle between the neutral axis and the positive direction of axis 1. The weighted principal moments of inertia may be computed from the values of J_x, J_y and J_{xy} above, by means of the expressions of rotation of reference axes of inertia

$$J_1 = J_x\cos^2\theta_1 + J_y\sin^2\theta_1 - 2J_{xy}\sin\theta_1\cos\theta_1 = 16.32Ec^4$$

$$J_2 = J_x\cos^2\theta_2 + J_y\sin^2\theta_2 - 2J_{xy}\sin\theta_2\cos\theta_2 = 3.675Ec^4 \ .$$

Since the position of the moment vector (direction x, Fig. VII.12) is in this case given by $\alpha = -9.22°$, we get for angle β the value (Fig. VII.12)

$$\beta = \arctan\left[\frac{J_1}{J_2}\tan(-9.22°)\right] = -35.79° \ .$$

The stresses could be obtained from (172). As an alternative, (169) may be used. In order to use the second possibility, the moment of inertia with respect to the neutral axis is needed

$$J_n = J_1\cos^2\beta + J_2\sin^2\beta = 12.00Ec^4 \ .$$

From Fig. VII.12 we conclude that the angle between the action and neutral axes is

$$\theta = 180° - (35.79° - 9.22°) - 90° = 63.43° \ .$$

The maximum distances to the neutral axis are

$$\begin{cases} v_a = 2c\cos(35.79° - 9.22°) = 1.789c \\ v_b = 2c\cos(35.79° - 9.22°) + c\sin(35.79° - 9.22°) = 2.236c \ , \end{cases}$$

respectively for materials a and b (points A and B, Fig. VII.12). The maximum stresses in the two materials are then (169)

$$J_\theta = \frac{12.00Ec^4}{\sin 63.43°} = 13.42Ec^4 \Rightarrow \begin{cases} \sigma_{a-\max} = \frac{M2E}{13.42Ec^4}1.789c = 0.267\frac{M}{c^3} \\ \sigma_{b-\max} = \frac{ME}{13.42Ec^4}2.236c = 0.167\frac{M}{c^3} \ . \end{cases}$$

VII.13. Consider a prismatic bar that has a rectangular cross-section with a height h, made of a material with elastic perfectly plastic behaviour characterized by the elasticity modulus E and the yielding stress σ_Y. A bending moment with an action axis parallel to the height h, with the magnitude $M = 1.3M_e$ (M_e is the maximum bending moment in the elastic regime) is applied and subsequently removed. Determine:

(a) the curvature of the bar in the loading phase;
(b) the residual curvature after unloading;
(c) the residual stresses.

Resolution

(a) The curvature in the loading phase is given directly by (179), yielding

$$\frac{1}{\rho} = \frac{2\sigma_Y}{hE} \frac{1}{\sqrt{3 - 2 \times 1.3}} = 1.581 \frac{2\sigma_Y}{hE} = 1.581 \frac{1}{\rho_e} \ ,$$

where ρ_e is the curvature radius for $M = M_e$.

(b) The deformation recovery in the unloading is elastic and proportional to the removed bending moment. Thus, we have

$$\frac{1}{\rho_{\text{unload}}} = -1.3 \frac{1}{\rho_e} \ \Rightarrow \ \frac{1}{\rho_{\text{residual}}} = (1.581 - 1.3) \frac{1}{\rho_e} = 0.281 \frac{1}{\rho_e} \ .$$

(c) To compute the residual stresses it is necessary to determine the height of the cross-section which remains in the elastic regime in the loading phase (h_e, Fig. 98). To this end, (178) may be used, yielding

$$M = M_e \left[\frac{3}{2} - \frac{1}{2} \left(\frac{h_e}{h} \right)^2 \right] \ \Rightarrow \ h_e = \sqrt{3 - 2\frac{M}{M_e}} h = \sqrt{0.4} h \approx 0.632 h \ .$$

According to Fig. 98, the stresses caused by unloading are $1.3\sigma_Y$ and $\sqrt{0.4} \times 1.3\sigma_Y \approx 0.822\sigma_Y$, respectively in the farthest fibers and in the fibres at distance $\frac{h_e}{2}$ from the neutral axis. Thus, the residual stresses distribution takes the form represented in the last diagram of Fig. 98, with a residual stress in the farthest fibres of $0.3\sigma_Y$, while in the fibres at distance $\frac{h_e}{2}$ from the neutral axis the residual stress takes the value $\sigma_Y - 0.822\sigma_Y = 0.178\sigma_Y$. The residual curvature may also be computed from this last stress (181), yielding

$$\frac{1}{\rho_{\text{residual}}} = \frac{2 \times 0.178\sigma_Y}{E \times 0.632h} = 0.281 \frac{2\sigma_Y}{Eh} = 0.281 \frac{1}{\rho_e} \ .$$

VII.14. Compute the shape factors of the I-beams INP200 and IPE200 [9].

Resolution

INP200

The plastic section modulus ((180) and following text) may be expressed as a function of the first area moment of half cross-section with respect to the neutral axis (S_x, [9], 7.1.1), i.e.,

$$S_x = 125 cm^3 \ \Rightarrow \ Z = 2S_x = 250 cm^3 \ .$$

The shape factor may be given by the relation between the plastic (Z) and the elastic $\left(\frac{I}{v}\right)$ section moduli, yielding

$$\frac{I}{v} = 214cm^3 \Rightarrow \varphi = \frac{Z}{\frac{I}{v}} = \frac{250}{214} \approx 1.168\ .$$

IPE200

The same procedure gives the result ([9], 7.1.2)

$$\begin{cases} S_x = 110cm^3 \\ \\ \dfrac{I}{v} = 194cm^3 \end{cases} \Rightarrow \varphi = \frac{2 \times 110}{194} \approx 1.134\ .$$

VII.15. Consider the cantilever beam with a rectangular cross-section and variable height represented in Fig. VII.15. Compare the exact value of the stress in point A, obtained from the solutions of the Theory of Elasticity, with the approximate solution furnished by the bending theory for the same stress. The cross-section has a constant width b.

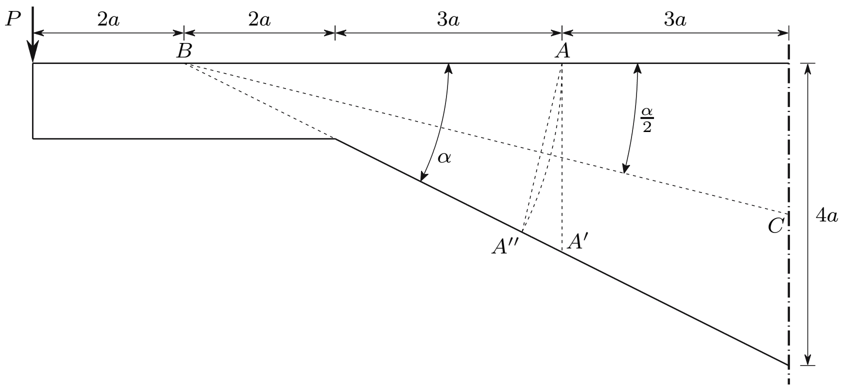

Fig. VII.15

Resolution

According to the theory of prismatic members, in cross-section AA' we have shear force and bending moment, which induces in point A the normal stress

$$\begin{cases} M = 7Pa \\ \\ \overline{AA'} = 2.5a \end{cases} \Rightarrow \sigma = \frac{7Pa}{\frac{b(2.5a)^2}{6}} = 6.72\frac{P}{ab}\ .$$

The solution of the Theory of Elasticity may be obtained by combining the solutions presented in Sects. VI.7.c and VII.8.b for the wedge shaped element.

To this end, it is necessary to consider the system of forces, which is statically equivalent to force P, but acting in point B. Force P is decomposed into two components, one (N) in the direction of the wedge axis (segment BC), and other (V) in the perpendicular direction. Thus, we have

$$M = 2Pa \qquad V = P\cos\frac{\alpha}{2} \qquad N = -P\sin\frac{\alpha}{2} \; .$$

In accordance with (165), the bending moment M causes the stress ($\alpha = \arccos\frac{1}{2}$, $r = 5a$)

$$\sigma_M = \frac{2Pa}{b\,(5a)^2}\,\frac{2}{1-\frac{\alpha}{\tan\alpha}} \approx 2.2007\frac{P}{ab} \; .$$

The stress caused by the shear force V may be computed from (164), yielding

$$\sigma_V = \frac{2}{\alpha - \sin\alpha}\,\frac{P\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}}{b5a} \approx 5.4425\frac{P}{ab} \; .$$

The axial force N induces the stress (cf. Subsect. VI.7.c)

$$\sigma_N = \frac{2}{\alpha + \sin\alpha}\,\frac{-P\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{b5a} \approx -0.0982\frac{P}{ab} \; .$$

Thus, the total stress in point A is

$$\sigma = \sigma_M + \sigma_V + \sigma_N = 7.545\frac{P}{ab} \; .$$

This value is about 12% larger than the solution of the bending theory, which exceeds the predictions indicated in Subsect. VII.8.b. This is because there, a perpendicular section to the wedge axis was considered. In fact, if we consider, instead of section AA', the section AA'', the solution of the bending theory becomes substantially closer to the exact solution, exceeding it by about 5%.

$$\overline{AA''} = 2\times 5a\sin\frac{\alpha}{2} \;\Rightarrow\; \sigma = \frac{7Pa}{\frac{b\left(10\sin\frac{\alpha}{2}a\right)^2}{6}} = 7.9566\frac{P}{ab} \; .$$

VII.16. The prismatic bar with the cross-section depicted in Fig. VII.16-a is made of two materials, a and b, and undergoes a uniform temperature increase Δt. The materials have linear elastic behaviour defined by the parameters

$$E_a = E \qquad E_b = 2E \qquad \alpha_a = \alpha \qquad \alpha_b = 2\alpha \; .$$

(a) Determine the elongation and the curvature introduced by Δt (the bar has length l).
(b) Determine the distribution of stresses in the cross-section.

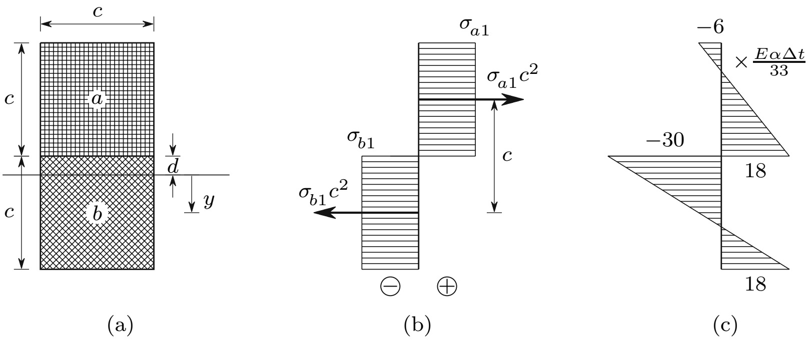

Fig. VII.16

Resolution

(a) The temperature variation causes bending in the bar, as explained in Sect. VI.6.d (Fig. 65). In order to compute the corresponding curvature, let us first suppose that the bending is prevented by applying adequate bending moments at both ends of the bar. Under these conditions, the stresses in the cross-section are given by (135), since the bar remains straight, yielding ($\Omega_a = \Omega_b = c^2$)

$$\begin{cases} \sigma_{a1} = \dfrac{2E^2c^4}{Ec^2 + 2Ec^2}\Delta t\dfrac{\alpha}{c^2} = \dfrac{2}{3}E\alpha\Delta t \\ \sigma_{b1} = \dfrac{2E^2c^4}{Ec^2 + 2Ec^2}\Delta t\dfrac{-\alpha}{c^2} = -\dfrac{2}{3}E\alpha\Delta t\ . \end{cases}$$

The couple of forces corresponding to the stress distribution in the cross-section (Fig. VII.16-b) is the bending moment needed to prevent bending. This moment takes the value

$$M = \sigma_{a1}c^2 \times c = \frac{2}{3}c^3E\alpha\Delta t\ .$$

The elongation of the bar is not affected by this bending moment, and so it may be computed from the stress in one of the materials. Using, for example, σ_{a1}, we get from (117)

$$\Delta l = l\left(\alpha\Delta t + \frac{\sigma_{a1}}{E_a}\right) = l\left(\alpha\Delta t + \frac{2}{3}\alpha\Delta t\right) = \frac{5}{3}\alpha\Delta tl\ .$$

If a bending moment $M' = -M$, is subsequently applied to the bar, the total bending moment vanishes and only the temperature variation remains. Thus, the curvature acquired by the bar in this second loading phase is the curvature caused by the temperature variation. This curvature may be obtained by the theory of bending of composite members described in Sect. VII.9. To this end, the position of the neutral axis must be computed ((167) and Fig. VII.16-a)

$$c^2\left(\frac{c}{2}+d\right)E = c^2\left(\frac{c}{2}-d\right)2E \Rightarrow d = \frac{c}{6}\,.$$

The weighted moment of inertia takes the value (168)

$$J_n = E\left[\frac{c^4}{12}+c^2\left(\frac{c}{2}+\frac{c}{6}\right)^2\right] + 2E\left[\frac{c^4}{12}+c^2\left(\frac{c}{2}-\frac{c}{6}\right)^2\right] = \frac{11}{12}Ec^4\,.$$

The curvature is then (169, $\theta = 90°$)

$$\frac{1}{\rho} = \frac{M'}{J_n} = \frac{\frac{2}{3}c^3E\alpha\Delta t}{\frac{11}{12}Ec^4} = \frac{8}{11}\frac{\alpha\Delta t}{c}\,.$$

(b) The stresses caused by the bending moment M' may by computed by means of (169), yielding

$$\begin{cases} \sigma_{a2} = \dfrac{ME_a}{J_n}y = \dfrac{\frac{2}{3}c^3E\alpha\Delta tE}{\frac{11}{12}Ec^4}y = \dfrac{8}{11}E\alpha\Delta t\dfrac{y}{c} \\[2ex] \sigma_{b2} = \dfrac{ME_b}{J_n}y = \dfrac{\frac{2}{3}c^3E\alpha\Delta t2E}{\frac{11}{12}Ec^4}y = \dfrac{16}{11}E\alpha\Delta t\dfrac{y}{c}\,. \end{cases}$$

By superposing the stresses caused by the temperature variation in the straight bar to the bending stresses, the total stresses are obtained

$$\begin{cases} \sigma_a = \sigma_{a1}+\sigma_{a2} = \left(\dfrac{2}{3}+\dfrac{8}{11}\dfrac{y}{c}\right)E\alpha\Delta t \\[2ex] \sigma_b = \sigma_{b1}+\sigma_{b2} = \left(-\dfrac{2}{3}+\dfrac{16}{11}\dfrac{y}{c}\right)E\alpha\Delta t\,. \end{cases}$$

Particularizing these stresses for the upper fibres ($y = -(c+d) = -\frac{7}{6}c$), to the interface between the two materials ($y = -\frac{c}{6}$) and to the bottom fibres ($y = c-d = \frac{5}{6}c$), the stress distribution represented in Fig. VII.16-c is obtained.

VII.17. Figure VII.17-a represents the cross-section of a beam made of a ductile material with elastic perfectly plastic behaviour. The bending moment M acts in the cross-section. Compute the shape factor of this cross-section.

Resolution

The shape factor is the relation between the plastic moment M_p and the maximum moment in the elastic regime M_e, which is equivalent to the relation between the plastic and elastic section moduli, Z and $\frac{I}{v}$, respectively

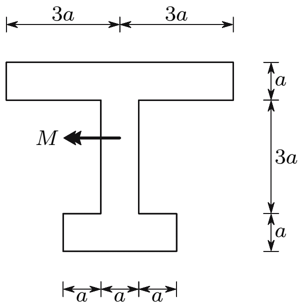

Fig. VII.17-a

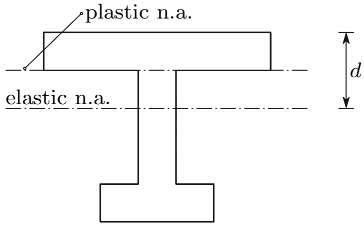

Fig. VII.17-b

$$\varphi = \frac{M_p}{M_e} = \frac{Z}{\left(\frac{I}{v}\right)} \ .$$

Since the cross-section does not have a horizontal axis of symmetry, the neutral axis does not have the same position in the elastic and plastic phase, suffering a displacement during the elasto-plastic phase, from the centroid of the cross-section, to a position which divides the cross-section into two equal areas.

The centroid's position may be defined by the distance d (Fig. VII.17-b), which takes the value

$$d = \frac{6a^2 \frac{a}{2} + 3a^2 (\frac{5}{2}a + \frac{9}{2}a)}{12a^2} = 2a \ .$$

The moment of inertia, with respect to the elastic neutral axis, is

$$I = \frac{6a^2}{12} + 6a^2 \left(\frac{3}{2}a\right)^2 + \frac{a\,(3a)^3}{12} + 3a^2 \left(\frac{1}{2}a\right)^2 + \frac{3a^4}{12} + 3a^2 \left(\frac{5}{2}a\right)^2 = 36a^4 \ .$$

The elastic section modulus is then

$$\frac{I}{v} = \frac{36a^4}{3a} = 12a^3 \ .$$

The plastic section modulus may be computed from (180), with the neutral axis in the position indicated in Fig. VII.17-b (plastic n.a.), yielding

$$Z = \frac{\Omega}{2}\left(y_{G1} + y_{G2}\right) = 6a^2 \left(\frac{a}{2}\right) + 3a^2 \left(\frac{3}{2}a + \frac{7}{2}a\right) = 18a^3 \ .$$

The shape factor is then

$$\varphi = \frac{Z}{\left(\frac{I}{v}\right)} = \frac{18a^3}{12a^3} = 1.5 \ .$$

VII.18. Figure VII.18 represents the cross-section of a bar made of a material with elastic perfectly plastic behaviour with a yielding stress σ_Y. Determine the maximum values of the bending moment which can be applied to this cross-section in elastic and in elasto-plastic regime. What is the shape factor of this cross-section?

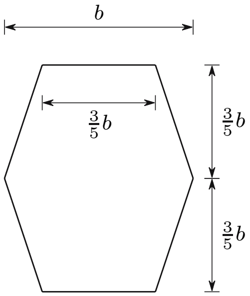

Fig. VII.18

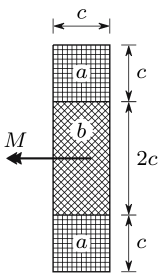

Fig. VII.20

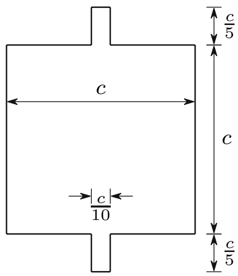

Fig. VII.22

VII.19. A steel wire with a yielding stress ε_Y and rectangular cross-section with dimensions $a \times 3a$ is wound around a cylindrical drum. Determine the minimum diameter the drum can have in order to avoid permanent deformations in the wire. enddescription

VII.20. Figure VII.20 represents the cross-section of a bar made of two materials, a and b, which have elasticity moduli $E_a = 2E$ and $E_b = 3E$ and coefficients of thermal expansion $\alpha_a = 3\alpha$ and $\alpha_b = 2\alpha$. The bar undergoes a uniform temperature increase Δt and supports the bending moment M. Determine:

(a) the elongation of the bar, knowing that it has an initial length l;
(b) the curvature of the bar;
(c) the distribution of stresses in the cross-section.

VII.21. To the bar considered in example VI.15 a bending moment M is applied, whose action axis makes a 30° angle with the vertical.

(a) Determine the curvature of the bar, if only elastic deformations occur.
(b) If M is gradually increased, which of the two materials yields at first? Justify the answer and determine the relation between the maximum stress in this material and moment M, considering only elastic deformations.

VII.22. Supposing that the bar with the cross-section represented in Fig. VII.22 is made of a brittle material with linear elastic behaviour until rupture, ascertain if it is possible to increase its bending strength by removing the two small rectangles, so that a square cross-section $c \times c$ is obtained. Justify the answer.

Answer the same question, supposing now that the material is ductile and that the bar does not undergo cyclic loading.

VII.23. The bar with the cross-section represented in Fig. VII.23 is made of a material with elastic perfectly plastic rheological behaviour, defined by the yielding stress σ_Y and by the elasticity modulus E. Determine:
(a) the bending moment which is needed to plastify the flanges, while the web remains in the elastic regime;
(b) the residual stresses, when this bending moment is removed;
(c) the residual curvature of the bar.

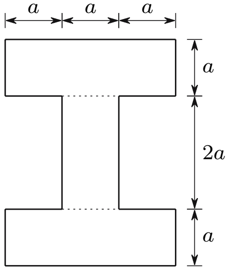

Fig. VII.23

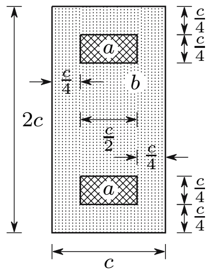

Fig. VII.24

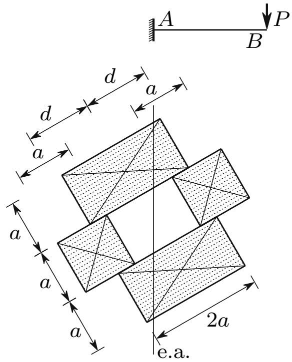

Fig. VII.25

VII.24. The bar whose cross-section is represented in Fig. VII.24 is made of two materials, a and b, with linear elastic behaviour defined by the elasticity moduli $E_a = 4E$ and $E_b = E$, respectively. Determine the maximum stresses in each material caused by a bending moment M with a vertical action axis.

VII.25. The cantilever beam $\overline{AB}$ (Fig. VII.25) is made of a material with linear elastic behaviour and is obtained by assembling four bars, so that the cross-section represented in Fig. VII.25 is obtained. Determine the distance d, so that the displacement of point B has the same direction as the line of action of the force P. Justify the procedure used.

VII.26. Determine and compare the plastic section moduli of the cross-section depicted in Fig. VII.23 and of a rectangular cross-section with the same area and the same height.

Resolution

The cross-section represented in Fig. VII.23 has a area $\Omega = 8a^2$. A rectangular cross-section with the same area and the same height has a width $2a$. The plastic moduli of the cross-section depicted in Fig. VII.23 (Z_1) and of the rectangular cross-section (Z_2) are, respectively

$$Z_1 = 2 \times \left(3a^2 \times \frac{3}{2}a + a^2 \times \frac{a}{2}\right) = 10a^3$$

and

$$Z_2 = 2 \times 4a^2 \times a = 8a^3 \ .$$

We confirm that, even in the case of constant tensile and compressive stresses, the cross-section with less material in the region around the neutral axis has a larger resisting moment, as mentioned in Sect. VII.3 (see Footnote 38).

VII.27. Show that the solution obtained for the pure bending of a prismatic bar made of a material with linear elastic behaviour obeys every condition of equilibrium and compatibility.

Resolution

Considering a reference system, where axis x coincides with the neutral axis and axis z is the centroidal axis of the bar, the solution of the problem may be described by the expressions

$$\sigma_z = \frac{Ey}{\rho};\ \varepsilon_z = \frac{y}{\rho};\ \sigma_x = \sigma_y = \tau_{xy} = \tau_{yz} = \tau_{xz} = \varepsilon_x = \varepsilon_y = \gamma_{xy} = \gamma_{yz} = \gamma_{xz} = 0 \ .$$

These expressions obey the constitutive law of the material ($\sigma = E\varepsilon$). Substituting them in the differential equations of equilibrium (5), we find at once that they are satisfied ($\frac{\partial \sigma_z}{\partial z} = 0$). The same happens with the conditions of equilibrium at the lateral boundary (8, $n = 0 \ \Rightarrow \ n\sigma_z = 0$). In the end cross-sections we have $n = 1$, i.e., $\overline{Z} = n\sigma_z = \sigma_z$, which means that the equilibrium conditions are satisfied only if the bending moments are applied by means of forces distributed as defined by the linear law defining σ_z. If this does not happen, the solution is only valid for points which are sufficiently far from the end cross-sections for Saint-Venant's principle to be considered valid.

The local conditions of strain compatibility (53) are automatically satisfied, since the only non-zero stress (σ_z) is a linear function of coordinate y. In the case of a multiply connected cross-section, the integral conditions of compatibility would have to be verified, which would require the analysis of the displacement functions corresponding to the above strain distribution. This analysis is, however, not presented here (see, e.g. [1] or [4]). We may also conclude that the integral conditions of compatibility are satisfied by the analysis explained in Sect. VII.6.