V

Fundamental Concepts of Strength of Materials

V.1 Introduction

In Chaps. I to IV expressions were developed for the mathematical description of the relations between external forces, stresses, strains and displacements in a body made of a material which may be considered as continuous. The computation of the stresses and strains induced in the body by a given system of external forces, or by imposed displacements, demands the direct or indirect computation of the solution of systems of equations based on those expressions. Generally, we deal with differential equations (note that many of the main expressions presented in Chaps. I to IV are in differential form) with a degree of complexity which depends on the geometry of the body, on the rheological behaviour of the materials it is made of and on the magnitude of the deformations and rotations. For these reasons, analytical solutions are obtained only for those cases where deformations and rotations are sufficiently small to be considered as infinitesimal, the material is isotropic and has linear elastic behaviour and the geometry of the body has a simple analytical description.

Traditionally the solutions were obtained in two ways:

- *Theory of Elasticity* – this science mainly uses mathematical tools to get analytical solutions for the problems of the Mechanics of Materials. Since the differential equations describing those problems generally have a high degree of complexity, only the simpler problems could be solved. Thus, solutions have been obtained for two-dimensional problems with a simple description for the body geometry and for the loading distribution, in rectangular coordinates (bodies with a rectangular or right triangular border under concentrated and uniformly or linearly distributed external forces), or in polar coordinates (bodies with a circular and/or radial border, problems with some axisymmetry conditions in the stress distribution, etc.). Solutions have also been obtained for some three-dimensional problems, by using rectangular, cylindrical and spherical coordinates [4].

The great advantage of the Theory of Elasticity is that it gives analytical solutions, which allow, for example, a simple investigation about the way a solution changes when the parameters included in it change, which cannot be achieved directly with numerical solutions.

The bigger disadvantage is that it yields solutions only for the simplest cases under ideal conditions. Furthermore, as a consequence of the mathematical approach employed, it is not easy to use physical considerations to get approximate generalizations of its solutions for cases where those conditions do not apply exactly (for example, a slight non-linearity of the material constitutive law).

- *Strength of Materials* – this science favours a more physically grounded, phenomenological and praxis oriented approach. Traditionally, its focus has been on the computation of stresses and deformations in the special case of slender members, although other kinds of structures are also analysed. In fact, these cases belong to the class of problems which can be solved without exaggerated use of mathematical formalism and their solutions were developed prior the appearance of powerful numerical tools. The phenomenological approach and the relatively simple geometry of the problems allow the treatment of a broader spectrum of constitutive laws, as some particular cases of non-linear elasticity, plasticity, etc., the consideration of some material discontinuities, as, for example, slender members made of two or more materials and even non-infinitesimal rotations.

Summarizing, we can make the highly simplified statement that the Theory of Elasticity furnishes mathematical solutions for problems whose geometry is relatively complex (two- or three-dimensional problems), but whose material behaviour is the simplest possible, while the Strength of Materials yields physical solutions for problems with a simpler geometry (slender members), but with some incursions into more complex aspects of material behaviour and non-infinitesimal displacements and rotations. These two sciences are complementary. In fact, the first frequently starts from solutions obtained by means of the second, to develop analytical solutions, and the Strength of Materials often uses solutions obtained by the Theory of Elasticity for particular problems, either for testing a simplifying hypothesis, or to investigate the possibility of generalizing some solutions to problems where the starting conditions are only approximately satisfied.

With the appearance of machines for automatic computation – the computer – it became possible to solve algebraic systems of equations with a large number of unknowns, which made the development of a third method possible: the numerical simulation of structures. This method, by discretizing the continuum and thus allowing the transformation of the differential equations into algebraic equations, took only a few decades to become the most powerful tool for solving problems of Solid Mechanics and, more generally, all continuum problems. Of all the computational tools, the Finite Element Method deserves a special reference, since its flexibility and modularity has allowed it to be successfully applied to practically all kinds of problems of Continuum Mechanics.

This second part of the book introduces structural analysis and the theory of slender members, using the approach which is traditionally called Strength of Materials. In fact, despite of the success of numerical analysis, this subject is still a core part of the Engineering Sciences that deal with Solid Mechanics, since it yields a large number of directly applicable expressions for practical problems. These expressions concern the computation of the effects of the axial and shear forces and of the bending and torsion moments in slender members. Furthermore, as a consequence of the physical approach and of the large number of simple exercises which are solved, it develops in the student a greater capacity for intuitively evaluating the way as a structure behaves.

The notions of stress, strain and rheological behaviour are explained again at the beginning of this chapter. This is because the subjects are introduced differently for the Strength of Materials, compared with Solid Mechanics, and because its is intended that the reader of the second part is able to understand it, without a deep study of the subjects dealt with in the first chapters. However, the reader should already have some knowledge about the two-dimensional analysis of a second order tensor, at least, especially in relation to the transposition of reference axes and Mohr's circle, and the computation of reaction forces in statically determinate structures and internal forces in slender members.

V.2 Ductile and Brittle Material Behaviour

The main characteristics of the rheological behaviour of materials are usually investigated by means of simple experimental tests, in which the relations between forces and deformations in a body with appropriate geometry, made of the material to be studied (test specimen) are measured. The one-dimensional tensile test is the most widely used way to study the behaviour of current structural materials, such as metals. This test determines the relation between an axial force N and the corresponding elongation Δl, as represented schematically in Fig. 47-a.

Consider a prismatic specimen with a doubly symmetric cross-section made of mild steel. If the elongation Δl is gradually increased from zero until the value which causes the rupture of the specimen, and the corresponding axial force N is measured, a relation between these two quantities is obtained. This relation may be represented by a diagram, like that in Fig. 47.[1]

The diagram is typical for a ductile material and is characterized by a zone of purely deformation plastic (irreversible) or *yielding zone*, where deformation

[1] The test is carried out with *displacement control*, i.e., by defining a value for the elongation and measuring the corresponding value of the axial force. If *force control* is used instead, the shape of the diagram in the descending zone, where the value of force decreases as the deformation increases (softening), is not correctly captured.

suddenly increases, without a significant increase of the axial force N (line BC). In this diagram distinct zones may be identified. The first corresponds to the straight line OA, where the elongation Δl is completely recoverable and proportional to the axial loading N. Since recoverable deformations are defined as *elastic*, this region is called the linear elastic zone. Usually, the deformations of structural materials are in this zone under service loads.

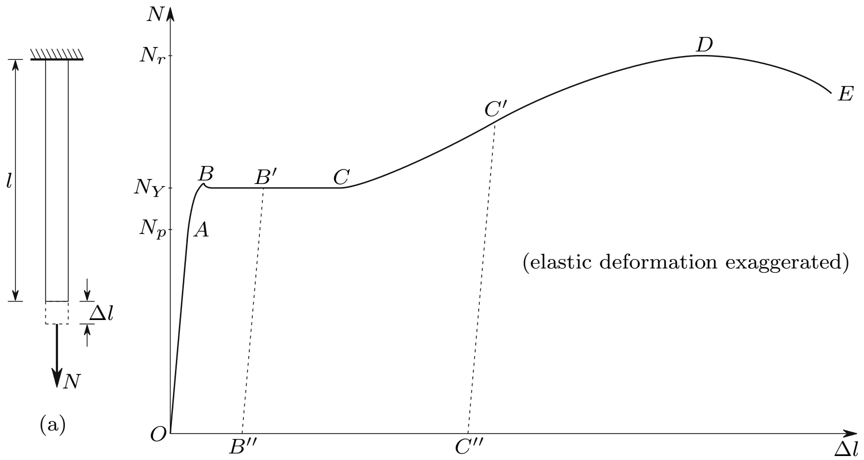

Fig. 47. Force-elongation diagram of mild steel, obtained by means of a one-dimensional tensile test

In the region AB the deformation is still elastic, but there is no proportionality between forces and deformations anymore. Axial force N_p indicates the transition from the linear elastic to the non-linear elastic deformations and is therefore called the *limit of proportionality*. When the loading attains the value N_Y, yielding starts. Region BC is the yielding zone defined above and N_Y is the yielding force. Between these two values (N_p and N_Y) an *elasticity limit* N_e may be defined. This value indicates the maximum value of N, which causes purely elastic deformation, i.e., the maximum value that N can reach, so that the N-Δl diagrams in the loading and unloading phases coincide. In practical terms, the difference between N_p and N_Y is very small so they may be considered to take the same value. In point C the *hardening* of the material starts. In region DE a decrease of the axial force with deformation increase (softening) takes place. This softening is only apparent, since it is a consequence of a reduction of the cross-section (necking) which takes place prior to rupture: in fact, the force per area unit in the necking cross-section increases until rupture (cf. Sect. V.3). If the loading process is stopped at any stage after point B and the axial force is reduced until zero, the N-Δl relation follows a linear path, with the same angle as the initial straight line OA, and a residual deformation remains, as represented by the line $B'B''$. In a subsequent reloading the N-Δl diagram follows the path $B''B'CDE$. If the unloading occurs when the load is already in the hardening region (point C'), the material behaviour is the same: the unloading is linear (line $C'C''$) and the reloading follows the path $C''C'DE$. This means that, in the reloading, the first plastic deformations appear for a higher value of the axial force than in the first loading. This is why the curve $CC'D$ is called the hardening region.

If, instead of a tensile force N, a compressive one is applied, the obtained N-Δl diagram is approximately the same until point C'. Since no necking occurs in compression, the hardening continues indefinitely and no rupture takes place, even for very large deformations. The proportionality and yielding forces take the same value as in the tensile test. This behaviour is characteristic of ductile materials.

In the case of brittle materials as cast iron, concrete, glass, rock, ceramic materials, etc., the obtained force-elongation diagram takes a form of the type represented in Fig. 48. The main differences between the diagrams for brittle and ductile materials are: the linear elastic zone is less defined, i.e., the tangent to the curve decreases steadily until rupture, which occurs with little plastic deformations, and the behaviour under tensile and compressive forces is different. Generally, these materials display more stiffness and strength under compressive loading.

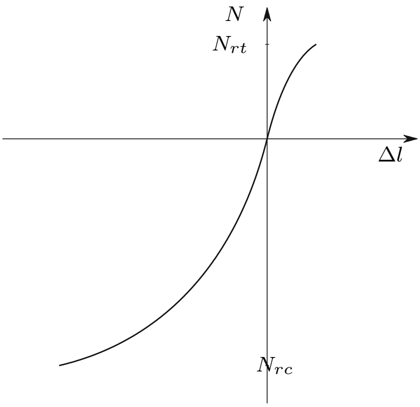

Fig. 48. Force-elongation diagram in a brittle material

In many materials, especially metallic materials, ductility changes with temperature, with less ductility for low temperatures.

V.3 Stress and Strain

In the previous section the relation between axial force and elongation was described. This relation obviously depends on the dimensions of the test specimen. To be more specific, the larger the cross-section area and the smaller the specimen's length, the smaller the elongation will be, for the same axial force N. It is, however, more convenient to express the material properties independently of the specimen's dimensions. This objective may be achieved by means of the *stress* and *strain* definitions. Thus, we may define stress σ as the force per cross-section unit Ω, $\sigma = \frac{N}{\Omega}$[2] and strain ε as the elongation per unit length l, $\varepsilon = \frac{\Delta l}{l}$. It is evident that the stress-strain relation has the same shape as the N-Δl diagram if Ω and l are approximately constant, which happens while the specimen's deformation is small. This is true in the diagrams presented above, with exception of the necking zone (curve DE) in Fig. 47. In the necking cross-section, although the axial force decreases with the deformation, the stress increases.

The coefficient of proportionality E between stress and strain in the linear elastic region (line OA, Fig. 47) is a *rheological parameter* of the material and is called the *longitudinal modulus of elasticity* or *Young's modulus*. In all kinds of steel this parameter takes the value $E = 206 \times 10^9 N/m^2$. In the curved regions of the σ-ε diagram the stress is no longer proportional to the strain. In this case a *tangent elasticity modulus* may be defined: $E_t = \frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon}$. A longitudinal elongation is usually accompanied by a reduction of the transversal dimensions and vice versa. The relation ν between the the transversal and longitudinal strains, ε_t and ε, respectively, multiplied by -1 ($\varepsilon_t = -\nu\varepsilon$), defines another rheological parameter and is called the *Poisson's coefficient* of the material. In steel, as in most metals, this parameter takes a value of 0.3 in elastic deformations and 0.5 in plastic deformations.

If we now consider an inclined section at an angle θ with the cross-section, we can define a stress $T = \frac{N}{\Omega_{inc}}$ (Ω_{inc} is the area of the inclined section). This stress must have the direction of the axial force N, in order to be able to balance it, as represented in Fig. 49. Thus, this stress has a normal component σ_θ (to the inclined section) and a tangential or shearing component τ_θ. Denoting the cross-section area by Ω, we get

$$\Omega_{inc} = \frac{\Omega}{\cos\theta} \Rightarrow T = \frac{N}{\Omega_{inc}} = \frac{N}{\Omega}\cos\theta.$$

The normal and shearing components are then, respectively,

$$\sigma_\theta = T\cos\theta = \frac{N}{\Omega}\cos^2\theta \quad \text{and} \quad \tau_\theta = T\sin\theta = \frac{N}{\Omega}\sin\theta\cos\theta = \frac{1}{2}\frac{N}{\Omega}\sin 2\theta\ .$$

The maximum value of σ_θ clearly occurs for $\theta = 0$ (cross-section). The shearing stress attains its maximum value in an inclined section at a 45° angle with the cross-section, as may be easily verified[3]

[2]In this expression a uniform distribution of the stress in the cross-section is assumed. This hypothesis will be proved in Sect. VI.1).

[3]These conclusions may also be drawn by means of the Mohr circle or from (35) and (36) ($\sigma_x = \frac{N}{\Omega}$, $\sigma_y = \tau_{xy} = 0$, $\theta = 45°$).

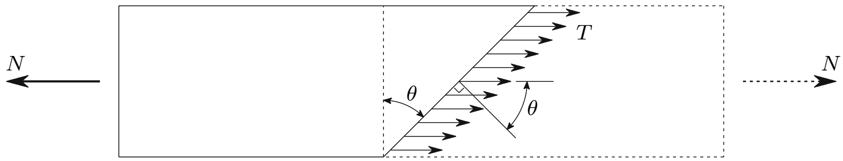

Fig. 49. Stresses in an inclined section

$$\frac{\mathrm{d}\tau_\theta}{\mathrm{d}\theta} = \frac{N}{\Omega}\cos 2\theta = 0 \Rightarrow \theta = \frac{\pi}{4} \ .$$

If, in the one-dimensional experimental test of a ductile material, a specimen with a flat and polished lateral surface is used, careful observation shows lines at a 45° angle with the longitudinal direction, when yielding takes place. These lines, called Lüder-Hartman's lines, have the directions corresponding to the maximum shearing stress and indicate that the plastic deformation is mainly a shearing deformation. This explains the same material behaviour observed in tensile and compressive experiments, especially for the yielding stress $\sigma_Y = \frac{N_Y}{\Omega}$. In fact, there is no physical difference between the shearing deformation in compressive and tensile tests. In brittle materials deformation and rupture are mainly influenced by cohesion, contact and friction forces between the material particles. These forces are obviously different under tensile and compressive loadings. Concrete is one example. In this material the tensile strength is mainly influenced by the cohesion properties of the cement paste, while in compression the properties of the aggregates play an important role, because of the contact and friction forces between the rock particles.

V.4 Work of Deformation. Resilience and Tenacity

When a body deforms under the action of external forces, their points of application suffer displacements and the forces do work. Physics defines the work of a constant force in a straight displacement as the scalar product of the vectors defining the force and the displacement of its point of application.

In the example of the prismatic steel bar under a tensile axial force (Fig. 47-a) the work W_0 done by force N, for a given elongation Δl_i, may be given by the expression

$$W_0 = \int_0^{\Delta l_i} N(\Delta l)\,\mathrm{d}(\Delta l) \ . \tag{112}$$

The integral is necessary because the force N is not constant, but varies during the deformation as a function of Δl, as shown in Fig. 47. In fact, the definition of work given above is valid only for an infinitesimal displacement $\mathrm{d}(\Delta l)$. By introducing the definitions of stress and strain into (112), the work W done per volume unit may be obtained

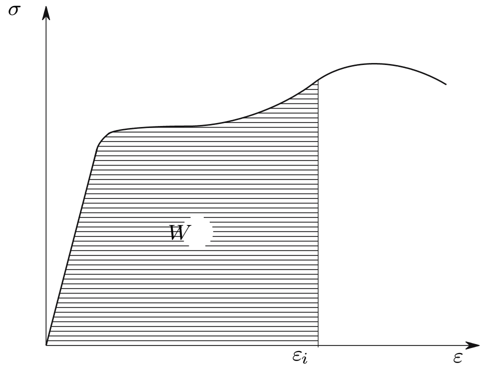

Fig. 50. Work done per volume unit in the deformation ε_i

$$W_0 = \int_0^{\varepsilon_i} \underbrace{\sigma\Omega}_{N} \overbrace{l\,\mathrm{d}\varepsilon}^{\mathrm{d}(\Delta l)} = V \int_0^{\varepsilon_i} \sigma\,\mathrm{d}\varepsilon \;\Rightarrow\; W = \frac{W_0}{V} = \int_0^{\varepsilon_i} \sigma\,(\varepsilon)\;\mathrm{d}\varepsilon \ ,$$

where $V = \Omega l$ represents the volume of the bar. This quantity takes the same value as the area under the stress-strain diagram, as represented in Fig. 50.

From Physics we know that for the production of a given amount of work an equal amount of energy U must be spent. If the strain ε_i is in the elastic region of the stress-strain diagram, this energy is totally stored by the deformed material as *elastic potential energy*. This energy is recovered during unloading. However, if the strain is larger than the value corresponding to the elasticity limit, the energy is partly dissipated (transformed in heat) during the plastic deformation. In this case, the elastic potential energy is only a fraction of the work done in the deformation and is given by the expression

$$U_e = \int_{\varepsilon_r}^{\varepsilon_i} \sigma'\,(\varepsilon)\;\mathrm{d}\varepsilon \ ,$$

where ε_r is the residual strain and $\sigma'\,(\varepsilon)$ is the stress corresponding to the strain ε in the unloading. As stated above, in the unloading of a steel bar the stress-strain relation is linear, even when the stress is larger than the proportionality limit. The dissipated and potential elastic parts of the deformation energy per volume unit (energy density) are represented in Fig. 51.

The amount of energy per volume unit needed to start plastic deformation, is called *resilience*. The amount of energy per volume unit needed to cause rupture is called *tenacity*. These quantities play important roles in the shock-absorbing capacity of a structure. Ductile materials have high tenacity, as opposed to brittle materials, which display low tenacity. In a ductile material the tenacity is much larger than the resilience, while in brittle materials these quantities are similar since the plastic deformations are small. Ductile materials usually have a much higher tenacity than brittle materials.

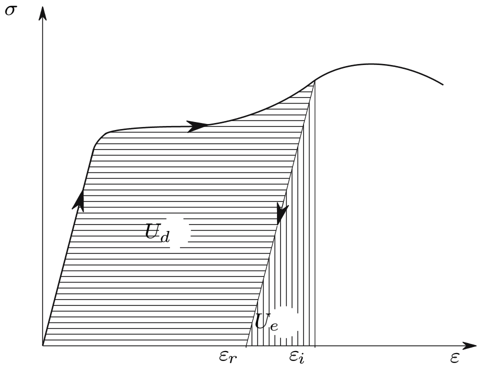

Fig. 51. Dissipated energy (U_d) and elastic potential energy (U_e) in the deformation of a steel bar

V.5 High-Strength Steel

As described in Sect. V.2, if a mild steel is deformed until the strain reaches the hardening zone and the loading is subsequently removed, this steel displays a higher elasticity limit in a later reloading, i.e., a larger linear elastic region in the stress-strain diagram. Since the elasticity limit is usually considered as the limit load under service conditions, the loading capacity of a steel may be increased in this way. This method of increasing the elasticity limit of a steel by means of pre-deformation, with the objective of increasing the admissible stress, is called *strain hardening.* Obviously this process causes a loss of tenacity, since part of the energy dissipation capacity of the material is consumed by the pre-deformation in the hardening process. Conversely, the resilience is increased since the elasticity limit is higher and the elasticity modulus remains unchanged, as depicted in Fig. 52.

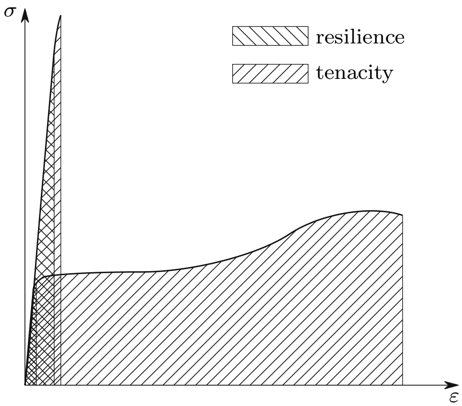

Fig. 52. Resilience and tenacity of a mild steel and of a high strength steel

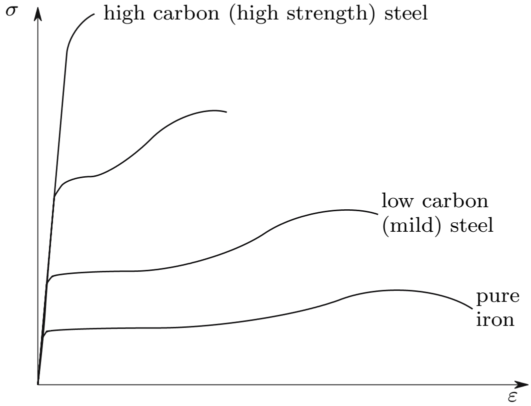

Fig. 53. Variation of the stress-strain diagram with the carbon percentage

The hardened steel is therefore more brittle. Furthermore, the yielding stress for a compressive axial force decreases in a steel bar that is hardened by means of a tensile axial force. This is why the strain hardened bars used in reinforced concrete are pre-deformed by torsion, which increases the tensile and compressive limits of elasticity to the same extent.

The elasticity limit of steel can also be increased by increasing the quantity of carbon added during the metallurgical process of steel production. This process, called *natural hardening*, has the advantage of not disturbing the isotropy of the material. Just as in the strain hardening process, the capacity of plastic deformation (ductility) decreases as the elastic limit increases, as indicated in Fig. 53.

High strength steels do not have a yielding zone where only plastic deformations take place (cf. Figs. 52 and 53). As a consequence, the onset of the plastic deformation is not clearly shown by the stress-strain diagram. For this reason, the elastic limit is defined with the help of a convention, which states that the elastic limit is the stress which causes an unrecoverable strain with the value $0.2\% = 0.002$ ($\sigma_{0.2}$, Fig. 54).

V.6 Fatigue Failure

In structural elements subjected to rapidly changing internal forces, such as bridge elements under vibration loads caused by traffic or wind loads, machine parts performing cyclic motions, aircraft structural elements, etc., *fatigue failure* may occur. This kind of failure usually takes place for substantially lower stresses than in a monotonically increasing loading, as in the experimental tests described in Sect. V.2.

The behaviour of structural materials under the action of loads varying with great frequency is investigated by means of fatigue tests. In these tests a

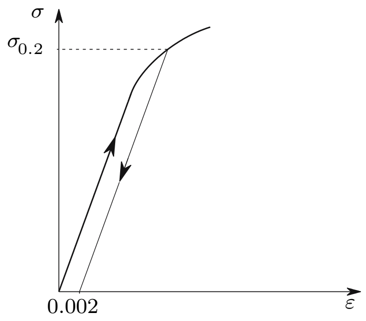

Fig. 54. Conventional elastic limit

specimen undergoes a generally one-dimensional loading which causes stresses varying cyclically between two given values, usually a compressive and a tensile stress. The loading cycles are repeated a large number of times until rupture takes place. The higher the given stress values are, the smaller is the number of cycles needed to cause rupture. In the simplest and commonest of these experimental tests equal compressive and tensile stresses ($\sigma_{\min} = -\sigma_{\max}$) are cyclically applied.

If, for example, several specimens of steel are tested under the same stress level, the results are generally found to be widely dispersed, i.e., the number of cycles necessary to cause rupture varies substantially from one test to another. However, if the number of tests is sufficiently high and the experiments are performed for different stress levels, results are obtained which may be approximated by a curve with the shape depicted in Fig. 55 [3]. This curve tends asymptotically for a stress value σ_f, which means that for a sufficiently low stress no fatigue failure occurs, irrespective of the number of loading cycles. This stress value is called *fatigue limit stress.* In iron-carbon steel this stress is approximately half the rupture stress, which is lower than the elastic limit stress. In other more ductile materials like lead, copper, zinc or pure iron the fatigue limit stress is higher then the elastic limit [3].

The value of the fatigue failure stress depends strongly on the specimen's surface, with a higher failure stress obtained when the surface is polished. This is a consequence of the fact that rupture is initiated by a crack. The crack starts at the surface and propagates to the interior of the specimen as the cyclic loading goes on, until the uncracked part of the specimen's cross-section becomes too small to carry the applied loading and failure takes place. The crack starts at a lower stress in the unpolished surface, since its imperfections cause higher stress concentration, as will be seen later (Sect. VI.9). This crack initiation mechanism explains the above-mentioned dispersion of the number of cycles required to cause fatigue failure for the same stress level. It is also due to the sensitivity to the imperfections that fatigue failure takes place for fewer

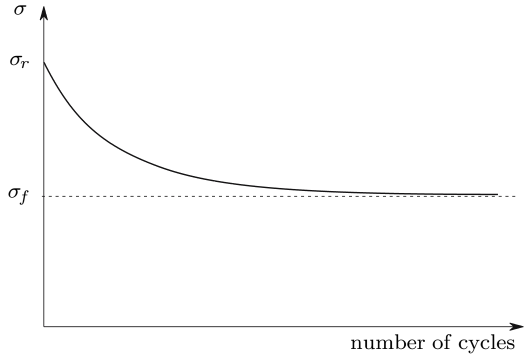

Fig. 55. Relation between the maximum stress σ and the number of loading cycles needed to cause rupture in fatigue tests with $\sigma_{\min} = -\sigma_{\max}$

cycles (or lower stresses) in the case of corrosive environments, since corrosion initially affects the surface of the material and causes larger imperfections.

From these considerations we conclude that fatigue failure has the same character as brittle failure, since it takes place without being preceded by large plastic deformations. It is a dangerous kind of failure, since there are no visible signs of the fatigue crack before rupture. Sensitivity to imperfections is also a characteristic of brittle failure. As will be seen later (Sect. VI.5), ductile structures are safer than structures made of brittle materials, since the capacity for plastic deformation allows a redistribution of internal forces, which automatically optimizes the distribution of internal forces until failure occurs. However, if there is a risk of fatigue failure, the advantages of structures made of ductile materials are lost, since fatigue-induced rupture is of a brittle nature, even when it occurs in ductile materials.

V.7 Saint-Venant's Principle

Saint-Venant's principle states that *in a body under the action of a system of forces which are applied in a limited region of its boundary, the stresses and strains induced by those forces in another region of the body, located at a large distance from the region where the forces are applied, do not depend on the particular way the forces are applied, but only on their resultant.* This "large distance" may be considered, in most cases, as the largest dimension of the region where the forces are applied.

This principle does not have a formal, general and exact demonstration as yet, but it has been verified in so many cases, both experimentally and numerically, that it is accepted as valid by the generality of authors on this subject. It is a very useful principle, since complex force systems may be reduced to their resultants, which substantially simplifies and reduces the computation effort in practical problems. Besides, it is a very helpful tool in

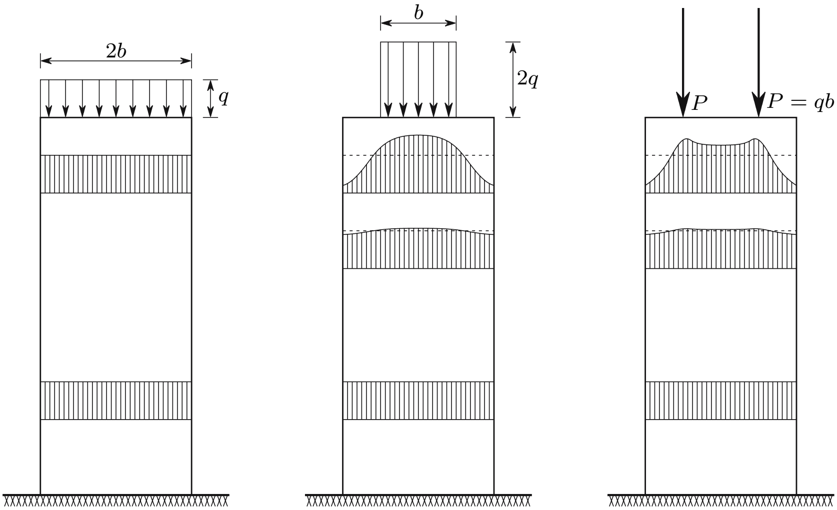

Fig. 56. Stress distribution in different cross-sections of a prismatic bar, caused by three force systems with the same resultant

the theoretical development of solutions for problems in Theory of Elasticity and Strength of Materials, as will be seen later.

As an example, let us consider the prismatic bar represented in Fig. 56 under the action of three systems of forces with equal resultants: the stresses at a grater distance than the transversal dimension $2b$ from the upper end of the bar may be accepted as equal in the three cases.

This principle is also valid in the cases of non-isotropic materials, non-linear material behaviour, plastic and viscous deformations and material heterogeneity. Furthermore, the validity of this principle is not limited to small deformations.

V.8 Principle of Superposition

In structures where the applied loading causes deformations and rotations which are sufficiently small to be considered as infinitesimal and where the rheological behavior of the material is linear (i.e., the proportionality limit stress is not exceeded), the relation between the intensity of a force and the effects it causes (stresses, strains, displacements) is linear, i.e., the effects are proportional to the intensity of the force which causes them.

The increase of displacement corresponding to the increase of the force which causes it is therefore independent of the intensity of the force before the increment. Furthermore, as the geometry of the structure, after the application of loading, is only infinitesimally different from the undeformed configuration, the initial geometry of the structure may always be used, regardless of the existence or not of other previously applied loads (geometrical linearity). Under these conditions (material and geometrical linearity) the *Principle of Superposition* is valid: *the effect of the application of a force to a structure is independent of the existence or not of other forces applied to the structure.* As a consequence, the effects of applying different loading systems to the structure may be computed separately and added.

This principle has a simple analytical demonstration. To this end, it suffices to take into consideration that, in the case of infinitesimal deformations and rotations, all the conditions relating applied forces, stresses, strains and displacements are linear. These conditions are:

- the differential equations of equilibrium (5),
- boundary balance equations (8),
- the relations between strains and displacements (50),
- the local (53) and integral conditions of compatibility (the integration of strains is a linear operation),
- the constitutive law (74), (75), (79), (81), (83) and (85).

As a consequence, the sum of two sets of forces, stresses, strains and displacements also obeys these conditions.

A temperature variation is also a kind of loading whose effect is generally defined by a linear law: the strain induced by a temperature variation is generally proportional to the value of that variation, with the thermal expansion coefficient playing the role of the proportionality constant. The effect of the temperature variation may be quantified by adding another element to the expressions, allowing the computation of the longitudinal strains for given stresses (e.g., (74)). Taking, for example, the longitudinal strain in direction x, we have $\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] + \alpha\Delta T$, where α is the coefficient of thermal expansion.

From a physical (and practical) point of view we may make the simplified statement that the effect of the application of a force to a supported body depends only on two components: the constitutive law of the material and the geometry of the body.

If the constitutive law is linear and if we can admit that its rheological behaviour does not depend on temperature, the first component does not change with the application of forces or with a temperature variation.

If, in addition, the deformations are small enough for it to be acceptable that the geometry of the body does not change, the second component does not change either. Thus, the effect of the force is independent of the previous application of forces and of temperature variations, which leads to the principle of superposition as stated above.[4]

[4]It must be noted that possible interactions between the deformations and the internal forces caused by the external loads are not taken into account in these considerations. This interaction may occur in presence of infinitesimal deformations and cause structural instability. As it will be seen later (Chap. XI), in the analysis of this phenomenon the principle of superposition is not valid.

This principle has many useful applications, both from a practical point of view, since it allows the separate consideration of single loading cases and any linear combination of their effects, and in theoretical analysis, as, for example, in the demonstration of energy theorems for linear elastic structures. It must, however, be noted that *it is only valid for structures with linear elastic force-displacement behaviour.*

V.9 Structural Reliability and Safety

V.9.a Introduction

A structure must resist all the loadings that will act on it, in its expected life and conditions of use. In the context of structural reliability "resist" means that the structure must be able to carry out safely all the functions for which it is designed.

We consider that the structure ceases to be able to carry out those functions when a *limit state* occurs. Two kinds of limit states are usually considered: *ultimate limit states* and *serviceability limit states.*

The first are associated with the rupture, collapse or failure of the entire structure or parts of it, such as failure of structural elements caused by material rupture, structural instability of compressed members, fatigue failure, displacements leading to loss of supports, etc.

The serviceability limit states include everything that may cause structural malfunction, although not inducing collapse. Examples of serviceability limit states in Civil Engineering structures include: excessive deformation of a structural frame which may cause cracking in non-structural walls, excessive vibration of a pedestrian bridge which may cause discomfort to the users, too large cracks in reinforced concrete members which may lead to corrosion of the reinforcing steel bars, etc.

V.9.b Uncertainties Affecting the Verification of Structural Reliability

When verifying the reliability of a structure it is not possible to use a totally deterministic approach, since the quantification of the problem data – the different kinds of loading (actions) and the rheological properties of the materials – is always affected by some uncertainty, which makes it impossible to define exact values. The main sources of uncertainty are:

– uncertainty in the *value of the actions*: all actions are characterized by a smaller or larger dispersion in relation to their mean value; besides, it is often impossible to define limiting values. Examples of loadings are those caused by wind, snow, temperature variation, earthquakes, etc. Furthermore, it is generally not economically defensible to use the limiting values

of the actions, when they exist, since the probability of their occurrence is generally very small;
- *statistical dispersion of the rheological properties of structural materials*, especially rupture stress, elasticity modulus, resilience, tenacity, etc.;
- uncertainty introduced by the *dimensional tolerance* of pre-moulded structural elements;
- *execution imperfections*, especially in Civil Engineering structures as in concrete elements for example, introduce uncertainty into the geometrical dimensions, on the position of the reinforcing bars, the verticalness of columns, etc.;
- uncertainty introduced by the *methods of analysis and computation*, since they are always based on idealized models, resulting from *simplifying hypotheses*, like the consideration of a linear stress-strain relation and of the undeformed geometry of the structure, non-consideration of time-dependent effects, such as viscous deformation, etc.

As a consequence of these random factors, the verification of structural reliability necessarily has a probabilistic basis, since a zero probability of failure can never be guaranteed. The criteria of dimensioning and safety evaluation are based on the definition of a sufficiently low probability for the structure to reach a limit state. For ordinary Civil Engineering constructions the following values are considered acceptable [3]:

- serviceability limit states $< 5 \times 10^{-2}$;
- ultimate limit states $< 10^{-5}$.

In the case of structures with special safety requirements, like large dams or nuclear power plants, the maximum values of these parameters are much lower.

The above considerations lead to the conclusion, that a structure is safe if the probability of it reaching a limit state is sufficiently low.

V.9.c Probabilistic Approach

The probabilistic approach consists of the direct computation of the probability that the structure reaches a limit state. Thus, from a theoretical point of view, we can accept that either the actions and the resistance properties of the structure (strength) may be represented by parameters A and R which are described by probabilistic density curves $f_A(A)$ and $f_R(R)$, as represented in Fig. 57.

The probability of the simultaneous occurrence of a value of the action A within the interval $\mathrm{d}A$ and of a value of the strength R within the interval $\mathrm{d}R$ ($\mathrm{d}A$ and $\mathrm{d}R$ are infinitesimal quantities) is given by

$$\mathrm{d}\,(\mathrm{d}P_f) = f_A\,\mathrm{d}A\,f_R\,\mathrm{d}R\ .$$

The probability of strength R taking a lower value than action A, or, more precisely, of having the action in the interval $\mathrm{d}A$ and inferior values of the

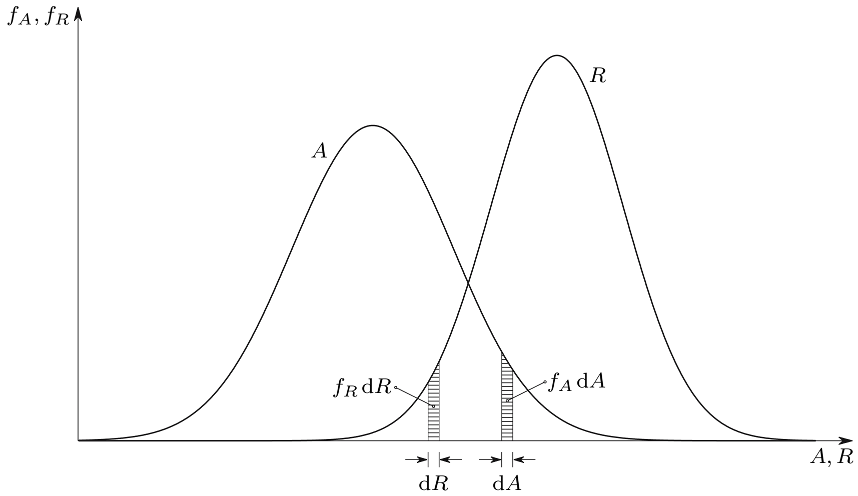

Fig. 57. Probabilistic density curves for action A and for strength R

strength R, may be obtained by integrating the previous expression for all values of $R < A$, which yields

$$\mathrm{d}P_f = f_A\,\mathrm{d}A \int_{R=0}^{A} f_R\,\mathrm{d}R \ .$$

By integrating this expression for all possible values of the action, we get

$$P_f = \int_{A=0}^{\infty} \mathrm{d}P_f = \int_{A=0}^{\infty} f_A \int_{R=0}^{A} f_R\,\mathrm{d}R\,\mathrm{d}A \ .$$

This expression represents the probability of the action exceeding the strength, that is, the probability of the structure reaching a limit state.

Using this methodology to verify the reliability of a structure is, however, not easy in practice, since it is generally very difficult and laborious to define and combine the multiple laws of probabilistic distribution for actions and strength parameters for a particular structure. In the practical verification of the reliability of ordinary structures, therefore, a *semi-probabilistic approach* is used instead, as described in the next sub-section.

V.9.d Semi-Probabilistic Approach

The semi-probabilistic approach is based on the definition, with a probabilistic basis, of nominal values for the actions and strength parameters, so that a sufficiently low probability of failure is guaranteed, without the need for the explicit computation of this probability.

The first step consists of defining *characteristic values* for the actions and strength parameters.

For actions these values are defined as values with a very low probability of being exceeded (upper quantile of the probabilistic density curve), except in the case of permanent actions with an advantageous effect on the safety of the structure. In the latter case the lower quantiles are used (values which are exceeded with a very high probability). However, in the case of permanent actions with low dispersion values, i.e., actions with close upper and lower quantiles, as the self-weight of structural materials, the mean value may be used. This considerably simplifies reliability verification, since it is not necessary to distinguish between permanent actions with beneficial and adverse effects on the structural safety.

For the strength parameters only the inferior quantiles of the probabilistic density curve are used as a rule (values which are exceeded with a very high probability). However, cases may be imagined where the failure of a structural element might be beneficial for the global failure safety. For these elements the use of the upper quantiles would be the logical choice.

Common values for the probabilities corresponding to the lower and upper quantiles are 5% and 95%, respectively. Generally, the values corresponding to these quantiles are defined in the official standards relating to the different types of structures and structural materials.

The second step consists of defining *nominal values* which are obtained from the characteristic values by multiplying them by *partial factors*.

In the case of actions, these factors take into account the probability of the characteristic values being exceeded, the reduced probability of all the actions present in a given loading case simultaneously reaching their characteristic values, the probability that the distribution of external forces resulting from a particular action (wind, for example) may be different from the assumed distribution, etc.

In the case of the strength parameters, the partial factors are intended to cover the reduction of the material's strength due to accidental material defects, the reduction of the strength parameters with the time (aging), small time-dependent deformations, the simplifying hypotheses used in the definition of the material's constitutive law, etc.

V.9.e Safety Stresses

Traditionally, the structural safety, used to be verified on the basis of *safety stresses*, especially in the fields of Civil and Mechanical Engineering. This method has been gradually abandoned and replaced by the semi-probabilistic approach. However, a short description of it is entirely justified, both for historical reasons and because it is still used.

The safety stress method has the same probabilistic basis as the semi-probabilistic approach, since the safety stresses are defined on the basis of the same characteristic values for the material strength parameters. The safety verification is performed by computing the stresses induced by the same characteristic values of the actions, which must not exceed the characteristic value of the strength (the yield stress in ductile metals), multiplied by a *safety coefficient*. This parameter plays the same role as the partial factors for the actions and strength parameters, simultaneously, in the semi-probabilistic approach.

Obviously, this method leads to the same degree of safety as the semi-probabilistic approach only if the structure behaves linearly until it reaches the limit states: in this case, multiplying the actions by a factor leads to the same result as dividing the allowable stress by the same factor. However, as seen earlier, a linear stress-strain relation is generally only acceptable in the initial loading phase and not until the limit states. For this reason, the results yielded by the semi-probabilistic approach are generally better, since the increase in safety due to multiplying the actions by a factor, whose objective is to guarantee that the structure resists a larger loading than that expected, is not "distorted" by the non-linear character of the relation between the external forces and the stresses in a close to the limit state loading situation.

Furthermore, treating the different actions separately allows different factors to be considered for each one, in accordance with the degree of uncertainty associated with it. As an example, let us consider two actions: the self-weight of the structure of a building and the wind acting on it. The uncertainty associated with the self-weight is very low, since, once the characteristic value of the density of the material has been defined, the computation of the corresponding internal forces is not affected by significant uncertainties. In the case of the wind action, on the other hand, the probabilistic analysis leading to the characteristic value takes only the statistical dispersion of the wind velocity into account, with a large degree of uncertainty remaining in relation to the distribution of pressures caused by a wind with that velocity on the surface of the building. This is why it is advisable to use a larger partial factor for the wind than for the self-weight.

V.10 Slender Members

V.10.a Introduction

As mentioned in Sect. V.1, the relations between external forces, stresses, strains and displacements are generally complex. The degree of complexity depends on two components: the rheological behaviour of the structural material and the geometry of the structure. For this reason, analytical solutions for these relations are only possible if both components have particularly simple forms, such as isotropy and linearity of the constitutive law, and a geometry (and loading) with a simple description in a given reference system (rectangular, spherical, cylindrical or polar coordinates). If the structure, or structural component, does not obey these conditions, the solution must be obtained numerically using, for example, the finite element method.

Slender members are an exception, since in these structural elements it is possible to find relatively simple analytical relations between the internal forces which act symmetrically in relation to the cross-sections (constant axial force and bending moment) and the corresponding stresses, especially if the stress-strain relation is linear. For the shear force and torsional moment this relation is not so simple, except for particular cross-section shapes: thin-walled sections under shear force and closed thin-walled and circular sections under torsion.

In Chaps. VI to VIII and X, these cases, where the solution may be considered as exact, are analysed and approximate solutions for other cases are indicated: cross-sections with a symmetry axis under shear force and rectangular and open thin-walled cross-sections under torsion.

Slender members are very often used as structural elements in the fields of Civil Engineering (structures in buildings and bridges contain very often slender members), Mechanical Engineering (many machine parts may be analysed as slender members) and Aeronautical Engineering (the wings of gliders and low speed airplanes, for example, may be considered as slender members). Structures made of slender members are called *framed structures*.

V.10.b Definition of Slender Member

A slender member is a bar-shaped body, i.e., a three-dimensional body, in which one dimension (the length of the bar) is considerably larger than any of the other two (at least five times). More precisely, a slender member may be understood as a solid body generated by a plane geometrical figure, when it moves along a straight line (or a curved line with a large curvature radius, in comparison to the dimensions of the figure), remaining perpendicular to that line. The shape and dimensions of the plane figure – which represents the cross-section of the slender member – may change during that motion, but only gradually. The discontinuities corresponding to a sudden change of the cross-section, or to a corner of the longitudinal line may be regarded as a link between two slender members. The theory of slender members is not valid in the region around these singularities.

Only prismatic bars are considered in the development of the theory of slender members. The generalization of the theory to curved bars or to members with a variable cross-section is only possible for bars with a large curvature radius (as compared with the dimensions of the cross-section), or for a gradually varying cross-section (cf. Sects. VI.7, VII.8 and VIII.5).

V.10.c Conservation of Plane Sections

The cross-sections of prismatic bars under the action of a constant axial force and a constant bending moment remain plane and perpendicular to the axis of the bar during the deformation. In order to demonstrate this statement, let us consider a piece of the bar, whose ends are sufficiently far from the ends of the bar and from the sections where the external forces are applied for Saint-Venant's principle to be valid (Fig. 58).

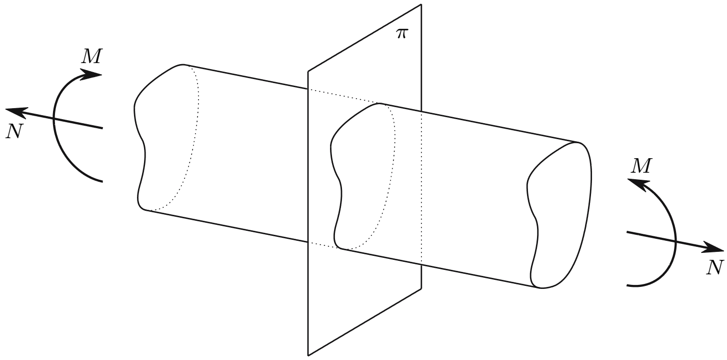

Fig. 58. Symmetrical internal force resultants in a piece of a prismatic bar

Plane π (Fig. 58) is a symmetry plane, both in relation to the geometry of the piece, and in relation to the applied forces (N and M). If, in addition, the material is isotropic, or has, at least, a rheological plane of symmetry parallel to plane π, then the problem is completely symmetric in relation to that plane.

Using the symmetry principle, we conclude that the deformation of the piece must also be symmetric in relation to plane π. Thus, the points of the piece which are on the plane π, will remain there after the deformation, which means, that this cross-section remains plane. The symmetry of the deformation also leads to the conclusion that, in an infinitesimal neighbourhood of the plane, the bar axis (and any longitudinal axis parallel to it) remains perpendicular to plane π. Furthermore, by choosing the piece of prismatic bar properly, any cross-section may be considered as the middle section of a piece. Thus, the above conclusions are valid for any cross-section which is sufficiently far from the above-mentioned singularities. Therefore, the following conclusion may be drawn: *in a prismatic bar under constant axial force and constant bending moment, the cross-sections remain plane and perpendicular to the axis of the bar during the deformation.*

This statement was formulated as an hypothesis by J. Bernoulli in 1705, [3], for the case of bending and is still known by his name in the literature on Strength of Materials. With the demonstration above, it may be considered as a law, which *is valid independently of any considerations about the material properties, with the exception of the symmetry considerations. It is also independent of the size of the deformation.* However, it not valid for non-symmetrical internal force resultants, such as the shear force, torsional moment and varying axial force and bending moment.