III

The Strain Tensor

III.1 Introduction

When the material points inside a solid body or a liquid mass suffer a displacement, this may be a consequence of a rigid body motion or of a deformation. Forces are not necessarily involved in a rigid body motion, unless the displacement is accompanied by acceleration. On the contrary, the deformation is almost always a consequence of internal forces. Other causes may be a temperature variation or similar phenomena, like the retraction of a concrete mass during the curing process.

In Solid Mechanics consideration of the deformation associated with the displacement field is generally unavoidable, since, unless the case under consideration fits into the rare category of fully statically determinate problems, the way the material deforms influences the way the internal forces are distributed inside the body.

When the validity of the continuum hypothesis is accepted, the internal forces may be defined by the stress tensor, as explained in the previous chapter. In the same way, as mentioned in Chap. I, the deformations of a continuous material may be defined independently of the geometrical dimensions of the continuum, by means of the strain definition. If these strains are defined appropriately, they define a symmetrical second order tensor, with exactly the same mathematical characteristics as the stress tensor. This chapter analyses both the properties of this tensor and the relations of its components with the functions that represent the displacement field describing the motion of the material points.

III.2 General Considerations

The deformation caused in a body by external forces or other actions generally varies from one point to another, i.e., it is not homogeneous. In fact, a homogeneous deformation is rare. It occurs, for example, in a body with isostatic supports under a uniform temperature variation or in a slender member under constant axial force.

A non-homogeneous deformation may by more clearly understood by imagining small line segments in several places of the body, which, before the deformation, have the same infinitesimal length, $\mathrm{d}s$, and are parallel. As a rule, the deformation causes various rotations and elongations in the different line segments, as represented schematically in Fig. 16.

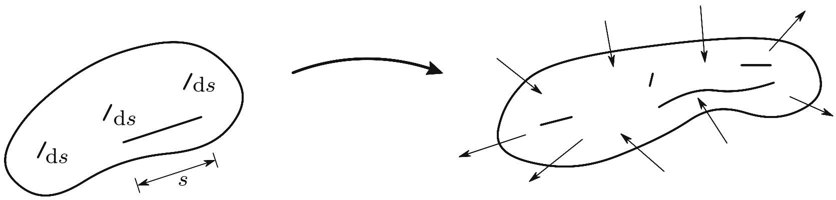

Fig. 16. Non-homogeneous deformation of a body

If the deformation is homogeneous, however, i.e., if it does not vary from point to point, the elongation and the rotation are the same all along the line segments, which means that two parallel straight lines of equal length (now not necessarily infinitesimal) remain straight, parallel and with the same length after the deformation. As a consequence, a homogeneous deformation transforms triangles into triangles, rectangles into parallelograms, tetrahedra into tetrahedra and rectangular parallelepipeds into, generally non-rectangular, parallelepipeds (Fig. 17).

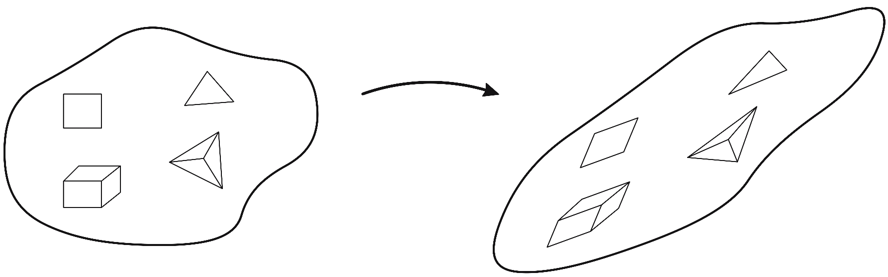

Fig. 17. Homogeneous deformation of a body

From these considerations we conclude that a homogeneous deformation may be fully defined by the six quantities that are required to define the shape and dimensions of a non-rectangular parallelepiped (e.g. the length of the three sides and the three independent angles between non-parallel sides) or of a tetrahedron (e.g. the length of its six sides). In the two-dimensional case the three quantities needed to define a parallelogram or a triangle, are enough.

As mentioned above, the deformation is not usually homogeneous, but varies from point to point. But it may be treated as such *if we limit the deformation analysis to an infinitesimal neighbourhood of a point.* This statement may be easily proved by developing in series the functions that define the coordinates of the material points after the deformation [1].

A physical visualization is, however, more indicative. To this end, let us consider the shapes which result from the non-homogeneous deformation of a rectangle, a triangle and a rectangular parallelepiped (Fig. 18).

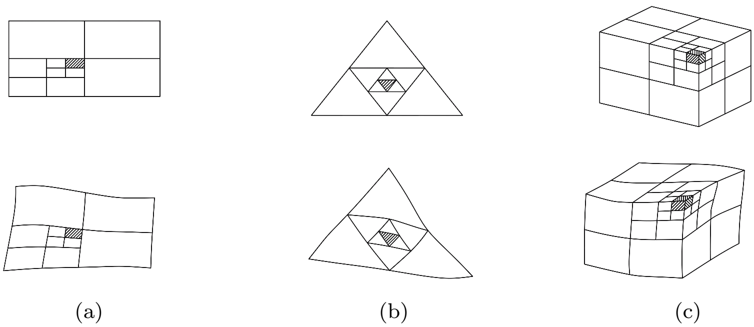

Fig. 18. Homogeneous deformation of an infinitesimal region

The non-homogeneous deformation results in the initially simple geometrical shapes transforming into complex shapes, which cannot be described by means of a reduced number of parameters. However, by subdividing the initial geometrical shapes into others of the same type, we find that the finer the subdivision, the closer the deformation gets to a homogeneous deformation. Ultimately, when the dimensions of the smallest shape go to zero, the deformation is homogeneous, since we can accept that the shaded rectangle in Fig. 18-a has become a parallelogram, the shaded triangle in Fig. 18-b remains a triangle and the rectangular parallelepiped with shaded faces in Fig. 18-c changes into a (non-rectangular) parallelepiped. From these considerations, we conclude that the definition of the *state of deformation* in an infinitesimal neighbourhood around a point needs six parameters, and these are the six quantities necessary to define a homogeneous deformation in the three-dimensional case (or three, in the two-dimensional case).

The above considerations are valid irrespective of the size of the deformation. However, as will be seen later, the expressions which relate the functions describing the displacement of the material points with the strain may be greatly simplified if the deformations and rotations are sufficiently small to be considered as infinitesimal quantities. Furthermore, the restriction on infinitesimal deformations and rotations allows the superposition of the strains associated with different displacement fields.

III.3 Components of the Strain Tensor

The general considerations discussed in the previous section will now be quantified by using a rectangular Cartesian reference frame, xyz. It will be seen later that, in this reference system, the strain tensor has components, which, for infinitesimal deformations, correspond to the elongation per unit length of line segments having the direction of the reference axes, and to half the angular variation of what were initially right-angles between these line segments (three pairs). The three elongations – the *longitudinal strains* – and the three angular variations – the *shearing strains* – are the six quantities necessary (and sufficient) to define the state of deformation around a point.

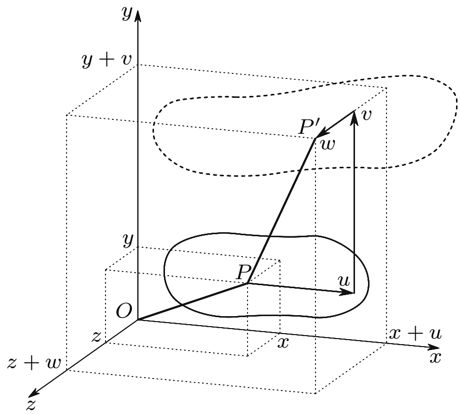

Fig. 19. Displacement of a material point P inside a body: —— before the deformation; ------ after the deformation

Then the initial position of the material points of the body is described by the coordinates x, y, z of the generic point P and its displacement is defined by vector $\overrightarrow{PP'}$ with components u, v, w in the reference directions x, y, z, respectively (Fig. 19). The position of the point after the deformation is therefore given by the coordinates $x + u$, $y + v$, $z + w$. If the material is continuous before and after the deformation, the functions $u(x, y, x)$, $v(x, y, z)$, $w(x, y, z)$ are continuous functions of the position coordinates of the body before the deformation, x, y, z.

Now, let us consider a straight line of infinitesimal length $\mathrm{d}x$, which is parallel to axis x in the undeformed configuration and is defined by the two close points $P_0(x_0, y_0, z_0)$ and $P_1(x_0 + \mathrm{d}x, y_0, z_0)$, as represented in Fig. 20.

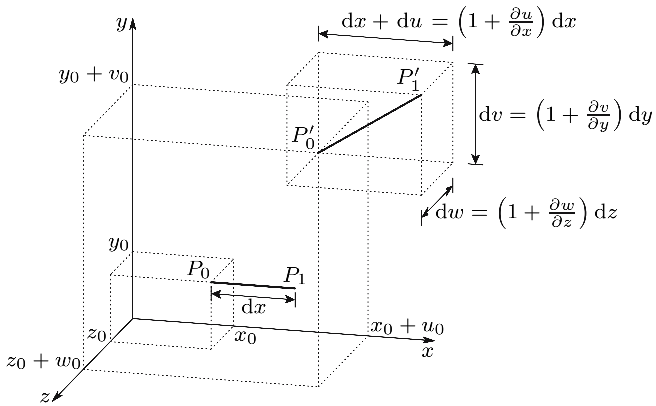

Fig. 20. Computation of the strain ε_x

After the deformation, these points will occupy the positions defined by the coordinates $P_0'(x_0+u_0, y_0+v_0, z_0+w_0)$ and $P_1'(x_0+\mathrm{d}x+u_1, y_0+v_1, z_0+w_1)$. As point P_1 is at an infinitesimal distance from P_0 and, in going from P_0 to P_1 only the coordinate x suffers an increment $\mathrm{d}x$ (undeformed configuration), we have

$$\begin{cases} u_1 = u_0 + \mathrm{d}u = u_0 + \dfrac{\partial u}{\partial x}\,\mathrm{d}x \\ v_1 = v_0 + \mathrm{d}v = v_0 + \dfrac{\partial v}{\partial x}\,\mathrm{d}x \\ w_1 = w_0 + \mathrm{d}w = w_0 + \dfrac{\partial w}{\partial x}\,\mathrm{d}x \ . \end{cases}$$

The deformation transforms the line segment $\overline{P_0P_1}$ into the line segment $\overline{P_0'P_1'}$, which is generally no longer parallel to axis x and suffers an elongation. The projections of $\overline{P_0'P_1'}$ in the reference directions are then

$$\begin{cases} \underbrace{x_0 + \mathrm{d}x}_{x_1} + \underbrace{u_0 + \dfrac{\partial u}{\partial x}\,\mathrm{d}x}_{u_1} - (x_0 + u_0) = \left(1 + \dfrac{\partial u}{\partial x}\right)\mathrm{d}x \\ y_0 + \underbrace{v_0 + \dfrac{\partial v}{\partial x}\,\mathrm{d}x}_{v_1} - (y_0 + v_0) = \dfrac{\partial v}{\partial x}\,\mathrm{d}x \\ z_0 + \underbrace{w_0 + \dfrac{\partial w}{\partial x}\,\mathrm{d}x}_{w_1} - (z_0 + w_0) = \dfrac{\partial w}{\partial x}\,\mathrm{d}x \ . \end{cases} \tag{43}$$

Defining longitudinal strain (or simply strain) as the elongation per unit length, the strain in direction x (strain of the line segment $\mathrm{d}x$) takes the value

$$\varepsilon_x = \frac{\overline{P_0'P_1'} - \overline{P_0P_1}}{\overline{P_0P_1}} = \frac{\overline{P_0'P_1'} - \mathrm{d}x}{\mathrm{d}x} \Rightarrow \overline{P_0'P_1'} = (1 + \varepsilon_x)\,\mathrm{d}x \ . \tag{44}$$

Using the Pythagorean theorem, the length $\overline{P_0'P_1'}$ may be computed from its projections in the coordinate axes, yielding, from expressions 43 and 44

$$\begin{aligned}\left(\overline{P_0'P_1'}\right)^2 &= (1+\varepsilon_x)^2\,\mathrm{d}x^2\\ &= \left[\left(1+\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2\right]\mathrm{d}x^2\\ &\Rightarrow \varepsilon_x+\frac{\varepsilon_x^2}{2}=\frac{\partial u}{\partial x}+\frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2\right].\end{aligned} \tag{45}$$

In the same way, the expressions relating strains in the directions y and z to the displacement functions may be established, yielding

$$\begin{aligned}\varepsilon_y+\frac{\varepsilon_y^2}{2}&=\frac{\partial v}{\partial y}+\frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]\\ \varepsilon_z+\frac{\varepsilon_z^2}{2}&=\frac{\partial w}{\partial z}+\frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right].\end{aligned} \tag{46}$$

Let us consider now two straight lines of infinitesimal lengths, $\mathrm{d}x$ and $\mathrm{d}y$, which, in the undeformed configuration, are parallel to axes x and y and are defined by points P_0, P_1 and P_2 (Fig. 21). The deformation transforms these straight lines into $\overline{P_0'P_1'}$ and $\overline{P_0'P_2'}$. Following the same line of reasoning as above, these line segments have the components

$$\overline{P_0'P_1'}\longrightarrow\left\{\begin{array}{c}\left(1+\dfrac{\partial u}{\partial x}\right)\mathrm{d}x\\ \dfrac{\partial v}{\partial x}\mathrm{d}x\\ \dfrac{\partial w}{\partial x}\mathrm{d}x\end{array}\right\}\quad\text{and}\quad\overline{P_0'P_2'}\longrightarrow\left\{\begin{array}{c}\dfrac{\partial u}{\partial y}\mathrm{d}y\\ \left(1+\dfrac{\partial v}{\partial y}\right)\mathrm{d}y\\ \dfrac{\partial w}{\partial y}\mathrm{d}y\end{array}\right\}.$$

In accordance with the definition of strain given earlier, the line segments $\overline{P_0'P_1'}$ and $\overline{P_0'P_2'}$ have the lengths $(1+\varepsilon_x)\,\mathrm{d}x$ and $(1+\varepsilon_y)\,\mathrm{d}y$, respectively. The scalar product of vectors $\overrightarrow{P_0'P_1'}$ and $\overrightarrow{P_0'P_2'}$ may be expressed by $(\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta)$

$$\begin{aligned}&(1+\varepsilon_x)\,\mathrm{d}x\,(1+\varepsilon_y)\,\mathrm{d}y\cos\left(\frac{\pi}{2}-\theta_{xy}\right)\\ &=\left(1+\frac{\partial u}{\partial x}\right)\mathrm{d}x\frac{\partial u}{\partial y}\mathrm{d}y+\frac{\partial v}{\partial x}\mathrm{d}x\left(1+\frac{\partial v}{\partial y}\right)\mathrm{d}y+\frac{\partial w}{\partial x}\mathrm{d}x\frac{\partial w}{\partial y}\mathrm{d}y\\ &\Rightarrow\sin\theta_{xy}=\frac{\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}}{(1+\varepsilon_x)(1+\varepsilon_y)}.\end{aligned} \tag{47}$$

Angle θ_{xy} represents the decrease of the initially right-angle between the line segments $\mathrm{d}x$ and $\mathrm{d}y$. It therefore defines the *distortion* (double shearing

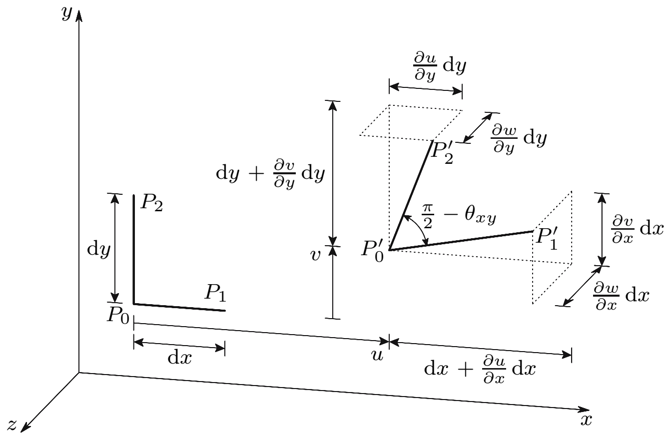

Fig. 21. Motion of the line segments $\mathrm{d}x = \overline{P_0P_1}$ and $\mathrm{d}y = \overline{P_0P_2}$ during the deformation (not considering the displacement of P_0 in direction z)

strain) of directions x and y after the deformation. In the same way, the distortions θ_{xz} and θ_{yz} may be related to the derivatives of the displacement functions, yielding

$$\sin\theta_{xz} = \frac{\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z}}{(1+\varepsilon_x)\,(1+\varepsilon_z)}$$
$$\sin\theta_{yz} = \frac{\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}}{(1+\varepsilon_y)\,(1+\varepsilon_z)} \,. \tag{48}$$

The quantities ε_x, ε_y, ε_z, θ_{xy}, θ_{xz} and θ_{yz}, described by expressions 45, 46, 47 and 48, fully define the state of deformation around point P_0, since they are enough to define the shape and dimensions of the generally non-rectangular parallelepiped resulting from the homogeneous deformation of the rectangular parallelepiped, defined by the line segments $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$, which are parallel to the coordinate axes in the undeformed configuration. However, the analytical treatment of these expressions is not simple, since they contain those quantities implicitly.[1] Moreover, they are not linear.

In most problems of Solid Mechanics that arise in structural Engineering the longitudinal and shearing strains are small enough to be considered as infinitesimal quantities, which allows the simplifications $\varepsilon^2 + 2\varepsilon \approx 2\varepsilon$ and

[1]For this reason, when the deformations are too large to be considered as infinitesimal, the strains are defined in a different way. Instead of considering the elongation per unit length $\varepsilon = \frac{l-l_0}{l_0}$, half the relative variation of the square of the length $E = \frac{1}{2}\frac{l^2-l_0^2}{l_0^2}$ is considered, which considerably simplifies the expressions, since we have $E = \varepsilon + \frac{\varepsilon^2}{2}$.

$\sin\theta \approx \theta$ and makes it possible to disregard the strains in the denominators of expressions 47 and 48. The strains may therefore be defined by explicit expressions of the type (γ_{xy} is the infinitesimal distortion of directions x and y)

$$\begin{aligned}\varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \\ \frac{\theta_{xy}}{2} = \frac{\gamma_{xy}}{2} = \varepsilon_{xy} &= \frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right].\end{aligned} \tag{49}$$

Furthermore, if the rotations are sufficiently small to be considered as infinitesimal quantities, the squares and the products of the derivatives contained in expressions 45 to 49 may be disregarded, since, with infinitesimal strains and rotations, these derivatives will also be infinitesimal, as may easily be concluded from Fig. 21. In this case, the strains may be defined by the linear expressions[2]

$$\begin{cases}\varepsilon_x = \dfrac{\partial u}{\partial x} \\ \varepsilon_y = \dfrac{\partial v}{\partial y} \\ \varepsilon_z = \dfrac{\partial w}{\partial z}\end{cases} \quad \text{and} \quad \begin{cases}\dfrac{\gamma_{xy}}{2} = \varepsilon_{xy} = \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) \\ \dfrac{\gamma_{xz}}{2} = \varepsilon_{xz} = \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) \\ \dfrac{\gamma_{yz}}{2} = \varepsilon_{yz} = \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) .\end{cases} \tag{50}$$

We shall now consider that the strain-displacement relations are defined by these simple expressions. One of the most useful consequences of this simplification is that it makes it possible to superpose the deformations associated with distinct displacement fields. This is quite evident if we consider the displacements u_1, v_1, w_1 and u_2, v_2, w_2, with which the strains ${}^1\varepsilon$ and ${}^2\varepsilon$ are respectively associated. From expressions 50 we immediately conclude

$$\begin{cases}\varepsilon_x = \dfrac{\partial (u_1 + u_2)}{\partial x} = \dfrac{\partial u_1}{\partial x} + \dfrac{\partial u_2}{\partial x} = {}^1\varepsilon_x + {}^2\varepsilon_x \\ \quad\vdots \\ \varepsilon_{yz} = \dfrac{1}{2}\left[\dfrac{\partial (v_1 + v_2)}{\partial z} + \dfrac{\partial (w_1 + w_2)}{\partial y}\right] \\ \quad = \dfrac{1}{2}\left(\dfrac{\partial v_1}{\partial z} + \dfrac{\partial w_1}{\partial y}\right) + \dfrac{1}{2}\left(\dfrac{\partial v_2}{\partial z} + \dfrac{\partial w_2}{\partial y}\right) = {}^1\varepsilon_{yz} + {}^2\varepsilon_{yz} .\end{cases} \tag{51}$$

In the case of a flowing liquid (Fluid Mechanics), the deformations and rotations in relation to the original configuration obviously cannot be considered

[2]It must, however, be pointed out that, as a consequence of the last simplification (infinitesimal rotations), the force-displacement relations obtained from these linearized strain-displacement relations (cf. Sect. I.3) cannot capture instability phenomena, in which sudden rotations of parts of the body occur.

as infinitesimal. However, taking the position of the points constituting the liquid mass in the instant t_0 as reference configuration, the rotations and deformations in instant $t_0 + \mathrm{d}t$ ($\mathrm{d}t$ is an infinitesimal time step) may be considered as infinitesimal and therefore expressions 50 may be used. Obviously, this also holds in the case of large deformations of solid bodies.

Expressions 50 furnish the components of the strain tensor in a rectangular Cartesian reference system for infinitesimal deformations and rotations. As we shall see below (Sect. III.6), the factor $\frac{1}{2}$ in the expressions concerning the shearing strains is necessary so that the quantities defining the state of deformation, ε_x, ε_y, ε_z, ε_{xy}, ε_{xz} and ε_{yz}, can form a tensor in the Cartesian space xyz, in the mathematical sense of the term.[3] The main advantage of having the deformation state defined as a tensor is that tensor mathematics can be used in its analytical treatment, which is obviously the same, regardless of the particular tensor under consideration: the stress tensor, the strain tensor or any other symmetric second order tensor. This fact allows conclusions to be drawn about the properties of the strain tensor, which are taken by analogy with the stress tensor, as will be seen later.

III.4 Pure Deformation and Rigid Body Motion

In Sect. III.3 we developed expressions to compute the elements of the strain tensor from the displacement functions u, v, w. The infinitesimal neighbourhood around a point suffers not only pure deformation but a rigid body motion, too, due to the deformation of other regions of the body. In the case of infinitesimal deformations and rotations, the motions associated with different displacement fields may be superposed, which allows the rigid body motion of the infinitesimal region around a point to be identified.

The motion of the infinitesimal region is fully defined by the quantities u, v, w, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial u}{\partial z}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial z}$, $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$ and $\frac{\partial w}{\partial z}$. Quantities u, v and w obviously represent the translation motion. The remaining nine quantities contain the deformation and the rigid body rotation. In order to identify the latter, we should point out that, in a rigid body motion, longitudinal and shearing strains vanish, which means (cf. (50))

[3]Following the mathematical definition of a second order tensor, its components transform as defined by expression 15 (stress tensor), when the reference system rotates. As we have already seen, the state of deformation may be defined by any set of six quantities, enabling the quantification of the deformation of the elementary parallelepiped. However, only the six quantities defined as represented in expression 50 define the components of a symmetric second order tensor in the Cartesian reference frame xyz.

$$\begin{cases} \varepsilon_x = \dfrac{\partial u}{\partial x} = 0 \\[2ex] \varepsilon_y = \dfrac{\partial v}{\partial y} = 0 \\[2ex] \varepsilon_z = \dfrac{\partial w}{\partial z} = 0 \end{cases} \quad \text{and} \quad \begin{cases} \gamma_{xy} = 0 \Rightarrow \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \\[2ex] \gamma_{xz} = 0 \Rightarrow \dfrac{\partial u}{\partial z} = -\dfrac{\partial w}{\partial x} \\[2ex] \gamma_{yz} = 0 \Rightarrow \dfrac{\partial v}{\partial z} = -\dfrac{\partial w}{\partial y} \ . \end{cases}$$

A displacement field u_r, v_r, w_r, where we have $\frac{\partial v_r}{\partial x} = -\frac{\partial u_r}{\partial y} = \omega_{xy}$, $\frac{\partial u_r}{\partial z} = -\frac{\partial w_r}{\partial x} = \omega_{xz}$ and $\frac{\partial w_r}{\partial y} = -\frac{\partial v_r}{\partial z} = \omega_{yz}$ thus describes a rigid body rotation of the infinitesimal region, with ω_{yz}, ω_{xz} and ω_{xy} representing positive rotations around the reference axes x, y and z, respectively. In this motion the pairs of line segments (dy,dz), (dz,dx) and (dx,dy) rotate around axes x, y and z, respectively, and remain perpendicular to each other.

If, in addition to the rigid body motion, the infinitesimal region suffers a deformations, this does not take place anymore, as seen in Sect. III.3. However, we can define the rigid body rotations around the reference axes, under the action of the deformation field u, v, w, as the *mean rotations of the line segments* dx, dy, *and* dz *around those axes*

$$\begin{cases} \omega_x = \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z}\right) = \omega_{yz} \\[2ex] \omega_y = \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial x}\right) = \omega_{xz} \\[2ex] \omega_z = \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}\right) = \omega_{xy} \ . \end{cases} \tag{52}$$

By subtracting the rigid body rotation from the total motion of the infinitesimal region, the pure deformation is obtained, which is defined by a motion where the mean rotations of the three pairs of line segments vanish. Figure 22 illustrates these situations, with the example of the rotation around axis z. Obviously this decomposition eliminates the rigid body rotation only in the infinitesimal region under consideration, since it generally varies from point to point.

We will see later (Sect. III.6) that this definition of rigid body motion is independent of the spatial orientation of the reference frame.

The additive decomposition of the strain tensor presented in this section is only valid in the case of infinitesimal deformation, where the simplified linear form of the strain-displacement relations (50) can be used. However, it is also possible to define a rigid body rotation in the case of large deformations and rotations, by using the *polar decomposition theorem*, which is based on a multiplicative decomposition of the deformation gradient (cf., e.g. [15]). Nevertheless, in this case, the mean rotation of two orthogonal line segments (in the initial configuration) does not define the rigid body rotation anymore. This

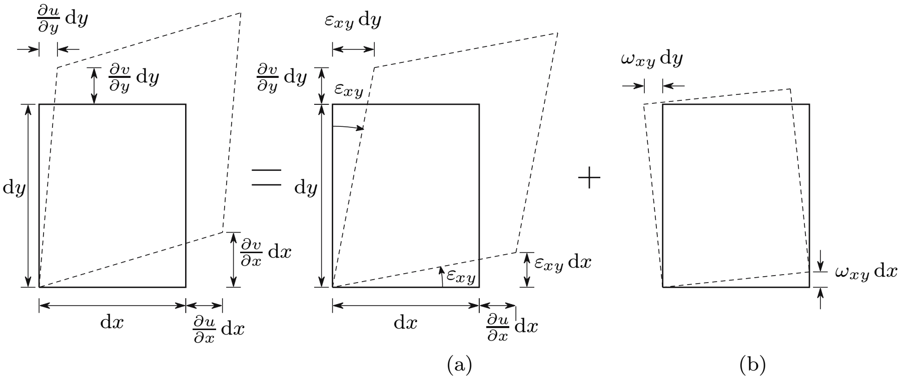

Fig. 22. Decomposition of the motion of an infinitesimal region in pure deformation and rigid body motion (infinitesimal deformations and rotations): $\varepsilon_{xy}\,\mathrm{d}x + \omega_{xy}\,\mathrm{d}x = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\mathrm{d}x + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathrm{d}x = \frac{\partial v}{\partial x}\,\mathrm{d}x$ $\varepsilon_{xy}\,\mathrm{d}y - \omega_{xy}\,\mathrm{d}y = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\mathrm{d}y - \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathrm{d}y = \frac{\partial u}{\partial y}\,\mathrm{d}y$

theorem is not given here, since a deeper insight into the finite deformation theory is beyond the scope of this introductory text.

III.5 Equations of Compatibility

As mentioned in the first chapter, we accept that the material is continuous before the deformation and remains continuous after it. This continuity condition will be satisfied if the displacement functions u, v and w are continuous, since the coordinates of the material points after the deformation are given by the expressions $x' = x + u$, $y' = y + v$ and $z' = z + w$. Therefore, if u, v and w are continuous, two points, which lie at an infinitesimal distance from each other before the deformation, will remain at an infinitesimal distance after it.

The question of the deformation's compatibility arises when, given six strain functions $\varepsilon_x(x, y, z), \ldots, \gamma_{yz}(x, y, z)$, we want to know if they represent a *compatible deformation*, i.e., a deformation where the material remains continuous. It is expected that some conditions exist between the six strain functions in a compatible deformation, as this is completely defined by the three displacement functions, u, v and w, which means that the system of equations formed by Expressions 50 has only three unknowns.

The existence of compatibility conditions may also be understood by means of geometrical considerations. For this purpose, let us imagine the continuum divided into very small parallelepipeds, so that the deformation of each one may be considered as homogeneous. A compatible deformation will be a deformation, in which the deformed parallelepipeds fit perfectly together. An incompatible deformation, however, will lead either to gaps between the parallelepipeds, or to other material discontinuities.

The compatibility conditions are obtained by eliminating the displacements u, v and w from the system formed by (50).

A first group is obtained from the relations between the longitudinal strains ε_x, ε_y, ε_z and the displacement functions. Taking, for example, the longitudinal strains in the xy-plane, we get

$$\begin{cases} \dfrac{\partial^2 \varepsilon_x}{\partial y^2} = \dfrac{\partial^3 u}{\partial x \partial y^2} \\ \dfrac{\partial^2 \varepsilon_y}{\partial x^2} = \dfrac{\partial^3 v}{\partial x^2 \partial y} \\ \dfrac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \dfrac{\partial^3 u}{\partial x \partial y^2} + \dfrac{\partial^3 v}{\partial x^2 \partial y} \end{cases} \Rightarrow \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} .$$

Proceeding in the same way with the other two pairs of reference directions, x, z and y, z, two other similar equations are obtained.

Another group of three equations may be obtained by derivation of the shearing strains with respect to the coordinate, which is absent from its indexes, and combining the obtained relations as follows

$$\begin{cases} \dfrac{\partial \gamma_{xy}}{\partial z} = \dfrac{\partial^2 u}{\partial y \partial z} + \dfrac{\partial^2 v}{\partial x \partial z} \\ \dfrac{\partial \gamma_{xz}}{\partial y} = \dfrac{\partial^2 u}{\partial y \partial z} + \dfrac{\partial^2 w}{\partial x \partial y} \\ \dfrac{\partial \gamma_{yz}}{\partial x} = \dfrac{\partial^2 v}{\partial x \partial z} + \dfrac{\partial^2 w}{\partial x \partial y} \end{cases} \Rightarrow \left| \begin{array}{l} \dfrac{\partial}{\partial x}\left(\dfrac{\partial \gamma_{xy}}{\partial z} + \dfrac{\partial \gamma_{xz}}{\partial y} - \dfrac{\partial \gamma_{yz}}{\partial x}\right) \\ = 2\dfrac{\partial^3 u}{\partial x \partial y \partial z} = 2\dfrac{\partial^2 \varepsilon_x}{\partial y \partial z} . \end{array} \right.$$

Two other relations of this type are obtained by a similar process.

The complete set of conditions, to which the strain functions $\varepsilon_x(x, y, z), \ldots,$ $\gamma_{yz}(x, y, z)$ must obey, in order to define a compatible deformation, in the sense that the deformed infinitesimal parallelepipeds fit perfectly together are then

$$\begin{aligned} &\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad && 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{yz}}{\partial x}\right) \\ &\frac{\partial^2 \varepsilon_x}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial x^2} = \frac{\partial^2 \gamma_{xz}}{\partial x \partial z} && 2\frac{\partial^2 \varepsilon_y}{\partial x \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{yz}}{\partial x}\right) \\ &\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} && 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{yz}}{\partial x}\right) . \end{aligned} \tag{53}$$

These conditions are necessary to ensure deformation compatibility *at the local level*, i.e., at the level of the infinitesimal neighbourhood of a point, since they contain only derivatives of the strain functions.[4] However, these conditions are

[4]Using the model of the infinitesimal parallelepipeds, local compatibility means that each parallelepiped fits perfectly with those in contact with it.

sufficient only in the case of *simply-connected bodies.* In a *multiply-connected body* supplementary conditions are needed to ensure compatibility. These are the *integral conditions of compatibility.*

The mathematical demonstration of these considerations is rather long and time-consuming, so it is not presented here (cf. e.g. [2]). However, a physical explanation on the basis of geometrical considerations is substantially simpler and more indicative.

A simply-connected body is a body where any closed line, fully contained in the body, can be shrunk to a point without leaving the body. Thus a two-dimensional region will be simply-connected if its boundary is defined by only one closed line, i.e., if it has no holes. A three-dimensional body may have holes and be simply-connected: for example a body defined by the space between two concentric spheres (a hollow sphere) is simply-connected, since any closed lined defined in it can shrink to a point without touching the boundaries of the body. An o-ring (torus), on the contrary, is not simply-connected, since a closed line around the hole cannot shrink to a point, without leaving the body.

The degree of connection may be defined as the number of cuts required to render the body simply-connected plus one (the intersection of the cut with the boundary of the body must be a closed line). It can also be defined as the maximum number of cuts which can be made without dividing the body into two, plus one. Some examples of the determination of the degree of connection are presented in Fig. 23.

The fact that the local compatibility conditions are sufficient to ensure the continuity of a simply-connected body after the deformation may be easily understood with the help of the two-dimensional example presented in Fig. 24-a. Clearly, if the deformed parallelepipeds fit perfectly with the neighbouring ones, i.e., if the local compatibility conditions are satisfied, the deformed body will be continuous.

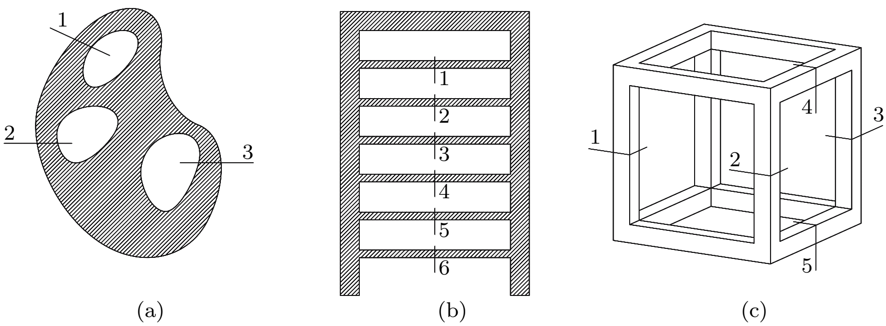

Fig. 23. Cuts required to render a body simply-connected Degrees of connection: (**a**) 4, (**b**) 7, (**c**) 6

On the other side, in the doubly-connected example presented in Fig. 24-b, despite the fact that in every point the local compatibility conditions are satisfied (the line $a'b'$ fits perfectly with line $a''b''$, so that every infinitesimal parallelepiped fits with the neighbouring ones), the deformation is not compatible, since the deformed body displays a discontinuity in the points belonging to line $\overline{ab}$. This is only possible, because the hole exists, i.e., because the degree of connection is superior to one.

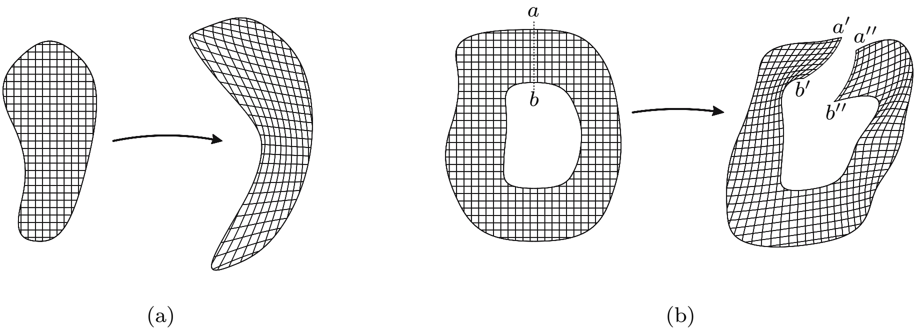

Fig. 24. Local compatibility conditions: (**a**) simply-connected body: necessary and sufficient condition; (**b**) doubly-connected body: necessary, but not sufficient condition

The expressions corresponding to the integral conditions of compatibility are not shown here. They will be studied in the second part (Strength of Materials), in the particular case of the computation of internal forces in hyperstatic (statically indeterminate) frames.

III.6 Deformation in an Arbitrary Direction

In the preceding sections the deformations of line segments, which are parallel to the reference axes in the undeformed configurations have been analysed. These deformations define the elements of the strain tensor, so they therefore allow the computation of the longitudinal and shearing deformations in arbitrary directions.

To this end, let us consider a line segment with infinitesimal unit length, whose orientation in relation to the coordinate axes is defined by the direction cosines l, m, n, which simultaneously define the components of vector $\overrightarrow{OP}$. Figure 25 illustrates the motion of the infinitesimal region around the line segment. Excluding the translations, this motion may be defined by the displacement $\overrightarrow{PP'}$ of the tip P of the unit vector $\overrightarrow{OP}$ (Fig. 25-a). The projection in the xy-plane of the non rectangular parallelepiped, which resulted from

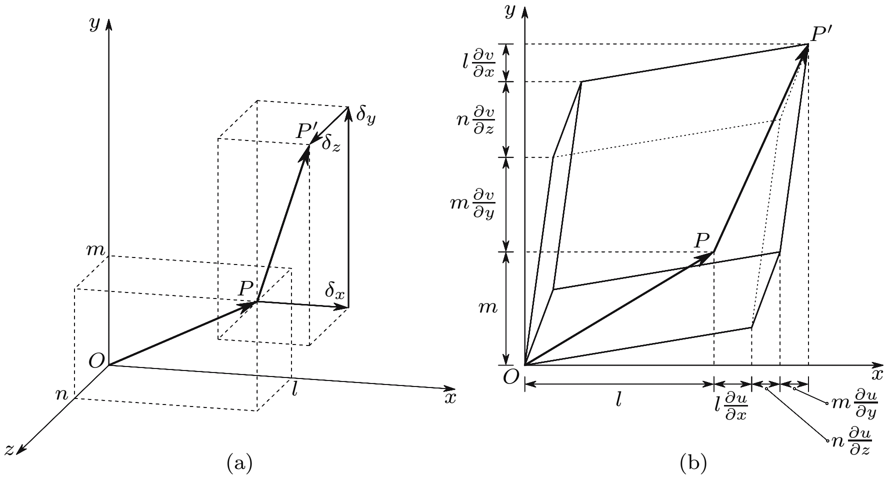

Fig. 25. Motion of the infinitesimal region around a point

the initial rectangular parallelepiped defined by the vector $\overrightarrow{OP}$, is depicted in Fig. 25-b.

The components in directions x and y of the displacement vector $\overrightarrow{PP'}$ may be obtained directly from Fig. 25-b. The projection of the deformed parallelepiped in the yz or in the zx plane would show the three contributions of the displacement in direction z. The components of vector $\overrightarrow{PP'}$ are then.

$$\begin{cases} \delta_x = l\dfrac{\partial u}{\partial x} + m\dfrac{\partial u}{\partial y} + n\dfrac{\partial u}{\partial z} \\ \delta_y = l\dfrac{\partial v}{\partial x} + m\dfrac{\partial v}{\partial y} + n\dfrac{\partial v}{\partial z} \\ \delta_z = l\dfrac{\partial w}{\partial x} + m\dfrac{\partial w}{\partial y} + n\dfrac{\partial w}{\partial z} \end{cases} \quad \Leftrightarrow \quad \begin{Bmatrix} \delta_x \\ \delta_y \\ \delta_z \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix}. \tag{54}$$

These expressions could also be obtained by simple differentiation of the displacement functions u, v and w, since the displacements δ_x, δ_y and δ_z, represent the difference between the displacements of points O and P in the reference directions, and l, m and n are the increments of coordinates x, y and z from point O to point P.

Since vector $\overrightarrow{OP}$ has unit length and only infinitesimal deformations and rotations are considered, the longitudinal strain in its direction may by obtained by the projection of vector $\overrightarrow{PP'}$ in the direction $\overline{OP}$, yielding

$$\begin{aligned}\varepsilon &= l\delta_x + m\delta_y + n\delta_z \\ &= l^2\frac{\partial u}{\partial x} + m^2\frac{\partial v}{\partial y} + n^2\frac{\partial w}{\partial z} \\ &\quad + lm\underbrace{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)}_{\gamma_{xy}=2\varepsilon_{xy}} + ln\underbrace{\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)}_{\gamma_{xz}=2\varepsilon_{xz}} + mn\underbrace{\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)}_{\gamma_{yz}=2\varepsilon_{yz}} \\ &= l^2\varepsilon_x + m^2\varepsilon_y + n^2\varepsilon_z + 2lm\varepsilon_{xy} + 2ln\varepsilon_{xz} + 2mn\varepsilon_{yz}\ .\end{aligned} \tag{55}$$

This expression is perfectly analogous to Expr. 11, which, in the stress state around a point, gives the normal stress in a facet whose semi-normal has the direction cosines l m and n. This analogy arises, because the elements of the strain tensor have been used to define the homogeneous deformation of the infinitesimal region around point O.

The transversal component δ_t of vector $\overrightarrow{PP'}$ gives the rotation of vector $\overrightarrow{OP}$. This rotation generally has a rigid body rotation and a shearing strain component. The rigid body rotation may be eliminated by considering, instead of the total displacements u, v and w, the displacements associated with the pure deformation of the infinitesimal region under consideration u', v' and w'. In this case we have (cf. Sect. III.4, Expr. 52 and Fig. 22)

$$\omega_x = \omega_y = \omega_z = 0 \Rightarrow \begin{cases} \dfrac{\partial u'}{\partial y} = \dfrac{\partial v'}{\partial x} = \varepsilon_{xy} \\ \dfrac{\partial u'}{\partial z} = \dfrac{\partial w'}{\partial x} = \varepsilon_{xz} \\ \dfrac{\partial v'}{\partial z} = \dfrac{\partial w'}{\partial y} = \varepsilon_{yz}\ . \end{cases}$$

With this modification Expr. 55 does not change, since $\frac{\partial u'}{\partial x} = \varepsilon_x, \ldots, \frac{\partial v'}{\partial z} + \frac{\partial w'}{\partial y} = \gamma_{yz}$ and Expr. 54 takes the symmetrical form

$$\underbrace{\begin{Bmatrix} \delta'_x \\ \delta'_y \\ \delta'_z \end{Bmatrix}}_{\{\delta'\}} = \underbrace{\begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_z \end{bmatrix}}_{[\varepsilon]} \underbrace{\begin{Bmatrix} l \\ m \\ n \end{Bmatrix}}_{\{l\}}. \tag{56}$$

This expression is perfectly analogous to Expr. 10 of the strain tensor, with the difference that it contains the elements of the strain tensor instead of the elements of the stress tensor. As a matter of fact, the operations performed in the analysis of the stress tensor after Expr. 10 are solely tensorial operations on a second order symmetric tensor (note that no equilibrium conditions were used in that development). These operations are therefore also valid in the case of the deformation state, since it is also described by a symmetrical second order tensor, although with different physical quantities. In this sense, Expr. 12, which furnishes the shearing stress in an arbitrary oriented facet, is analogous to the expression of the transversal displacement in the pure deformation displacement field δ'_t, which is given by($\vec{\delta'} = \varepsilon + \vec{\delta'}$)

$$\begin{cases} \delta'_{tx} = \delta'_x - l\varepsilon \\ \delta'_{ty} = \delta'_y - m\varepsilon \\ \delta'_{tz} = \delta'_z - n\varepsilon \ . \end{cases}$$

This analogy and the reciprocity of the shearing stresses allow the conclusion, that the rotation of a line segment $\overrightarrow{OQ}$, which has the same direction as δ'_t, in the plane defined by vectors $\overrightarrow{OP}$ and $\vec{\delta'}$, is equal and has the opposite direction to the rotation of vector $\overrightarrow{OP}$. In fact, if $\overrightarrow{OQ}$ is a unit vector, then $|\overrightarrow{OQ}|=|\overrightarrow{OP}|$ and, as a consequence of the analogy, $\delta'_{tP} = \delta''_{tQ}$ (cf. Fig. 26). Therefore, δ'_t actually represents the maximum shearing strain between direction $\overrightarrow{OP}$ and orthogonal directions, i.e., $\delta'_t = \frac{\gamma}{2} = \sqrt{\delta'^2_x + \delta'^2_y + \delta'^2_z - \varepsilon^2}$.

This fact leads to the conclusion that the definition of pure deformation, which was stated in Sect. III.4 for a reference system x, y, z, is independent of the coordinate axes, since, once the rigid body rotation is eliminated for those directions, it is also eliminated for all other directions.

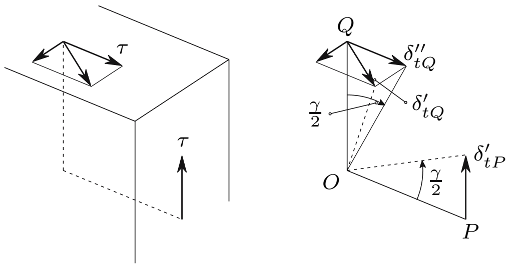

Fig. 26. Analogy between the reciprocity of the shearing stresses and the rotation of two orthogonal directions in the pure deformation

The complete analogy between the tensor operations on the stress and strain tensors, represented by the analogy between Expressions 10 and 56, allows some immediate conclusions to be drawn about the strain tensor representing a deformation state, as follows:

– The reference axes of the strain tensor may be transposed by means of the matrix operation

$$[\varepsilon'] = [l]^t[\varepsilon][l], \tag{57}$$

– where $[\varepsilon']$ contains the tensor components in the new reference axes and the orthogonal matrix $[l]$ contains the same direction cosines as in Expr. 15.

– In the deformation state around a point there are at least three orthogonal directions which do not undergo shearing strain, i.e., where $\delta_t' = 0$. These are the *principal directions* of the deformation state. The (longitudinal) strains in these directions are the *principal strains*, ε_1, ε_2 and ε_3; as a rule, a descending order is adopted, $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$.
– The *characteristic equation of the strain tensor* is given by the expression

$$-\varepsilon^3 + I_1\varepsilon^2 - I_2\varepsilon + I_3 = 0 \ ,$$

– where I_1, I_2 and I_3 are the invariants of the strain tensor and take the values given by

$$I_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$I_2 = \begin{vmatrix} \varepsilon_x & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_y \end{vmatrix} + \begin{vmatrix} \varepsilon_x & \varepsilon_{xz} \\ \varepsilon_{xz} & \varepsilon_z \end{vmatrix} + \begin{vmatrix} \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{yz} & \varepsilon_z \end{vmatrix}$$

$$I_3 = \begin{vmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_z \end{vmatrix} \ .$$

– A Lamé's ellipsoid may be drawn for the strain tensor, in the same way as for the stress tensor (Fig. 9, with principal semi-axes $\overline{OA} = \varepsilon_1$, $\overline{OB} = \varepsilon_2$ and $\overline{OC} = \varepsilon_3$).
– The strain tensor may be decomposed into isotropic and deviatoric (distortional) components ($\varepsilon_m = \frac{\varepsilon_x+\varepsilon_y+\varepsilon_z}{3}$)

$$\begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_z \end{bmatrix} = \underbrace{\begin{bmatrix} \varepsilon_m & 0 & 0 \\ 0 & \varepsilon_m & 0 \\ 0 & 0 & \varepsilon_m \end{bmatrix}}_{\text{isotropic tensor component}} + \underbrace{\begin{bmatrix} \varepsilon_x - \varepsilon_m & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_y - \varepsilon_m & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_z - \varepsilon_m \end{bmatrix}}_{\text{distortional tensor component}} .$$

– The octahedral longitudinal and shearing strains are defined by the expressions (cf. (29) and (31))

$$\varepsilon_{oct} = \frac{\varepsilon_x + \varepsilon_y + \varepsilon_z}{3}$$

$$\frac{\gamma_{oct}}{2} = \frac{1}{3}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + (\varepsilon_x - \varepsilon_z)^2 + (\varepsilon_y - \varepsilon_z)^2 + 6\left(\varepsilon_{xy}^2 + \varepsilon_{xz}^2 + \varepsilon_{yz}^2\right)} \ .$$

– A Mohr's representation of the strain tensor, similar to that displayed in Fig. 14 for the stress tensor, may be made, with the longitudinal strains ε in the axis of abscissas and the shearing strain $\frac{\gamma}{2}$ in the axis of ordinates.

III.7 Volumetric Strain

The deformation usually causes a volume change. We define *volumetric strain* as the volume change per unit of initial volume. Since, at this point, only small deformations are considered, the changes to the initially right angles of the infinitesimal parallelepiped, $\gamma_{xy} = 2\varepsilon_{xy}$, $\gamma_{yz} = 2\varepsilon_{yz}$ and $\gamma_{xz} = 2\varepsilon_{xz}$, may be considered as infinitesimal quantities and, therefore, they do not cause volume change. Thus, the volume V of the generally non-rectangular parallelepiped, which results from the initial rectangular parallelepiped defined by the infinitesimal distances $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$, may be computed as though it were rectangular, yielding ($V_0 = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ is the initial volume of the infinitesimal parallelepiped)

$$\begin{aligned} V &= (1+\varepsilon_x)\,\mathrm{d}x\;(1+\varepsilon_y)\,\mathrm{d}y\;(1+\varepsilon_z)\,\mathrm{d}z \\ &= (1+\varepsilon_x)\,(1+\varepsilon_y)\,(1+\varepsilon_z)\,V_0\;. \end{aligned}$$

Bearing in mind, that the longitudinal strains are also infinitesimal quantities, the products of these strains are infinitesimal quantities of higher order, so they may be disregarded. Thus, the volumetric strain is given by

$$\begin{aligned} \varepsilon_v &= \frac{V - V_0}{V_0} = \varepsilon_x + \varepsilon_y + \varepsilon_z + \varepsilon_x\varepsilon_y + \varepsilon_x\varepsilon_z + \varepsilon_y\varepsilon_z + \varepsilon_x\varepsilon_y\varepsilon_z \\ &\approx \varepsilon_x + \varepsilon_y + \varepsilon_z = I_1\;. \end{aligned} \tag{58}$$

We may conclude that, in the case of small deformations, the first invariant of the strain tensor takes the value of the volumetric strain.

III.8 Two-Dimensional Analysis of the Strain Tensor

III.8.a Introduction

In the same way as in the case of the stress tensor, a two-dimensional analysis of the strain tensor can also be performed, if one of the principal directions is known. If the principal strain associated with this direction is zero, we have a state of *plane strain.*

As noted in Sect. II.9, for the plane state of stress, the two-dimensional analysis of the strain tensor could be performed by particularizing the expressions developed for the general three-dimensional case to the stresses contained in the plane defined by two principal directions. However, for the same reasons as explained in that section, an independent development of the two-dimensional expressions is preferable.

Only the general considerations presented in Sects. III.1 and III.2 are needed to understand the following explanation. The expressions developed here are only valid in the linear case, where both the deformations and the rotations take infinitesimal values. For simplicity, we consider that the known principal direction is direction z, so that the two-dimensional analysis is performed in the xy plane.

III.8.b Components of the Strain Tensor

As discussed in Sect. III.2, the homogeneous deformation of a rectangle may be defined by the elongation of its sides and by the variation of the initially right-angle between two sides. These three quantities (and the initial dimensions) define the parallelogram, which results from the rectangle.

Let us now consider an infinitesimal rectangle, whose sides are parallel to the Cartesian reference axes x, y and have the infinitesimal lengths $\mathrm{d}x$ and $\mathrm{d}y$. The elongation of its sides, $\Delta\,\mathrm{d}x$ and $\Delta\,\mathrm{d}y$, divided by the initial lengths, gives the longitudinal strains $\varepsilon_x = \frac{\Delta\,\mathrm{dx}}{\mathrm{dx}}$ and $\varepsilon_y = \frac{\Delta\,\mathrm{dy}}{\mathrm{dy}}$. The variation of the initially right-angle between $\mathrm{d}x$ and $\mathrm{d}y$, $\gamma_{xy} = 2\varepsilon_{xy}$, defines the double shearing strain or distortion. These three dimensionless quantities, ε_x, ε_y and γ_{xy}, fully define the state of deformation around a point in the two-dimensional case, since they allow the computation of the strain in any arbitrary direction of plane xy, as will be seen in the following Sub-section.

As in the three-dimensional case, a sign convention is used, in which a positive longitudinal strain corresponds to an increase in length and a positive shearing strain corresponds to a decrease in the angle defined by the positive directions of the reference axes (cf. Fig. 27).

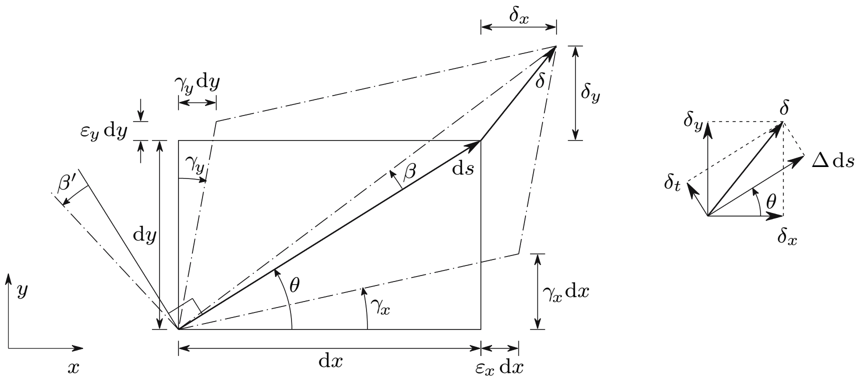

Fig. 27. Components of the deformation of a line segment with arbitrary direction

III.8.c Strain in an Arbitrary Direction

Let us consider an infinitesimal line segment with infinitesimal length $\mathrm{d}s$ and orientation defined by the angle θ, measured from axis x in the positive direction (from x to y), as represented in Fig. 27. As a consequence of the longitudinal and shearing strains ε_x, ε_y and γ_{xy} this line segment undergoes a longitudinal strain and a rotation. Denoting the rotations of the line segments $\mathrm{d}x = \mathrm{d}s\cos\theta$ and $\mathrm{d}y = \mathrm{d}s\sin\theta$ by γ_x and γ_y, respectively, positive if they lead to a decrease in the angle between the positive semi-axes x and y, the geometrical considerations depicted in Fig. 27 may be established.

In the displacements represented in this Figure the products of longitudinal and shearing strains have been disregarded, since they are infinitesimal quantities of higher order, because only infinitesimal deformations and rotations are considered (for example, we have considered $\gamma_x\,(\mathrm{d}x + \varepsilon_x\,\mathrm{d}x) \approx \gamma_x\,\mathrm{d}x$). Furthermore, as all the rotations are infinitesimal the simplifications $\cos\gamma \approx 1$ and $\sin\gamma \approx \tan\gamma \approx \gamma$ have been made.

The displacement δ of the tip of vector $\mathrm{d}s$ may be defined by its components (cf. Fig. 27)

$$\begin{cases} \delta_x = \varepsilon_x\,\mathrm{d}x + \gamma_y\,\mathrm{d}y \\ \delta_y = \varepsilon_y\,\mathrm{d}y + \gamma_x\,\mathrm{d}x\ . \end{cases} \tag{59}$$

The projection of δ_x and δ_y in the direction of $\mathrm{d}s$ gives the elongation of this line segment

$$\Delta\,\mathrm{d}s = \delta_x \cos\theta + \delta_y \sin\theta\ , \tag{60}$$

or, substituting (59) in (60) and dividing by the initial length $\mathrm{d}s$

$$\varepsilon_\theta = \frac{\Delta\,\mathrm{d}s}{\mathrm{d}s} = \varepsilon_x \frac{\mathrm{d}x}{\mathrm{d}s}\cos\theta + \gamma_y \frac{\mathrm{d}y}{\mathrm{d}s}\cos\theta + \varepsilon_y \frac{\mathrm{d}y}{\mathrm{d}s}\sin\theta + \gamma_x \frac{\mathrm{d}x}{\mathrm{d}s}\sin\theta\ .$$

Taking into consideration that $\frac{\mathrm{d}x}{\mathrm{d}s} = \cos\theta$ and $\frac{\mathrm{d}y}{\mathrm{d}s} = \sin\theta$, we get

$$\varepsilon_\theta = \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + 2\frac{\gamma_{xy}}{2}\sin\theta\cos\theta\ , \tag{61}$$

since $\gamma_x + \gamma_y = \gamma_{xy}$. This expression is analogous to Expr. 33, which furnishes the stress in a facet, whose orientation is defined by angle θ (cf. Fig.11). By projecting δ_x and δ_y on the normal direction to $\mathrm{d}s$, the transversal displacement δ_t of the tip of $\mathrm{d}s$ is obtained

$$\delta_t = -\delta_x \sin\theta + \delta_y \cos\theta\ .$$

In this expression, the displacement δ_t is considered positive if it corresponds to a rotation of $\mathrm{d}s$ in the counterclockwise direction. By dividing by $\mathrm{d}s$ and taking (59) into consideration, as well as the relations $\mathrm{d}x = \mathrm{d}s\cos\theta$ and $\mathrm{d}y = \mathrm{d}s\sin\theta$, the rotation β (cf. Fig. 27) is obtained

$$\beta = \frac{\delta_t}{\mathrm{d}s} = (\varepsilon_y - \varepsilon_x)\sin\theta\cos\theta + \gamma_x\cos^2\theta - \gamma_y\sin^2\theta\ . \tag{62}$$

The rotation β' of a line segment $\mathrm{d}s'$, which makes a right-angle, in the positive (counterclockwise) direction, with $\mathrm{d}s$ (cf. Fig. 27) may be computed by substituting θ by $\theta + \frac{\pi}{2}$ in (62), yielding

$$\beta' = -(\varepsilon_y - \varepsilon_x)\sin\theta\cos\theta + \gamma_x\sin^2\theta - \gamma_y\cos^2\theta,$$

since $\sin\left(\theta+\frac{\pi}{2}\right)=\cos\theta$ and $\cos(\theta+\frac{\pi}{2})=-\sin\theta$. The double shearing strain γ_θ between the directions defined by the angles θ and $\theta+\frac{\pi}{2}$ is then given by (cf. Fig. 27).

$$\gamma_\theta=\beta-\beta'=(\varepsilon_y-\varepsilon_x)\,2\sin\theta\cos\theta+\underbrace{\left(\gamma_x+\gamma_y\right)}_{=\gamma_{xy}}\left(\cos^2\theta-\sin^2\theta\right). \qquad (63)$$

Taking into consideration that an infinitesimal rotation of the reference axes causes only infinitesimal changes in the components of the strain tensor (this may be easily verified by substituting in (61) θ by $\mathrm{d}\theta \Rightarrow \varepsilon_\theta \approx \varepsilon_x$, or by $\frac{\pi}{2}+\mathrm{d}\theta \Rightarrow \varepsilon_\theta \approx \varepsilon_y$ and in (63) θ by $\mathrm{d}\theta \Rightarrow \gamma_\theta \approx \gamma_{xy}$). Therefore, we may consider in Fig. 27 and in (62) that $\gamma_x=\gamma_y=\frac{\gamma_{xy}}{2}$. In this case, (62) immediately gives the shearing strain $\frac{\gamma_\theta}{2}$ in the orthogonal directions θ and $\theta+\frac{\pi}{2}$, yielding[5]

$$\begin{aligned}&\gamma_x=\gamma_y=\frac{\gamma_{xy}}{2}\\&\quad\Rightarrow\ \beta=-\beta'=\frac{\gamma_\theta}{2}=(\varepsilon_y-\varepsilon_x)\sin\theta\cos\theta+\frac{\gamma_{xy}}{2}\left(\cos^2\theta-\sin^2\theta\right)\ .\end{aligned} \qquad (64)$$

Equation (64) is formally analogous to (34). By transforming (61) and (64) in the same way as (33) and (34) were transformed in Subsect. II.9.b, we get, from (61)

$$\varepsilon_\theta=\frac{\varepsilon_x+\varepsilon_y}{2}+\frac{\varepsilon_x-\varepsilon_y}{2}\cos 2\theta+\frac{\gamma_{xy}}{2}\sin 2\theta\ . \qquad (65)$$

In the same way, we get from (64)

$$\frac{\gamma_\theta}{2}=-\frac{\varepsilon_x-\varepsilon_y}{2}\sin 2\theta+\frac{\gamma_{xy}}{2}\cos 2\theta\ . \qquad (66)$$

These two expressions (65 and 66) are formally analogous to those obtained in the two-dimensional analysis of the stress tensor for the normal and shearing stresses in an arbitrary oriented facet ((35) and (36), respectively). As the further developments based on these expressions were based solely on mathematical considerations, they are also valid for the strain tensor, if we substitute σ_x, σ_y and τ_{xy} by ε_x, ε_y and $\varepsilon_{xy}=\frac{\gamma_{xy}}{2}$, respectively. The following conclusions may therefore be drawn:

– There are two orthogonal directions, which do not suffer shearing strain during the deformation. These are the principal directions of the strain tensor and may be computed by an expression analogous to (37)

$$\theta=\frac{1}{2}\arctan\frac{\gamma_{xy}}{\varepsilon_x-\varepsilon_y}. \qquad (67)$$

[5] This conclusion confirms the considerations established in Sects. III.4 and III.6 about the decomposition of the motion of the material points in an infinitesimal region in pure deformation and rigid body rotation. The infinitesimal rotation of the reference axes, so that $\gamma_x=\gamma_y$, is equivalent to the elimination of the rigid body motion.

– The longitudinal strain ε reaches extreme values (maximum and minimum) in the principal directions. These are the principal strains, which may be computed by the expression

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{Bmatrix} = \frac{\varepsilon_x + \varepsilon_y}{2} \begin{Bmatrix} + \\ - \end{Bmatrix} \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} .$$

– A Mohr's circle may be drawn for the two-dimensional strain tensor, where the axis of abscissas contains the longitudinal strains and the axis of ordinates the shearing strains.

III.9 Conclusions

In this chapter we have mainly analysed the physical aspects of the deformation. This has been possible because the conclusion that the state of deformation, as the state of stress, may be described by a symmetric second order tensor, allows a full analogy between the purely mathematical tensor transformations in the two cases. Thus, we have concluded that the tensorial operations described in Sects. II.5 to II.8 for the stress tensor are also valid in the case of the strain tensor.

As in Chap. II for the stress theory, the analysis is mainly performed in an infinitesimal neighbourhood around a point. The functions describing the evolution of the elements of the strain tensor in the continuum were taken into consideration only for developing the equations of compatibility. Here it should be noted that, while the six elements of the strain tensor are completely independent of each other, the six functions defining the elements of the strain tensor must obey the compatibility conditions.[6]

The two restrictions used in the development of the mathematical expressions for the deformation state are of completely different nature.

The first – restriction of the analysis to an infinitesimal neighbourhood around a point, so that the deformation may be considered as homogeneous – has consequences on the level of the mathematical tools used: the simplifications made possible by the consideration of a a homogeneous deformation impose the use of integral and differential calculus.[7]

The second – consideration of infinitesimal deformations and rotations – has consequences on the level of the problem's physics. As a consequence, no matter how good the mathematical or numerical tools used are, an error

[6]The same conclusion may be drawn in relation to the functions defining the stresses in the continuum: the six elements of the stress tensor are independent of each other, but the six functions, which define the same stresses as functions of the coordinates x, y, z must obey the differential equations of equilibrium (5).

[7]The corresponding restriction in the stress state is the consideration of infinitesimal facets. The consequences of this restriction are, as in the deformation state, only on the level of the mathematical formulation of the problem.

is always present, and this becomes larger when deformations and rotations grow. As mentioned in Footnote 6, the restriction to small rotations even excludes the capacity to consider structural instability phenomena. For this reason, the analysis of the buckling of slender members (Chap. XI) is based on the bending theory, where, as Chap. VII will show, the validity of the relation between the motion of cross sections and strains is not limited to small rotations.

III.10 Examples and Exercises

III.1. Displacements were measured in a deformed body, which may be approximated by the expressions

$$\begin{cases} u = 5x^2 + 3xy + 4 + 4y^2 + 3yz \\ v = 6xy + 4y^2 + 5 + 2z^2 \\ w = 4xz + 2y^2 + 3y + 6z^2 \ . \end{cases}$$

Knowing that both deformations and rotations are sufficiently small to be considered as infinitesimal, determine the functions describing the strains and the rotations in the body.

III.2. Displacements were measured in the deformation of a body, which may be approximated by the expressions ($A, B, \ldots, H$ are constants)

$$\begin{cases} u = Ax^3 + By^2 + Cyz \\ v = Dx^2y + Ey^3 + Fy^2z \\ w = Gxz^2 + Hyz^2 \ . \end{cases}$$

Knowing that, although the rotations are of considerable magnitude, the deformations are sufficiently small to be considered as infinitesimal, compute the longitudinal strain of an infinitesimal line segment which is parallel to axis x and located in an infinitesimal neighbourhood of the point of coordinates (2,-3,5).

III.3. What are the degrees of connection of the following bodies:
(a) a body composed by six bars linked like the edges of a tetrahedron;
(b) a prism with a square base and an interior cavity, which intersects the four side faces and does not intersect the top and bottom faces;
(c) a ring with a tubular cross section.

III.4. Write a computation sequence to verify the reciprocity of the rotations in a pure deformation (cf. Fig. 26).

Resolution

Given data: elements of the strain tensor, ε_x, ε_y, ε_z, ε_{xy}, ε_{xz} and ε_{yz}; direction cosines of direction OP, l, m and n.

Computation Sequence

1. components of the displacement of the tip of vector $\overrightarrow{OP}$:

$$\begin{cases} \delta'_{px} = l\varepsilon_x + m\varepsilon_{xy} + n\varepsilon_{xz} \\ \delta'_{py} = l\varepsilon_{xy} + m\varepsilon_y + n\varepsilon_{yz} \\ \delta'_{pz} = l\varepsilon_{xz} + m\varepsilon_{yz} + n\varepsilon_z \ ; \end{cases}$$

2. longitudinal strain in direction OP:

$$\varepsilon_p = l\delta'_{px} + m\delta'_{py} + n\delta'_{pz};$$

3. transversal displacement of the tip of vector $\overrightarrow{OP}$:

$$\delta'_{tp} = \sqrt{\delta'^2_{px} + \delta'^2_{py} + \delta'^2_{pz} - \varepsilon^2_p};$$

4. direction cosines of direction OQ $(\overrightarrow{OQ} \|)\vec{\delta'_{tp}}$:

$$l_q = \frac{\delta'_{px} - l\varepsilon_p}{\delta'_{tp}} \qquad m_q = \frac{\delta'_{py} - m\varepsilon_p}{\delta'_{tp}} \qquad n_q = \frac{\delta'_{pz} - n\varepsilon_p}{\delta'_{tp}};$$

5. components of the displacement of the tip of vector $\overrightarrow{OQ}$:

$$\begin{cases} \delta'_{qx} = l_q\varepsilon_x + m_q\varepsilon_{xy} + n_q\varepsilon_{xz} \\ \delta'_{qy} = l_q\varepsilon_{xy} + m_q\varepsilon_y + n_q\varepsilon_{yz} \\ \delta'_{qz} = l_q\varepsilon_{xz} + m_q\varepsilon_{yz} + n_q\varepsilon_z; \end{cases}$$

6. longitudinal strain in direction OQ:

$$\varepsilon_q = l_q\delta'_{qx} + m_q\delta'_{qy} + n_q\delta'_{qz};$$

7. components of the transversal displacement of the tip of vector $\overrightarrow{OQ}$:

$$\delta'_{tqx} = \delta'_{qx} - l_q\varepsilon_q \qquad \delta'_{tqy} = \delta'_{qy} - m_q\varepsilon_q \qquad \delta'_{tqz} = \delta'_{qz} - n_q\varepsilon_q;$$

8. projection of δ'_{tq} in direction OP:

$$\delta''_{tq} = l\delta'_{tqx} + m\delta'_{tqy} + n\delta'_{tqz};$$

9. verification:

$$\delta''_{tq} = \delta'_{tp}.$$