II

The Stress Tensor

II.1 Introduction

Some physical quantities, like the mass of a body, its volume, its surface, etc., are mathematically represented by a scalar, which means that only one parameter is necessary to define them. Others, like forces, displacements, velocities, etc., are vectorial entities, which need three quantities to be defined in a three-dimensional space, or two in the case of a two-dimensional space. Other physical entities, like the *states of stress and strain* around a material point inside a body under internal forces, are tensorial quantities, which may be described by nine components in a three-dimensional space, or by four in a two-dimensional space.

In a more general and systematic way, a scalar may be defined as a tensor of order zero with $3^0 = 1$ components, and a vector as a first order tensor with $3^1 = 3$ components. A second order tensor, or simply, tensor, has $3^2 = 9$ components. Higher order tensors may also be defined. An n^{th} order tensor will have 3^n components in a three-dimensional space (or 2^n in a two-dimensional space). As will be seen later, the tensor components are not necessarily all independent.

Below, the stress tensor is defined and some of its properties are analysed.

II.2 General Considerations

Consider a solid body under a system of self-equilibrating forces, as shown in Fig. 1-a. Imagine that the body is divided in two parts by the section represented in the same Figure. Internal forces act in the left surface of the section, representing the action of the right part of the body on the left part. Similarly, as a consequence of the equilibrium condition, in the right surface forces act with the same magnitude and in opposite directions, as shown in Fig. 1-b. The force F and the moment M represent the resultant of the internal forces distributed in the section, which generally vary from point to point. However, by considering an infinitesimal area, $\mathrm{d}\Omega$, in the surface (Fig. 2-a), we may consider a homogeneous distribution of the internal force in this area. Dividing the infinitesimal force $\mathrm{d}F$, which acts in the infinitesimal area $\mathrm{d}\Omega$, we get the internal force per unit of area or *stress*.

$$T = \frac{\mathrm{d}F}{\mathrm{d}\Omega} \ . \tag{1}$$

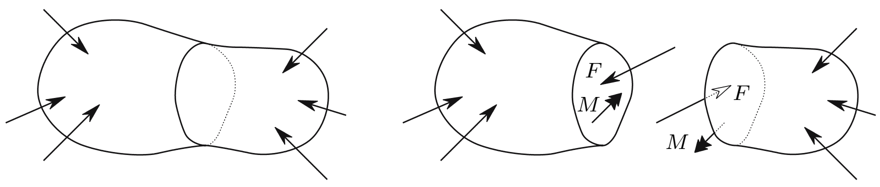


Fig. 1. Internal forces in a solid body under a self-equilibrating system of forces

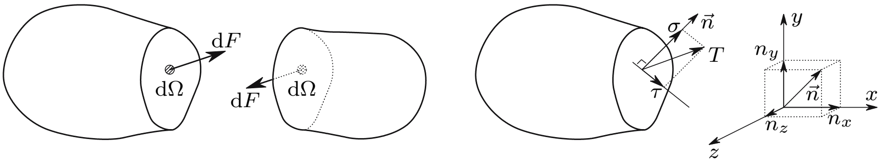


Fig. 2. Stress in an infinitesimal surface (facet)

The orientation of the infinitesimal surface of area $\mathrm{d}\Omega$ (facet) in a rectangular Cartesian reference frame xyz may be defined by a unit vector $\vec{n}$, which is perpendicular to the facet and points to the outside direction in relation to the part of the body considered (Fig. 2-b). This vector $\vec{n}$, is the *semi-normal of the facet* and, as a unit vector, its components are the cosines of the angles between the vector and the coordinate axes – the *direction cosines* of the facet

$$\begin{cases} n_x = \cos(n, x) = l \\ n_y = \cos(n, y) = m \\ n_z = \cos(n, z) = n \ . \end{cases}$$

As the vector has a unit length, we have

$$l^2 + m^2 + n^2 = 1 \ . \tag{2}$$

The stress acting on the facet may be decomposed into two components: a *normal* one, with the direction of the semi-normal of the facet $\sigma = T \cos\alpha$, and a *tangential* or *shearing* component $\tau = T \sin \alpha$, where α is the angle between the semi-normal $\vec{n}$ and the total stress vector T (Fig. 2-b).

In the right surface of the section we may define a facet, which is coincident with the left one, but has an opposite semi-normal with direction cosines $-l, -m, -n$ and stresses σ and τ with the same magnitude as in the left facet, but opposite directions. In the case of a facet which is perpendicular to a coordinate axis, it will be a positive facet if its semi-normal has the same direction as the axis to which it is parallel, and it will be negative in the opposite case. As the normal stress σ in these facets is parallel to one of the coordinate axes, the shearing stress τ may be decomposed in the directions of the other two coordinate axes.

In the presentation that follows the Von-Karman convention will be used for the stresses. According to this convention, the stresses are positive if they have the same direction as the coordinate axis to which they are parallel, in the case of a positive facet. In the case of a negative facet, the stresses will be positive, if they have the direction opposite to the corresponding coordinate axis. We will denote the normal stresses parallel to the axes x, y and z by σ_x, σ_y and σ_z, respectively. The shearing stresses are represented by the notation τ_{ij}, where the first index represents the direction of the semi-normal of the facet and the second one the direction of the shearing stress vector. For example τ_{yz} denotes the shearing stress component which is parallel to the z coordinate axis and acts in a facet whose semi-normal is parallel to the y axis.

External force components are positive if they have the same direction as the coordinate axes to which they are parallel.

Figure 3 shows the stresses acting in a rectangular parallelepiped defined by three pairs of facets, which are perpendicular to the three coordinate axis and are located in an infinitesimal neighborhood of point P.

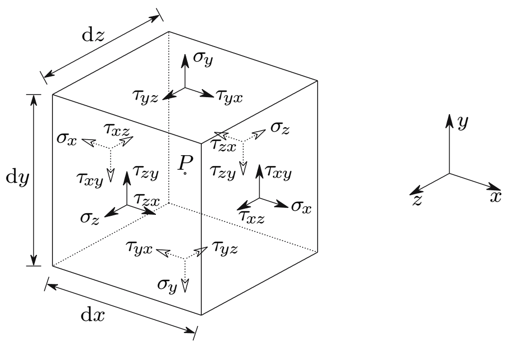


Fig. 3. Positive normal and shearing stresses

II.3 Equilibrium Conditions

Stresses and external forces must obey static and dynamic equilibrium conditions. Using these conditions, some relations may be established in the interior of the body, as well as in its boundary. These fundamental relations are deduced in the following two sub-sections.

II.3.a Equilibrium in the Interior of the Body

The static equilibrium of a body, or a part of it, under the action of a system of forces demands that both its resulting force and its resulting moment vanish. If the resulting moment is zero, we have rotation equilibrium; if the resulting force is zero, equilibrium of translation is attained.

The forces acting in the rectangular parallelepiped defined by the three pairs of facets in Fig. 3 are in equilibrium of translation, since the stress vectors in each pair of facets are equal (more precisely, the difference between them is infinitesimal) and have opposite directions. The external body forces are therefore equilibrated by the infinitesimal difference between the stresses in the negative and positive facets of the pair. The corresponding expressions are presented later. We will first analyse the rotation equilibrium conditions.

Equilibrium of Rotation

Assuming that the translation equilibrium is guaranteed, the resulting moment will be zero or a couple. The latter will vanish if the moments of the forces in relation to three axes, which have a common point, are non-parallel and do not lie along to the same plane, are zero. For simplicity, we consider axes, which are parallel to the reference system and contain the geometrical center of the infinitesimal parallelepiped (Fig. 3). Considering, for example, the axis x' parallel to x, the only forces which have a non-zero moment in relation to this axis are the resultants of τ_{yz} and τ_{zy}, as it can be confirmed by looking at Fig. 3 and as represented in Fig. 4.

The condition of zero moment of the forces which result from the stresses represented in Fig. 4, around the axis x', may be expressed by the equation

$$2\left(\tau_{yz}\,\mathrm{d}x\,\mathrm{d}z\,\frac{\mathrm{d}y}{2}\right) - 2\left(\tau_{zy}\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}z}{2}\right) = 0 \Rightarrow \tau_{zy} = \tau_{yz}\ . \tag{3}$$

The conditions which express the equilibrium of rotation around the axes y' and z', parallel to the global axes y and z, respectively lead to the conclusion that $\tau_{xy} = \tau_{yx}$ and $\tau_{xz} = \tau_{zx}$. These expressions, together with expression 3, represent the so-called *reciprocity of shearing stresses* in perpendicular facets. Since the reference axes may have any spatial orientation, the reciprocity may be expressed in the following way, which is independent of reference axes: *considering two perpendicular facets, the components of the shearing stresses*

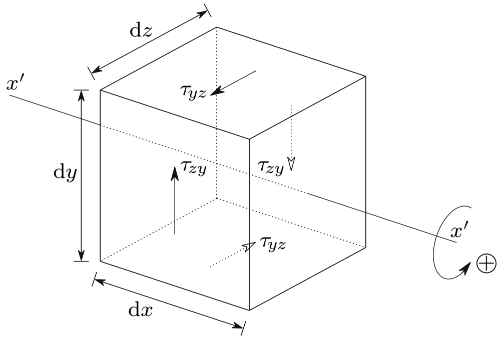


Fig. 4. Equilibrium of rotation around axis x'

which are perpendicular to the common edge of the two facets have the same magnitude and either both point to that edge or both diverge from it.[1]

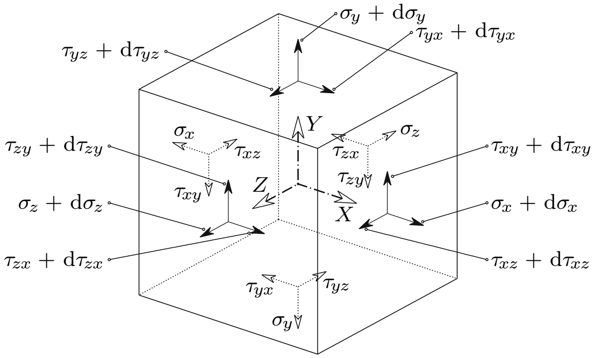
Equilibrium of Translation

As stated above, the translation equilibrium, in terms of the forces, which act on the faces of the infinitesimal parallelepiped (Fig. 3) is verified. These forces are infinitesimal quantities of the second order: for example, the force corresponding to the stress σ_y is $\sigma_y \,\mathrm{d}x\,\mathrm{d}z$. The body forces acting in the parallelepiped are infinitesimal quantities of the third order: for example, the force corresponding to the body force per unit of volume in the direction x, X, is $X\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. For these reasons, the body forces can be related to the forces corresponding to the variation of the stress, which are also infinitesimal quantities of third order. Since $\sigma_x, \ldots, \tau_{zy}$ are the mean values of the stresses in the facet, it is only necessary to compute the variation of the stress in the direction of the coordinate corresponding to the semi-normal of the facet, on which the stress acts. Figure 5 displays the forces acting on the infinitesimal parallelepiped, including the body forces and the variations of the stress functions.

The condition of equilibrium of the forces acting in direction x leads to the expression

$$\mathrm{d}\sigma_x\,\mathrm{d}y\,\mathrm{d}z + \mathrm{d}\tau_{yx}\,\mathrm{d}x\,\mathrm{d}z + \mathrm{d}\tau_{zx}\,\mathrm{d}x\,\mathrm{d}y + X\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0\,. \qquad (4)$$

[1] If the external loading were to include moments M_X, M_Y, M_Z, distributed in the volume of the body, instead of equation (3) we would obtain the expression $\tau_{yz} - \tau_{zy} + M_X = 0$ and there would be no reciprocity of the shearing stresses. However, this kind of loading does not usually have physical significance, except in problems which are beyond the scope of this text, such as the case of the influence of a strong magnetic field on the stress distribution in a magnetized body. For this reason, in the discussion below, the reciprocity of the shearing stresses will will always be considered valid.



$$
\begin{cases}
\mathrm{d}\sigma_x = \frac{\partial \sigma_x}{\partial x}\,\mathrm{d}x \\
\mathrm{d}\sigma_y = \frac{\partial \sigma_y}{\partial y}\,\mathrm{d}y \\
\mathrm{d}\sigma_z = \frac{\partial \sigma_z}{\partial z}\,\mathrm{d}z \\
\mathrm{d}\tau_{xy} = \frac{\partial \tau_{xy}}{\partial x}\,\mathrm{d}x \\
\mathrm{d}\tau_{yx} = \frac{\partial \tau_{yx}}{\partial y}\,\mathrm{d}y \\
\mathrm{d}\tau_{xz} = \frac{\partial \tau_{xz}}{\partial x}\,\mathrm{d}x \\
\mathrm{d}\tau_{zx} = \frac{\partial \tau_{zx}}{\partial z}\,\mathrm{d}z \\
\mathrm{d}\tau_{yz} = \frac{\partial \tau_{yz}}{\partial y}\,\mathrm{d}y \\
\mathrm{d}\tau_{zy} = \frac{\partial \tau_{zy}}{\partial z}\,\mathrm{d}z
\end{cases}
$$

Fig. 5. Forces acting on the infinitesimal parallelepiped

By substituting the stress variations with their values as defined in Fig. 5 and eliminating the product $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, which appears in every element of the resulting expression, we get the first of the *differential equations of equilibrium*, which are

$$
\begin{cases}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0 \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0 \, .
\end{cases} \tag{5}
$$

The last two expressions are obviously obtained from the conditions of equilibrium of translation in directions y and z, respectively.

Expressions 5 have been obtained by using the equilibrium conditions in a solid body in static equilibrium or in uniform motion. But it is very easy to generalize them to solids or liquids in non-uniform motion, by including the *inertial forces* in the body forces.

To this end, let us consider the situation represented in Fig. 5, for the case of no static balance. In this case, the resulting force is not zero, but induces an acceleration, which, in the most general case, has components in the three coordinate axes. Taking the direction x, for example, instead of expression 4, the fundamental equation of dynamics yields the relation

$$
\underbrace{\mathrm{d}\sigma_x\,\mathrm{d}y\,\mathrm{d}z + \mathrm{d}\tau_{yx}\,\mathrm{d}x\,\mathrm{d}z + \mathrm{d}\tau_{zx}\,\mathrm{d}x\,\mathrm{d}y + X\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}_{\text{force}}
$$

$$
= \underbrace{\rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}_{\text{mass}} \overbrace{a_x}^{\text{acceleration}} \;\Rightarrow\; \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X \underbrace{-\rho a_x}_{X_i} = 0 \, , \tag{6}
$$

where a_x represents the acceleration component in direction x and ρ is the density of the material. If we define the inertial forces

$$\begin{cases} X_i = -\rho a_x \\ Y_i = -\rho a_y \\ Z_i = -\rho a_z \ , \end{cases}$$

these may be treated as body forces in a body in static equilibrium, as stated by expression 6.

II.3.b Equilibrium at the Boundary

The balance conditions of the forces acting in the infinitesimal neighborhood of a point belonging to the boundary of the body may be established by considering an infinitesimal tetrahedron defined by three facets, whose semi-normals are parallel to the coordinate axes and by a facet on the boundary. Figure 6 shows this tetrahedron and the stresses and boundary forces per area unit ($\overline{X}$, $\overline{Y}$, $\overline{Z}$) acting on its faces. Since stresses and boundary forces may be considered as uniformly distributed, their resultants act on the centroids of the facets.

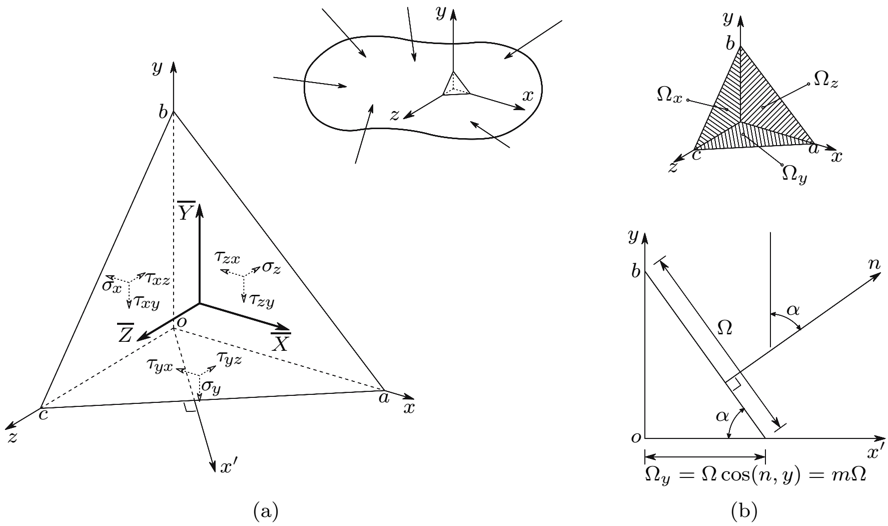


Fig. 6. Infinitesimal tetrahedron defined at the boundary of a body

The conditions expressing the rotation equilibrium around the axis of $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ confirm the reciprocity of the shearing stresses, since the moments of the body forces acting on the tetrahedron do not need to be considered, because they are infinitesimal quantities of the fourth order, while the moments of the stress resultants are infinitesimal quantities of the third order (note that boundary forces and normal stresses are on the same lines).

The balance equation for the translation in direction x yields the expression (Fig. 6-a)

$$\overline{X}\Omega - \sigma_x\Omega_x - \tau_{yx}\Omega_y - \tau_{zx}\Omega_z = 0 \, , \tag{7}$$

where Ω_x, Ω_y, Ω_z, Ω represent the areas of the triangles *obc*, *oac*, *oab*, *abc*, respectively. Denoting the direction cosines of the semi-normal of the facet *abc* by l, m, n, the following relations are easily stated (cf. Fig. 6-b)

$$\Omega_x = l\Omega \qquad \Omega_y = m\Omega \qquad \Omega_z = n\Omega \, .$$

By substituting these relations in equation (7), we get the first of the *boundary balance equations*, which are

$$\begin{cases} l\sigma_x + m\tau_{yx} + n\tau_{zx} = \overline{X} \\ l\tau_{xy} + m\sigma_y + n\tau_{zy} = \overline{Y} \\ l\tau_{xz} + m\tau_{yz} + n\sigma_z = \overline{Z} \, . \end{cases} \tag{8}$$

The last two equations are obviously obtained from the conditions of equilibrium in the directions y and z, respectively. Expressions 8 are also valid in presence of inertial forces, since these, as body forces, lead to infinitesimal quantities of the higher order in the balance equations, so that they do not need to be considered.

II.4 Stresses in an Inclined Facet

The stresses acting on an inclined facet (a facet whose semi-normal is not parallel to any of the coordinate axes) may be obtained from the balance equations of the forces acting in an infinitesimal tetrahedron similar to the one in Fig. 6, with the difference that the triangle *abc* represents the inclined facet inside the body (Fig. 7-a).

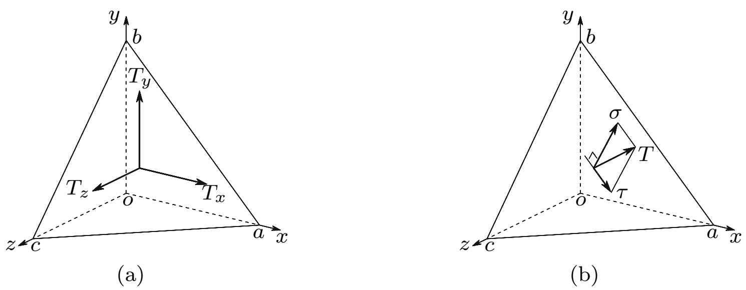


Fig. 7. Stresses in an inclined facet

Denoting by T_x, T_y, T_z the components in the reference directions of the stress vector acting on the facet *abc* and by l, m and n the direction cosines of its semi-normal, expression 8 directly gives the *Cauchy equations*

$$\begin{cases} T_x = l\sigma_x + m\tau_{yx} + n\tau_{zx} \\ T_y = l\tau_{xy} + m\sigma_y + n\tau_{zy} \\ T_z = l\tau_{xz} + m\tau_{yz} + n\sigma_z \ . \end{cases} \tag{9}$$

Using matrix notation, we may write

$$\underbrace{\begin{Bmatrix} T_x \\ T_y \\ T_z \end{Bmatrix}}_{\{T\}} = \underbrace{\begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}}_{[\sigma]} \underbrace{\begin{Bmatrix} l \\ m \\ n \end{Bmatrix}}_{\{l\}} . \tag{10}$$

We may conclude that the elements of matrix $[\sigma]$ are sufficient to compute the stress in any inclined facet around point $\underline{o}$, which means that they completely define the *state of stress around point* $\underline{o}$. This matrix therefore defines the *stress tensor.* As a consequence of the reciprocity of the shearing stresses, only six of its nine components are independent, which means that six quantities are generally necessary (and sufficient) to define the stress state around a point.

The normal stress component is the projection of vector T in the direction of the semi-normal to the facet. Taking into consideration the reciprocity of the shearing stresses ($\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$ and $\tau_{yz} = \tau_{zy}$), we get

$$\begin{aligned} \sigma &= lT_x + mT_y + nT_z \\ &= l^2\sigma_x + m^2\sigma_y + n^2\sigma_z + 2lm\tau_{xy} + 2ln\tau_{xz} + 2mn\tau_{yz} \ . \end{aligned} \tag{11}$$

The magnitude of the shearing stress may be found by means of Pythagoras' theorem, $\tau^2 = T^2 - \sigma^2$ (Fig. 7-b). The components τ_x, τ_y and τ_z of the shearing stress in the reference directions may be obtained by subtracting the components of the normal stress σ to the components of the total stress T yielding

$$\begin{cases} \tau_x = T_x - l\sigma \\ \tau_y = T_y - m\sigma \\ \tau_z = T_z - n\sigma \ . \end{cases} \tag{12}$$

II.5 Transposition of the Reference Axes

Rotating the reference axes obviously causes a change in the components of the stress tensor. These are the stresses that act in facets, which are perpendicular to the new reference axes as shown in Fig. 8. Next we develop an expression to compute the new components of the tensor when the Cartesian rectangular reference system rotates.

Let us first consider the stress $T_{x'}$, which acts on the facets with a semi-normal x' and has the components $T_{x'x}$, $T_{x'y}$ and $T_{x'z}$ in the original reference

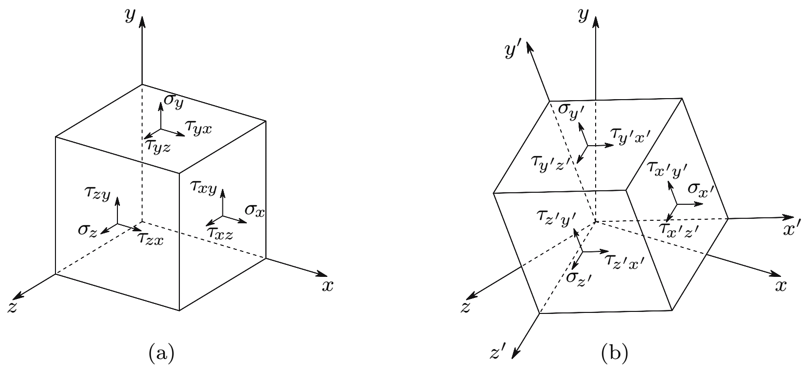


Fig. 8. Transposition of the reference axes

system (xyz). Changing the notation used for the direction cosines, expressions 10 give

$$\begin{cases} l &= (x',x) \\ m &= (x',y) \\ n &= (x',z) \end{cases} \Rightarrow \begin{Bmatrix} T_{x'x} \\ T_{x'y} \\ T_{x'z} \end{Bmatrix} = \begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{Bmatrix} (x',x) \\ (x',y) \\ (x',z) \end{Bmatrix} .$$

Proceeding in the same way in relation to the stresses acting in the facets with semi-normals y' and z', we get, in matrix notation

$$\underbrace{\begin{bmatrix} T_{x'x} & T_{y'x} & T_{z'x} \\ T_{x'y} & T_{y'y} & T_{z'y} \\ T_{x'z} & T_{y'z} & T_{z'z} \end{bmatrix}}_{[T]} = \underbrace{\begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}}_{[\sigma]} \times \underbrace{\begin{bmatrix} (x',x) & (y',x) & (z',x) \\ (x',y) & (y',y) & (z',y) \\ (x',z) & (y',z) & (z',z) \end{bmatrix}}_{[l]} . \tag{13}$$

The elements of matrix $[T]$ are the stresses acting in the new facets (semi-normals x', y' and z'), but still represented by their components in the original xyz reference system. The components of the tensor in the new reference system $x'y'z'$ are the projections of the stresses $[T]$ in the directions $x'y'z'$. These components may be obtained by the matrix operation

$$\underbrace{\begin{bmatrix} \sigma_{x'} & \tau_{y'x'} & \tau_{z'x'} \\ \tau_{x'y'} & \sigma_{y'} & \tau_{z'y'} \\ \tau_{x'z'} & \tau_{y'z'} & \sigma_{z'} \end{bmatrix}}_{[\sigma']} = \underbrace{\begin{bmatrix} (x',x) & (x',y) & (x',z) \\ (y',x) & (y',y) & (y',z) \\ (z',x) & (z',y) & (z',z) \end{bmatrix}}_{[l]^t} \times \underbrace{\begin{bmatrix} T_{x'x} & T_{y'x} & T_{z'x} \\ T_{x'y} & T_{y'y} & T_{z'y} \\ T_{x'z} & T_{y'z} & T_{z'z} \end{bmatrix}}_{[T]=[\sigma][l]} . \tag{14}$$

Combining expressions 13 and 14, we get

$$[\sigma'] = [l]^t [\sigma][l] . \tag{15}$$

As the vectors in matrix $[l]$ are orthogonal and have unit length and since the scalar product of orthogonal vectors is zero, we get

$$[\,l\,]^t[\,l\,] = [\,I\,] = [\,l\,][\,l\,]^t \Rightarrow [\,\sigma\,] = [\,l\,]\,[\,\sigma'\,]\,[\,l\,]^t \,, \tag{16}$$

where $[\,I\,]$ represents the identity matrix.

II.6 Principal Stresses and Principal Directions

The stress tensor $[\sigma]$ may be seen as a linear operator, which transforms the unit vector represented by the semi-normal of the facet, with components l, m and n, in the vector of components T_x, T_y, T_z (the stress on the facet), as described by expression 10.

Since it is a symmetrical linear operator, it is known from the linear Algebra that it can always be diagonalized, that the three roots of its characteristic equation are all real and, if they are all different, its eigenvectors are orthogonal. Transposing these conclusions to the stress state around a point, this means that there are always three facets, perpendicular to each other, where the stress vector has the same direction as the normal to the facet. As a consequence, the shearing stress vanishes. The stresses in those *principal facets* are the *principal stresses* and their normals are the *principal directions* of the stress state.

In the following exposition, these notions are analysed and expressions for their computation from the components of the stress tensor in a rectangular Cartesian system are deduced. As far as possible, a physical analysis of the stress state will be preferred to a mathematical analysis of the linear operator $[\sigma]$, since, for the student of engineering, the physical understanding of the underlying phenomena is of crucial importance.

Let us consider a principal facet. The stress acting on it has only the normal component σ, so that the components of the stress vector are $T_x = l\sigma$, $T_y = m\sigma$ and $T_z = n\sigma$. Substituting these values in expression 10, we get the homogeneous system of linear equations

$$\underbrace{\begin{bmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma \end{bmatrix}}_{[\,C\,]} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \,. \tag{17}$$

Such a system of equations has the trivial solution $l = m = n = 0$, and has other non-zero solutions only if there is a linear dependency between the equations, that is, if the determinant of the system matrix, $[\,C\,]$, vanishes. The direction cosines l, m and n cannot be zero simultaneously, since they are the components of a unit vector. Thus, the second possibility (zero determinant) must yield, as expressed by the condition

$$\begin{vmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma \end{vmatrix} = -\sigma^3 + I_1\sigma^2 - I_2\sigma + I_3 = 0 \,. \tag{18}$$

In this expression the quantities I_1, I_2 and I_3 take the values

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xz} \\ \tau_{xz} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{yz} & \sigma_z \end{vmatrix}$$

$$= \sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z - \tau_{xy}^2 - \tau_{xz}^2 - \tau_{yz}^2$$

$$I_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{vmatrix} = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{xz}\tau_{yz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2 \ .$$

The roots of equation (18) are the stresses, which satisfy equation (17), with non-simultaneous zero direction cosines l, m and n.[2] They represent the normal stresses in facets, where the shearing stress is zero, which means that they are principal stresses. The direction cosines of the normals to these facets – the principal directions – may be computed by substituting in Expression 17 σ for one of the roots of equation (18) and considering the supplementary condition $l^2 + m^2 + n^2 = 1$, since, with that substitution, equations (17) become linearly dependent ($|C| = 0$). Usually the principal stresses are denoted by σ_1, σ_2 and σ_3 with $\sigma_1 > \sigma_2 > \sigma_3$ (cf. example II.1).

The roots of equation (18) must not vary when the reference system is rotated, since they represent the principal stresses, which are intrinsic values of the stress state and therefore must not depend on the particular reference system used to describe the stress tensor. For this reason, equation (18) is designated as the *characteristic equation of the stress tensor*. The roots of this equation will be independent of the reference system if the coefficients I_1, I_2, I_3 are insensitive to coordinate changes. These coefficients are therefore *invariants of the stress tensor*.

Sometimes (for example in elasto-plastic computations) it is more convenient to define the invariants in the following way

$$J_1 = \sum_{i=1}^{3} \sigma_{ii} = I_1$$

$$J_2 = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \sigma_{ij}\sigma_{ij} = \frac{1}{2}I_1^2 - I_2$$

$$J_3 = \frac{1}{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \sigma_{ij}\sigma_{jk}\sigma_{ki} = \frac{1}{3}I_1^3 - I_1 I_2 + I_3 \ , \qquad (19)$$

where $\sigma_{11} = \sigma_x$, $\sigma_{22} = \sigma_y$, $\sigma_{33} = \sigma_z$, $\sigma_{12} = \sigma_{21} = \tau_{xy}$, $\sigma_{13} = \sigma_{31} = \tau_{xz}$ and $\sigma_{23} = \sigma_{32} = \tau_{yz}$. These relations may be verified by direct substitution. The last verification is, however, rather time-consuming. Obviously, if I_1, I_2 and I_3 are invariant, J_1, J_2 and J_3 will also be.

[2] As components of a unit vector these direction cosines must obey the condition $l^2 + m^2 + n^2 = 1$.

II.6.a The Roots of the Characteristic Equation

The characteristic equation always has three real roots. In order to prove this statement, let us first remember that a third order polynomial equation always has at least one real root, since an odd-degree polynomial may take arbitrary high values, positive or negative, by assigning sufficiently high positive or negative values to the variable. Now, let us assume that one of the reference axes (for example axis z) is parallel to the principal direction, which corresponds to that real root. For simplicity, we will consider $\sigma_z \equiv \sigma_3$ (in this section we abandon the convention $\sigma_1 > \sigma_2 > \sigma_3$). In this case, the shearing stresses τ_{xz} and τ_{yz} will vanish and expression 17 takes the form

$$\begin{bmatrix} \sigma_x - \sigma & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y - \sigma & 0 \\ 0 & 0 & \sigma_z - \sigma \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} . \tag{20}$$

The characteristic equation is therefore

$$\begin{aligned} &(\sigma_z - \sigma) \begin{vmatrix} \sigma_x - \sigma & \tau_{xy} \\ \tau_{xy} & \sigma_y - \sigma \end{vmatrix} = 0 \\ &\Rightarrow \; (\sigma_z - \sigma) \left[x\sigma_y - (\sigma_x + \sigma_y)\, \sigma + \sigma^2 - \tau_{xy}^2 \right] = 0 . \end{aligned} \tag{21}$$

One of the roots is obviously $\sigma = \sigma_z = \sigma_3$, as expected, since z is a principal direction. The other two roots may be obtained by solving the second degree equation

$$\sigma^2 - (\sigma_x + \sigma_y)\, \sigma + \left(\sigma_x \sigma_y - \tau_{xy}^2\right) = 0 .$$

The solution of this equation may be written as follows

$$\begin{aligned} \sigma &= \frac{\sigma_x + \sigma_y}{2} \pm \frac{1}{2} \sqrt{\sigma_x^2 + 2\sigma_x\sigma_y + \sigma_y^2 - 4\sigma_x\sigma_y + 4\tau_{xy}^2} \\ &= \frac{\sigma_x + \sigma_y}{2} \pm \frac{1}{2} \sqrt{\underbrace{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}_{\geq 0}} . \end{aligned} \tag{22}$$

The roots of this equation are always real, since the binomial under the square root cannot take negative values. Therefore, there are always three real roots of the characteristic equation. The roots can, however, be double or even triple. For example, if the binomial is zero, we have for any pair of reference axes x, y of a plane perpendicular to axis z

$$\begin{aligned} &(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = 0 \; \Rightarrow \left\{ \begin{array}{l} \sigma_x = \sigma_y \\ \tau_{xy} = 0 \end{array} \right. \\ &\Rightarrow \; \sigma_1 = \sigma_2 = \frac{\sigma_x + \sigma_y}{2} = \sigma_x = \sigma_y . \end{aligned} \tag{23}$$

From this expression the conclusion may be drawn that, if two roots are equal (double root) and the third is different, then all the normal stresses of the plane, which is perpendicular to the principal direction corresponding to the third root (in this case the direction z and the plane x, y, respectively), are principal stresses and take the same value, since $\sigma_1 = \sigma_2 = \sigma_x = \sigma_y$ and $\tau_{xy} = 0$. We have, in this case, a stress state, which is axis-symmetric, i.e. symmetric in relation to an axis (the z axis, in this case).

If the three roots are equal (triple root), the shearing stress vanishes in every facet, as a similar analysis in any plane containing the z axis easily shows. Furthermore, the normal stress has the same value in every facet. Since the stresses do not vary with the orientation of the facet, we have an *isotropic stress state*. The components of this stress tensor are $\sigma_x = \sigma_y = \sigma_z = \sigma$ and $\tau_{xy} = \tau_{xz} = \tau_{yz} = 0$, regardless of the orientation of the reference system.

II.6.b Orthogonality of the Principal Directions

In the case of three different principal stresses, the corresponding principal directions are perpendicular to each other. This has already been implicitly demonstrated in the previous considerations, since the plane xy is perpendicular to direction z, which coincides with one of the principal directions. The orthogonality may, however be proved more clearly from expression 20.

The last equation in this expression is linearly independent of the other two, unless $\sigma = \sigma_z = \sigma_3$. In this last case, we must have

$$\begin{vmatrix} \sigma_x - \sigma & \tau_{xy} \\ \tau_{xy} & \sigma_y - \sigma \end{vmatrix} \neq 0 \, ,$$

since the value of σ, for which this determinant vanishes, is different from σ_3 (cf. (21)). Thus, the direction cosines must take the values $l = m = 0$ and $n = 1$, to obey equations (20). These are the direction cosines of direction z, as expected.

In the case of $\sigma \neq \sigma_3$, equations (20) are satisfied only if

$$n = 0 \qquad \text{and} \qquad \begin{vmatrix} \sigma_x - \sigma & \tau_{xy} \\ \tau_{xy} & \sigma_y - \sigma \end{vmatrix} = 0 \, ,$$

since, in this case, $l^2 + m^2 = 1$. As $n = 0$, the principal directions corresponding to the principal stresses σ_1 and σ_2 are in the plane xy, i.e. they are perpendicular to z. As axis z is parallel any of the three principal directions, they must be all be perpendicular to each other.

II.6.c Lamé's Ellipsoid

In the previous section we have demonstrated that there are always three orthogonal principal directions in a stress state. It is therefore always possible to choose a rectangular Cartesian reference system which coincides with the three principal directions. In this case, the shearing components of the stress tensor vanish and it takes the form

$$\begin{cases} \sigma_x &= \sigma_1 \\ \sigma_y &= \sigma_2 \\ \sigma_z &= \sigma_3 \end{cases} \quad \text{and} \quad \begin{cases} \tau_{xy} &= 0 \\ \tau_{xz} &= 0 \\ \tau_{yz} &= 0 \end{cases} \Rightarrow [\sigma] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} . \tag{24}$$

In an inclined facet, with a semi-normal defined by the direction cosines l, m, n, the relation between the components of the stress vector and the principal stresses may be deduced from expression 9, yielding

$$\begin{cases} T_1 = l\sigma_1 \\ T_2 = m\sigma_2 \\ T_3 = n\sigma_3 \end{cases} \Rightarrow \begin{cases} l = \dfrac{T_1}{\sigma_1} \\ m = \dfrac{T_2}{\sigma_2} \\ n = \dfrac{T_3}{\sigma_3} . \end{cases} \tag{25}$$

Since the direction cosines must obey the condition $l^2+m^2+n^2 = 1$, expression 25 gives

$$\frac{T_1^2}{\sigma_1^2} + \frac{T_2^2}{\sigma_2^2} + \frac{T_3^2}{\sigma_3^2} = 1 . \tag{26}$$

If we consider a Cartesian reference system T_1, T_2, T_3, this expression represents the equation of an ellipsoid, whose principal axes are the reference system and where the points on the ellipsoid are the tips P of the stress vectors $\overrightarrow{OP}$ (T_1, T_2, T_3) acting in facets containing the point with the stress state defined by expression 24 (point O, Fig. 9)

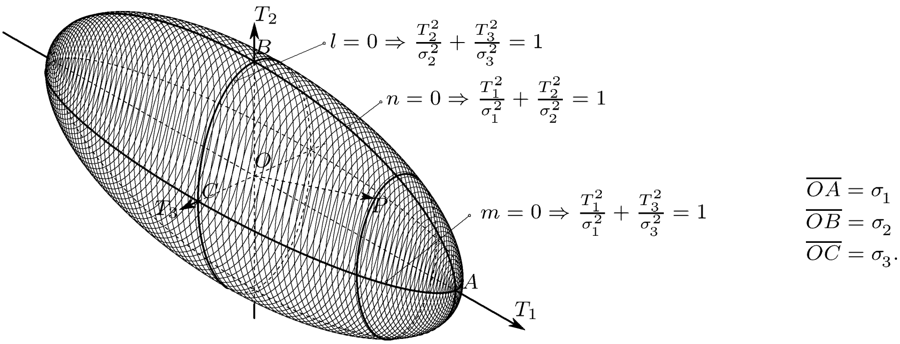


Fig. 9. Lamé's Ellipsoid or stress ellipsoid

This ellipsoid is a complete representation of the magnitudes of the stress vectors in facets around point O. It allows an important conclusion about the stress state: the magnitude of the stress in any facet takes a value between the maximum principal stress σ_1 and the minimum principal stress σ_3. It must be mentioned here that this conclusion is only valid for the absolute value of the stress, since in expression 26 only the squares of the stresses are considered.

From Fig. 9 we conclude immediately that if the absolute values of two principal stresses are equal the ellipsoid takes a shape of revolution around the third principal direction and if the three principal stresses have the same absolute value the ellipsoid becomes a sphere.

In the first case, the stress $\vec{T}$ acting in facets, which are parallel to the third principal direction have the same absolute value. Besides, if these two principal stresses have the same sign, we have an axisymmetric stress state, as concluded in Sect. II.6.a.

In the second case ($|\sigma_1| = |\sigma_2| = |\sigma_3|$), the stress $\vec{T}$ has the same magnitude in all facets. Furthermore, if $\sigma_1 = \sigma_2 = \sigma_3$, we have an isotropic stress state (cf. Sect. II.6.a).

II.7 Isotropic and Deviatoric Components of the Stress Tensor

The stress tensor may be considered as a system of forces in equilibrium, acting on an infinitesimal parallelepiped. Such a system may be decomposed in subsystems of forces in equilibrium.

When applying the stress theory to isotropic materials it is often necessary to separate the component of the stress tensor, which induces only volume changes in the material, from the component, which causes distortions. For example, as will be seen later in the study of the strain tensor and the constitutive law, the volume change in an isotropic material depends only on the isotropic component of the stress tensor

$$\begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} \quad \text{with} \quad \sigma_m = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = \frac{I_1}{3} \,. \tag{27}$$

The decomposition of the stress tensor may be described by the expression

$$\begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}}_{\substack{\text{isotropic tensor} \\ \text{component}}} + \underbrace{\begin{bmatrix} \sigma_x - \sigma_m & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y - \sigma_m & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma_m \end{bmatrix}}_{\substack{\text{deviatoric tensor} \\ \text{component}}} \,. \tag{28}$$

In isotropic materials the deviatoric component of the stress tensor does not cause volume change, as will be seen later. In this tensor component the first invariant vanishes ($I_1' = \sigma_x + \sigma_x + \sigma_z - 3\sigma_m = 0$), which means that $J_2' = -I_2'$ and $J_3' = I_3'$ (cf. (19)).

II.8 Octahedral Stresses

Octahedral stresses are stresses acting in facets which are equally inclined in relation to the principal directions. Considering a reference system, where the axes lie in the principal directions of the stress state, the semi-normals of these facets have direction cosines with equal absolute values. Since there are eight facets obeying this condition (one in each of the eight trihedrons), they define one octahedron, which is symmetrical in relation to the principal planes (Fig. 10).

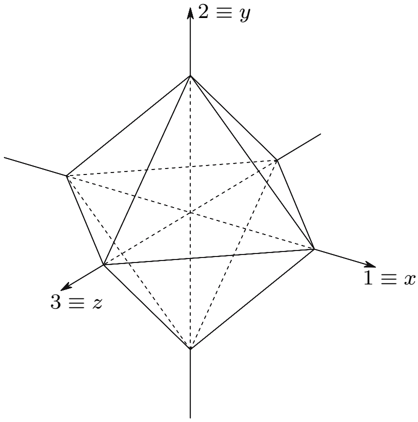


Fig. 10. Octahedron defined by equally inclined facets in relation to the principal directions 1, 2, 3

The direction cosines of the octahedral semi-normals take the values

$$\begin{cases} |l| = |m| = |n| \\ l^2 + m^2 + n^2 = 1 \end{cases} \Rightarrow \begin{cases} l = \pm\dfrac{1}{\sqrt{3}} \\ m = \pm\dfrac{1}{\sqrt{3}} \\ n = \pm\dfrac{1}{\sqrt{3}}\,. \end{cases}$$

As the reference system is a principal one, the shearing components of the stress tensor vanish. Therefore, the Cauchy equations (9) furnish the stress components

$$\begin{cases} T_x = l\sigma_1 = \pm\dfrac{\sigma_1}{\sqrt{3}} \\ T_y = m\sigma_2 = \pm\dfrac{\sigma_2}{\sqrt{3}} \\ T_z = n\sigma_3 = \pm\dfrac{\sigma_3}{\sqrt{3}}\,. \end{cases}$$

The normal component of the octahedral stress is then

$$\begin{aligned}\sigma_{oct} &= lT_x + mT_y + nT_z = l^2\sigma_1 + m^2\sigma_2 + n^2\sigma_3 \\ &= \frac{\sigma_1+\sigma_2+\sigma_3}{3} = \frac{I_1}{3} = \frac{\sigma_x+\sigma_y+\sigma_z}{3} \, .\end{aligned} \tag{29}$$

This stress coincides with the isotropic stress (cf. (27)).

The magnitude of the shearing component of the octahedral stress may be computed by using Pythagoras' theorem (cf. Sect. II.4), yielding

$$\begin{aligned}\tau_{oct}^2 &= T_{oct}^2 - \sigma_{oct}^2 = T_x^2 + T_y^2 + T_z^2 - \sigma_{oct}^2 = l^2\sigma_1^2 + m^2\sigma_2^2 + n^2\sigma_3^2 - \sigma_{oct}^2 \\ &= \frac{1}{3}\underbrace{\left(\sigma_1^2+\sigma_2^2+\sigma_3^2\right)}_{I_1^2-2I_2} - \frac{I_1^2}{9} = \frac{2}{9}\left(I_1^2 - 3I_2\right) \, .\end{aligned} \tag{30}$$

As the quantities I_1 and I_2 are insensitive to changes in the reference coordinates, the octahedral shearing stress may be expressed directly as a function of the components of the stress tensor in any rectangular Cartesian reference system xyz

$$\begin{aligned}\tau_{oct} &= \frac{\sqrt{2}}{3}\sqrt{I_1^2 - 3I_2} \\ &= \frac{\sqrt{2}}{3}\sqrt{(\sigma_x+\sigma_y+\sigma_z)^2 - 3\left(\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z - \tau_{xy}^2 - \tau_{xz}^2 - \tau_{yz}^2\right)} \\ &= \frac{1}{3}\sqrt{(\sigma_x-\sigma_y)^2 + (\sigma_y-\sigma_z)^2 + (\sigma_x-\sigma_z)^2 + 6\left(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2\right)} \, .\end{aligned} \tag{31}$$

By substituting in the last expression σ_x, σ_y and σ_z for $\sigma_x - \sigma_m$, $\sigma_y - \sigma_m$ and $\sigma_z - \sigma_m$, respectively, we conclude immediately that the octahedral shearing stresses of the complete stress tensor and of its deviatoric component (28) are equal. As we shall see later (Sect. IV.7.b.v), the octahedral shearing stress plays an important role in one of the plastic yielding theories.

An even more simple expression of the octahedral shearing stress in terms of the invariants (cf. (30)) may be obtained by considering only the deviatoric tensor. For this purpose, we establish a relation between the second invariant of the deviatoric tensor, I_2', and the two first invariants of the complete stress tensor I_1 and I_2 (cf. (28))

$$\begin{aligned}I_2' &= \begin{vmatrix}\sigma_x-\sigma_m & \tau_{xy}\\ \tau_{xy} & \sigma_y-\sigma_m\end{vmatrix} + \begin{vmatrix}\sigma_x-\sigma_m & \tau_{xz}\\ \tau_{xz} & \sigma_z-\sigma_m\end{vmatrix} + \begin{vmatrix}\sigma_y-\sigma_m & \tau_{yz}\\ \tau_{yz} & \sigma_z-\sigma_m\end{vmatrix} \\ &= \begin{vmatrix}\sigma_x & \tau_{xy}\\ \tau_{xy} & \sigma_y\end{vmatrix} + \begin{vmatrix}\sigma_x & \tau_{xz}\\ \tau_{xz} & \sigma_z\end{vmatrix} + \begin{vmatrix}\sigma_y & \tau_{yz}\\ \tau_{yz} & \sigma_z\end{vmatrix} \\ &\qquad + \underbrace{\sigma_m^2 - (\sigma_x+\sigma_y)\,\sigma_m + \sigma_m^2 - (\sigma_x+\sigma_z)\,\sigma_m + \sigma_m^2 - (\sigma_y+\sigma_z)\,\sigma_m}_{-3\sigma_m^2}\end{aligned}$$

$$\Rightarrow \qquad I_2' = I_2 - \frac{I_1^2}{3} \qquad \Leftrightarrow \qquad I_1^2 - 3I_2 = -3I_2' \ .$$

Substituting in Expression 30, we get

$$\tau_{oct}^2 = -\frac{2}{3} I_2' \ . \tag{32}$$

From this expression we conclude that the second invariant of the deviatoric stress tensor always takes a negative value.

The third invariant of the deviatoric stress tensor, I_3', may also be expressed in terms of the invariants of the complete tensor, as follows

$$I_3' = \begin{vmatrix} \sigma_x - \sigma_m & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma_m & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma_m \end{vmatrix} = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{vmatrix}$$

$$- \underbrace{\left(\sigma_y\sigma_z + \sigma_x\sigma_z + \sigma_x\sigma_y - \tau_{yz}^2 - \tau_{xz}^2 - \tau_{xy}^2\right)}_{I_2} \sigma_m + \underbrace{\left(\sigma_x + \sigma_y + \sigma_z\right)}_{3\sigma_m} \sigma_m^2 - \sigma_m^3$$

$$= I_3 - I_2\sigma_m + 2\sigma_m^3 = I_3 - \tfrac{1}{3} I_1 I_2 + \tfrac{2}{27} I_1^3 \ .$$

II.9 Two-Dimensional Analysis of the Stress Tensor

II.9.a Introduction

In many applications of the stress theory, one of the principal directions is known. As examples, we may refer the stress state at the surface of a body (in the very common case of no tangential surface loads), the stress state in a thin plate under in-plane forces, the stress states induced by the normal and shear forces and by the bending and torsion moments in bars, etc. In many cases, the principal stress, which corresponds to the known principal direction, is zero, as in the referred case of the thin plate, or in the surface of a body, where there are no external forces applied. In this case we have a *plane stress state.*

In any of these cases, a two-dimensional analysis of the stress tensor is enough to compute the remaining two principal stresses and directions. Since the three principal directions are perpendicular to each other, the remaining two principal directions act in facets, which are parallel to the known principal direction. Therefore, only this family of facets needs to be considered. As this two-dimensional analysis is considerably simpler than a three-dimensional one, a deeper insight into the stress state is possible.

The two-dimensional analysis could be performed by particularizing the expressions developed for the three-dimensional case and by developing them further in the simplified two-dimensional form. However, in the following account, the two-dimensional expressions will be deduced from scratch, i.e. without using the three-dimensional framework described in the previous sections. This option is useful because it allows the two-dimensional case to be understood, without first having to learn the more demanding three-dimensional one. As a side effect, some of the conclusions obtained in the general case will be repeated in the two-dimensional analysis, although they are obtained in a different way.

For simplicity, we will consider that the known principal direction is direction 3, and that that direction coincides with axis z. Thus the two-dimensional analysis is performed in plane xy, by considering facets which are perpendicular to this plane, and in which there are no shearing stresses with a z-component, since z is a principal direction.

II.9.b Stresses on an Inclined Facet

Let us consider a triangular prism, where two of the lateral faces are perpendicular to the coordinate axes x and y and the third lateral face has an orientation defined by the angle θ between its semi-normal and axis x. Figure 11 illustrates this prism and the stresses acting in its facets.

The equilibrium condition of the forces acting in direction θ yields

$$\sigma_\theta \,\mathrm{d}z\,\mathrm{d}s = \sigma_x \,\mathrm{d}z\,\mathrm{d}y \cos\theta + \sigma_y \,\mathrm{d}z\,\mathrm{d}x \sin\theta + \tau_{xy} \,\mathrm{d}z\,\mathrm{d}y \sin\theta + \tau_{yx} \,\mathrm{d}z\,\mathrm{d}x \cos\theta \ ,$$

or, as $\mathrm{d}x = \mathrm{d}s \sin\theta$ and $\mathrm{d}y = \mathrm{d}s \cos\theta$,

$$\sigma_\theta = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta\cos\theta \ . \tag{33}$$

Similarly, the equilibrium condition in the perpendicular direction $(\theta \pm \frac{\pi}{2})$ yields the relation

$$\tau_\theta \,\mathrm{d}z\,\mathrm{d}s + \sigma_x \,\mathrm{d}z\,\mathrm{d}y \sin\theta + \tau_{yx} \,\mathrm{d}z\,\mathrm{d}x \sin\theta = \tau_{xy} \,\mathrm{d}z\,\mathrm{d}y \cos\theta + \sigma_y \,\mathrm{d}z\,\mathrm{d}x \cos\theta \ .$$

Simplifying, we get

$$\tau_\theta = (\sigma_y - \sigma_x) \sin\theta\cos\theta + \tau_{xy} \left(\cos^2\theta - \sin^2\theta\right) \ . \tag{34}$$

Expressions 33 to 34 show that the stresses σ_x, σ_y and τ_{xy} allow the computation of the stresses in an arbitrary facet, whose orientation is defined by angle θ. They thus fully define the two-dimensional stress state around point P (Fig. 11). These stresses are the components of the stress tensor in the reference system xy.

The expressions 33 and 34 may be given another form, if we take into account the trigonometric relations

$$\sin\theta\cos\theta = \frac{\sin 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2} \ .$$

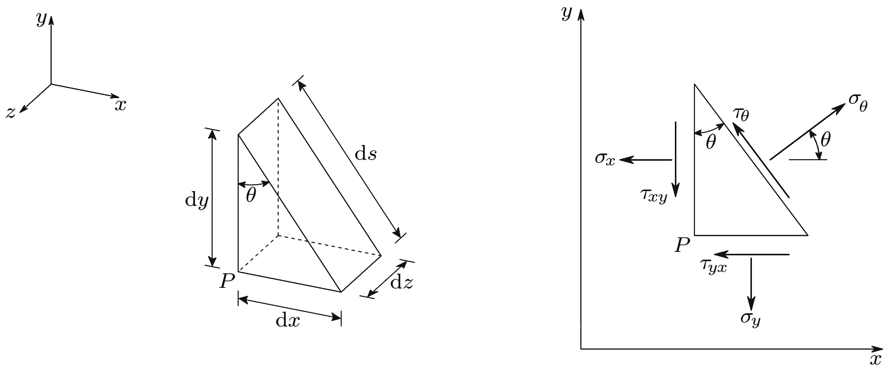


Fig. 11. Infinitesimal prism used in the two-dimensional analysis of the stress state

Substituting these relations in expression 33, we get

$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \ . \tag{35}$$

In the same way, expression 34 becomes

$$\tau_\theta = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \ . \tag{36}$$

II.9.c Principal Stresses and Directions

Expressions 35 and 36 furnish the normal and shearing components of the stress acting in facet θ, as functions of the stress tensor components σ_x, σ_y and τ_{xy}. With these expressions, the evolution of σ_θ and τ_θ with the facet orientation θ may be analysed. Differentiating expression 35 in relation to θ and equating to zero gives

$$\begin{aligned} \frac{\partial \sigma_\theta}{\partial \theta} &= -\left(\sigma_x - \sigma_y\right) \sin 2\theta + 2\tau_{xy} \cos 2\theta = 0 \\ \Rightarrow \tan 2\theta &= \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \Rightarrow \theta = \frac{1}{2} \arctan \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \ . \end{aligned} \tag{37}$$

Expression 37 yields two values of θ (θ_1 and $\theta_2 = \theta_1 + \frac{\pi}{2}$), which correspond to a maximum and a minimum of σ_θ. By substituting expression 37 in expression 36, we get $\tau_\theta = 0$. This means that, in a two-dimensional stress state, there are always two orthogonal directions which define facets where the shearing stress takes a zero value and where the normal stress takes its minimum and maximum values. These directions are the *principal directions* and the corresponding values of the stress are the *principal stresses*. Usually, the maximum principal stress is denoted by σ_1 and the minimum by σ_2.[3]

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[3] As we have seen in the three-dimensional analysis of the stress state, there is a third principal stress in a parallel facet to the plane xy. A descending ordering of the

These stresses may be computed by substituting in Expression 35 θ with the values of θ_1 and θ_2 given by (37). To this end, the following trigonometric relations are used

$$\cos 2\theta = \frac{1}{\pm\sqrt{1+\tan^2 2\theta}} \qquad \sin 2\theta = \frac{\tan 2\theta}{\pm\sqrt{1+\tan^2 2\theta}} \; .$$

By substituting the last but one of Equations 37 in these expressions and the result in Expression 35, after some manipulation we get

$$\left\{ \begin{matrix} \sigma_1 \\ \sigma_2 \end{matrix} \right\} = \frac{\sigma_x + \sigma_y}{2} \left\{ \begin{matrix} + \\ - \end{matrix} \right\} \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \; . \tag{38}$$

It should be noted here that σ_1 does not necessarily corresponds to the direction of θ_1, as defined above, since in Equation 38 the convention $\sigma_1 \geq \sigma_2$ is used. These values are not known when the directions θ_1 and θ_2 are obtained from Expression 37.

The value of θ, which corresponds to each of the principal stresses given by expression 38 may, however, be computed easily by using a relation which is deduced directly from the equilibrium condition, in direction x or y, of the forces acting in the prism shown in Fig. 11. By considering $\theta = \theta_1$ and, as a consequence, $\sigma_\theta = \sigma_1$ and $\tau_\theta = 0$, the equilibrium condition in direction x yields

$$\sigma_1 \, \mathrm{d}s \cos\theta_1 = \sigma_x \overbrace{\mathrm{d}s \cos\theta_1}^{\mathrm{d}y} + \tau_{xy} \underbrace{\mathrm{d}s \sin\theta_1}_{\mathrm{d}x} \;\Rightarrow\; \tan\theta_1 = \frac{\sigma_1 - \sigma_x}{\tau_{xy}} \; .$$

The equilibrium condition in direction y gives the relation $\tan\theta_1 = \frac{\tau_{xy}}{\sigma_1 - \sigma_y}$.

If the reference system coincides with the principal directions, the shearing stress is zero and the normal and shearing stresses acting on an arbitrary facet are

$$\left\{ \begin{array}{l} \sigma_x = \sigma_1 \\ \sigma_y = \sigma_2 \\ \tau_{xy} = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \sigma = \dfrac{\sigma_1 + \sigma_2}{2} + \dfrac{\sigma_1 - \sigma_2}{2} \cos 2\alpha \\ \\ \tau \;\; = -\dfrac{\sigma_1 - \sigma_2}{2} \sin 2\alpha \; , \end{array} \right. \tag{39}$$

where α is the angle between the principal direction 1 and the semi-normal to the facet, as shown in Fig. 11 (with the principal direction 1 in the place of axis x).

principal stresses could demand a different ordering (for example, $\sigma_z = \sigma_1$ or $\sigma_z = \sigma_2$ instead of $\sigma_z = \sigma_3$). However, for the sake of simplicity, in the two-dimensional case we adopt the descending order only for the principal stresses lying in the plane xy.

II.9.d Mohr's Circle

In the last two sections we have implicitly adopted a sign convention for the shearing stresses, where a positive stress corresponds to the y-direction in a referential system, which is obtained by a rotation θ in the direct direction (counterclockwise), of the xy referential system represented in Fig. 11. This means that a positive shearing stress in the inclined facet corresponds to a rotation, in the direct direction, around point P.

If we adopt the opposite convention – the shearing stress is positive, when it defines a clockwise rotation – the negative sign in the second of Expressions 39 disappears. In this case, these expressions are the parametric equations of a circle in a rectangular Cartesian reference system σ-τ. This circle can be used to represent the whole stress state graphically, since each point in the circle represents the stress vector in a facet, whose orientation is defined by angle α (cf. (39)). This representation of the stress tensor was developed by the end of the 19^{th} century by Otto Mohr and it still remains very popular, despite the decline of the graphic methods with the emergence of computational tools, because of its simplicity and capacity for visualizing the whole stress state.

Representing the normal stresses in the axis of abscissas (horizontal direction), and the shearing stress in the axis of ordinates (with the second sign convention defined above) Expressions 39 define a circle with radius $\frac{\sigma_1-\sigma_2}{2}$, whose center is the point of abscissa $\frac{\sigma_1+\sigma_2}{2}$ and zero ordinate, as shown in Fig. 12. Point A represents the facet with a semi-normal, whose orientation is defined by an angle α measured from the principal direction 1, positive in the counterclockwise direction. (Fig. 11, with $\theta = \alpha$). Orthogonal facets are represented by opposite points in the Mohr's circle, since an α-rotation of the facet corresponds to a 2α-rotation of its representation.

From a quick glance at Fig. 12, the following conclusions may immediately be drawn:

– the maximum value of the shearing stress is $\tau_{\max} = \frac{\sigma_1-\sigma_2}{2}$ (radius of the Mohr's circle);
– the maximum shearing stress occurs in facets with a 45°-orientation, in relation to the principal directions ($2\alpha = 90°$ – point B – and $2\alpha = 270°$ – point C);
– in the facets where the normal stress attains its extreme values the shearing stress takes a zero value (points on the axis of abscissas).

Irradiation Poles

The *irradiation poles* enable a graphic relation to be established between the Mohr's circle and the facet representation (Fig. 11). Irradiation poles for the facets and for the normals to the facets may be defined. Figure 13 presents the graphical construction leading to the facet irradiation pole.

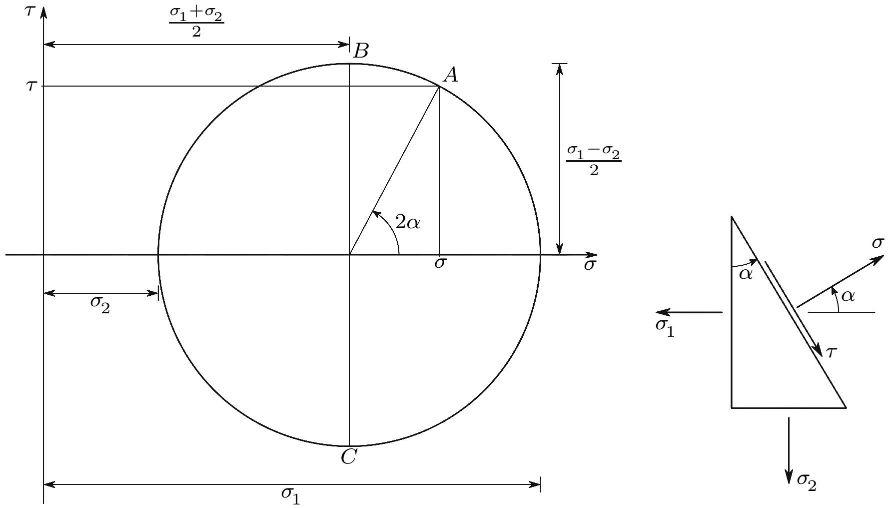

Fig. 12. Mohr's representation for the two-dimensional stress state

In this figure, the position of the facet irradiation pole I_f is first obtained by drawing a parallel line to facet a, which contains the point representing this facet in the Mohr's circle (obviously, facet b could also be used). The point representing the generic facet c on the Mohr's circle may then be obtained by drawing a line passing by the irradiation pole I_f, which is parallel to facet c. This line intersects the Mohr's circle in the point which represents facet c: σ_c and τ_c are the normal and shearing stresses acting on facet c. The rightness of this procedure is easily demonstrated: as the angle between facets a and c is β, their representations on the Mohr's circle (a and c on the circle, Fig. 13) are at the distance defined by the central angle 2β. As a consequence, the circumferential angle (a, I_f, c) measures β, since it must take half the value of 2β. Thus, if the line $\overline{aI_f}$ is parallel to the facet a, then the line $\overline{cI_f}$ is parallel to facet c.

If the direction of the normals to the facets is used instead of the facet direction, the irradiation pole of the normals, I_f, is obtained (Fig. 13). Most times, the irradiation pole of the facets is used. For simplicity, it is usually denoted by I.

The principal directions may be obtained directly from the irradiation poles:

- if the irradiation pole of the facets is used, the line joining this point with the one representing the facet where the principal stress σ_1 acts (point B), is parallel to this facet; thus, it corresponds to principal direction 2; in the same way the line $\overline{I_f\,A}$ gives the principal direction 1;

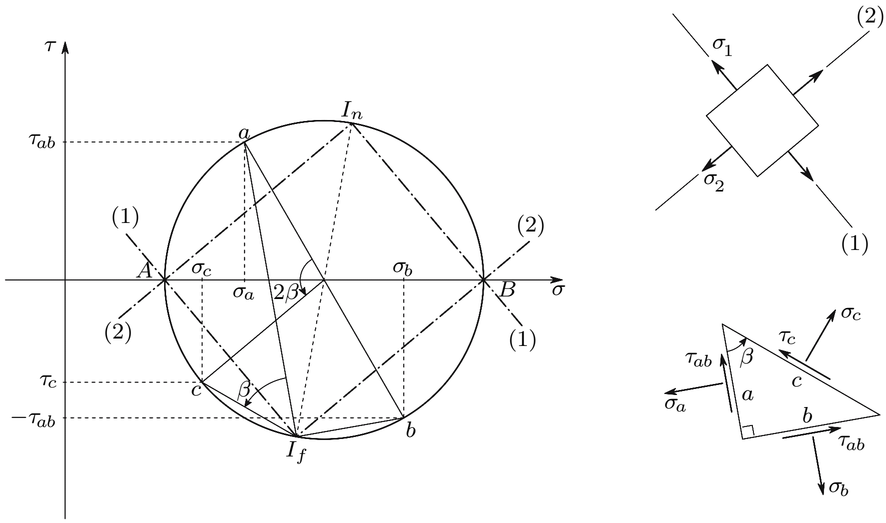

Fig. 13. Irradiation poles

– if the irradiation pole of the normals is used, as principal direction 1 is the normal to the facet, where σ_1 acts (point B), then the line $\overline{I_n\,B}$ is parallel to the principal direction 1.

II.10 Three-Dimensional Mohr's Circles

If a two-dimensional analysis is performed in each of the principal planes (planes defined by the principal directions), it is easily concluded that the stresses in the three families of facets that are parallel to each of the three principal directions may be represented in the Mohr's plane by the three circles defined by the three pairs of principal stresses, as shown in Fig. 14.

The facets which are not parallel to any of the principal directions are represented by points contained in the shaded area of Fig. 14.[4] The demonstration of this statement is based on the solution of the system of equations (cf. (2), (11) and (12))

$$\left\{\begin{array}{l} l^2 + m^2 + n^2 = 1 \\ l^2\sigma_1 + m^2\sigma_2 + n^2\sigma_3 = \sigma \\ l^2\sigma_1^2 + m^2\sigma_2^2 + n^2\sigma_3^2 = \sigma^2 + \tau^2 \end{array}\right. \Leftrightarrow \begin{bmatrix} 1 & 1 & 1 \\ \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_1^2 & \sigma_2^2 & \sigma_3^2 \end{bmatrix} \left\{\begin{array}{c} l^2 \\ m^2 \\ n^2 \end{array}\right\} = \left\{\begin{array}{c} 1 \\ \sigma \\ \sigma^2 + \tau^2 \end{array}\right\} . \tag{40}$$

[4]Only the upper half is considered, since it is not possible to make a general distinction between positive and negative shearing stresses in an inclined facet in a three-dimensional space.

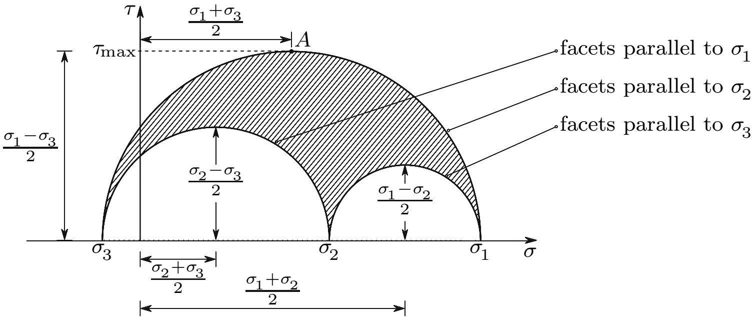

Fig. 14. Mohr's representation of the stress state in the three-dimensional case

The solution of this system may be obtained by means of determinants, yielding

$$l^2 = \frac{1}{D}\begin{vmatrix} 1 & 1 & 1 \\ \sigma & \sigma_2 & \sigma_3 \\ \sigma^2+\tau^2 & \sigma_2^2 & \sigma_3^2 \end{vmatrix} = \frac{\tau^2 + (\sigma-\sigma_2)(\sigma-\sigma_3)}{(\sigma_1-\sigma_2)(\sigma_1-\sigma_3)}$$

$$m^2 = \frac{1}{D}\begin{vmatrix} 1 & 1 & 1 \\ \sigma_1 & \sigma & \sigma_3 \\ \sigma_1^2 & \sigma^2+\tau^2 & \sigma_3^2 \end{vmatrix} = \frac{\tau^2 + (\sigma-\sigma_1)(\sigma-\sigma_3)}{(\sigma_2-\sigma_1)(\sigma_2-\sigma_3)}$$

$$n^2 = \frac{1}{D}\begin{vmatrix} 1 & 1 & 1 \\ \sigma_1 & \sigma_2 & \sigma \\ \sigma_1^2 & \sigma_2^2 & \sigma^2+\tau^2 \end{vmatrix} = \frac{\tau^2 + (\sigma-\sigma_1)(\sigma-\sigma_2)}{(\sigma_3-\sigma_1)(\sigma_3-\sigma_2)} ,$$

where $D = (\sigma_1-\sigma_2)(\sigma_2-\sigma_3)(\sigma_3-\sigma_1)$ is the system's determinant (Expr. 40). After some algebraic manipulations, these expressions may be given the forms (cf. e.g. [1])

$$\begin{aligned}
\tau^2 + \left(\sigma - \frac{\sigma_2+\sigma_3}{2}\right)^2 &= \left(\sigma_1 - \frac{\sigma_2+\sigma_3}{2}\right)^2 l^2 + \left(\frac{\sigma_2-\sigma_3}{2}\right)^2 (1-l^2) \\
\tau^2 + \left(\sigma - \frac{\sigma_1+\sigma_3}{2}\right)^2 &= \left(\sigma_2 - \frac{\sigma_1+\sigma_3}{2}\right)^2 m^2 + \left(\frac{\sigma_1-\sigma_3}{2}\right)^2 (1-m^2) \\
\tau^2 + \left(\sigma - \frac{\sigma_1+\sigma_2}{2}\right)^2 &= \left(\sigma_3 - \frac{\sigma_1+\sigma_2}{2}\right)^2 n^2 + \left(\frac{\sigma_1-\sigma_2}{2}\right)^2 (1-n^2) \ .
\end{aligned} \tag{41}$$

Considering the first of these equations, for example, we easily confirm that it represents, in the (σ, τ)-plane, a family of circles with center in the point of coordinates $\sigma = \frac{\sigma_2+\sigma_3}{2}$ and $\tau = 0$. The radius of each circle depends on the value of l^2, which varies between 0 and 1. As this equation depends linearly on l^2, the extreme values of the radius are $\frac{\sigma_2-\sigma_3}{2}$ and $\sigma_1 - \frac{\sigma_2+\sigma_3}{2}$, respectively for $l^2 = 0$ and $l^2 = 1$. The other two equations represent the other two families of circles, as shown in Fig. 15.

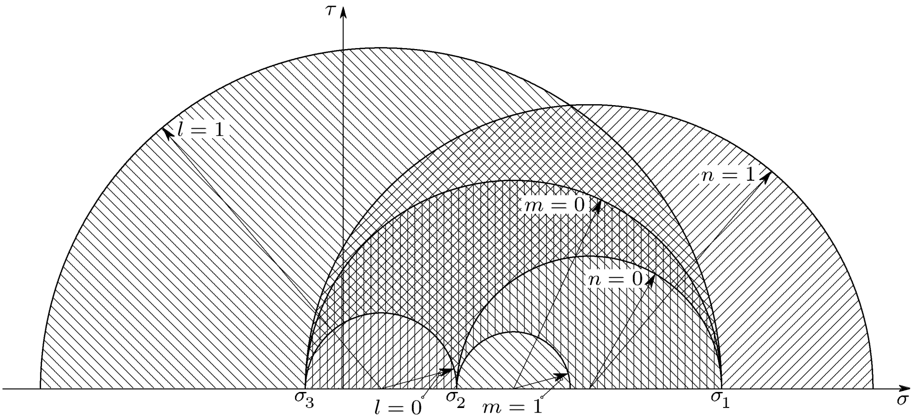

Fig. 15. Families of circles described by Expressions 41 in the Mohr's plane

The normal and shearing stresses, σ and τ, in a facet, whose semi-normal has an orientation defined by the direction cosines l, m and n in a principal reference system (axes parallel to the principal directions o the stress tensor), must obey the three Expressions 41. As we have, simultaneously,

$$0 \leq l^2 \leq 1 \qquad 0 \leq m^2 \leq 1 \qquad 0 \leq n^2 \leq 1 \; , \tag{42}$$

the points representing facets of the stress state defined by σ_1, σ_2 and σ_3 in the Mohr's plane must be on the surface containing the points whose coordinates obey the conditions 41 and 42. They are therefore on the shaded area of Fig. 14, which corresponds to the triple shaded area in Fig. 15.

The point representing a facet defined by a set of direction cosines may be found by the intersection of two of the three circles defined by Equations 41, for the corresponding values of l, m and n. The three circles intersect in this point, since the three Equations 41 must be satisfied simultaneously.

The position of this point can also be obtained graphically. However, as the explanation of the corresponding procedure is relatively lengthy and the importance of the quantitative graphical methods has substantially declined since the appearance of the computer, this method is not described here. Quite a detailed description of this procedure can be found in reference [1].

The actual importance of the Mohr's representation of the stress tensor resides in the fact, that it provides a simple global visualization of the stress state, making some conclusions obvious whose demonstration would be more difficult by other methods. From it we conclude, for example, that the maximum shearing stress occurs in facets which are parallel to the middle principal direction (direction of σ_2) and make a 45°-angle with the directions of the maximum and minimum principal stresses (point A in Fig. 14). In these facets the normal and shearing stress take the values $\frac{\sigma_1+\sigma_3}{}$

Lamé's ellipsoid, that σ_1 and σ_3 are the extreme values attained by the total stress acting in the family of facets, passing through the point, whose stress tensor has the principal stresses σ_1, σ_2 and σ_3.

II.11 Conclusions

In the theory presented in this Chapter, we have mainly analysed the stress state around a point, i.e. in an infinitesimal neighborhood of a point inside or on the surface of a solid body, or of a liquid mass, under the action of forces. This spatial restriction makes it possible to treat the stress state as homogeneous, i.e., as if it would not vary from point to point.

The expressions defining the components of the stress tensor as functions of the coordinates x, y and z were used only to develop the differential equations of equilibrium (5). In the rest of the Chapter, only the elements of the stress tensor at a given point were considered Those functions, however, play an important role in the analytical solutions for the stress distribution inside a body, which are obtained using the Theory of Elasticity (cf. e.g. Reference [4]).

In relation to the sign conventions used for the normal and shearing stresses, it should be mentioned, that, while the same convention could always be retained for the normal stresses, in the case of the shearing stresses it was necessary to abandon the initial convention (the Von-Karman convention), when studying the Mohr's representation. This is a consequence of the fact that the sign convention for the normal stresses is based on the physical concept of the tensile force as a force which causes an increase of the distance between two points, while such a physically grounded convention does not exist for the shearing stresses. In fact, in the Von Karman convention the positive stress corresponds to the positive direction of the reference axes, which have arbitrary directions; in the Mohr's circle, the positive shearing stress is defined by a direction of rotation, which depends on the observer's position. Finally, it is not possible to define a positive direction for the shearing stress in a facet in the three-dimensional case. For these reasons, a physical distinction between positive and negative stresses only makes sense for the normal stresses.

The validity of the theory expounded in this chapter is only limited by the hypothesis of continuity. Thus, it is valid in a solid with small or large deformations, in static equilibrium or in dynamic motion, or in a fluid in steady or unsteady motion. However, in the case of a solid body under finite deformations (deformations which are not small enough to be considered as infinitesimal quantities), it should be noted, that the coordinates x, y, z of the points of the body refer to the deformed configuration and not to the initial geometry of the body.

There are, however, tensors which describe the stress state using the coordinates corresponding to the undeformed geometry of the body, even in the case of large deformations (Lagrange and Piola-Kirchhoff stress tensors, cf., e.g. [2]). However, the deformations of the structures used in the engineering problems, which are solved by means of the Solid Mechanics (Civil, Mechanical, Aeronautical Engineering, etc. structures) are mostly small enough, to be treated as infinitesimal quantities. Furthermore, as the study of these tensors is rather involved and fairly abstract, they are not included in this introduction to the Mechanics of Materials.

II.12 Examples and Exercises

II.1. Using the theory described in Sects. II.6 to II.8, derive expressions for the direct computation of the principal stresses and directions.

Principal Stresses

Considering only the deviatoric component of the stress tensor, the corresponding characteristic equation takes the form

$$\sigma'^3 + I_2'\sigma' - I_3' = 0 \ ,$$

where $I_2' < 0$, as demonstrated in Sect. II.8. As the three principal stresses always exist (Sect. II.6.a), we can compute the roots of this equation using the algorithm (cf., e.g. [5], Sect. 2.4.2.3, or [2], prob. 3.5)

$$\begin{aligned} \sigma_1' &= 2\sqrt[3]{\alpha}\cos\beta \\ \sigma_2' &= 2\sqrt[3]{\alpha}\cos\left(\beta + \frac{4\pi}{3}\right) \\ \sigma_3' &= 2\sqrt[3]{\alpha}\cos\left(\beta + \frac{2\pi}{3}\right) \end{aligned} \quad \text{with} \quad \begin{cases} \alpha = \sqrt{-\dfrac{I_2'^3}{27}} \\ \beta = \dfrac{1}{3}\arccos\left(\dfrac{I_3'}{2\alpha}\right) \ . \end{cases}$$

α is always a real quantity, since $I_2' < 0$. The expression of parameter β shows that it takes values between 0 and $\frac{\pi}{3}$, since $0 \le 3\beta \le \pi$. With these limits, it is easily verified that we always have

$$\begin{cases} 0.5 \le \cos\beta \le 1 \\ -0.5 \le \cos\left(\beta + \dfrac{4\pi}{3}\right) \le 0.5 \\ -1 \le \cos\left(\beta + \dfrac{2\pi}{3}\right) \le -0.5 \end{cases} \Rightarrow \cos\beta \ge \cos\left(\beta + \frac{4\pi}{3}\right) \ge \cos\left(\beta + \frac{2\pi}{3}\right) \ .$$

From this we conclude that $\sigma_1' \ge \sigma_2' \ge \sigma_3'$.

The principal stresses of the total stress state may then be found by adding the isotropic stress (mean normal stress) to the principal stress of the deviatoric tensor component

$$\sigma_i = \sigma_i' + \frac{\sigma_x + \sigma_y + \sigma_z}{3} \quad \text{with} \quad i = 1, 2, 3 \ .$$

It is obvious that $\sigma_1' \ge \sigma_2' \ge \sigma_3'$ implies $\sigma_1 \ge \sigma_2 \ge \sigma_3$.

Principal Directions

An expression for the computation of the direction cosines of principal direction i may be obtained by ascertaining that the values of l, m and n given by

$$l = KL \qquad m = KM \qquad n = KN \,,$$

where K is an arbitrary constant and L, M and N take the values

$$L = \begin{vmatrix} \sigma_y - \sigma_i & \tau_{yz} \\ \tau_{yz} & \sigma_z - \sigma_i \end{vmatrix} \qquad M = -\begin{vmatrix} \tau_{xy} & \tau_{yz} \\ \tau_{xz} & \sigma_z - \sigma_i \end{vmatrix} \qquad N = \begin{vmatrix} \tau_{xy} & \sigma_y - \sigma_i \\ \tau_{xz} & \tau_{yz} \end{vmatrix} \,,$$

satisfy (17). This is easily confirmed, since the product of the first line of matrix $[\,C\,]$ with this vector of direction cosines $\{l, m, n\}$ corresponds to the product of K with the determinant of matrix $[\,C\,]$ computed by decomposition, using the elements of the first line and their complementary minors. This determinant vanishes, when σ takes the value of a principal stress, as is the case. The products of the second and third lines of matrix $[\,C\,]$ by the same vector are zero as well, since they represent determinants of matrices with two equal lines.

The value of K may then be computed by means of Expression 2, yielding

$$L^2K^2 + M^2K^2 + N^2K^2 = 1 \;\Rightarrow\; K = \pm\frac{1}{\sqrt{L^2 + M^2 + N^2}} \,.$$

The two vectors obtained in this way, corresponding to the two possible signs of K, represent the two opposite senses of principal direction i.

As an example, consider the stress state defined by the tensor

$$\sigma_x = 30 \quad \sigma_y = -40 \quad \sigma_z = 60 \quad \tau_{xy} = -20 \quad \tau_{xz} = 25 \quad \tau_{yz} = 50 \,.$$

Using the previously developed expressions on this tensor, the principal stresses and directions are obtained

$$\begin{array}{llll} \sigma_1 = 85.4719 & l_1 = 0.294148 & m_1 = 0.312980 & n_1 = 0.903062 \\ \sigma_2 = 33.3299 & l_2 = -0.914713 & m_2 = 0.366114 & n_2 = 0.171057 \\ \sigma_3 = -68.8017 & l_3 = 0.277086 & m_3 = 0.876358 & n_3 = -0.393978 \,. \end{array}$$

The exactness of these values is easily verified by using (15). By transposing the reference axes to the principal directions of the stress state, we get a diagonal tensor with the principal stresses

$$\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \times \begin{bmatrix} 30 & -20 & 25 \\ -20 & -40 & 50 \\ 25 & 50 & 60 \end{bmatrix} \times \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} = \begin{bmatrix} 85.47 & 0 & 0 \\ 0 & 33.33 & 0 \\ 0 & 0 & -68.80 \end{bmatrix} .$$

II.2. Verify the differential equations of equilibrium in the stress field installed in a still liquid under its own weight.

Resolution

Considering a reference system with the origin on the free surface of the liquid, whose axis y is vertical and points downwards, the stress field has the components

$$\sigma_x = \sigma_y = \sigma_z = \rho y \qquad \text{and} \qquad \tau_{xy} = \tau_{xz} = \tau_{yz} = 0 \ ,$$

where ρ represents the mass density of the liquid. Since the only body force is the gravity force, we have

$$X = Z = 0 \qquad \text{and} \qquad Y = -\rho \ .$$

Substituting these values in Expressions 5, we immediately see that they are satisfied.

II.3. In a body under a plane stress state the body forces are zero and the stresses have been approximated by the expressions (η, ρ, H and λ are constants)

$$\sigma_x = \sigma_y = \eta\rho \left(H - y - \frac{x}{\lambda} \right) \qquad \tau_{xy} = 0 \ .$$

(a) Verify that these functions cannot represent the stress distribution in the body.

(b) Determine the conditions which the body forces have to obey so that these expressions can represent a possible stress distribution.