The Heritage of Fourier

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1 Purpose of the Article

The heritage of Fourier is many-sided.

First of all Fourier is a physicist and a mathematician. The name Fourier is familiar to mathematicians, physicists, engineers and scientists in general. The Fourier equation, meaning the heat equation, Fourier series, Fourier coefficients, Fourier integrals, Fourier transforms, Fourier analysis, Fast Fourier Transforms, are everyday terms. The Analytical Theory of Heat is recognized as a landmark in science.

But Fourier is known also as an Egyptologist. He wrote an extensive introduction to the series of books entitled "Description de l'Egypte". He was in Egypt when the Rosetta stone was discovered, and Jean-François Champollion, who deciphered the hieroglyphs, was introduced in the subject by Fourier.

He was also an administrator and a politician. He took part in the French Revolution (Arago said that he was a pure product of the French Revolution, because he was supposed first to become a priest), he followed Bonaparte and Monge in Egypt as "secrétaire perpétuel de l'Institut d'Egypte", then Bonaparte elected him as prefect in Grenoble where he led a very important action in health and education, and he became a member of both Académie des sciences and Académie Française when he settled back in Paris after the fall of Napoleon. He was elected as Secrétaire perpétuel de l'Académie des sciences and played a role for the recognition of statistics in France.

His scientific work does not consist only in the analytical theory of heat and the tools that he created for this theory. He was interested in algebraic equations and his work on the localization of the roots is the transition from Descartes to Sturm; unfortunately he neglected Galois. He himself was neglected for his work on inequalities, what he called "Analyse indéterminée". Darboux considered that he gave the subject an exaggerated importance and did not publish the papers on this question in his edition of the scientific works of Fourier. Had they been published, linear programming and convex analysis would be included in the heritage of Fourier.

Fourier was a learned man and a philosopher in the sense of the eighteenth century. In a way he is a late representative of the Age of Enlightenment. On the other hand he is the main reference for Auguste Comte, a starting point for the French "positivism" of the nineteenth century.

I shall concentrate on a narrow but important part of his scientific heritage, namely the expansion of a function into a trigonometric series and the formulas for computing the coefficients. It is a way to enter the way of thinking of Fourier and its relation to physics and natural philosophy, as well as to explore the purely mathematical continuation of his work.

About the way of thinking of Fourier my general theme is that it has been disregarded for a long time, in France if not in Germany, and that it became very popular quite recently. This will be explained by a few facts and quotations.

About the continuation of his work on trigonometric series I shall focus on a very few topics according to their historical and present interest. A good part of the Conference on "Perspectives in Analysis" can be considered as a illustration of the heritage of Fourier.

The main part of the article is made of quotations and comments (2.1 Victor Hugo, 2.2 Jacobi, 2.3 Fourier, 2.4 Dirichlet and Riemann). Section 3 is devoted to the Riemann theory of trigonometric series and Sect. 4 to the convergence problem (4.1 The Carleson Theorem, 4.2 Variations About Convergence). The end, Sect. 5, is about the coming back of Fourier.

2 A Few Quotations

2.1 Victor Hugo

Let me begin with Victor Hugo. In 1862, he was in exile in the island of Guernsey, where he wrote a large part of his work. It is the year when his novel "Les misérables" was published. This novel contains a lot of information on life in France at the beginning of the nineteenth century. One chapter is entitled "1817". Joseph Fourier appears in this chapter with a short sentence:

> Il y avait à l'académie des sciences un Fourier célèbre que la postérité a oublié, et dans je ne sais quel grenier un Fourier obscur dont la postérité se souviendra.
>
> (*Les Misérables*, Victor Hugo 1862)

> There was at the Académie des sciences a celebrated Fourier whose name is forgotten now, and in some attic an obscure Fourier who will be remembered in times to come.

The first is Joseph Fourier and the second is the utopist Charles Fourier. Clearly Victor Hugo did not consider Joseph Fourier as a “gloire nationale”. He was a friend of François Arago, who succeeded Fourier as “secrétaire perpétuel de l’Académie des sciences”. After the death of Fourier in 1830, Arago wrote an obituary in this quality, and Victor Cousin as member of the Académie Française. Both obituaries are very rich and interesting about the life of Joseph Fourier, but both ignore or underestimate his work as a mathematician. Arago, who was elected very young and long before Fourier as a member of the Academy of sciences, likely remembered that Lagrange was very reluctant towards the decomposition of a function into a trigonometric series and knew that the Prize given to Fourier in 1811 for the Analytical Theory of Heat expressed reservations on this theory, both from the point of view of generality and of rigor. Fourier had competitors like Cauchy and Poisson. All that can explain why the obituary by Arago looked as a beautiful burial.

Actually, the French did not recognize the importance of Joseph Fourier until recently. There is a “rue Charles Fourier” in Paris, no street Joseph Fourier. In the first editions of Encyclopaedia Universalis, the French equivalent to Encyclopaedia Britannica, there was no article on Joseph Fourier; it was still the case in the sixth edition in 1974. I already said that Darboux published only a part of his work in mathematics, essentially the Analytical Theory of Heat, and the Collected Works were never published. Until recently, the life and works of Joseph Fourier did not attract much attention in France, at least outside Grenoble; in Grenoble, the Institute of Mathematics was called Institut Fourier a long time ago, and the whole university to which it belongs was called Université Joseph Fourier in 1987. In 1998, an excellent book appeared: “Fourier, créateur de la physique mathématique”, by a mathematician, Jean Dhombres, and a physicist, Jean-Bernard Robert. There are signs that the name of Joseph Fourier is not forgotten anymore in France.

2.2 Jacobi

More important is the appreciation of Carl Gustav Jacobi, then 26, a few weeks after the death of Fourier. In a letter to Legendre he wrote:

> M. Poisson n’aurait pas dû reproduire dans son rapport une phrase peu adroite de feu M. Fourier, où ce dernier nous reproche, à Abel et à moi, de ne pas nous être occupés de préférence du mouvement de la chaleur. Il est vrai que M. Fourier avait l’opinion que le but principal des mathématiques était l’utilité publique et l’explication des phénomènes naturels; mais un philosophe comme lui aurait dû saisir que le but unique de la science, c’est l’honneur de l’esprit humain, et que, sous ce titre, une question de nombres vaut autant qu’une question de système du monde.
>
> (Jacobi, lettre à Legendre, 2 juillet 1830)

> M. Poisson should not have reproduced an ill-timed appreciation of the late M. Fourier, reproaching Abel and me for not paying enough attention to the movement of heat. In truth, M. Fourier thought that public interest and explanation of natural phenomena were the main purpose of mathematics. But, as a philosopher, he should have known that the unique purpose of science is the honour of the human mind, and that, in this respect, a question about numbers is as valuable as a question about the universe.

The key word "l'honneur de l'esprit humain" became a motto for pure mathematics, in opposition with Fourier's point of view. In particular, it gave the title of a famous book of Jean Dieudonné.

2.3 Fourier

The point of view of Fourier is expressed very clearly in a few sentences of the general introduction ("Discours préliminaire") of his Analytical Theory of Heat. Here is a selection of such sentences:

> Les équations du mouvement de la chaleur, comme celles qui expriment les vibrations des corps sonores, ou les dernières oscillations ds liquides, appartiennent à une des branches de la science du calcul les plus récemment découvertes... Après avoir établi ces équations différentielles, il fallait en obtenir les intégrales; ce qui consiste à passer d'une expression commune à une solution propre assujettie à toutes les conditions données. Cette recherche difficile exigeait une analyse spéciale, fondée sur des théorèmes nouveaux... La méthode qui en dérive ne laisse rien de vague ni d'indéterminé dans les solutions. Elle les conduit jusqu'aux dernières applications numériques, condition nécessaire de toute recherche, et sans lesquelles on n'arriverait qu'à des transformations inutiles...
>
> (Joseph Fourier [2], Discours préliminaire)

> L'étude approfondie de la nature est la source la plus féconde des découvertes mathématiques...
>
> Les équations analytiques...s'étendent à tous les phénomènes généraux. Il ne peut y avoir de langage plus universel et plus simple, plus exempt d'erreurs et d'obscurités, c'est-à-dire plus digne d'exprimer les rapports invariables des êtres naturels.
>
> Considérée de ce point de vue, l'analyse mathématique est aussi étendue que la nature elle même... Son attribut principal est la clarté. Elle n'a point de signes pour exprimer les notions confuses. Elle rapproche les phénomènes les plus divers et découvre les analogies secrètes qui unissent... Elle nous les rend présents et mesurables, et semble être une faculté de la raison humaine, destinée à suppléer à la brièveté de la vie et à l'imperfection des sens. (ibid)

Let me begin with the second excerpt:

> The thorough study of nature is the most productive source of mathematical discoveries.
>
> Analytic equations apply to all general phenomena. There is no simpler and more universal language, more free from errors, and more able to express permanent relations between natural bodies.
>
> From this point of view mathematical analysis is as large as nature itself. Its main feature is clarity. It has no sign for confuse notions. It connects the most diverse phenomena and expresses their hidden analogies. It makes them accessible and measurable, and it seems to be a faculty of the human brain, making up for the brevity of life and imperfection of our senses.

This is a glorious definition of mathematical analysis. By the way, Fourier was interested also by the human life and industry, and this, parallel to "the thorough study of nature", was another important source of his mathematical investigations and discoveries.

Here is an approximative translation of the first excerpt:

> The heat equation, as well as the equations concerning vibrating strings or motions of liquids, belongs to a quite recent brand of analysis [namely, PDE]. After establishing the equations one has to find the solutions, that is, go from a general expression to a particular solution subject to prescribed conditions. This investigation was difficult and needed a new kind of analysis, based on new theorems... The corresponding method leaves nothing vague in the solutions. It leads to final numerical applications, as any investigation should do in order to be useful.

The Fourier approach is very well described in the Discours préliminaire: start from natural phenomena and end with numerical conclusions. In between "a new kind of analysis" is needed, what we call now Fourier analysis. This sounds more modern now, with modeling and computers, than 50 years ago, and explains the comeback of Fourier's views.

2.4 Dirichlet and Riemann

The main continuators of Fourier were Dirichlet and Riemann. Dirichlet met Fourier when he stayed in Paris, between 1822 and 1825, not yet 20 years old. In 1829, he published the first general and correct statement about the convergence of Fourier series. The beginning of this article was a tribute payed to Fourier ("l'illustre géomètre qui a ouvert une nouvelle carrière aux applications de l'analyse") and a criticism of the approach of the convergence problem by Cauchy. For Dirichlet as for Fourier, the starting point was a function given in some way. Then came integral formulas providing the coefficients of a trigonometric series. The problem was to show that this trigonometric series converges to the function.

The treatment of the problem by Dirichlet was a masterpiece of analysis, and it led to the celebrated Dirichlet conditions. It led also to an important remark: the given function needs to satisfy some conditions in order that the integral formulas make sense. As a comment, Dirichlet introduced his famous example of a function taking a value on rational points and another on irrational points: for him as later for Riemann, such a function is not integrable on any interval.

Since Dirichlet the *convergence problem* of Fourier series (that is, trigonometric series whose coefficients are given by the Fourier integral formulas) is linked to two major questions: what do you mean by a *function*? What do you mean by an *integral*?

Actually, the term *Fourier series* ("Fouriersche Reihe") appears for the first time in the dissertation of Riemann on trigonometric series, written in 1854. The beginning of the dissertation is a history of the subject from the controversy on vibrating strings in the 18$^{\text{th}}$ century to the article by Dirichlet, with comments and remarks, on analytic and harmonic continuation on one hand, and ordinary and absolute convergence of numerical series on the other, inspired by the mistakes of Cauchy. A few pages are devoted to ordinary and generalized integrals. The Riemann integral of a bounded function on a bounded interval is defined in a few lines, and (before Lebesgue!) Riemann gives an explicit necessary and sufficient condition for a function to be integrable.

A leitmotiv of Riemann's dissertation is the recognition of the role of Fourier. After the controversy on vibrating strings in which d'Alembert, Euler, Daniel Bernoulli and Lagrange took part, "for almost 50 years the question of representing an arbitrary function by an analytic expression did not make any essential progress. Then a remark of Fourier gave this question a new look, and a new epoch began in this part of mathematics, which proved soon of exceptional importance in the development of mathematical physics. Fourier remarked that, given a trigonometric series $f(x) = \dots$, the coefficients are well-defined by the formulas $a_n = \dots$, $b_n = \dots$. He observed that this definition of the series makes sense for quite arbitrary functions $f(x)$; taking for $f(x)$ a so-called discontinuous function (the ordinate of a broken line above the abscissa x), he obtained a series that actually gave always the value of the function."

Then, after discussing why Lagrange was reluctant and Poisson hostile, Riemann states that "it was Fourier who actually recognized the nature of trigonometric series in a complete and correct way; since then they were applied many times in mathematical physics for representing arbitrary functions, and one can be easily convinced in each particular case that the Fourier series converges indeed to the value of the function."

A general proof was needed. After the attempt by Cauchy, this was realized by Dirichlet for a large class of functions, covering (Riemann said) all possible needs of physics. Nevertheless "the application of Fourier series is not restricted to researches in physics; they are now applied successfully in a domain of pure mathematics, the theory of numbers, where it seems that the most important functions are not those considered by Dirichlet." Here are the exact quotations of Riemann.

> Fast fünfzig Jahre vergingen, ohne dass in der Frage über die analytische Darstellbarkeit willkürlicher Functionen ein wesenticher Fortschritt gemacht wurde. Da warf eine Bemerkung Fourier's ein neues Licht auf diesen Gegenstand; eine neue Epoche in der Entwicklung dieses Teils der Mathematik begann, die sich bald auch äusserlich in grossartigen Erweiterungen der mathematischen Physik kund tat. Fourier bemerkte, dass in der trigonometrischen Reihe
>
> $$f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + \tfrac{1}{2} b_0 + b_1 \cos x + b_2 \cos 2x + \cdots \ ,$$
>
> die Coefficienten sich durch die Formeln
>
> $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, \mathrm{d}x \ , \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, \mathrm{d}x \ ,$$
>
> bestimmten lassen. Er sah, dass diese Bestimmungsweise auch anwendbar bleibe, wenn die Function $f(x)$ ganz willkürlich sei; er setzte für $f(x)$ eine so genannte discontinuirliche Function (die Ordinate einer gebrochenen Linie für die Abcissa x) and erhielt eine Reihe, welche in der That stets den Wert der Function gab.
>
> (Bernhard Riemann [9])

> Durch Fourier was nun zwar die Natur der trigonometrischen Reihen vollkommen richtig erkannt; sie wurden seitdem in der mathematischen Physik zur Darstellung willkürlicher Functionen vielfach angewandt, und in jedem einzelnen Falle überzeugte man sich leicht, dass die Fouriersche Reihe wirklich gegen den Werth der Function convergire. (ibid)

> Zweitens aber ist die Anwendbarkeit der Fourierschen Reihen nicht auf physikalische Untersuchungen beschränkt; sie ist jetzt auch in einem Gebiete der seinen Mathematik, der Zahlentheorie, mit Erfolg angewandt, and hier scheinen gerade diejenigen Funktionen, deren Darstellbarkeit durch eine trigonometrische Reihe Dirichlet nicht untersucht hat, von Wichtigkeit zu sein. (ibid)

3 The Riemann Theory of Trigonometric Series

However, the main and most original part of Riemann's dissertation does not follow the lines traced by Fourier and Dirichlet. Instead of starting from a function and looking for the properties of the Fourier series, in particular the convergence problem, Riemann started from an everywhere convergent trigonometric series and looked for the properties of the sum as a function. The machinery that he created is known as the Riemann theory of trigonometric series. It implies double formal integration of the given series and formal multiplication by sufficiently regular series (or functions): viewed from now, it is an anticipation of the treatment of Schwartzs distributions. More directly, it anticipates the "smooth functions" of Zygmund and the "pseudomeasures" and "pseudofunctions" of Kahane and Salem.

The Riemann theory of trigonometric series played a major role in the history of mathematics because of two questions that Riemann left open and that were solved by Georg Cantor: prove that, if an everywhere convergent trigonometric series converges everywhere, its coefficients tend to zero, and, more difficult, that if it converges everywhere to zero it is the null series. This last statement is the "uniqueness theorem" of Cantor. Then Cantor looked for an "extension": is the result valid when the assumption is weakened in the form that the series converges to zero out of a given set? Cantor proved that it was the case for "reducible" sets (with countable closure). It was his first opportunity to develop his theory of real numbers and real sets, so that trigonometric series appear as the first historical source of the theory of sets.

Sets of uniqueness (for which the answer to the question is positive) and sets of multiplicity (the opposite) became a favorite field for applying methods coming from real and functional analysis, probability, number theory, and logics. The main contributors were Lebesgue, Marcel Riesz, Young, Menchoff, Rajchman, Bari, Zygmund, Marcinkiewicz, Salem, then R. Kaufman, T. Körner, J.-P. Kahane, B. Mandelbrot, Y. Katznelson, B. Connes, and more recently A.S. Kechris and A. Louveau, G. Debs and J. Saint-Raymond, J. Bourgain, M. Ash and G. Wang. A brief history of the subject can be found in [4].

The uniqueness theorem of Cantor means that, given the sum of an everywhere convergent trigonometric series, the coefficients are well-defined. In order to compute them, a new kind of Fourier formulas is needed, with a new meaning for the integral. It is the purpose of the second "totalization" theory of A. Denjoy, and his "Leçons sur le calcul des coefficients d'une série trigonométriques" (1941, 1949) is a complete exposition of the subject, in four books.

4 The Convergence Problem After Dirichlet

The Riemann theory and the Cantor uniqueness theorem are part of the heritage of Fourier, but not in the main direction.

The main direction was well described by Riemann himself. It is the direction explored by Dirichlet, the convergence problem.

Dirichlet believed that the Fourier series of a continuous function should converge pointwise to the function, though he was not able to prove it. This was disproved only in 1873, by Paul du Bois-Raymond. He constructed a continuous function whose Fourier series diverges at a given point. The construction is a kind of "condensation of singularities". Variations were given by Lebesgue and by Fejer. Today it appears as a standard application of the Banach–Steinhaus theorem, using the fact that the L^1–norms of the Dirichlet kernel (the so-called "Lebesgue constants") are not bounded.

Is it possible to construct a continuous function whose Fourier series diverges everywhere? The question was still open in 1965, before the Carleson theorem, and Katznelson and I proved that either it is the case, or the Fourier series of any continuous function converges almost everywhere to the function. The key was to construct a continuous function whose Fourier series diverges on a given null-set.

4.1 The Carleson Theorem

The question was settled by the Carleson theorem of 1966: the Fourier series of a continuous function converges to the function almost everywhere and nothing better can be said for continuous functions [1]. But much more can be said, by enlarging the class of functions. Carleson proved that the result holds for L^2 (1966), and R. Hunt for L^p with $p > 1$ (1967). On the other hand, the result does not hold for L^1, since Kolmogorov constructed an L^1–function whose Fourier series diverges almost everywhere (1922) and even everywhere (1926). The situation near L^1 was investigated recently and raises some puzzling questions, as we shall see later.

In order to understand the importance of Carleson–Hunt in 1966–67, let us consult the successive editions and impressions of Zygmund's book "Trigonometric series" [11]. For general L^2–functions, the best known result before 1966 was due to Kolmogorov (1922): given a Hadamard lacunary sequence (n_k), for example $n_k = 2^k$, the partial sums of order n_k tend to the function almost everywhere. For general L^p–functions, $p > 1$, the same holds but the proof relies on a highly sophisticated machinery, the Littlewood–Paley theory (1938). Two chapters of the second edition of Zygmund's book are devoted to the Littlewood–Paley theory and to what was considered as its main application, the almost everywhere convergence of partial sums of order n_k. The discovery of Carleson changed the landscape in a drastic way. First, Kolmogorov's result was not the best anymore. Second, due to Hunt's extension of Carleson's theorem for all L^p, $p > 1$, the Littlewood–Paley theorem became obsolete as a means to study the convergence problem. This was a shock for Zygmund. There were several new "impressions" of the second edition of "Trigonometric series" during his life-time, but no "third edition". A "third edition" would have been a complete rewriting of the book: Carleson–Hunt would have been included, and many partials results and related methods dropped; there would have been no reason to keep the Littlewood–Paley theory. Fortunately, Zygmund gave up such a rewriting and the book was kept with its beautiful exposition of the Littlewood–Paley theory, without Carleson–Hunt.

The situation near L^1 is in close relation with the asymptotic behaviour of the partial sums $S_n(f,x)$ when f belongs to L^1. The first important result is due to Hardy (1913): if $f \in L^1$, $S_n(f,x) = o(\log n)$ almost everywhere. Hardy conjectured that it was a best possible result. In the opposite direction, the Kolmogorov example of 1926 establishes the existence of $f \in L^1$ such that $\lim |S_n(f,x)| = \infty$ everywhere. For which increasing sequences $\ell(n)$ is it true that there exists $f \in L^1$ such that $\lim(|S_n(f,x)|/\ell(n)) = \infty$ everywhere? Until a few years ago, the best result was that $\ell(n) = o(\log\log n)$ works (Chen 1962). In 1999, Konyagin [8] went as far as $\ell(n) = o((\log n/\log\log n)^{1/2})$, and Bochkarev proved in 2003 that $\ell(n) = o((\log n)^{1/2})$ is sufficient when the circle is replaced by the Cantor group. It is a challenging problem now to improve either Hardy or Konyagin. A test case is $\ell(n) = (\log n)^p$ with $1/2 < p < 1$.

4.2 Variations About Convergence

The revival of Fourier series in the twentieth century is due in a large part to other ways to consider the convergence problem.

In the first place one can introduce summability processes instead of ordinary convergence. This way was opened by Fejer in 1900 and it led to positive kernels, approximate identities, multipliers and convolution. The most important notion, convolution, was formalized rather late and actually no formal definition covers the real range of the notion. It became better understood with the convolution algebras of Wiener, one of the sources of the normed rings (Banach algebras) of Gelfand.

On the other hand convergence can be considered in spaces of functions, and summability as well. This direction created a strong link between Fourier series and the beginning of functional analysis. The initial impulse was the Lebesgue integral (1901) and its application to Fourier series (1906), followed by the Riesz–Fischer theorem (1907). The simple statement "L^p is complete" (needing first the definitions of L^p–spaces and complete metric spaces) belongs to the heritage of Fourier.

Fourier series are a prototype of orthogonal series, and orthogonal series appear in all parts of analysis. The L^2–convergence is guaranteed, but problems on almost everywhere convergence are quite interesting, parallel to those on classical Fourier series.

Thousands of papers and hundreds of books being written on these subjects, I shall not insist on them any more.

5 Fourier Comes Back

Let us go back to Fourier.

His first and typical use of trigonometric series was a solution of a problem on the distribution of temperatures in a given solid body, a cylinder based on the half strip $-\pi/2 \le x \le \pi/2$, $0 \le y < \infty$, with $-\infty < z < \infty$. The horizontal basis ($y = 0$) is at temperature 1, the vertical edges ($x = \pm\pi/2$) at temperature 0. At the state of equilibrium the temperature $u(x, y)$ satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 ,$$

with $u(\pm\pi/2, y) = 0$ and $u(x, 0) = 1$ ($-\pi/2 < x < \pi/2$). A formal solution is

$$u(x, y) = a_1 \mathrm{e}^{-y} \cos x + a_3 \mathrm{e}^{-3y} \cos 3x + a_5 \mathrm{e}^{-5y} \cos 5x + \cdots \qquad (1)$$

and the last requirement is expressed by the condition

$$1 = a_1 \cos x + a_3 \cos 3x + a_5 \cos 5x + \cdots \qquad (-\pi/2 < x < \pi/2) . \qquad (2)$$

Before writing the integral formulas for the coefficients, Fourier computed them in a strange way: differentiation, truncation, solution of a linear system. Darboux said that this was a natural method and he was right. What Fourier did was to look at trigonometric polynomials $1 - \sum a_n \cos nx$ (n odd) of a given degree that are as flat as possible at 0, and compute the limits of the a_n when the degree increases to infinity, namely $(-1)^{n+1} 4/\pi n$. The derivatives of these trigonometric polynomials are as flat as possible at 0, odd, and their integral on $(-\pi/2, 0)$ is 1. In the sense of distributions they converge to $2(\delta_{-\pi/2} - \delta_{\pi/2})$, and this is enough in order to establish (1) when $y > 0$. The proof of (2) is more delicate.

Of course, Fourier did not know the theory of distributions. But he had a flair for the meaning of computations, and it may be wise to try to understand what he did before condemning his methods and statements.

After giving the integral formulas Fourier observes that, due to the exponential factors, the series (1) is very rapidly convergent when $y > 0$ ("très convergente", "extrèment convergente"). About series (2) he says that convergence can be proved ("on démontre rigoureusement") and actually he gives a correct proof later in the chapter.

Then he extends the procedure to other functions, gives the Fourier formulas for general 2π–periodic functions, and states that this applies to arbitrary functions and gives trigonometric series which always converge to the functions. In a formal way this is wrong, and we just discussed some of the main contributions of mathematicians in order to find the correct notions and statements. Fourier was criticized, by Darboux in particular, for being inaccurate in some of his statements, and that explains in part why he was so long in disfavour with French people. But Fourier deserves to be appreciated not because he proved theorems and made perfect statements, but in the way he launched a long-term program in mathematics.

Examples can be found in all parts of The Analytical Theory of Heat, of statements that were considered as absurd and may also sound prophetic. Here is one ([2] Chap. III, pp. 235). From the integral formulas for coefficients Fourier derives the formula

$$F(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} F(\alpha)\mathrm{d}\alpha \left(\frac{1}{2} + \sum_i \cos i(x-\alpha)\right) .$$

The comment of Darboux is that the parenthesis has no possible meaning. However, here is Fourier's explanation:

> L'expression (...) représente une fonction de x et de α telle que, si on la multiplie par une fonction quelconque $F(\alpha)$ et si, après avoir écrit dα, on intégre entre les limites $\alpha = -\pi$ et $\alpha = \pi$, on aura changé la fonction proposée $F(\alpha)$ en une pareille fonction de x, multipliée par la demi-circonférence. On verra par la suite quelle est la nature de ces quantités, telles que $1/2 + \sum \cos i(x-\alpha)$, qui jouissent de la propriété que l'on vient d'énoncer.

Clearly, I believe, Fourier had the intuition of the Dirac measure and the way to use it (in particular, in later sections, derivation and representation as a Fourier integral).

It should be noted that the first exposition by Laurent Schwartz of his theory of distributions is an article (1946) entitled "Généralisation de la notion de fonction, de dérivation, de transformation de Fourier, et applications mathématiques et physiques" [10]. In a way, the theory of distributions belongs to the heritage of Fourier.

I shall be brief on more recent and spectacular reincarnations of Fourier series. The Fast Fourier Transform of Cooley and Tuckey (1965), presented as "An algorithm for the machine calculation of complex Fourier series", has invaluable applications in all parts of science, from astrophysics to biology.

The wavelets of Yves Meyer (1985) came from physicists and engineers, and soon created a common ground for specialists of different fields of science and industry. The story is well known and still in progress[1] and there is no point in telling it again.

As far as the heritage of Fourier is concerned, the main point is that FFT and wavelets testify that the philosophy of Fourier, expressed in the excerpts of the Discours préliminaire I gave above, makes a spectacular come-back.

I indicated at the beginning that I would restrict the heritage of Fourier to his work as a mathematician and, to be more specific, to Fourier series. I chose, actually, a very few points of interest in the history of Fourier series, and I hardly mentioned *one* open problem. My oral communication was not organized exactly in that way. But I feel better now to contribute to "Perspectives in Analysis" by a reflection on the past than by a list of open problems according to my taste. The past is very rich and the way in which it echoes the most recent research in mathematics is actually part of "Perspectives in Analysis".

[1] See Jaffard et al. [3] for example.

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