Open Questions on the Mumford–Shah Functional

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1 Introduction

The Mumford–Shah functional was introduced in [20] as a tool for image segmentation. In this context, we are given a simple domain Ω (the screen) and a bounded function g on Ω (representing the image), and we look for a simplified approximation u of g. Here simplified means that we would like u to have only slow variations on Ω, except that we want to allow jumps on a nice singular set K, which we think of as a set of boundaries. In the good cases, it is hoped that the pair (u, K), or even K or u alone, will retain important information on g and drop less interesting details or noise.

Mumford and Shah proposed to get image segmentations by minimizing the following functional, which we already define on $\mathbb{R}^n$ to save time (but so far image processing corresponds to $n = 2$). Set

$$J(u, K) = H^{n-1}(K) + \int_{\Omega \setminus K} |\nabla u|^2 + \int_{\Omega \setminus K} |u - g|^2 \, , \tag{1}$$

where we restrict to pairs (u, K) such that K is a closed set in Ω, with finite Hausdorff measure $H^{n-1}(K)$ of codimension 1, and u lies in the Sobolev space $W^{1,2}(\Omega \setminus K)$, which means that it has a derivative in L^2 on that open set. For the purpose of this lecture, it would be enough to restrict to $u \in C^1(\Omega \setminus K)$, because this is the case for minimizers, but this would be less natural.

The three terms in the functional correspond to the three constraints on u that were mentioned above, except perhaps that $H^{n-1}(K)$ does not really measure how simple K is. We shall return to this soon. In principle one should multiply two of the three terms by tuning constants, but we can normalize out one of them by multiplying u and g by a constant, and the other one by dilating everything.

The existence of pairs (u, K) for which $J(u, K)$ is minimal (we shall call them *minimizers*) is far from trivial, because we can easily find minimizing sequences $\{(u_k, K_k)\}$ that converge to pairs (u, K), but for which

$H^{n-1}(K) > \limsup_{k\to+\infty} H^{n-1}(K_k)$. Nevertheless, minimizers always exist; this was proved by Ambrosio [2] and De Giorgi, Carriero, Leaci [15].

Then we can ask what minimizing pairs look like. Let us first observe that when K is fixed, minimizing $J(u,K)$ in terms of u is rather simple. The two last terms of the functional are convex in u, so there is a unique minimizer; in addition this minimizer is rather regular (away from K), for instance because it satisfies the elliptic equation $\Delta u = u - g$ on $\Omega \setminus K$. As for regularity near K, the best way to get is to show first that K is nice. In this respect, it may help the reader to think about the special case of dimension 2 and the simpler local minimization of $\int_{\Omega\setminus K} |\nabla u|^2$ in $\Omega \setminus K$. In this case, we can use the fact that energy integrals are conformally invariant to map K locally into something nicer (like a line) by a conformal mapping ψ; the regularity of u will then depend mostly on the size of ψ'. The general case (with the extra term $\int |u-g|^2$ and in higher dimensions) is not very different.

So the main problem is the regularity of the singular set K. Observe that if we add a set of H^{n-1}–measure 0 to K, we do not change $J(u,K)$ but our description of K may become more complicated. For this reason we shall restrict to *reduced minimizers*, i.e., minimizers (u,K) for which we cannot find $K_1 \subset K$, $K_1 \neq K$, such that u has an extension in $W^{1,2}(\Omega \setminus K_1)$. It is rather easy to see that for each minimizer (u,K) there is a reduced minimizer (u_1,K_1) such that $K_1 \subset K$ and u_1 is an extension of u.

In dimension $n = 2$, Mumford and Shah conjectured that if (u,K) is a reduced minimizer for J and Ω is bounded and smooth, K is a finite union of C^1 arcs of curves, that may only meet by sets of three, at their ends, and with angles of 120°. The reader should not pay too much importance to the precise statement, with the C^1 curves. The point is to get some minimal amount of regularity; once we know that K is a C^1 curve near a point, it is easy to get additional regularity (like $C^{1,1}$, see [5], and even better if g is smooth).

In higher dimensions, it is reasonable to expect that there is a set of codimension 2 in K out of which K is locally a C^1 hypersurface, but there is no very precise conjecture on the shape of that set of codimension 2. We shall return to this in Sect. 6.

Many partial results are known. We shall try to describe a few in the next sections, but the main goal of this text is rather to convince the reader that there are many interesting open questions besides the celebrated Mumford–Shah conjecture above; in addition, some of them could even be easy.

Before we move to a rapid discussion of known regularity results, let us observe that they are also good news for image segmentation. First they mean that even though we only put the term $H^{n-1}(K)$ in the functional, singular sets of minimizers are actually smoother than this term suggest, and in particular they look like what we would normally think boundaries in an image should look like. If minimizers for J had often been unrectifiable Cantor sets, J would probably have been much less used for image segmentation. Also, good regularity properties for minimizers probably mean better resistance to noise (because the functional cannot render noise, even if it tries to), and should help with the computations of minimizers, both because it should make them more stable and because we already know what to look for.

Nonetheless, it seems fair to say that J and its variants are a little less used in image segmentation nowadays, probably because people prefer to use algorithms that include some a priori knowledge, often of a statistical nature, on the images to be treated. See [18] and the references therein for more information on image processing.

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2 First Regularity Results

Let (u, K) be a reduced minimizer for J in a domain $\Omega \subset \mathbb{R}^n$. A first remarkable property of K is its local Ahlfors–regularity: there is a constant C, that depends only on n and $||g||_\infty$, such that

$$C^{-1} r^{n-1} \leq H^{n-1}(K \cap B(x, r)) \leq C r^{n-1} \tag{2}$$

when $x \in K$, $0 < r \leq 1$, and $B(x, r) \subset \Omega$. The second inequality is very easy (and does not require $B(x, r)$ to be contained in Ω): simply consider the competitor (v, G) obtained by keeping $G = K$ and $v = u$ out of $\overline{B}(x, r)$, setting $K = \emptyset$ and $v = 0$ in $B(x, r)$, and adding $\partial B(x, r) \cap \Omega$ to G to allow jumps across $\partial B(x, r)$. Thus we take $G = [K \setminus \overline{B}(x, r)] \cup [\partial B(x, r) \cap \Omega]$. When we write that $J(u, K) \leq J(v, G)$, we get that

$$\begin{aligned} H^{n-1}(K \cap B(x, r)) &+ \int_{\Omega \cap B(x,r) \setminus K} |\nabla u|^2 \\ &\leq H^{n-1}(\partial B(x, r) \cap \Omega) + \int_{B(x,r) \cap \Omega} |g|^2 \leq C r^{n-1} + C r^n ||g||_\infty^2 , \end{aligned} \tag{3}$$

which yields the second inequality in (2) when $r \leq 1$.

The first inequality in (2) is due to Dal Maso, Morel, and Solimini [8] when $n = 2$, and to Carriero and Leaci [7] in higher dimensions, and it is more subtle. It means that it does not help to put just a small amount of K somewhere, because it will not be enough to create big jumps across K and release the tension. That is, suppose we have a ball B such that $H^{n-1}(K \cap B)$ is very small; we want to say that you could do better by removing $K \cap B$ from the singular set. This will allow you to save $H^{n-1}(K \cap B)$ in the first term of J, but of course you will have to replace u in B with a function $\widetilde{u}$ that does not jump across $K \cap B$. A good choice is to take the harmonic extension of the values of u on ∂B. Then we can estimate how much larger $\int |\nabla \widetilde{u}|^2$ has to be (one shows that the contributions that come from the $|u-g|$ term can be neglected). An integration by parts shows that the price we have to pay is essentially the integral on $K \cap B$ of a product $\mathrm{Jump}(u) \times \frac{\partial \widetilde{u}}{\partial n}$, of the jump of u across $K \cap B$ times the derivative of $\widetilde{u}$ in a direction perpendicular to K. The computations can essentially be done like this when $n = 2$, and this is the essence of the proof in [8]. The point is that the jump of u can be estimated, and it is at most $CH^1(K \cap B)^{1/2}$. Then the integral of $\mathrm{Jump}(u) \times \frac{\partial \widetilde{u}}{\partial n}$ is much smaller than $H^1(K \cap B)$, and we get the desired contradiction. When $n > 2$, complications appear, in particular because we may not be able choose B so that $\partial B \cap K$ is empty, and the proof of [7] uses a compactness argument.

Notice that (2) is already good to know for image processing.

We talked a lot about (2) because it is a very useful tool for proving other estimates, for instance because it says that K is a space of homogeneous type. Let us just give an example. Set

$$\omega_p(x,r) = r^{1-2n/p} \left\{ \int_{B(x,r) \setminus K} |\nabla u|^p \right\}^{2/p}, \tag{4}$$

for $1 \leq p \leq 2$, $x \in K$, and $B(x,r) \subset \Omega$; do not pay too much attention to the normalizations, the main point is to have a dimensionless quantity. The trivial estimate (3) says that $\omega_2(x,r) \leq C$ for $r \leq 1$, but with (2), (3), Hölder, and Fubini, one can readily show the Carleson measure estimate

$$\int_{y \in K \cap B(x,r)} \int_{t=0}^{r} \omega_p(y,t) \, \frac{dH^{n-1}(y)\mathrm{d}t}{t} \leq C_p r^{n-1}, \tag{5}$$

for $x \in K$ and $0 < r \leq 1$ such that $B(x,3r) \subset \Omega$, provided that we take $p < 2$ [12]. It often turns out that replacing 2 with a smaller exponent p is a very small price to pay, compared to the fact that (5) gives lots of pairs such that $\omega_p(y,t)$ is as small as we want. For instance (5) is a very good starting point for the results quoted just below; the good balls mentioned below are precisely balls where $\omega_p(y,Ct)$ is small.

Next (and omitting quite a few interesting results), K is locally uniformly rectifiable and contains big pieces of Lipschitz graphs ([12], [14]). But even more is true ([9], [3], [21]): we can find lots of balls where K is C^1. More precisely, there is a constant C such that, for each $x \in K$ and $r \in (0,1]$ such that $B(x,r) \subset \Omega$, we can find $y \in K \cap B(x,r/2)$ and $t \in [r/C, r/2]$ such that $K \cap B(y,t)$ is the intersection of $B(y,t)$ with the graph of a 10^{-2}–Lipschitz and C^1 function.

As a consequence there is a small set, in fact of Hausdorff dimension less than $n-1$, out of which K is locally nice and C^1. But we would like to know more about that small set, and the way the different pieces are attached to each other.

See [4], [10], [13], [18] and their references for proofs and lots of other regularity results on K. Here we shall just say a little more about a result of Bonnet [5]:

Fact 2.1. In dimension 2, every *isolated* connected component of K is a finite union of C^1 curves.

The reason why we mention this is because the approach that led to it is very important. Notice that Fact 2.1 is much more precise than the C^1 result mentioned above, but only if we can find an isolated component.

3 Blow-up Limits and Global Minimizers

We start with a very simple observation on dilation invariance. Pick a point $x \in \Omega$ (typically, $x \in K$) and a small radius t, and dilate everything so that $B(x,t)$ becomes the unit ball. That is, set $\Omega_{x,t} = t^{-1}(\Omega - x)$, $K_{x,t} = t^{-1}(K - x)$, $g_{x,t}(y) = \frac{1}{\sqrt{t}}\, g(x+ty)$ for $y \in \Omega_{x,t}$, and $u_{x,t}(y) = \frac{1}{\sqrt{t}}\, u(x+ty)$ for $y \in \Omega_{x,t} \setminus K_{x,t}$. We divided by $\sqrt{t}$ so that the two first terms of J in (1) would scale the same way. A simple computation shows that if (u,K) minimizes J, $(u_{x,t}, K_{x,t})$ is a minimizer for $J_{x,t}$, where

$$J_{x,t}(v,G) = H^{n-1}(G) + \int_{\Omega_{x,t}\setminus G} |\nabla v|^2 + t^2 \int_{\Omega_{x,t}\setminus G} |v - g_{x,t}|^2 \,. \tag{6}$$

Notice that when t gets small, the third term becomes less and less important, even though $||g||_\infty^2$ is divided by t. This corresponds to the (desired for image segmentation) effect that the third term of J in (1) should have little influence at small scales, and is one of the basic facts of the theory. It is thus surprising that we had to wait till [5] before one actually decided to take blow-up sequences and study their limits, as follows.

Fix a point $x \in K$, and take a sequence $\{t_k\}$ that tends to 0. Set $(u_k, K_k) = (u_{x,t_k}, K_{x,t_k})$ to simplify notation. One hopes that limits of such pairs are simpler, and that they will help us study (u,K).

The first thing to check is that given any sequence $\{(u_k, K_k)\}$ as above, we can extract subsequences that converge. Note that Ω_{x,t_k} converges to $\mathbb{R}^n$, so the limits should be pairs (v,G) that live on $\mathbb{R}^n$.

We do not have too much choice when defining the convergence of $\{K_k\}$ to a limit K_∞; if we want to be sure to have convergent subsequences, we have to use the local convergence in Hausdorff distance, which is also the most reasonable notion. The convergence of $\{u_k\}$ to v is a little more delicate to define, because the L^∞ norms of the u_k will normally tend to $+\infty$. On the other hand, we have a good uniform control on the gradients away from G, by (3), so we just require the convergence of the gradients ∇u_k in some L^p, on compact subsets of $\mathbb{R}^n \setminus G$. Equivalently (and this is the definition used in [10]) we require the following property.

Property 3.1. For each connected component V of $\mathbb{R}^n \setminus G$, there exists constants $\alpha_{V,k}$ such that $\{u_k - \alpha_{V,k}\}$ converges to v in $L^1(H)$ for every compact subset H of V.

This looks a little strange at first, because we have to subtract constants (and then v in only known up to an additive constant in each component of $\mathbb{R}^n \setminus G$), but this is the right way to deal with the fact that $||u_k||_\infty$ tends to $+\infty$.

With these definitions, it is not hard to extract convergent subsequences from any $\{(u_k, K_k)\}$ as above. One of the main points of [5] is the following.

Fact 3.2. If (u, K) is a reduced minimizer for J and $\{(u_k, K_k)\}$ converges to (v, G), then (v, G) is a reduced global minimizer in $\mathbb{R}^n$.

We shall see the definition in a moment, but let us first comment a little. First of all, Fact 3.2 is not trivial, once again because its verification involves checking that

$$H^{n-1}(G \cap U) \leq \lim_{k \to +\infty} H^{n-1}(K_k \cap U) , \tag{7}$$

for every open set $U \subset \Omega$, and this would brutally fail for a general sequence of sets K_k. But it turns out that since the K_k come from Mumford–Shah minimizers, they have a very nice property, the so-called uniform concentration property.

To simplify things slightly, we shall say that this property holds when, for every $\varepsilon > 0$, we can find $C > 0$ such that, for each ball $B(x, r)$ contained in Ω, centered on a K_k, and with radius $r \leq 1$, we can find another ball $B(y, t) \subset B(x, t)$, centered on the same K_k, with radius $t \geq C^{-1} r$, and for which we have the nearly optimal concentration

$$H^{n-1}(K_k \cap B(y, t)) \geq (1 - \varepsilon)\, \omega_{n-1}\, t^{n-1} , \tag{8}$$

where ω_{n-1} is the H^{n-1}–measure of the unit ball in $\mathbb{R}^{n-1}$. Thus the measure in (8) is at least almost as large as if K_k were a hyperplane through y. Then (7) holds as soon as the uniform concentration property holds.

We simplified things a little, because the uniform concentration property of [8] also allows us to modify slightly the quantifiers above (but the important thing is still to keep C independent of k); also their result still works in (integer) dimensions $d < n - 1$. See [8] or [18] for details.

Uniform concentration was introduced in [8], and proved for minimizers in [8] when $n = 2$ and [22] when $n > 2$. Also see in [10] for a proof with uniform rectifiability.

Once we get the semicontinuity of Hausdorff measure as in (7), the rest of the proof of Fact 3.2 is a technical, but not surprising argument. See [5] for the initial proof in two dimensions, [16] for a first generalization, and [17], [10] for extensions to higher dimensions and different limiting situations.

It is time to define global minimizers. Let us only consider “acceptable pairs” (v, G) such that G is a closed subset of $\mathbb{R}^n$, $v \in W^{1,2}_{loc}(\mathbb{R}^n \setminus G)$, and in addition

$$H^{n-1}(G \cap B(0,R)) + \int_{B(0,R)\setminus G} |\nabla v|^2 < +\infty \,,$$

for every $R > 0$. A competitor for (v, G) is another acceptable pair $(\widetilde{v}, \widetilde{G})$ such that for R large enough, the pair $(\widetilde{v}, \widetilde{G})$ coincides with (v, G) out of $\overline{B}(0, R)$, and in addition G satisfies the following topological condition.

Property 3.3. If $x, y \in \mathbb{R}^n \setminus [G \cup \overline{B}(0,R)]$ lie in different connected components of $\mathbb{R}^n \setminus G$, then they lie in different connected components of $\mathbb{R}^n \setminus \widetilde{G}$.

We say that the acceptable pair (v, G) is a *global minimizer* if

$$H^{n-1}(G \cap \overline{B}(0,R)) + \int_{B(0,R)\setminus G} |\nabla v|^2 \leq H^{n-1}(\widetilde{G} \cap \overline{B}(0,R)) + \int_{B(0,R)\setminus \widetilde{G}} |\nabla \widetilde{v}|^2 \,, \quad (9)$$

for every competitor $(\widetilde{v}, \widetilde{G})$ for (v, G), and R as above. Note that (9) does not depend on R large.

The topological constraint of Property 3.3 on competitors may seem a little strange, but it is imposed to us by the fact that we needed to work modulo constants when taking limits. If we did not ask for Property 3.3, taking for G a straight line and v a locally constant function on $\mathbb{R}^2 \setminus G$ would not give a global minimizer. This would be bad, because we know that such a pair shows up each time you take the blow-up limit of a Mumford–Shah minimizer (u, K) at a point where K is a C^1 curve.

So we have a definition of global minimizers, and we define reduced global minimizers as we did for Mumford–Shah minimizers in a domain. Thus Fact 3.2 makes sense.

Our big hope, once we have Fact 3.2, is that global minimizers in $\mathbb{R}^2$, for instance, will turn out to be so simple that we can list them completely (and later return to the Mumford–Shah minimizers in a domain if we are still interested). Of course the definition of global minimizer is a little more complicated, because of this strange Dirichlet condition at infinity where (v, G) itself gives the Dirichlet data, but at least we lost the auxiliary function g and we have just two terms to worry about.

4 Global Minimizers in the Plane

Here is a list of global minimizers in the plane. For the first three, we take $v = 0$, or equivalently v constant on each component of $\mathbb{R}^2 \setminus G$. This works if and only if G is a minimal set. Here our definition of minimal set, suggested by Property (3.3) and (9), is that you cannot make it shorter by compact modifications that do not merge the connected components near infinity. But it is easy to see that it is equivalent to the more usual notion, where we say that we cannot make G shorter by deforming it in a bounded region. The only options for G are the empty set, or a line, or the union of three half lines with a same origin, and that make angles of $120°$ with each other (we shall call this a Y). In these three cases the verification that (v, G) is a reduced global minimizer is rather easy.

There is a fourth known type of global minimizer, where G is a half line. By translation and rotation invariance, we may assume that G is the negative first axis $\{(x, 0)\,;\, x \leq 0\}$; then we take

$$v(r\cos\theta, r\sin\theta) = \pm\sqrt{\frac{2r}{\pi}}\,\sin(\theta/2) + C\ , \tag{10}$$

for $r > 0$ and $-\pi < \theta < \pi$, where the choice of constants $\pm$ and C does not matter.

It is not too hard to see that (when G is the half line) the only choices of v for which (v, G) is a global minimizer must given by (10), but the verification that (10) gives a global minimizer is long and painful [6].

Here is the natural analogue of the Mumford–Shah conjecture in the present context.

Conjecture 4.1. Every reduced global minimizer in the plane is of one of the four types described just above.

Conjecture 4.1 implies the Mumford–Shah conjecture. See [10], but this was essentially known by Bonnet [5], who only wrote down the part he needed to show that the result quoted below (when G is connected) implies Fact 2.1.

A priori Conjecture 4.1 is a little stronger, because there could be global minimizers in the plane that do not arise as blow-up limits of Mumford–Shah minimizers in a domain. But it is hard to imagine that the Mumford–Shah conjecture will be proved without Conjecture 4.1.

Only partial results are known so far. Here is a short list; see [10] and [17] for more.

If (v, G) is a reduced global minimizer in $\mathbb{R}^n$, then G is Ahlfors–regular, uniformly rectifiable, and we can even find lots of balls centered on G where G is a 10^{-2}–Lipschitz and C^1 graph. The statements and proofs are the same as in the local case in the domain, except that we no longer need the restrictions that $r \leq 1$ or $B(x, r) \subset \Omega$.

Conjecture 4.1 holds when

- G is connected [5],
- G is contained in a countable union of lines [16],
- G is symmetric with respect to the origin [11], [17],
- $\mathbb{R}^2 \setminus G$ has at least two connected components, or when G contains two disjoint connected unbounded sets [11].

Many of the results listed here use a monotonicity argument, in addition to the usual techniques of finding alternative competitors. Again the trend started in [5], where one of the main ingredients of the proof was to show the following.

Fact 4.2. If (v, G) is a global minimizer in $\mathbb{R}^2$ and G is connected, then

$$\Phi(r) = \frac{1}{r} \int_{B(x,r)\setminus G} |\nabla v|^2$$

is a nondecreasing function of r, for each $x \in \mathbb{R}^2$.

This was proved by computing Φ', integrating by parts, and then using an inequality of Wirtinger. But (unfortunately) the connectedness of K was used to remove the mean value of u in some intervals, and it seems to be really needed in the argument.

Things become easier once you have Fact 4.2, because the proof of Fact 4.2 also gives the form of (u, K) when Φ is constant. This can be used to control blow-up and blow-in limits of (v, G), because the analogue of Φ for those is constant. Then we get a better local description of (u, K), and we can use it to conclude. The proof takes some time though. The improvement of the situation when we have a monotonicity result like Fact 4.2 should probably be compared with what happens with minimal surfaces, where an analogous monotonicity result implies that the tangent objects to the set are minimal cones.

In [11] also one shows monotonicity, but for the different function $\Psi(r) = 2\Phi(r) + r^{-1}H^1(G \cap B(x,r))$. The proof uses a combination of the Bonnet estimate and direct comparison with various competitors constructed by hand.

J.-L. Léger [16] found a nice formula that allows you to compute v in terms of G (modulo the obvious invariance of the problem) when (v, G) is a global minimizer in $\mathbb{R}^2$. Identify $\mathbb{R}^2$ with the complex plane, and set

$$F(z) = \frac{\partial v}{\partial x} - \mathrm{i}\frac{\partial v}{\partial y} = 2\frac{\partial v}{\partial z} ,$$

for $z \in \mathbb{R}^2 \setminus G$. Notice that F is holomorphic in $\mathbb{R}^2 \setminus G$, because v is real and harmonic. Léger says that F^2 is the Beurling transform of $H^1_{|G}$, i.e.,

$$F(z)^2 = -\frac{1}{2\pi} \int_G \frac{dH^1(w)}{(z-w)^2} \quad \text{for } z \in \mathbb{R}^2 \setminus G . \tag{11}$$

Finally, let us mention that global minimizers in a half plane $\mathbb{P}$ are easier to characterize: the only possibilities for G are the empty set, or a half line starting from $\partial\mathbb{P}$ and perpendicular to it, and in both cases v is constant on each component of $\mathbb{P} \setminus G$. We can use this to return to Mumford–Shah minimizers in a bounded smooth domain and prove that near $\partial\Omega$, K is only composed of C^1 curves that meet $\partial\Omega$ perpendicularly. See [17] or [11] and [10].

See [10] and its references for more results about global minimizers.

5 General Questions

We start with a few questions in the plane. Besides the Mumford–Shah conjecture (or partial questions), of course.

There is no uniqueness in general for minimizers of the Mumford–Shah functional. For instance, if you take $\Omega = \mathbb{R}^2$ and g equal to the characteristic function of $B(0,r)$, there is one value of r for which J has exactly two minimizers, one with $K = \partial B(0,r)$ and one with $K = \emptyset$. See for instance [13] or [10]. It is easy to construct lots of examples of the same type, where the lack of uniqueness comes from a jump in the solution. But maybe we still have uniqueness for generic data g. This question also holds in $\mathbb{R}^n$.

We do not know yet that the global minimizer in (10) ever shows up as the blow-up limit of some Mumford–Shah minimizer in a domain. Even if we restrict to $\Omega = B(0,1)$ and take $g = v$, we do not know how to show that the restriction of (v,G) to Ω minimizes J. See [1] for the corresponding positive answer for a Y.

Suppose that the minimizer in (10) is a blow-up limit at x of the Mumford–Shah minimizer (u,K) in a domain Ω. We know that in a pointed neighborhood of x, K is a C^1 curve that looks flatter as we get near x (see for instance Sect. 69 of [10]), but we do not know precisely how K behaves at x (is the curve C^1 up to x, or can it turn infinitely many times around x, for instance?).

The easier case of a half plane suggests that it may be interesting to look at global minimizers in other planar domains, or even in larger 2–dimensional surfaces. So far we only know about sectors with aperture $< 3\pi/2$ [11].

Many of the partial results in the plane become questions in higher dimensions. For instance, it is not known whether every connected component of $\mathbb{R}^n \setminus G$ is a John domain with center at infinity; we just know that they are unbounded. Or whether v can be constant in a component of $\mathbb{R}^n \setminus G$ without being constant on every component. When $n = 2$, these things are true, and we even know that $\mathbb{R}^2 \setminus G$ is connected in the interesting cases.

We would also like to generalize some of our perturbations results. We know from [3] that if (u,K) is a Mumford–Shah minimizer or a global minimizer, in any dimension, and if $B(x,r)$ is a small ball centered on K where K is very flat (i.e., close to a plane) and $r^{1-n}\int_{B(x,r)} |\nabla u|^2$ is very small, then $K \cap B(x,r/2)$ is a nice C^1 hypersurface, i.e., a C^1 perturbation of a plane. We would like to know whether in such a statement, we can replace planes with other minimal sets. We know about the case of the Y in the plane [9], and we would like to know about the product of the Y with a line (in $\mathbb{R}^3$), for instance. This would also be an excuse for understanding these minimal sets better.

A little more difficult would be to obtain analogues of the result that says that if some blow-in limit of the global minimizer (v,G) is the cracktip minimizer from (10), then (v,G) itself is as in (10) [6]. All these things should be useful if we want study global minimizers systematically, or go from results on global minimizers to their counterpart for minimizers in a domain.

Can a same set G correspond to two really different global minimizers (i.e., that would not be obtained from each other by multiplying v by ± 1 and adding a constant to it in each component of $\mathbb{R}^n \setminus G$)? When $n = 2$, this is impossible because of the formula (11).

Could it be that when we take complex or vector-valued functions, we get fundamentally new global minimizers? We do not know this even in the plane, but it seems even more unlikely there.

6 What is the Mumford–Shah Conjecture in Dimension 3?

We would also like to have a description of all global minimizers in $\mathbb{R}^3$, even if we can't prove it. Let us already enumerate the ones we know.

We start with the case when g is locally constant, and hence G is a minimal set. Jean Taylor essentially showed that there are only four possibilities: the empty set, a plane, the product of a Y with an orthogonal line (i.e., three half planes that meet along their common boundary with 120° angles), and a set composed of six faces bounded by four half lines that start from the same origin and make maximal equal angles with each other. [Think about dividing a regular tetrahedron into for equal parts that all touch the center and three vertices.] These are the same sets that show up as tangent objects to soap films.

We also get a global minimizer in $\mathbb{R}^3$ by taking the product of the cracktip in a plane (as in (10)) by an orthogonal line (so that v, for instance, will not depend on the last variable); the verification is rather easy.

Now it seems that there should be at least one other global minimizer, and it is not too obvious to guess what it should be.

Let us first try to say why we may think that the list above is not complete. The argument will be vague and indirect, but there are not many arguments in the other direction either.

Consider the Mumford–Shah functional J in the domain $\Omega = B(0,1) \times (-N, N) \subset \mathbb{R}^3$ and with a function g of the form $g(x,y,z) = \varphi(z)\, g_0(x,y)$, where φ is a smooth function such that $0 \leq \varphi \leq 1$, $\varphi(z) = 1$ for $z \leq -1$, and $\varphi(z) = 0$ for $z \geq 1$. To define g_0, let Y be the intersection with $B(0,1) \subset \mathbb{R}^2$ of the union of three half lines starting from the origin and making 120° angles. Call D_1, D_2, and D_3 the three connected components of $B(0,1) \setminus Y$, and set $g_0(x,y) = 2j - 2$ on D_j. We take N reasonably large, to make sure that the last term of J will have some importance in the discussion.

If we restricted to the smaller domain $B(0,1) \times (-N, -1)$, (u, K) would coincide with $(g, Y \times (-N, -1))$. This is because (g_0, P) is the only minimizer of the 2–dimensional analogue of J where $g = g_0$, by [1]; the argument for the reduction to the plane is the same as for the cracktip minimizer just above.

It is reasonable to think that $K \cap [B(0,1) \times (-N, -N/2)]$, say, looks a lot like $Y \times (-N, -N/2)$, with three almost vertical walls that meet along a curve γ with 120° angles. If there is no other global minimizers than the five mentioned above, all the blow-up limits of (u, K) are among the five. If in addition we have a perturbation theorem for the case when K is close to the product of a Y by a line, we should be able to follow the curve γ (where the blow-up limits of K are products of Y's by lines), as long as no other type of blow-up limit shows up on γ. We cannot really exclude the possibility that this never happens, because γ may go all the way to $\partial\Omega$. However, it seems very unlikely that γ will go all the way to the top boundary (because $g = 0$ on the top part of Ω; there would be no good reason for the existence of three high walls), and still unlikely that γ will turn and hit the side of $\partial\Omega$ (even though our function g is not symmetric).

So let us assume that γ ends somewhere in Ω, at a point x_0 where some other blow-up limit shows up. If we get the minimal set with six faces, and we have a perturbation theorem near this minimal set, there will be three new curves like γ that leave from x_0, and we can try to follow them again. The case when all curves eventually end up on $\partial\Omega$ also seems unlikely: it is not clear why it would be needed to isolate three components near the bottom, rather than allowing gentle variations of u. But this is definitely a weak point of the argument. Anyway, if this does not happen, some other blow-up limit shows up, with a global minimizer that is not in the list above. Obviously some numerical experiments could be useful here.

Let us even try to guess what the set G for this new global minimizer should look like. The simplest bet would be to take something like $Y \times (-\infty, 0]$, but this may be a little too naive. For one thing, even though $Y \times (-\infty, 0]$ is nicely invariant under rotations of 120°, the corresponding function v could not be (the increments of v along an orbit of 3 points always add up to 0, so they cannot all be equal or opposite). So, if there is a global minimizer (v, G) with $G = Y \times (-\infty, 0]$, we are sure to lose the essential uniqueness of v given G. This is not so bad in itself, but since the function g will not be rotation invariant anyway, it does not seem so important to require G to be invariant. Probably making one of the walls a little higher than the others could help accommodate the corresponding slightly larger variations of v along that wall.

The author's bet is that there is indeed one more global minimizer, where G is the cone over three arcs of great circle $\Gamma_i \subset \mathbb{S}^2$; the Γ_i are vertical, start from the south pole where they make angles of 120°, and there are two shorter ones and a longer one. The lengths should be determined by all sorts of constraints, the main one being that we also guess that v will be homogeneous of degree 1/2, and we know it must be harmonic with Neumann boundary conditions $\frac{\partial v}{\partial n} = 0$. All these suggestions are based on essentially nothing, so the reader may find it fun to make any other bet. For instance, that there is a global minimizer for which function $\lambda \to \mathrm{e}^{-\lambda/2} v(\mathrm{e}^{\lambda} x)$ is almost periodic (but not periodic), or has a period $\sqrt{2}$. [Added in proof: since the lecture, Benoit Merlet did computations that indicate that the most obvious scenario suggested by the author is impossible.]

Our difficulty with rotations of order 3 leads us to wonder whether the situation would be different with functions v valued in $\mathbb{R}^2$, because then v could be rotation invariant. This seems tempting because we like symmetry, but I don't think this is too serious.

The interested reader may consult Sects. 76 and 80 of [10] for a few additional questions, or details about the questions above.

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