Analysis on Lie Groups: An Overview of Some Recent Developments and Future Prospects

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1 Introduction

In this paper I want to present some typical recent results in the subject to the general public. A serious effort has been made to make this paper accessible to the nonexpert. The only prerequisite is the definition of a Lie group and its Haar measure, and of the convolution product. Even the definition of the Lie algebra will not be essential and it will be given in Sect. 3 below. Some specialized notions, such as the semigroup $T_t = \mathrm{e}^{-t\Delta}$ generated by the closure of a subelliptic operator $\Delta = -\sum X_j^2$, where X_j are left-invariant vector fields on the real connected Lie group G (the letter G will be reserved throughout to denote such a group), will crop up. The nonspecialist can ignore these, or any other unknown word for that matter, and simply read on, this paper is so structured that this is possible.

The heart of the matter lies in Sect. 2.2 below, Sect. 4 makes comments on the previous sections and Sect. 5 gives some further results. There were several possibilities for the *further results* presented in Sect. 5. My choice was dictated by the following two considerations. On one hand I wanted to present some natural problems in the subject that sooner or later will have to be addressed and which I am sure, have a satisfactory answer. On the other hand I wanted to highlight some much more speculative prospects that emerge from the results presented here. These “speculations” have to do with combinatorial group theory.

Convention

I use throughout the convention that, in a formula, the letters C or c, possibly with suffixes, indicate, possibly different, positive constants that are independent of the important parameters of the formula.

2 The Analysis and the Geometry

2.1 Riemannian Structures on a Lie Group G

For every prescribed inner product $\langle \cdot , \cdot \rangle$ on the tangent space $T_e(G)$, where $e \in G$ is the neutral element of G, we can define a left invariant Riemannian structure on G, simply by left translating ($L_g : x \longmapsto gx$) $\langle \cdot , \cdot \rangle$ to an inner product on $T_g(G)$ ($g \in G$). A different inner product $\langle \cdot , \cdot \rangle^{\text{new}}$ gives rise to a new Riemannian structure that is *quasi-isometric* to the initial one in the sense that:

$$C^{-1}|x|^{\text{old}} \leq |x|^{\text{new}} \leq C|x|^{\text{old}} , \qquad \forall x \in G ,$$

where $|x| = d(x, e)$ denotes the Riemannian distance to the origin.

2.2 Convolution Powers

Let $\mu \in \mathbb{P}(G)$ be some probability measure on G, with continuous and compactly supported density $\varphi \in C_0(G)$ with respect to the right Haar measure $\mathrm{d}^r x$ on G: $\mathrm{d}\mu(x) = \varphi(x)\mathrm{d}^r x$. We introduce the following notation, for any integer $n \geq 1$:

$$\mu^{*n} = \mu * \cdots * \mu \quad (n \text{ times}) , \quad \mathrm{d}\mu^{*n}(x) = \varphi_n(x)\mathrm{d}^r x ,$$
$$\text{and} \quad \Phi(n) = \Phi(n; G, \mu) = \varphi_n(e) .$$

To avoid the obvious pathology (e.g. $G = \mathbb{R}$, $\operatorname{supp}\mu \subset$ positive axis with $\Phi(n) \equiv 0\ \forall n \geq 0$), we shall suppose that μ is *symmetric* i.e. $\mathrm{d}\mu(x^{-1}) = \mathrm{d}\mu(x)$.

2.3 Spectral Gap and Amenability

I shall denote $\|\mu\|_{\text{op}}$ ($\mu \in \mathbb{P}(G)$) the $L^2 \to L^2$ operator norm of $f \longmapsto f * \mu$ on $L^2(G, \mathrm{d}^r x)$ and define $\lambda = \lambda(G, \mu)$ by $\mathrm{e}^{-\lambda} = \|\mu\|_{\text{op}}$. It is important to recall that, for G fixed, either $\lambda(G, \mu) = 0$ for all $\mu \in \mathbb{P}(G)$ as above, and then we say that G is *amenable*, or $\lambda(G, \mu) > 0$ for all $\mu \in \mathbb{P}(G)$ as above, and then we say that G is *not amenable* (cf. [11]).

2.4 Basic Analytic Definitions [16]

We shall say that G is a *B–group* if, for all $\mu \in \mathbb{P}(G)$ and $\lambda = \lambda(G, \mu)$ as in Sect. 2.2 and Sect. 2.3, there exist $C_1, C_2, c_1, c_2 > 0$ such that:

$$C_2 \exp(-\lambda n - c_2 n^{1/3}) \leq \Phi(n) \leq C_1 \exp(-\lambda n - c_1 n^{1/3}) .$$

We shall say that G is an *NB–group* if, for all $\mu \in \mathbb{P}(G)$ and $\lambda = \lambda(G, \mu)$ as above, there exist $\nu = \nu(G, \mu) \geq 0$ and $C > 0$ such that:

$$C^{-1} n^{-\nu} \mathrm{e}^{-\lambda n} \leq \Phi(n) \leq C n^{-\nu} \mathrm{e}^{-\lambda n} .$$

2.5 The Basic Geometric Definition ([4], [18])

We shall say that G (here G is equipped with a Riemannian structure as in Sect. 2.1) admits the *polynomial homotopy property* (*PHP* in short) if the following holds: Let $2 \leq n \leq \dim G$, let $\alpha : \dot{e}^n = S^{n-1} \longrightarrow G$ be some C^∞ mapping of the unit Euclidean sphere into G ($\dot{e}^n$ stands for the boundary of the unit Euclidean n – cell e^n) and let us assume that $\alpha(\dot{e}^n)$ is *homotopic to zero* in G i.e. that α extends to a continuous mapping $\widehat{\alpha} : e^n \longrightarrow G$ (which simply means that $[\alpha]$ is zero in the $(n-1)^{\text{th}}$ homotopy group $\pi_{n-1}(G)$ of G [6]). Then the extension $\widehat{\alpha} : e^n \longrightarrow G$ can be chosen in order that

$$\mathrm{Vol}_n\big[\widehat{\alpha}(e^n)\big] \leq C\big(1 + \mathrm{Vol}_{n-1}\big[\alpha(\dot{e}^n)\big]\big)^c \ ,$$

for some positive constants C and c depending only on G.

$\mathrm{Vol}_r(\cdot)$ denotes here the r–dimensional Hausdorff measure with respect to the Riemannian structure (counted with multiplicity, cf. [10]). Observe that the above conditions are vacuous if $\dim G = 1$ and that then G admits the *PHP*.

2.6 The Classification: Analytic–Geometric

Theorem A–G. *A group G as above is a* B–group*, resp. an* NB–group *if and only G does not, resp. does admit the* PHP*.*

The above definitions and results extend to measures $\mu \in \mathbb{P}(G)$ that are symmetric but not necessarily compactly supported, provided that they satisfy Gaussian estimates at infinity, which amount grosso modo to imposing $\mu\{x \in G \mid |x| \geq R\} = \mathrm{O}\big(\mathrm{e}^{-cR^2}\big)$ (cf. [16], [19] for the exact definition). The importance of these Gaussian measures lies in the fact that the heat diffusion kernel of the semigroup $T_t = \mathrm{e}^{-t\Delta}$ (cf. Sect. 1) is Gaussian (cf. [16], [22]).

3 The Algebra

3.1 Review of Lie Algebra and Definitions ([9], [12])

Let $\mathfrak{g} = T_e(G)$ be the (real) Lie algebra of G. Recall that every vector $\xi \in T_e(G)$ extends by left translations (cf. Sect. 2.1) to a (left invariant) vector field X on G:

$$X(x) = (\mathrm{d}_e L_x)(\xi) \ ,$$

and that the Lie bracket on $\mathfrak{g}$ is induced by the bracket of vector fields, viewed as first order differential operators:

$$[\xi_1, \xi_2] = [X_1, X_2](e) = (X_1 \circ X_2 - X_2 \circ X_1)\big|_{x=e} \ .$$

Recall furthermore that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ denotes the complexified Lie algebra and

$$\mathrm{ad} : \mathfrak{g}_{(\mathbb{C})} \longrightarrow \mathcal{L}(\mathfrak{g}_{(\mathbb{C})})$$

the (*complexified*) *adjoint representation*, defined by $(\mathrm{ad}\,\xi)(\zeta) = [\xi, \zeta]$. The Lie algebra $\mathfrak{g}$ is *solvable* if the complexified adjoint representation can be simultaneously *triangularized*:

$$\mathrm{ad}\,\xi = \begin{pmatrix} \lambda_1(\xi) & & \star \\ & \ddots & \\ 0 & & \lambda_n(\xi) \end{pmatrix} .$$

Let us decompose $\lambda_j = \mathrm{Re}\,\lambda_j + \mathrm{i}\,\mathrm{Im}\,\lambda_j \in \mathfrak{g}^*_{\mathbb{C}} = \mathfrak{g}^* + \mathrm{i}\mathfrak{g}^*$ and let us consider the (possibly empty) subset $\Lambda = \{L_1, \ldots, L_k\}$ of $\mathfrak{g}^*$, consisting of all distinct $\mathrm{Re}\,\lambda_j \neq 0$.

3.2 The Algebraic Definition (Solvable Case) ([13], [18])

Let $\mathfrak{q}$ be some solvable real Lie algebra and let $\Lambda = \{L_1, \ldots, L_k\} \subset \mathfrak{q}^*$ be as above. We then say that $\mathfrak{q}$ is a *C–algebra* (*C* originally stands for *Condition*) if Λ is not empty and if its convex hull contains 0. We say that $\mathfrak{q}$ is *NC* (*Non–C*) otherwise.

3.3 Review of the Structure of Lie Groups ([7], [12])

It is the fundamental theorem in Lie theory that every real Lie algebra $\mathfrak{g}$ is the Lie algebra of a unique (up to isomorphism) simply connected real Lie group G. Furthermore every Lie group whose Lie algebra is isomorphic to $\mathfrak{g}$ is *locally isomorphic* to G. One basic but standard fact in that direction is that if $\mathfrak{g}$ is solvable then the corresponding simply connected group G is diffeomorphic to $\mathbb{R}^d$.

What is also standard is the following structure theorem: Let G be some simply connected real Lie group. Then

$$G = QK , \tag{1}$$

where Q and K are simply connected closed subgroups of G such that

(i) $Q \cap K = \{e\}$,
(ii) Q is solvable,
(iii) K contains a cocompact discrete closed subgroup Z which is central in G (*cocompact* meaning that K/Z is compact).

Remark 3.1. If G is amenable, Q is the radical and (1) is the Levi decomposition. If G is semisimple, then $Q = NA$ in the Iwasawa decomposition $G = NAK$. Unless G is amenable, Q cannot be normal and in general (1) is related to the Borel decomposition for algebraic groups [8].

3.4 The Algebraic Definition (General Case) [16]

Let $\mathfrak{g}$ be some real Lie algebra, let G_0 be the corresponding simply connected Lie group and let $G_0 = QK$ as in (1) We say that $\mathfrak{g}$ is a *B–algebra* if Q is a C–group. The fact that this definition does not depend on the particular decomposition (1) needs proving. If Q is NC, we say that $\mathfrak{g}$ is *NB* (*B* is the letter preceding *C* in the Latin alphabet and *NB* stands for *Non–B*, and not for the initials in the Greek alphabet of *N.Βαροπουλος*!). For any connected Lie group we say that G is algebraically B (resp. NB) if its Lie algebra is a B (resp. NB) algebra.

Example 3.2. All semisimple Lie groups, compact or not, are NB.

3.5 The Classification: Analytic–Algebraic

Theorem A–A. *G is a* B–group*, resp. an* NB–group *if and only if it is* algebraically–B*, resp.* algebraically–NB*.*

4 Comments

4.1 Unimodular Groups

When G is unimodular i.e. when the Haar measure on G is both left and right invariant, a different classification on the basis of $\Phi(n)$ was carried out in the 80's (cf. [22]). What is involved there are not homotopy considerations but the volume growth of G:

$$\gamma(n) = \text{Haar measure of a ball of radius } n \geq 1 \text{ in } G\,.$$

What we can say then is that when $\gamma(n) \approx n^D$ $(D = 0, 1, \dots)$ we have $\Phi(n) \approx n^{-D/2}$ and when $\gamma(n) \gtrsim n^a$ (for all $a \geq 0$) we have $\Phi(n) \leq \exp(-cn^{1/3})$.

This is a much coarser classification than the one we have here but it has the advantage that it generalizes to all the compactly generated locally compact groups, connected or not.

In terms of our classification given here, the unimodular NB–groups have the additional property that $\nu = \nu(G) \in \frac{1}{2}\mathbb{Z}$ i.e. $\nu = \nu(G)$ is a half integer which depends only on G and not on μ (cf. [14], [17] for a formula that gives ν; cf. [3], [22] for proofs of this when G is respectively semisimple or amenable).

4.2 The State of the Art on Theorem A–A

A complete proof of Theorem A–A when G is amenable can be found in [13]. In [16] one finds a proof for general groups of a slightly weaker result where the NB–condition in Sect. 2.2 is replaced by $C^{-1}\mathrm{e}^{-\lambda n}n^{-\nu_1} \leq \Phi(n) \leq C\mathrm{e}^{-\lambda n}n^{-\nu_2}$. To prove that $\nu_1 = \nu_2$ and to compute this index is a truly formidable task, and it takes from end to end several hundred pages to carry out. The reason is that, among other things, the solution of that problem relies on difficult estimates in potential theory (cf. [21]). It is a fortunate fact that these potential theoretic estimates have, very recently, come on their own and have given rise to interesting new results in classical probability (and potential) theory. Let me explain.

Let $\mu \in \mathbb{P}(\mathbb{R}^d)$ be some centered probability distribution with a high enough moment, and let $D = \{x = (x_1, x') \in \mathbb{R}^d = \mathbb{R} \times \mathbb{R}^{d-1} \mid x_1 > \varphi(x')\}$ be some Lipschitz domain in $\mathbb{R}^d$, where $|\varphi(x') - \varphi(y')| \leq A|x' - y'|$ $(x', y' \in \mathbb{R}^{d-1})$. Let

$$P(n, x) = \mathbb{P}_x[Z_j \in D; j = 1, 2, \dots, n]$$

be the probability of life (or the gambler's ruin estimate, depending on the point of view) of the random walk $Z_1, Z_2, \dots \in \mathbb{R}^d$ controlled by μ, i.e.:

$$\mathbb{P}[Z_{n+1} \in \mathrm{d}y // Z_n = x] = \mathrm{d}\mu(x - y) \ , \qquad x, y \in \mathbb{R}^d, \ n = 0, 1, 2, \dots \ .$$

Very precise estimates of $P(n, x)$ can be obtained in the above generality (cf. [21]). These estimates are essential for Theorem A–A, where D is then an appropriate conical domain in $\mathfrak{g}$ $(= \mathbb{R}^d$ as a linear space) defined by the positivity of the relevant roots (i.e. D is a generalized *Weyl chamber* [5], [13]).

Be that as it may, the complete proof of Theorem A–A will appear in a forthcoming paper [20].

4.3 The State of the Art on Theorem A–G

In [18] one finds a proof, among other things, of Theorem A–G when G is solvable and simply connected. What allows one to pass to a general connected Lie group G is that, if $T \subset G$ denotes the maximal normal torus of G, then $G/T \simeq Q \times K$ is quasi-isometrically diffeomorphic (but not necessarily homeomorphic) to the product of a simply connected solvable group Q and of a compact group K (cf. [18]; the proof of this is not difficult and it will appear in [20]). This allows us to reduce the problem to compact groups and what has to be proved is that every connected compact group admits the PHP.

I claim to have a proof of this fact on compact groups (which is quite involved). This is in the spirit of differential topology and Morse theory. The details of this have not yet been written out in full and, since in addition I am not an expert in differential topology, I feel that I have to warn the reader that unpleasant surprises in that direction are not to be excluded altogether (in plain terms my proof might collapse). Even without the above result for compact groups however, the results of [18] give a, perhaps less elegant to describe, but equally pertinent, B–NB geometric classification of Lie groups.

What emerges also from the results of [18] is a cohomological classification of Lie groups. (These results have been stated precisely in [18] but the proofs remain to be written out. These proofs are however easy to extract from [18].) Grosso modo, let us say that G, equipped with a Riemannian structure as in Sect. 2.1, has the *cohomological polynomial property* if, for every smooth differential form $\omega \in \wedge(G)$ that *grows polynomially* at infinity (cf. [18] for precise statements), that is closed (i.e. $\mathrm{d}\omega = 0$), and that represents the zero cohomology class of G, we can find $\theta \in \wedge(G)$, also with polynomial growth at infinity, such that $\mathrm{d}\theta = \omega$. The cohomological classification of Lie groups states then that G is an NB–group if and only if it has the polynomial cohomology property.

5 Further Results and Prospects

5.1 A Direct Proof of Theorem A–G

The most significant progress in the direction that I described in the previous sections would be to give a "direct" proof of Theorem A–G. By this I mean a proof that does not use the Algebraic classification of Sect. 3.2. This should be done in the spirit of [22] where in fact the above task is carried out for unimodular amenable G.

The reason why such a project is significant is that there would then be hope to extend the result to discrete groups. I shall discuss the problem of discrete groups in Sect. 5.2 below. But before I do that I wish to put the above project in perspective by considering another closely related, better posed, but also less interesting problem.

Let Q be some simply connected soluble Lie Group and let use assume that it is possible to assign such a group with a left invariant Riemannian structure with non positive curvature (necessary and sufficient conditions on the Lie Algebra for this to be possible exist [1]). We can then use the Geodesic Flow to show that then Q admits the PHP and even obtain optimal estimates in the definition e.g. for $n = 2$ in Sect. 2.5 we have

$$\mathrm{Vol}_2[\hat{\alpha}(e^2)] \ \leq C(1 + \mathrm{Vol}_1[\alpha(\dot{e}^2)])^2$$

and analogous results for every homotopy group π_n.

It is very likely and probably not even very hard to adapt [18] and go again via the Lie Algebra to show that the above optimal estimates on the volumes of the π_n's can be used to characterize the negatively curved groups Q.

Question 5.1. First of all is the above correct?

Question 5.2. If so, can one give a direct proof without going through [1]?

The interest of the above is that it would perhaps give some clue of how to go about a direct proof of the Theorem A–G.

5.2 Combinatorial Group Theory [2]

Let G be a finitely presented discrete group. We can glue to the Cayley graph of G associated to the generators, the 2–cells that correspond to the relations, and obtain thus a 2–dimensional G invariant CW–complex. 1–dim and 2–dim Hausdorff measures that are G invariant can clearly be assigned and the PHP for dimension $n = 2$ of Sect. 2.5 can clearly be defined. Analogous definitions for $n \geq 2$ can also be given, and all in all, the statement of Theorem A–G "makes sense".

Question 5.3. The issue is to decide if such a theorem in some appropriate form is provable?

The importance of the above PHP adapted to discrete groups has been recently stressed by several authors (e.g.[4]), and of course it can be restated and it gives an equivalent description of the word problem.

Example 5.4. Let $G = \pi_1(K)$ where K is a negatively curved compact manifold be a hyperbolic group. We can use then the geodesic flow on the universal cover $\tilde{K}$ to generate the homotopy and obtain the so called Dehn algorithm for the word problem.

5.3 More Concrete Problems

The project that I described in Sects. 5.1 and 5.2 may well be "daydreaming". There certainly does not seem to be much lead of how to start.

A much more realistic project, but also much more esoteric, is to examine in more detail the "polynomial property" of the various dimensions $n = 2, \cdots$, for the homotopies of Sect. 2.5 or the cohomologies of Sect. 4.3. Some dimensions may have the polynomial property and others not. Many examples can be given of all short of situations [4] [18]. A complete classification in terms of the GL_k–geometry (cf. Sect. 3.2) of the roots (like the C–condition but more refined) is no doubt within reach. Let us assume that we can write down such a classification. The question then arises whether we can read off these properties in terms of "Analytic" or rather Potential theoretic conditions as in Sect. 2.4? In other words:

Question 5.5. Can we refine Theorem A–G in terms of the polynomial behaviour of the "various homotopy groups $\pi_n(G)$"?

This also would be a first step towards the speculations of Sect. 5.1. But even here it is hard to make the right conjectures.

Observe that in Sect. 5.2 it is not the discreetness of the groups G that is the problem it is the lack of the Lie algebra. So we could simplify G and assume that it is, say, a cocompact lattice in some (non compact) semi-simple Lie group. What can we say about $\Phi(n, \mu)$ of Sects. 2.2 and 2.3 then. For instance:

Question 5.6. Can we assert, that $\Phi(n) \sim n^{-\alpha} e^{\lambda n}$ (cf. Sect. 4.1) where $\lambda > 0$ will have to depend on μ but $\alpha = \alpha(G)$ is a genuine group invariant?

This is what happens for the semi-simple Lie group itself [3]. The above is a concrete problem and it is probably within reach (there was a point that I even thought I could prove this. I have since drifted away from the subject and now I do not know any more). Problems like this may have an arithmetic significance. I am however very ignorant on arithmetic questions and any comment from me in that direction would be inappropriate.

5.4 Hardy–Littlewood Results ([14], [15])

An interesting direction where the final results remain to be worked out are the $L^p \to L^q$ mapping properties of

$$\Delta^{-\alpha/2} = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} T_t \mathrm{d}t \,, \tag{2}$$

where $\alpha > 0$ and $1 \leq p < q < +\infty$. Here I use the semigroup Laplace transform definition of the fractional powers of the Laplacian, and the notation of Sect. 1. Fairly satisfactory general results in that direction only exist in the unimodular case (cf. [14], [22]). What renders the above problem difficult is that the range of parameters (α, p, q), for which $\Delta^{-\alpha/2} : L^p \longrightarrow L^q$, depends in general (if G is not unimodular) on the particular Laplacian Δ and does not only depend on G (just as the $\nu(G,\mu)$ in Sect. 3 depends in general on μ).

A moment's reflexion shows that, for *amenable* groups, what is relevant for (2) are the following parameters:

$$\mathrm{d}^\ell x = \text{left Haar measure}\,, \qquad m(x) = \frac{\mathrm{d}^r x}{\mathrm{d}^\ell x}\,,$$
$$\widetilde{T}_t = m^{1/2} \circ T_t \circ m^{-1/2}\,, \qquad \ell(q) = \overline{\lim_{t\to\infty}} \frac{1}{t} \log \|\widetilde{T}_t\|_{1\to q}\,,$$
$$L = \inf\{q \geq 1 \mid \ell(q) = 0\}\,,$$

where we take here the $L^1 \to L^q$ operator norm w.r.t. $\mathrm{d}^\ell x$, and where we have to consider here $\widetilde{T}_t$, because the original semigroup T_t (cf. Sect. 1) is not symmetric. L is the *cut point* between the exponential and the subexponential growth of $\|\widetilde{T}_t\|_{1\to q}$ and it is easy to see that $1 \leq L \leq 2$. The following result is sharp, but unfortunately, very limited in scope and not easy to prove:

Fact 5.7. $L = L(G)$ only depends on G and is independent of the particular laplacian Δ. Furthermore $L = 1$ if and only if G is unimodular and $L = 2$ if and only if G is WNC.

WNC (*Weak–NC*) is a variant of the NB and NC definitions given in Sect. 3 (cf. [15]). The above can be reformulated by saying that, if the operator

(2) with T_t replaced by $\widetilde{T}_t$, is bounded for some $\alpha > 0$ and $1 \le p < q < 2$, then G is a *WNB–group*. This is very much in the spirit of the results that we expect to hold. Unfortunately what the correct conjectures are, is not clear.

Working out in full generality the above Hardy–Littlewood theory is challenging and, I feel, is within reach. The only problem is that it cannot be done by easy and superficial contributions. It would take a competent man (or woman. Why not !) several years to complete this project. I am probably too old to attempt this, but others should try. Just to qualify this last statement, I could also say that convincing applications, outside the subject itself (e.g. arithmetic results on lattices), would have to exist, before a *competent* man decides to spend several years on such a problem.

Acknowledgment

I would like to thank J.-Ph. Anker for many helpful suggestions and for the hard work he put in editing a first version of this paper.

References

1. R. Azencott, E.N. Wilson: Proc. A.M.S. Math. **215**, 323 (1976)
2. R.C. Lyndon, P.E. Schupp: *Combinatorial group theory* (Springer, Berlin New York 1977)
3. Ph. Bougerol: Ann. Sci. Éc. Norm. Sup. 4ᵉ Sér. **14**, 403 (1981)
4. M. Gromov: Asymptotic invariants of infinite groups. In: *Geometric group theory*, vol 2, ed by G.A. Niblo, M.A. Roller (Cambridge University Press 1993) pp 1–295
5. S. Helgason: *Differential geometry, Lie groups, and symmetric spaces* (Academic Press, New York London 1978)
6. P.J. Hilton: *An introduction to homotopy theory* (Cambridge University Press, Cambridge 1953)
7. G. Hochschild: *The structure of Lie groups* (Holden–Day, San Francisco London Amsterdam 1965)
8. J.E. Humphreys: *Linear algebraic groups* (Springer, New York Heidelberg 1975)
9. N. Jacobson: *Lie algebras* (Interscience, New York London 1962)
10. H.H. Lawson Jr: *Lectures on minimal submanifolds. Vol. I* 2nd edn (Publish or Perish, Wilmington 1980)
11. H. Reiter: *Classical harmonic analysis and locally compact groups* (Clarendon Press, Oxford 1968)
12. V.S. Varadarajan: *Lie groups, Lie algebras, and their representations* (Springer, New York 1984)
13. N.Th. Varopoulos: Canad. J. Math. **46**, 438, 1073 (1994), **48**, 641 (1996)
14. N.Th. Varopoulos: Ann. Inst. H. Poincaré Probab. Statist. **31**, 669 (1995)
15. N.Th. Varopoulos: Rev. Mat. Iberoamericana **12**, 147 (1996)
16. N.Th. Varopoulos: Rev. Mat. Iberoamericana **12**, 791 (1996)
17. N.Th. Varopoulos: The local theorem for symmetric diffusion on Lie groups. An overview. In: *Harmonic analysis and number theory*, vol 21, ed by S.W. Drury, M. Ram Murty (Amer. Math. Soc., Providence 1997) pp 143–152

18. N.Th. Varopoulos: Rev. Mat. Iberoamericana **16**, 1 (2000)
19. N.Th. Varopoulos: Canad. J. Math. **53**, 412 (2000)
20. N.Th. Varopoulos: *Analysis on Lie groups II*, to appear
21. N.Th. Varopoulos: Canad. J. Math. **53**, 1057 (2001)
22. N.Th. Varopoulos, L. Saloff-Coste, T. Coulhon: *Analysis and geometry on groups* (Cambridge University Press, Cambridge 1992)