GOCE Gravity Field Modeling: Computational Aspects – Free Kite Numbering Scheme

Christian Boxhammer and Wolf-Dieter Schuh

Institute of Theoretical Geodesy, University of Bonn, Germany
53115 Bonn, Nußallee 17, schuh@uni-bonn.de

Summary. The modelling of the Earth's gravity field by means of a high-resolving spherical harmonic analysis is a numerically demanding task, especially when realistic (non gridded) data sets are analysed. The *free kite numbering scheme*, presented in the current article, allows a flexible combination of models. It is focussed, in particular, on the combination of a model containing rotation-symmetrical, high-resolving data with a second model comprising fully correlated data, which allows the determination of the lower degrees. This kite scheme may, depending on the degree of conformance with rotation symmetry, be used both with a direct solver and to improve the convergence rate of an iterative solver.

Key words: GOCE mission, spherical harmonic analysis, preconditioner, kite numbering scheme

1 Introduction

GOCE gravity field determination constitutes a great challenge to stochastic and deterministic modelling. Obtaining optimal results from GOCE data has been the focus of research collaborations on both national and international levels. In addition to the challenge of mathematic modelling, the numerical implementation of the resulting large equation systems in particular, requires enormous efforts. As a joint venture between the Technical University of Graz, the Technical University of Munich, and the University of Bonn, a data processing chain was developed. This chain consists of three components: the *Quick-Look Tool*, which is based on the highly efficient semi-analytic approach, the *Tuning Machine* (a tailored iterative solver), and the *Final Solver*, which solves the normal equations directly. The development of this variety of methods is motivated by the complementary use of these three strategies, which all have distinct advantages as well as drawbacks.

The application of tailored numerical algorithms enabled the design of an efficient and flexible tool capable of processing the huge amount of data (approximately 58,000 parameters of a highly resolving model will be estimated

from some 50 million correlated observations per measurement period of 6 month). This solution technique is based on the method of preconditioned conjugate gradients and allows a strict one-step adjustment of heterogeneous data types both in terms of observation equations and normal equations. Although this method was developed especially for GOCE data processing, it may be applied within the general context of spherical harmonic analysis. The efficiency of this iterative method, named `pcgma` (Preconditioned Conjugate Gradient Multiple Adjustment), is underlined by its ability to solve high-order spherical harmonic models (degree and order 360 and more) within a reasonable processing time. `pcgma` consists of the following main components:

- Decorrelation of the data by discrete filtering in the time domain;
- Preconditioning by means of data-adaptive sparse matrices (Kite Scheme);
- Parallelization for use on massive parallel computer.

The current article focusses on the second component and outlines the results of the latest research regarding the generation, administration, and implementation of a data-adaptive preconditioner. The proposed preconditioner constitutes an extension of the traditional *kite numbering scheme* (Schuh, 1996b) for equal efficiency with greater flexibility. Therefore, this *free kite numbering scheme* may be adjusted to various problems. Possible fields of application will be demonstrated, using as examples typical problems from satellite geodesy.

2 Spherical Harmonic Analysis

Many physical processes of our *"System Earth"* may be represented mathematically by using the solid spherical harmonics

$$r^{-(\ell+1)}\, P_{\ell m}\,(\sin\varphi)\, \cos m\lambda \qquad \text{and} \qquad r^{-(\ell+1)}\, P_{\ell m}\,(\sin\varphi)\, \sin m\lambda$$

as base functions, where r, φ, λ denote the polar coordinates of a point and $P_{\ell m}\,(\sin\varphi)$ the Legendre functions of degree ℓ and order m (with $\ell \leq 0$ and $m \leq \ell$). As solution of Laplace's equation outside the sphere, these base functions possess a global support. On the other hand, they build up an orthogonal system when the data coverage is global and continuous (Heiskanen and Moritz, 2000, S.29). Furthermore, these base functions constitute a complete basis, i.e. any piecewise continuous function $f\,(\varphi, \lambda)$ defined on the unit sphere $(r = 1)$ may be represented by a linear combination of spherical harmonics as the infinite series

$$f\,(\varphi,\lambda) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \; C_{\ell m}\; P_{\ell m}(\sin\varphi)\, \cos m\lambda \;+\; S_{\ell m}\; P_{\ell m}(\sin\varphi)\, \sin m\lambda$$

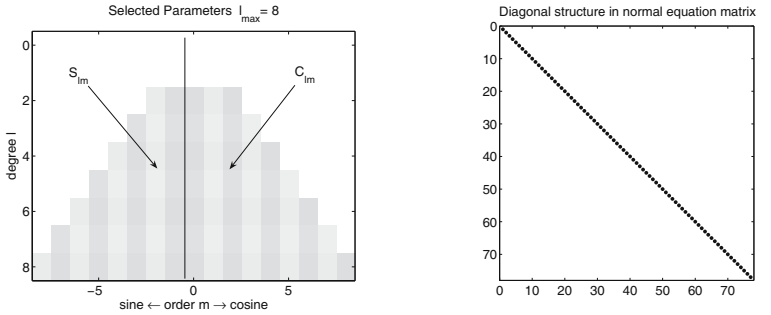
with $S_{\ell 0} = 0$. The coefficients $C_{\ell m}$ and $S_{\ell m}$ may be determined independently by integration over the unit sphere, yielding

$$C_{\ell m} = \frac{1}{\iint\limits_{\sigma} \left(P_{\ell m}\left(\sin\varphi\right)\cos m\lambda\right)^2 d\sigma} \iint\limits_{\sigma} f(\varphi,\lambda)\, P_{\ell m}(\sin\varphi)\, \cos m\lambda\, d\sigma$$

and

$$S_{\ell m} = \frac{1}{\iint\limits_{\sigma} \left(P_{\ell m}\left(\sin\varphi\right)\sin m\lambda\right)^2 d\sigma} \iint\limits_{\sigma} f(\varphi,\lambda)\, P_{\ell m}(\sin\varphi)\, \sin m\lambda\, d\sigma,$$

respectively, where $d\sigma = \cos\varphi\; d\varphi d\lambda$. As the representation of a band-limited function requires only a finite number of coefficients, the infinite sum above may be terminated at a finite degree ℓ_{max}. For a graphical representation, the coefficients $C_{\ell m}$ and $S_{\ell m}$ are usually arranged as a triangle in the following manner (Fig. 1): The ordinate is defined by the degree ℓ, which increases from top to bottom, the abscissa by the order m with the cosine coefficients $C_{\ell m}$ on the left and the sine coefficients $S_{\ell m}$ on the right hand side.



(a) Coefficient triangle (b) Normal equation matrix

Fig. 1. Representation of the coefficients $C_{\ell m}$, $S_{\ell m}$ and of their correlations

Figure 1b depicts the correlations between all of the spherical harmonic coefficients. Each non-vanishing correlation is represented by a black dot in the normal equation matrix. The fact that only the diagonal contains entries implies that all coefficients are estimable independently. Consequently, the computation of the coefficients, denoted as *spherical harmonic analysis*, constitutes, in the given case, an inverse problem which is easy to solve.

Unfortunately, the normal equation matrix will not, in general, be diagonal when discrete observations are used, because the orthogonality relations of the spherical harmonics hold only for very special discrete data distributions. In other words, the coefficients cannot, due to their correlations, be estimated independently. However, under certain assumptions regarding the local data distribution, some of the orthogonalities still hold. Especially the orthogonality relations for trigonometric functions may be exploited. As an illustration, if one full period of length 2π is sampled at $2L$ equidistant nodes, then

$$\sum_{i=0}^{2L-1} \cos m\lambda_i \ \cos k\lambda_i = (1 + \delta_{m0} + \delta_{mL})\ L\ \delta_{mk}$$

$$\sum_{i=0}^{2L-1} \sin m\lambda_i \ \sin k\lambda_i = (1 - \delta_{m0} - \delta_{mL})\ L\ \delta_{mk}$$

$$\sum_{i=0}^{2L-1} \cos m\lambda_i \ \sin k\lambda_i = 0.$$

where $\lambda_i = i\frac{2\pi}{2L}$ and δ_{mk} denotes the Kronecker symbol. As a consequence, all coefficients of different orders will be independent. Furthermore, the symmetries and asymmetries of Legendre's functions with respect to the equator, i.e.

$$P_{\ell m}(-\sin\varphi) = (-1)^{(\ell-m)} P_{\ell m}(\sin\varphi)\ ,$$

may be used to separate the even from the odd coefficients of a fixed order.

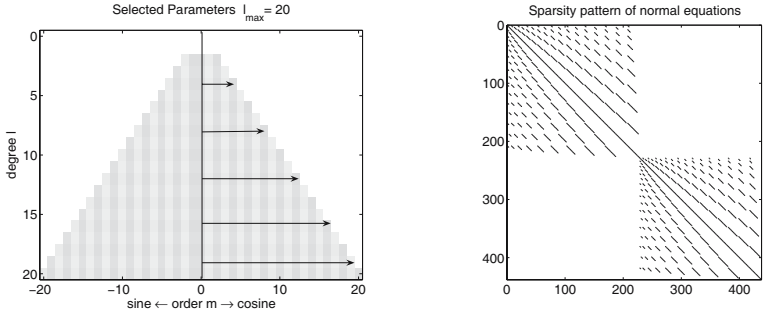
The standard numbering scheme, e.g. EGM96, is degree-by-degree (Rapp, 1994). Usually, the cosine coefficients $C_{\ell m}$ are enumerated first, and subsequently the sine coefficients $S_{\ell m}$ within a slightly modified loop (the coefficients $m = 0$ are omitted). The order of the coefficients arising from this numbering scheme can be demonstrated by the following loops:

Algorithm 1. Numbering scheme: degree-by-degree

```
for ℓ = 0 : ℓmax
    for m = 0 : ℓ
        Cℓm            % odd and even coefficients
    end
end
for ℓ = 1 : ℓmax
    for m = 1 : ℓ
        Sℓm            % odd and even coefficients
    end
end
```

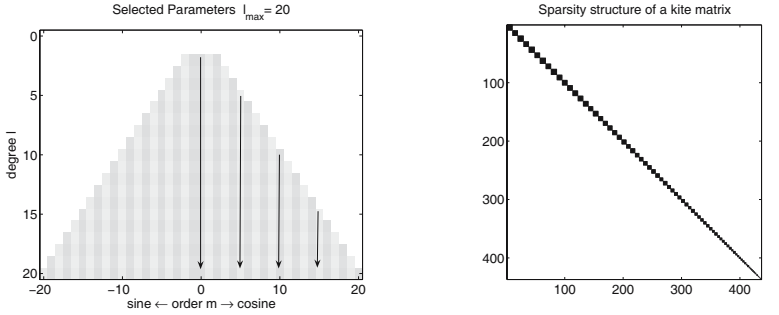
As can be seen in Fig. 2a, this numbering scheme is in accordance with moving through the coefficient triangle along a primary horizontal track. The corresponding normal equation matrix is characterized by diagonal stripes parallel to the main diagonal (Fig. 2b), which would necessitate an elaborate mechanism for its storing.

This procedure becomes extremely simplified by an order-wise enumeration, corresponding to a vertical movement inside the coefficient triangle (Fig. 3a). This scheme, which in the following will be referred to as *block numbering scheme*, produces a block-diagonal normal equation matrix (Fig. 3b), which is now much easier to store, and for which it is easy to see that the Cholesky reduction does not produce any additional fill-in elements. The same holds also for the standard numbering but with regard to the compact storage scheme the block numbering is clearly superior.



(a) Coefficient triangle (b) Normal equation matrix

Fig. 2. Standard numbering scheme



(a) Coefficient triangle (b) Normal equation matrix

Fig. 3. Block numbering scheme

The order of coefficients resulting from application of the *block numbering scheme* with additional consideration of equatorial symmetries is demonstrated in Algorithm 2. Thus, for each fixed order, coefficients of even and odd degrees are separated.

Algorithm 2. Block numbering scheme

for $m = 0 : m_{max}$
  for $\ell = m : 2 : \ell_{max}$
    $C_{\ell m}$ *% even/odd coefficients*
  end
  for $\ell = m + 1 : 2 : \ell_{max}$
    $C_{\ell m}$ *% odd/even coefficients*
  end
end

```
for m = 1 : m_max
    for ℓ = m : 2 : ℓ_max
        S_ℓm                    % odd/even coefficients
    end
    for ℓ = m + 1 : 2 : ℓ_max
        S_ℓm                    % even/odd coefficients
    end
end
```

Note that block-diagonality of the normal equation matrix holds only in the case that the nodes are, firstly, distributed rotation-symmetrically with respect to the North-South axis, secondly, aligned equidistantly along the parallels, and thirdly, located symmetrically regarding the equator. The density of the data coverage per parallel may be controlled via the grid width. Furthermore, it must be ensured that the data is of homogeneous accuracy. Although polar gaps downgrade the condition of the normal equation system, they do not destroy block-diagonality.

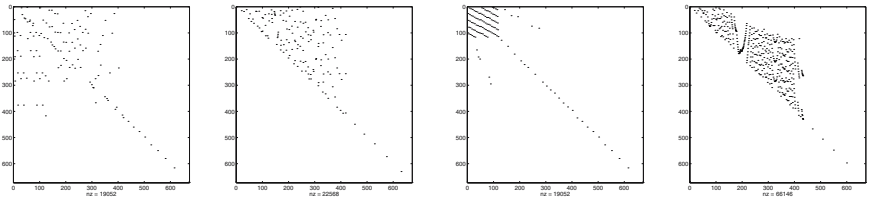
The orderwise independence of the coefficients consequently allows to process the data order by order. By this efficient mechanism for spherical harmonic analysis even very high-resolving models may be estimated (Colombo, 1981; Rummel *et al.*, 1993). For this reason, this block-based method is often denoted as *fast spherical harmonic analysis*.

3 Kite Numbering Scheme

Ever since satellite data became available for global gravity field determination, the idea of combining these, usually irregularly distributed measurements, with regularly distributed (gridded) data, such as gravity anomalies, has been nourished. While satellite data allows a precise determination of coefficients of lower degrees, gridded data is very suitable for estimating coefficients of higher degrees.

A simple modelling approach consists in a *"Patchwork"* technique, which, in the past, lead to a stepwise computation scheme and model refinements (confer e.g. OSU 91 (Rapp *et al.*, 1991), EGM 96 (Lemoine *et al.*, 1996)). With more complex models the correlations within the orders were strictly taken into account (Balmino, 1993). In the given context, the order-by-order-wise numbering scheme could be applied, producing the structures within the normal equation matrix as shown in Fig. 4a.

As can be seen in Fig. 4b, the Cholesky-reduced normal equation matrix contains numerous fill-in elements. The number of fill-ins increases quadratically with the maximal degree of the model. In order to simplify the structure of the reduced normal equation matrix, Bosch (1993) proposed a numbering scheme based on division of the coefficients into three zones (*three zone numbering scheme*). The first zone is built up by the fully correlated coefficients of lower degrees, the second zone by coefficients of higher degrees and



(a) normal eq. (b) red. normal eq. (c) normal eq. (d) red. normal eq.

Order-by-order numbering scheme *Three zone numbering scheme*

Fig. 4. Combined models

lower orders, and the third zone by coefficients of higher degrees and higher orders. Each zone is itself enumerated order-by-order. On the one hand, this scheme leads to a more compact occupation with non-zero elements within the normal equation matrix. On the other hand, Fig. 4d demonstrates that the Cholesky-reduction produces the same number of fill-in elements as for numbering order-by-order.

In the course of the GOCE studies a new numbering scheme was proposed by Schuh (1996a), that circumvents this fill-in effect. Reversing the order of the zones yields a scheme for which Cholesky reduction does not produce any fill-in elements. Since the structure of the normal equation matrix resembles a kite, this modified numbering scheme was termed *kite numbering scheme*. Thus, a strict combination of high-resolving, rotation-symmetrical data with arbitrarily distributed, fully correlated data is, for lower degrees, possible without fill-in elements. Therefore, this combination may be computed on a standard PC even for very high resolving models up to degree and order 720. Beside solving the equation system, partial inverses (i.e. strict inverse for selected elements) may be computed efficiently by means of this approach (Auzinger and Schuh, 1998).

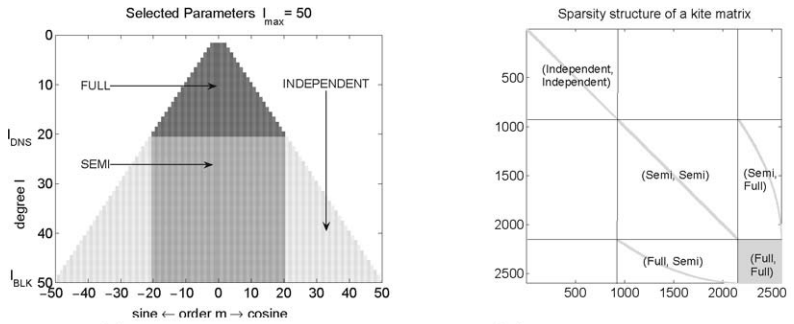
For a more detailed analysis of the properties of the kite structure, the relations between single parameters and zones need to be specified more thoroughly. For this purpose, the low-resolving, fully correlated data is denoted as `DNS (dense)`, while the high-resolving, block-wise correlated data will be referred to as `BLK (block)`. The correlations between the zones may be specified as follows:

- *zone:FULL*
  All parameters are fully correlated.
- *zone:SEMI*
  The BLK parameters are correlated only within the same order, parity (odd or even), and trigonometric function (sine or cosine). Similarly, correlations between type-DNS and type-BLK data exist only within the same order, parity, and trigonometric function.
- *zone:INDEPENDENT*
  Contains all BLK parameters for whose within the same order there are

no DNS parameters. The correlations within BLK data are according to the SEMI zone. However, there are no correlations between data of types DNS and BLK.

Figure 5a illustrates the three zones arising from an exemplary combination of type-DNS observations (degree/order 2-ℓ_{DNS}) with type-BLK data (degree/order 2-ℓ_{BLK}). The "FULL" zone comprises all the parameters of degrees and orders 2 up to ℓ_{DNS}. As all these coefficients are fully correlated, the corresponding part within the normal equation matrix is fully occupied. For the "SEMI" zone, correlations between observations of types DNS and BLK exist only within the same orders and between degrees of identical parity. There are no correlations between type-DNS and type-BLK observation inside the INDEPENDENT zone.

Recall that each parameter generates one row and one column within the normal equation matrix. The kite numbering scheme produces an order-by-order sorting within each zone, which leads to their typical arrangement (*kite structure*) (Schuh, 1996b). Figure. 5b depicts the normal equation matrix for the current example. For the upper-left blocks (INDEPENDENT,INDEPENDENT) it becomes evident that Cholesky reduction does not create any additional fill-ins. The same holds also for block (SEMI,SEMI) in the middle part. At first glance, fill-ins seem to occur for (SEMI,FULL) below its wings (better: below the off-diagonal blocks). However, it can be seen that each diagonal block of (SEMI,SEMI) corresponds to exactly one block of (SEMI,FULL). As there exists only one off-diagonal block per (block-) row, the scalar products of distinct (block-) columns vanish. Consequently, no additional fill-in elements are generated within the reduction step (e.g. Cholesky).



(a) correlation zones (b) normal equation matrix

Fig. 5. Combined models, *kite numbering scheme*

The *kite numbering scheme* may now be used for an efficient, yet strict solution of combined models with gridded, high-resolving data. In addition, it produces an excellent approximate solution in case the data is of almost

regular spatial distribution, i.e. not necessarily rotation-symmetrical. Here, the kite structure may be used as preconditioner for a strict and efficient iterative solver. The `pcgma` algorithm (Schuh, 1996b), which was developed especially for the analysis of GOCE data, makes use of the kite scheme for the combination of SST and SGG data.

Despite the benefits described above, the latest simulations reveal some deficiencies of the kite numbering scheme. The fully correlated coefficients were, for each order m, parameterized with degrees between $\ell_{\min} = \max(2, m)$ and some fixed $\ell_{\mathrm{DNS}_{\max}} = const.$ The degree of the high-resolving model also has a prescribed limit $\ell_{\mathrm{BLK}_{\max}} = const.$ for all orders. However, for many recent Earth gravity field models(e. g. EIGEN-1S (Reigber *et al.*, 2002), EIGEN-2 (Reigber *et al.*, 2003), EIGEN-3p (Reigber *et al.*, 2004)), the limits are kept variable, for instance to take resonance frequencies into account.

In the following section a method will be presented, that allows variable limits within the *kite numbering scheme*. The *free kite numbering scheme* consists essentially of the numbering scheme and a newly developed storage scheme with respect to the normal equations. As a first step we will generalize the *block numbering scheme* to introduce flexible borders and than we will apply this concept to the *free kite numbering scheme*.

4 Free Block Numbering Scheme

To improve the flexibility of Earth gravity field modelling, it is necessary to use numbering schemes that allow selection/deletion of certain coefficients. Fixed maximal degrees throughout all orders turn out to be too inflexible as to take into account particular strengths (resonance frequencies) and weaknesses (polar gap) of a model. To circumvent this problem, the minimal degree $\ell_{\min}$ and the maximal degree $\ell_{\max}$ is fixed for each order m. These limits are stored in three vectors $\boldsymbol{m}$, $\boldsymbol{\ell}_{\min}$ und $\boldsymbol{\ell}_{\max}$, which define the set of parameterized coefficients uniquely. These vectors are of equal lengths, each with as many elements as the number o of parameterized orders (Cosine and Sine coefficients are treated symmetrically) i.e.

$$\begin{aligned} \boldsymbol{m} &= \left(m_1, m_2, \ldots, m_{\mathrm{o}}\right) && \in \mathbb{R}^{o\times 1} \\ \boldsymbol{\ell}_{\min} &= \left(\ell_{\min_1}, \ell_{\min_2}, \ldots, \ell_{\min_o}\right) && \in \mathbb{R}^{o\times 1} \\ \boldsymbol{\ell}_{\max} &= \left(\ell_{\max_1}, \ell_{\max_2}, \ldots, \ell_{\max_o}\right) && \in \mathbb{R}^{o\times 1} \end{aligned}$$

As an example, let

$$\boldsymbol{m} = \left(0,1,2,3,4,5,6,7\right) \quad \boldsymbol{\ell}_{\min} = \left(2,2,2,3,5,5,6,7\right) \quad \boldsymbol{\ell}_{\max} = \left(4,6,7,7,7,6,7,7\right).$$

The resulting set of parameters is illustrated in Fig. 6. The order of the involved coefficents is generated by the loops in Algorithm 3:

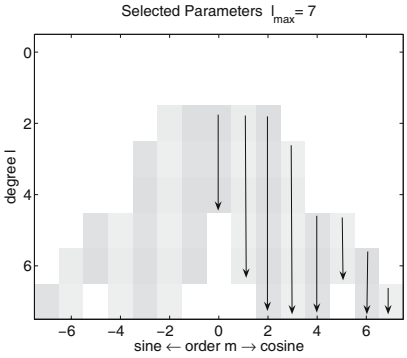


Fig. 6. Free block numbering scheme

Algorithm 3. Free block numbering scheme

for $m = 0 : o$
  for $\ell = \boldsymbol{\ell}_{\min}(m) : 2 : \boldsymbol{\ell}_{\max}(m)$
    $C_{\ell m}$ *% even/odd coefficients*
  end
  for $\ell = \boldsymbol{\ell}_{\min}(m) + 1 : 2 : \boldsymbol{\ell}_{\max}(m)$
    $C_{\ell m}$ *% odd/even coefficients*
  end
end
for $m = 1 : o$
  for $\ell = \boldsymbol{\ell}_{\min}(m) : 2 : \boldsymbol{\ell}_{\max}(m)$
    $S_{\ell m}$ *% even/odd coefficients*
  end
  for $\ell = \boldsymbol{\ell}_{\min}(m) + 1 : 2 : \boldsymbol{\ell}_{\max}(m)$
    $S_{\ell m}$ *% odd/even coefficients*
  end
end

This principle of variable block sizes will now be applied to data combination.

5 Free Kite Numbering Scheme

The newly developed *free kite numbering scheme* (FKN) allows combining two models, both having order-dependent, flexible (*"free"*) limits, as introduced in Sect. 4 for the *free block numbering scheme*. Note that it must be made sure that the normal equation matrix still is a kite matrix in order to avoid additional fill-in elements. For instance, the configuration depicted in Fig. 5 with fixed limits 2, ℓ_{DNS} and ℓ_{BLK} for *free block numbering*, is given by the following vectors:

$$\begin{aligned} \boldsymbol{m}_{\mathrm{DNS}} &= \big(0,1,2,3,4,\ldots,19,20\big) & \boldsymbol{m}_{\mathrm{BLK}} &= \big(0,1,2,3,4,\ldots,49,50\big) \\ \boldsymbol{\ell}_{\min} &= \big(2,2,2,3,4,\ldots,19,20\big) & \boldsymbol{\ell}_{\min} &= \big(2,2,2,3,4,\ldots,49,50\big) \\ \boldsymbol{\ell}_{\max} &= \big(20,20,\ldots,20\big) & \boldsymbol{\ell}_{\max} &= \big(50,50,\ldots,50\big)\,. \end{aligned}$$

From these vectors the symbolic parameter order and the positions of non-zero elements of the normal equation matrix can be computed. While the old algorithm for *kite numbering scheme* is based on rather complex programs with inflexible calculation rules, that do not allow any further flexibilization, the FKN scheme is built up on clearly structured, rule-based logic.

Rule-based processing

The algorithm for assembling the parameter order using the FKN scheme consists of two loops and one rule-based decision tree within the inner loop. In analogy to Algorithm 3 the outer loop is iterated over all orders and the inner loop from the minimal until the maximal degree corresponding to the current order. In contrast to the conditions of Algorithm 3, however, vectors containing the minimal and maximal degrees for both DNS and BLK data are at hand. The FKN algorithm uses, for its inner loop, order-wise the minimum of both minimal degrees and the maximum of both maximal degrees. Inside the loops three stacks are built up. Now each parameter runs through the decision tree and is, thereafter, either assigned to one of the stacks, or discarded. The resulting stacks contain, upon termination of the loop, the parameters of the zones FULL, SEMI, and INDEPENDENT. The decision tree is based on the following rules:

Algorithm 4.

Parameter $\{CS\}_{lm}$ *is analysed:*

a) If there are no observations of type DNS for order m*, then the parameter is assigned to the INDEPEDENT zone, otherwise to either the SEMI or the FULL zone.*

b) If the degree ℓ *of the current parameter lies within the type−DNS observations for this order, then the parameter is assigned to the FULL zone, otherwise to the SEMI zone.*

The assembling of the parameter order according to this rule goes as follows:

Algorithm 5.

Loop over all orders m
Loop over all degrees ℓ of the current order
Even cosine coefficients
Assign to FULL,SEMI or INDEPENDENT zone
Odd cosine coefficients
Assign to FULL,SEMI or INDEPENDENT zone
Even sine coefficients
Assign to FULL,SEMI or INDEPENDENT zone
Odd sine coefficients
Assign to FULL,SEMI or INDEPENDENT zone
end
end
String together zones

The vector of symbolic parameters is now saved as an ASCII file and may be used for data re-sorting. The elements are efficiently accessed by indirect adressing through index vectors.

The traditional graphical representing of the parameters based on the FKN scheme might generate unexpected structures if one actually exploits the full range of flexibility. Figure 7 give some unusual examples, which are, however, computable without any problems.

Rule-based computation of the correlations

Due to the definitions of the zones and their assumptions, some of the parameters are correlated, and others are not. The prescribed symbolic parameter order lead to the re-sorting of the normal equation matrix in such a way that certain operations from the domain of linear algebra are applicable to a kite matrix $\boldsymbol{K}$ without generating fill-ins. Examples are:

- Cholesky reduction;
- Partial inversion yielding $\boldsymbol{K}^{-1}_{\text{part.}}$ (strict inverse for selected elements cf. (Auzinger and Schuh, 1998));
- Solution of the equation system $\boldsymbol{K}\boldsymbol{X} = \boldsymbol{B}$ by means of Cholesky reduction $\boldsymbol{K} = \boldsymbol{R}^T\boldsymbol{R}$ ($\boldsymbol{B}$ contains multiple right hand sides).

In order to assemble a kite matrix, not only the symbolic parameter vector is required, but also the positions of its non-zero elements, which, in turn, represent the correlations between two particular parameters. The correlations are derived from the stored information regarding cosine/sine function, order, parity of degrees, and the vectors containing the minimal and maximal degrees (Algorithm 6).

Now let a kite matrix and a symbolic parameter vector in the FKN scheme be given. The symbolic structure of the kite matrix $\boldsymbol{K} \in \mathbb{R}^{n\times n}$ is stored in a symbolic parameter vector $\boldsymbol{p} \in \mathbb{R}^{n\times 1}$. $\boldsymbol{p}$ contains three entries, the degree ℓ, the order m, and the membership to one of the trigonometric functions CS

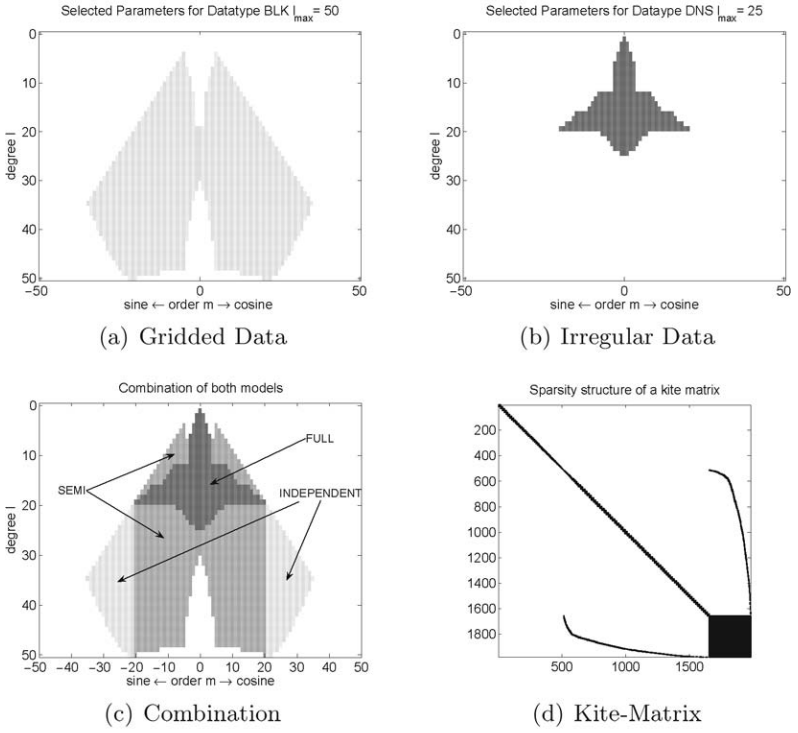


Fig. 7. Three possible correlation zones between BLK and DNS blocks

('C' for cosine and 'S' for sine). The entries of this object are accessed via the "dot" and an index variable. For instance, to test whether the fifth parameter of $\boldsymbol{p}$ has the cosine function, one would enter the statement $\boldsymbol{p}(5).CS == 'C'$. Furthermore, the kite matrix will have a non-zero element at position k_{ij} if the parameters $\boldsymbol{p}(i)$ and $\boldsymbol{p}(j)$ are correlated.

To determine the positions of the non-zero elements using the FKN scheme is computationally the most expensive step, because each of the n^2 possible positions must be evaluated. Due to the symmetry of the normal equation matrix, this number reduces to $n(n+1)/2$ positions. As can be seen from Algorithm 6, two nested loops are necessary to do the job. The outer loop is iterated over all parameters in $\boldsymbol{p}$, the inner loop over all parameters from the current position on.

Algorithm 6.

```
for i = 1 : n
    for j = i : n
        if correlated ( p(i), p(j) )
            save_indices (i, j)
        end
    end
end
```

The function correlated in line 3 of Algorithm 6 checks by means of Algorithm 7, if the two symbolic parameters currently under investigation are correlated.

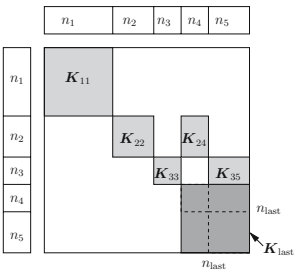
Algorithm 7.

```
function correlated
  if p(i) ∈ FULL and p(j) ∈ FULL
     return true                    % p(i), p(j) correlated
  if p(i).m == p(j).m               % same order
     and
     p(i).ℓ mod 2 == p(j).ℓ mod 2   % same parity in degree
     and
     p(i).CS == p(j).CS             % same trig. function
     return true                    % p(i), p(j) correlated
  return false
end
```

Algorithm 6 outputs the coordinates of the non-zero elements of the given kite matrix. These coordinates are saved internally by the software. Since the non-zero elements are always clustered in blocks, this process may be considerably simplified. For each such block, the row and column index of its first element and its total number of rows and columns is stored. Figure 8a illustrates the block structure of the kite matrix. Figure 8b shows an example for block-oriented storing of the shape of the kite matrix in an ASCII file. With the information contained in this file, the kite matrix can be assembled from the design matrix using level-3 BLAS routines.

6 Conclusion and Outlook

The innovative *free kite numbering scheme* renders the way to new options both for model parameterization and preconditioning. It is now possible to adapt the parameters to the information content of the data. For instance, it might be advisable to exclude certain poorly determinable zonal coefficients from the model. In addition, any coefficients of lower degrees could be eliminated in case the data turns out to be hardly sensitive to the corresponding parameters.



(a) Kite matrix

```
# created May 26 2005  6:37
# by class navigation (Compiled: May 20 2005,16:47:51)

# block  row    col    nrows  ncols filename
block      0      0    100    100   block0000.dat
block    100    100     99     99   block0001.dat
block    199    199     99     99   block0002.dat
block    298    298     99     99   block0003.dat
block    397    397     99     99   block0004.dat
block    496    496     99     99   block0005.dat
block    595    595    100    100   block0006.dat
block    695    695     99     99   block0007.dat
```

(b) ASCII file

Fig. 8. Storing the Kite-Matrix with block coordinates

Manifold are also the possibilities for the kite scheme to be used as a preconditioner. In that regard, one can focus on particularly correlated groups of parameters, which become included into the fully correlated part in order to improve the condition number and, therefore, the convergence rate of the iterative solver.

Consequently, kite numbering proves to be a very flexible and efficient method for a realistic modelling of the Earth's gravity field.

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References

Auzinger, T and Schuh, W-D (1998) High-degree spherical harmonic analysis combining gridded and random distributed data sets. *Phys. Chem. Earth*, 23:19–23

Balmino, G (1993) The spectra of the topography of the earth, venus and mars. *Geophys. Res. Lett.*, 20:1063–1066

Bosch, W (1993) A rigorous least squares combination of low and high degree spherical harmonics. In *Presented Paper on "IAG General Meeting"*. Beijing 1993

Colombo, O (1981) *Numerical methods for harmonic analysis on the sphere*. Reports of the Department of Geodetic Science. Ohio State University (OSU), Ohio. No. 310

Heiskanen, W and Moritz, H (2000) *Physical Geodesy*. Institute of Physical Geodesy, Technical University Graz, Reprint

Lemoine, F, Smith, D, Kunz, L, Smith, R, Pavlis, E, Pavlis, N, Klosko, S, Chinn, D, Torrence, M, Williamson, R, Cox, C, Rachlin, K, Wang, Y, Rapp, R, and Nerem, R (1996) The development of the nasa gsfc and nima joint geopotential

model. In *Proceedings of the "International Symposium on Gravity, Geoid, and Marine Geodesy"*. Tokyo, Japan

Rapp, R (1994) *The Use of Potential Coefficient Models in Computing Geoid Undulations*. Lecture Notes of the International School for the "Determination and Use of the Geoid" (Oct. 10-15, 1994), Milano

Rapp, R, Wang, Y, and Pavlis, N (1991) *The Ohio state 1991 geopotential and sea surface topography harmonic coefficient models*. Reports of the Department of Geodetic Science. Ohio State University (OSU), Ohio. No. 410

Reigber, C, Balmino, G, Schwintzer, P, Biancale, R, Bode, A, Lemoine, J-M, König, R, Loyer, S, Neumayer, H, Marty, J-C, Barthelmes, F, Perosanz, F, and Zhu, S (2002) A high quality global gravity field model from CHAMP GPS tracking data and accelerometry (EIGEN-1S). *Geophysical Research Letters*, 29(14):10.1029/2002GL015064

Reigber, C, Schwintzer, P, Neumayer, K-H, Barthelmes, F, König, R, Förste, C, Balmino, G, Biancale, R, Lemoine, J-M, Loyer, S, Bruinsma, S, Perosanz, F, and Fayard, T (2003) The CHAMP-only earth gravity field model EIGEN-2. *Advances in Space Research*, 31(8):1883–1888

Reigber, C, Jochmann, H, Wünsch, J, Petrovic, S, Schwintzer, P, Barthelmes, F, Neumayer, K-H, König, R, Förste, C, Balmino, G, Biancale, R, Lemoine, J-M, Loyer, S, and Perosanz, F (2004) Earth gravity field and seasonal variability from CHAMP. In C. Reigber, H. Lühr, P. Schwintzer, and J. Wickert, editors, *Earth Observation with CHAMP - Results from Three Years in Orbit*, pages 25–30, Berlin. Springer

Rummel, R, Sansò, F, van Gelderen, M, Brovelli, M, Koop, R, Miggliaccio, F, Schrama, E, and Scerdote, F (1993). *Spherical harmonic analysis of satellite gradiometry*. Netherlands Geodetic Commission, New Series, 39

Schuh, W-D (1996a) Least squares adjustment of high degree spherical harmonics. In m. Jacobsen, editor, *Inverse Methods - Interdisciplinary Elements of Methodology, Computation and Application*, Lecture Notes in Earth Sciences 63, pages 276–283, Heidelberg. Springer

Schuh, W-D (1996b) *Tailored numerical solution strategies for the global determination of the earth's gravity field*. Mitteilungen der Geodätischen Institute der TU, Graz. Folge 81