Chapter IV

Ferromagnetic Models on $\mathbb{Z}$

IV.1. Introduction

Phase transitions are a familiar aspect of nature. Water boils, becoming water vapor, or water vapor, under compression, liquefies. These are examples of a liquid–gas phase transition. The liquid and the gas are said to be two phases of the same substance. One of the most interesting problems in equilibrium statistical mechanics is to explain phase transitions in terms of the probability distributions on configuration space which describe the microscopic behavior of physical systems. The simplest systems for which this is possible are ferromagnetic models on a lattice. The present chapter introduces these models.

The phase transition for ferromagnetic systems has many similarities with the more common liquid–gas transition, although each is described by different variables.[1] Both phase transitions arise as a result of two competing microscopic effects. The first effect tends to order the system. It is caused by attractive forces of interaction and is measured by energy. The second effect tends to randomize the system. It is caused by thermal excitations and is measured by entropy. At sufficiently low temperatures, the energy effect predominates and a phase transition becomes possible.

This chapter develops the statistical mechanics of ferromagnetic models on the one-dimensional integer lattice $\mathbb{Z}$. Many of the results for models on $\mathbb{Z}$ generalize to ferromagnetic models on the D-dimensional integer lattice $\mathbb{Z}^D$, $D \in \{2, 3, \ldots\}$. These models will be treated in Chapter V. The next section discusses qualitatively the main features of ferromagnetic models as established by the theorems of this chapter and the next.

IV.2. An Overview of Ferromagnetic Models

The ultimate source of ferromagnetism is the quantum mechanical spinning of electrons. Because a small magnetic dipole moment is associated with the spin, the electron acts like a magnet with one north pole and one south pole. Both the spin and the magnetic moment can be represented by an arrow which defines the direction of the electron's magnetic field. The spin can point up (spin value 1) or down (spin value -1), and it flips between the two orientations. Ferromagnetic models were invented in order to represent, in simplified form, the interaction of electron spins in real ferromagnets. In this section we discuss the most popular ferromagnetic model which is the Ising model. Its properties are qualitatively the same as those of more complicated models to be discussed later in the book.[2]

Let Λ be a symmetric hypercube of the D-dimensional integer lattice $\mathbb{Z}^D$. To each site j of Λ there is assigned a variable ω_j which takes the value 1 (spin-up) or -1 (spin-down). Fix a number $\mathscr{J} > 0$. Associated with each configuration $\omega = \{\omega_j; j \in \Lambda\}$ of spins is a *Hamiltonian* or *interaction energy*

$$H_{\Lambda,h}(\omega) = -\frac{1}{2}\sum_{i,j \in \Lambda} J(i-j)\omega_i\omega_j - h\sum_{j \in \Lambda}\omega_j,$$

where $J(i-j)$ equals $\mathscr{J}$ if $\|i-j\| = 1$ and equals 0 if $\|i-j\| \neq 1$. Thus the first sum extends over all nearest neighbor pairs of sites in Λ. The number $\mathscr{J}$ is the strength of the nearest neighbor coupling and h is a real number which is the strength of an externally applied magnetic field. The configuration space is the set Ω_Λ of all sequences $\omega = \{\omega_j; j \in \Lambda\}$; thus $\Omega_\Lambda = \{1, -1\}^\Lambda$. Define $\mathscr{B}(\Omega_\Lambda)$ to be the set of all subsets of Ω_Λ. Let ρ be the measure $\frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$ and $\pi_\Lambda P_\rho$ the product measure on $\mathscr{B}(\Omega_\Lambda)$ with identical one-dimensional marginals ρ. The Ising model is defined by the probability measure $P_{\Lambda,\beta,h}$ on $\mathscr{B}(\Omega_\Lambda)$ which assigns to each $\{\omega\}$, $\omega \in \Omega_\Lambda$, the probability

$$P_{\Lambda,\beta,h}\{\omega\} = \exp[-\beta H_{\Lambda,h}(\omega)]\pi_\Lambda P_\rho\{\omega\} \cdot \frac{1}{Z}.$$

The parameter β represents the inverse absolute temperature $1/T$ and is positive. Z is the normalization $\int_{\Omega_\Lambda} \exp[-\beta H_{\Lambda,h}(\omega)]\pi_\Lambda P_\rho(d\omega)$. We call $P_{\Lambda,\beta,h}$ a *finite-volume Gibbs state* or a *finite-volume equilibrium state*. Z is called a *partition function*. Suppose that the external field h is nonzero and consider the configuration $\bar{\omega}$ whose spins are all aligned in the same direction as h. Since $\mathscr{J}$ is positive, this configuration has the smallest interaction energy, hence the largest probability $P_{\Lambda,\beta,h}\{\bar{\omega}\}$, of all configurations in Ω_Λ. Hence the positivity of $\mathscr{J}$ induces an alignment effect in the finite-volume Gibbs state. The effect becomes weaker as β is decreased. At $\beta = 0$ it disappears entirely as $P_{\Lambda,\beta,h}$ reduces to the product measure $\pi_\Lambda P_\rho$ which assigns equal probability to each configuration.

We now proceed to explain, without technicalities, the main properties of the Ising model. These themes are developed in detail in subsequent sections.

Magnetization. Let $S_\Lambda(\omega)$ be the observable $\sum_{j\in\Lambda}\omega_j$ which gives the total spin in Λ. We introduce two quantities

$$M(\Lambda,\beta,h) = \int_{\Omega_\Lambda} S_\Lambda(\omega)P_{\Lambda,\beta,h}(d\omega) \quad \text{and} \quad m(\beta,h) = \lim_{\Lambda\uparrow\mathbb{Z}^D}\frac{1}{|\Lambda|}M(\Lambda,\beta,h).$$

$M(\Lambda,\beta,h)$ is the average value of the total spin in Λ and is called the *magnetization*; $m(\beta,h)$ is the magnetization per site in the limit where Λ expands to fill $\mathbb{Z}^D$ and is called the *specific magnetization*. For $h > 0$, $M(\Lambda, \beta, h)$ is positive and because of the alignment effect built into the finite-volume Gibbs state, $M(\Lambda, \beta, h)$ is an increasing function of h. These properties persist for the specific magnetization after the limit $\Lambda \uparrow \mathbb{Z}^D$. However, the alignment effect becomes weaker as β is decreased. This is reflected in dramatically different behavior of the limit $\lim_{h \to 0^+} m(\beta, h)$ for different values of β.

Spontaneous magnetization. By symmetry, $M(\Lambda, \beta, 0)$ equals 0, and so $m(\beta, 0) = \lim_{\Lambda \uparrow \mathbb{Z}^D} |\Lambda|^{-1} M(\Lambda, \beta, 0)$ equals 0. There exists a critical value of β, called the *critical inverse temperature* and denoted by β_c, which has the following properties. If $0 < \beta < \beta_c$, then $m(\beta, h)$ converges to $m(\beta, 0) = 0$ as $h \to 0^+$. If $\beta > \beta_c$, then $m(\beta, h)$ converges to a positive number $m(\beta, +)$ as $h \to 0^+$. Thus for $\beta > \beta_c$, the system remains permanently magnetized after the external field is removed; $m(\beta, +)$ is known as the *spontaneous magnetization*. For $h < 0$, $m(\beta, h)$ behaves similarly: $m(\beta, h)$ is negative and as $h \to 0^-$, $m(\beta, h)$ converges to $m(\beta, 0) = 0$ or to the negative number $m(\beta, -) = -m(\beta, +)$ according to whether $0 < \beta < \beta_c$ or $\beta > \beta_c$. The value of the critical inverse temperature β_c depends upon $\mathscr{J}$ and upon the dimension D of the lattice. If $D = 1$, then β_c is infinite and spontaneous magnetization does not occur. By contrast, for any $D \geq 2$, β_c is finite. We have not discussed the value $m(\beta, +) = \lim_{h \to 0^+} m(\beta, h)$ at $\beta = \beta_c$. For the Ising model on $\mathbb{Z}^2$, $m(\beta_c, +)$ is 0. While $m(\beta_c, +)$ is believed to be 0 for any Ising model on $\mathbb{Z}^D$, $D \geq 3$, this has not been proved.

Curves showing m as a function of h for fixed β are depicted in Figure IV.1 for the Ising model on $\mathbb{Z}^2$. These curves are called *isotherms*, and the point $(\beta, h) = (\beta_c, 0)$ is called the *critical point*. Notice that $m(\beta, h)$ is an increasing, concave function of $h \geq 0$. The concavity represents a saturation effect. An increment $\Delta h > 0$ causes a change $\Delta m(\beta, h) = m(\beta, h + \Delta h) - m(\beta, h)$ in m. The larger the value of h, the smaller is $\Delta m(\beta, h)$. The quantity $\partial m(\beta, h)/\partial h$, which gives the slope of the isotherms, is called the *specific magnetic susceptibility* and is denoted by $\chi(\beta, h)$.

Infinite-volume Gibbs states. We now consider the limiting behavior of the finite-volume Gibbs states as $\Lambda \uparrow \mathbb{Z}^D$. First, we modify these states by means of *external conditions* or *boundary conditions*. An external condition is defined by fixing the values of the spins $\tilde{\omega}_j$ at each site j which is in $\Lambda^c = \mathbb{Z}^D \backslash \Lambda$ and has a nearest neighbor in Λ. The external condition $\tilde{\omega} = \{\tilde{\omega}_j\}$ changes the Hamiltonian of a configuration ω from $H_{\Lambda, h}(\omega)$ to

$$H_{\Lambda, h, \tilde{\omega}}(\omega) = -\frac{1}{2} \sum_{i, j \in \Lambda} J(i - j)\omega_i \omega_j - \sum_{i \in \Lambda} \left(h + \sum_{j \in \Lambda^c} J(i - j)\tilde{\omega}_j \right) \omega_i.$$

The corresponding finite-volume Gibbs state is defined by the probability measure $P_{\Lambda, \beta, h, \tilde{\omega}}$ on $\mathscr{B}(\Omega_\Lambda)$ which assigns to each $\{\omega\}$, $\omega \in \Omega_\Lambda$, the probability

$$P_{\Lambda, \beta, h, \tilde{\omega}}\{\omega\} = \exp[-\beta H_{\Lambda, h, \tilde{\omega}}(\omega)] \pi_\Lambda P_\rho\{\omega\} \cdot \frac{1}{Z},$$

where Z is a normalization. If each $\tilde{\omega}_j$ equals 1 (resp., -1), then the external

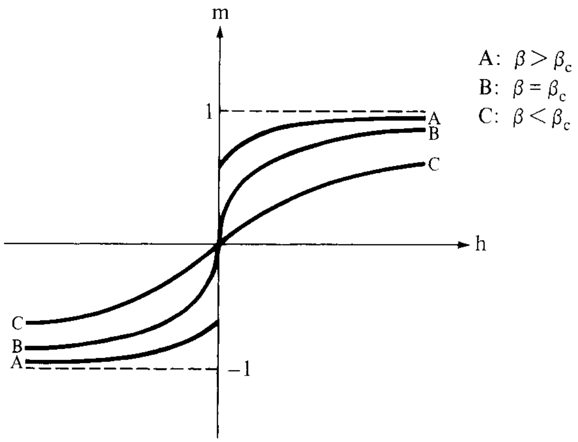


Figure IV.1. Isotherms for the Ising model on $\mathbb{Z}^2$. (Adapted from Figure 4.29 in L. E. Reichl, *A Modern Course in Statistical Physics*, University of Texas Press, Austin, 1980. Copyright © 1980 by the University of Texas Press.)

condition is called *plus* (resp., *minus*) and the measure is written as $P_{\Lambda,\beta,h,+}$ (resp., $P_{\Lambda,\beta,h,-}$). The external condition $\tilde{\omega}$ depends on Λ, and so we write $\tilde{\omega} = \tilde{\omega}(\Lambda)$. For fixed $\beta > 0$ and h real, let us consider the set of all weak limits

$$P = w\text{-}\lim_{\Lambda' \uparrow \mathbb{Z}^D} P_{\Lambda',\beta,h,\tilde{\omega}(\Lambda')}, \tag{4.1}$$

where $\{\Lambda'\}$ is any increasing sequence of symmetric hypercubes whose union is $\mathbb{Z}^D$ and $\tilde{\omega}(\Lambda')$ is any external condition for Λ'. Each weak limit P is a probability measure on the infinite-volume configuration space $\Omega = \{1, -1\}^{\mathbb{Z}^D}$. We call a probability measure P on Ω an *infinite-volume Gibbs state* or an *infinite-volume equilibrium state* if P belongs to the closed convex hull of the set of weak limits of the form (4.1).

Classification of infinite-volume Gibbs states. For the Ising model, various infinite-volume Gibbs states with different properties arise. They are listed in Table IV.1. The set of infinite-volume Gibbs states is denoted by $\mathcal{G}_{\beta,h}$. We now describe the structure of $\mathcal{G}_{\beta,h}$ for different values of β and h. Let β_c be the critical inverse temperature. For all $\beta > 0$, $h \neq 0$ and $0 < \beta < \beta_c$, $h = 0$, the finite-volume Gibbs states $\{P_{\Lambda,\beta,h,\tilde{\omega}(\Lambda)}\}$ have a unique weak limit for any choice of external conditions $\{\tilde{\omega}(\Lambda)\}$. Thus $\mathcal{G}_{\beta,h}$ consists of a unique measure $P_{\beta,h}$. The measure $P_{\beta,h}$ is translation invariant (strictly stationary with respect to the shift mappings on Ω) and ergodic. The ergodic phases are characterized among all translation invariant states by the property that macroscopic quantities are given definite values. For example, an experimenter might measure the spin per site $S_\Lambda(\omega)/|\Lambda| = \sum_{j \in \Lambda} \omega_j/|\Lambda|$ in a sample ω drawn from the magnet. The ergodic theorem implies that with respect to an ergodic phase such as $P_{\beta,h}$, $S_\Lambda(\omega)/|\Lambda|$ tends to a constant which is almost surely independent of the sample chosen. This justifies calling an ergodic phase *pure*.

Table IV.1. States of the Ferromagnet

Name	Definition
Infinite-volume Gibbs state	A member of the closed convex hull of the set of weak limits of $\{P_{\Lambda',\beta,h,\bar{\omega}(\Lambda')}; \Lambda' \uparrow \mathbb{Z}^D\}$.
Phase	Translation invariant infinite-volume Gibbs state.
Pure phase	Ergodic, translation invariant infinite-volume Gibbs state.
Mixed phase	Nontrivial convex combination of pure phases.

For $\beta > \beta_c$ and $h = 0$, the situation is radically different. $\mathscr{G}_{\beta,0}$ contains two distinct pure phases $P_{\beta,0,+}$ and $P_{\beta,0,-}$, which arise from the finite-volume Gibbs states $P_{\Lambda,\beta,0,+}$ and $P_{\Lambda,\beta,0,-}$ with plus and minus external conditions, respectively. The average value of the spin at any site j with respect to each of these measures is given by

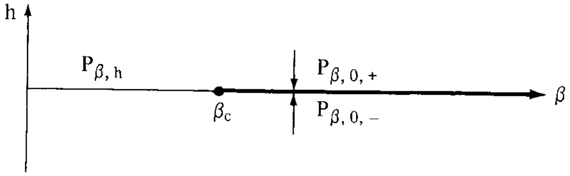
$$\begin{aligned} &\int_\Omega \omega_j P_{\beta,0,+}(d\omega) = m(\beta, +) > 0, \\ &\int_\Omega \omega_j P_{\beta,0,-}(d\omega) = -m(\beta, +) < 0, \end{aligned} \tag{4.2}$$

where $m(\beta, +)$ is the spontaneous magnetization. $P_{\beta,0,+}$ is called a *pure plus phase* and $P_{\beta,0,-}$ a *pure minus phase*. In addition, $\mathscr{G}_{\beta,0}$ contains all convex combinations $P_{\beta,0}^{(\lambda)} = \lambda P_{\beta,0,+} + (1-\lambda)P_{\beta,0,-}$, $0 < \lambda < 1$. These measures are translation invariant but not ergodic and are called *mixed phases*. Such a phase corresponds physically to the situation where an experimenter has an a priori choice of external condition. With probability λ he prepares each finite-volume Gibbs state to have the plus external condition and with probability $1 - \lambda$ to have the minus external condition. The existence of more than one pure phase for $\beta > \beta_c$ and $h = 0$ corresponds to a *phase transition*.

The phase transition reflects a crucial instability in the model. Choosing the plus or minus external condition outside Λ induces a slight preference for spin-up or spin-down at any fixed site inside Λ. For $\beta > \beta_c$ and $h = 0$, even this slight preference is strong enough, in the limit $\Lambda \uparrow \mathbb{Z}^D$, to push the infinite-volume system into a phase with net positive or negative specific magnetization. The phase transition is related to the notion of symmetry breaking [see page 116].

More is known about the structure of $\mathscr{G}_{\beta,0}$ for $\beta \geq \beta_c$. For the Ising model on $\mathbb{Z}^2$, $\mathscr{G}_{\beta_c,0}$ consists of a unique measure which is a pure phase. For $\beta > \beta_c$ and $h = 0$, $\mathscr{G}_{\beta,0}$ consists precisely of the measures $P_{\beta,0,+}$, $P_{\beta,0,-}$, and convex combinations [Aizenman (1979, 1980), Higuchi (1982)]. Thus all measures in $\mathscr{G}_{\beta,0}$ are translation invariant. For the Ising model on $\mathbb{Z}^D$ with $D \geq 3$, $\mathscr{G}_{\beta,0}$ contains nontranslation invariant states for any sufficiently large $\beta > \beta_c$ [Dobrushin (1972), van Beijeren (1975)].

The pure phases of the Ising model on $\mathbb{Z}^2$ are shown in Figure IV.2. The interval $h = 0$, $\beta > \beta_c$ is called the *coexistence interval* for the pure plus



$P_{\beta,h}$: unique pure phase ($\beta > 0$, $h \neq 0$ and $0 < \beta \leq \beta_c$, $h \neq 0$)
$P_{\beta,0,+}$: pure plus phase ($\beta > \beta_c$)
$P_{\beta,0,-}$: pure minus phase ($\beta > \beta_c$)

Figure IV.2. The phase diagram of the Ising model on $\mathbb{Z}^2$.

phase and pure minus phase. Crossing the interval at constant β by decreasing h through 0 gives an abrupt transition between phases characterized by the discontinuity in the specific magnetization ($m(\beta, h)$ jumps from $m(\beta, +)$ to $m(\beta, -)$).

Correlations.[3] Correlations in the Ising model are related to the phase transition just discussed. We consider the model on $\mathbb{Z}^2$. For $D \geq 3$ similar behavior is expected. The following discussion is heuristic; not all the statements have been proved. Fix $h = 0$. At infinite temperature ($\beta = 0$), the finite-volume Gibbs state $P_{\Lambda,\beta,0,\bar{\omega}}$ reduces to the product measure $\pi_\Lambda P_\rho$. The corresponding infinite-volume Gibbs state is the product measure P_ρ on Ω, with respect to which the spins are independent and thus uncorrelated. At small but nonzero β, there is a unique infinite-volume Gibbs state $P_{\beta,0}$. Spins begin to be positively correlated with their nearest neighbors which in turn begin to be positively correlated with the second nearest neighbors, and so on. Since $\langle \omega_i \rangle_{\beta,0} = \int_\Omega \omega_i P_{\beta,0}(d\omega)$ equals 0 for each i, the convariance (or as we will call it, the pair correlation) equals $\langle \omega_i \omega_j \rangle_{\beta,0} = \int_\Omega \omega_i \omega_j P_{\beta,0}(d\omega)$. The correlations decrease with distance, and in fact $\langle \omega_i \omega_j \rangle_{\beta,0}$ has roughly an exponential decay when the Euclidean distance $\|i - j\|$ is large. We write

$$\langle \omega_i \omega_j \rangle_{\beta,0} \sim \exp[-\|i - j\|/\xi(\beta, 0)] \qquad \text{as } \|i - j\| \to \infty.$$

This relation defines the (exponential) *correlation length* $\xi(\beta, 0)$. The number $\xi(\beta, 0)$ is a rough measure of the distance over which correlations between spins are significant. As β increases, $\xi(\beta, 0)$ increases, and correlations begin to extend over larger and larger distances. These correlations take the form of spin fluctuations, which are islands of a few spins each that mostly point in the same direction. As β approaches the critical inverse temperature β_c, the correlation length grows rapidly, but the smaller fluctuations are not suppressed. They become contained in areas of larger fluctuations which themselves can be distinguished only because they have an overall excess of one spin orientation. When β equals β_c, the correlation length is infinite. Spin fluctuations persist at all scales of length and are extremely sensitive to small perturbations in h. The infinite correlation length is reflected in the fact that $\langle \omega_i \omega_j \rangle_{\beta_c,0}$ no longer decays exponentially but decays like a power of $\|i-j\|^{-1}$:

$$\langle\omega_i\omega_j\rangle_{\beta_c,0} \sim \|i-j\|^{-z} \qquad \text{as } \|i-j\| \to \infty,$$

where z is some positive number ($z = \frac{1}{4}$ for the Ising model on $\mathbb{Z}^2$).

For β larger than β_c, we enter the region of positive spontaneous magnetization $m(\beta, +)$. We consider the pair correlation with respect to the pure plus phase $P_{\beta,0,+}$. By (4.2), $\langle\omega_i\rangle_{\beta,0,+} = \langle\omega_j\rangle_{\beta,0,+} = m(\beta, +)$. As in the case $\beta < \beta_c$, the pair correlation decays exponentially:

$$\langle\omega_i\omega_j\rangle_{\beta,0,+} - \langle\omega_i\rangle_{\beta,0,+}\langle\omega_j\rangle_{\beta,0,+}$$
$$= \langle\omega_i\omega_j\rangle_{\beta,0,+} - [m(\beta, +)]^2 \sim \exp[-\|i-j\|/\xi(\beta,0)] \qquad \text{as } \|i-j\| \to \infty,$$

where $0 < \xi(\beta,0) < \infty$. As β increases, $m(\beta, +)$ increases and the alignment effect becomes more rigid. Within a region of up-spins, islands of down-spins become, on the average, smaller and $\xi(\beta,0)$ decreases. As $\beta \to \infty$, $m(\beta, +)$ converges to 1, $\xi(\beta,0)$ converges to 0, and $P_{\beta,0,+}$ converges weakly to the state where all spins are oriented up. A similar discussion holds for the pure minus phase $P_{\beta,0,-}$.

The infinite correlation length at $\beta = \beta_c$ is related to the behavior of the specific magnetic susceptibility $\chi(\beta,h) = \partial m(\beta,h)/\partial h$ at $h = 0$. In Chapter V, we will prove that for $0 < \beta \le \beta_c$

$$\chi(\beta,0) = \beta \sum_{k\in\mathbb{Z}^D} \langle\omega_0\omega_k\rangle_{\beta,0}$$

($\langle\omega_0\rangle_{\beta_c,0} = \langle\omega_k\rangle_{\beta_c,0} = 0$ for the model on $\mathbb{Z}^2$). For $0 < \beta < \beta_c$, $\langle\omega_0\omega_k\rangle_{\beta,0}$ decays exponentially and $\chi(\beta,0)$ is finite. By contrast, at $\beta = \beta_c$, $\langle\omega_0\omega_k\rangle_{\beta_c,0}$ decays like $\|k\|^{-1/4}$ and $\chi(\beta_c,0)$ is infinite. These conclusions are confirmed by Figure IV.1, in which $\chi(\beta,0)$ is the slope at $h = 0$ of the isotherm $m(\beta,h)$. The infinite slope at $h = 0$ of the critical isotherm $m(\beta_c,h)$ anticipates the onset of spontaneous magnetization for $\beta > \beta_c$ ($m(\beta,h)$ discontinuous at $h = 0$). By definition, a large value of $\chi(\beta,h)$ implies a dramatic response of the magnetization to a small change in external field. The divergence of the specific magnetic susceptibility at the critical point is one way in which the extreme sensitivity of the spin fluctuations to small perturbations in h shows up in the macroscopic behavior of the ferromagnet.[4]

This completes our qualitative discussion of the Ising model. The study of ferromagnetic models on $\mathbb{Z}$ begins in the next section.

IV.3. Finite-Volume Gibbs States on $\mathbb{Z}$

The models will be defined on the symmetric intervals $\Lambda = \{j\in\mathbb{Z}: |j| \le N\}$, where N is a non-negative integer. To each site $j\in\Lambda$ there is assigned a spin ω_j which takes the value 1 (spin-up) or -1 (spin-down). The configuration space is the set Ω_Λ of all sequences $\omega = \{\omega_j; j\in\Lambda\}$; thus, $\Omega_\Lambda = \{1, -1\}^\Lambda$. The coordinate functions on Ω_Λ, defined by $Y_j(\omega) = \omega_j$, are called the *spin* *random variables* at the sites j. The presence of interactions distinguishes these models from the discrete ideal gas. The *Hamiltonian* or *interaction energy* of a spin configuration $\omega \in \Omega_\Lambda$ is defined as

(4.3) $$H_{\Lambda,h}(\omega) = -\frac{1}{2} \sum_{i,j\in\Lambda} J(i-j)\omega_i\omega_j - h \sum_{j\in\Lambda} \omega_j.$$

We assume that J is a non-negative function on $\mathbb{Z}$ which is symmetric; i.e., satisfies $J(i-j) = J(j-i)$ for each i and j. J is called a *ferromagnetic interaction.** The parameter h is a real number which gives the strength of an external magnetic field acting at each site in Λ. The term $-J(i-j)\omega_i\omega_j$ in the first sum in (4.2) gives the interaction energy between the spins at sites i and j. The interaction strength $J(i-j)$ is translation invariant; i.e., $J((i+k)-(j+k)) = J(i-j)$ for each k. The factor $\frac{1}{2}$ is included in (4.3) because each pair i, j with $i \neq j$ appears twice with equal weight $J(i-j) = J(j-i)$. The term $-h\omega_j$ in the second sum in (4.3) gives the interaction energy between the external field and the spin at site j. An interaction J is said to have *finite-range* if $J(k)$ equals 0 for all sufficiently large k. The *range* is the smallest number L such that $J(k) = 0$ whenever $|k| > L$.

We denote by $\mathscr{B}(\Omega_\Lambda)$ the set of all subsets of Ω_Λ. Let ρ be the measure $\frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$ and define $\pi_\Lambda P_\rho$ to be the product measure on $\mathscr{B}(\Omega_\Lambda)$ with identical one-dimensional marginals ρ. For each $\omega \in \Omega_\Lambda$, $\pi_\Lambda P_\rho\{\omega\}$ equals $2^{-|\Lambda|}$, where $|\Lambda| = 2N + 1$ is the number of sites in Λ. Let $\beta = 1/T > 0$ be the inverse absolute temperature. The ferromagnetic model is defined by the probability measure $P_{\Lambda,\beta,h}$ on $\mathscr{B}(\Omega_\Lambda)$ which assigns to each $\{\omega\}$, $\omega \in \Omega_\Lambda$, the probability

(4.4) $$P_{\Lambda,\beta,h}\{\omega\} = \exp[-\beta H_{\Lambda,h}(\omega)]\pi_\Lambda P_\rho\{\omega\} \cdot \frac{1}{Z(\Lambda,\beta,h)}.$$

$Z(\Lambda,\beta,h)$ is a normalization which is picked so that $\sum_{\omega\in\Omega_\Lambda} P_{\Lambda,\beta,h}\{\omega\} = 1$:

$$Z(\Lambda,\beta,h) = \int_{\Omega_\Lambda} \exp[-\beta H_{\Lambda,h}(\omega)]\pi_\Lambda P_\rho(d\omega) = \sum_{\omega\in\Omega_\Lambda} \exp[-\beta H_{\Lambda,h}(\omega)] \frac{1}{2^{|\Lambda|}}.$$
(4.5)

For A a subset of Ω_Λ, we have

$$P_{\Lambda,\beta,h}\{A\} = \sum_{\omega\in A} P_{\Lambda,\beta,h}\{\omega\}.$$

The measure $P_{\Lambda,\beta,h}$ is called a *finite-volume Gibbs state* on Λ. $Z(\Lambda,\beta,h)$ is called the *partition function.*[5] Here are some examples of ferromagnetic models.

Example IV.3.1. (a) *The Ising model on $\mathbb{Z}$.* Fix a number $\mathscr{J} > 0$. Define $J(i-j)$ to be $\mathscr{J}$ if $|i-j| = 1$ and to be 0 if $|i-j| \neq 1$. This interaction, which couples only nearest neighbors, has range 1.

* More general interactions are discussed in Appendix C.3.

(b) *The Curie–Weiss model.* Fix a number $\mathscr{J}_0 > 0$. Define $J(i-j)$ to be $\mathscr{J}_0/|\Lambda|$ if both i and j are in Λ and to be 0 if either i or j is not in Λ. This interaction, which depends on the set Λ, couples all pairs of spins in Λ with equal strength. The range tends to ∞ as $|\Lambda| \to \infty$.

(c) *Infinite-range models.* Fix a number $\alpha > 1$. Define $J(0) = 0$ and $J(i-j) = |i-j|^{-\alpha}$ for all $i \neq j$ in $\mathbb{Z}$. Since $\alpha > 1$, this interaction is summable: $\sum_{k \in \mathbb{Z}} J(k) < \infty$ [see Section IV.5].

In order to exclude trivial cases, we assume that the interaction $J(i-j)$ in (4.3) is positive for at least one distinct pair i, j in Λ. The finite-volume Gibbs state $P_{\Lambda,\beta,h}$ has the same form as the reduced canonical ensemble for the discrete ideal gas, which is defined in (3.26). The difference is that the kinetic energy $U_n(\omega)$ in (3.26) is replaced by the interaction energy $H_{\Lambda,h}(\omega)$.* While (3.26) can be written as a product measure (with identical one-dimensional marginals $\bar{\rho}_\beta$), $P_{\Lambda,\beta,h}$ cannot be written as a product measure because of the positivity assumption on J.

Let us examine the form of $P_{\Lambda,\beta,h}$ for different values of β and h. As $\beta \to 0$, $P_{\Lambda,\beta,h}$ converges weakly to the product measure $\pi_\Lambda P_\rho$ which gives equal probability to each ω. Thus, at $\beta = 0$ (infinite temperature) the magnet is completely random. For $\beta > 0$ an alignment effect comes into play. Since $P_{\Lambda,\beta,h}\{\bar{\omega}\} > P_{\Lambda,\beta,h}\{\omega\}$ if and only if $H_{\Lambda,h}\{\bar{\omega}\} < H_{\Lambda,h}\{\omega\}$, the smaller the energy of a configuration, the more probable it is. Thus the most likely configurations are those that minimize $H_{\Lambda,h}$ over Ω_Λ. These minimizing configurations, called *ground states*, are not hard to identify. Let $\bar{\omega}_+$ (resp., $\bar{\omega}_-$) be the configuration with $\bar{\omega}_{+,j} = 1$ (resp., $\bar{\omega}_{,j} = -1$) for each $j \in \Lambda$. If A is a subset of $\mathbb{Z}$, then the interaction J is said to be *irreducible on A* if for each pair of sites i, j in A either $J(i-j) > 0$ or there exists a finite sequence $i_1 = i, i_2, \ldots, i_{r-1}, i_r = j$ in A such that $J(i_{\alpha+1} - i_\alpha) > 0$, $\alpha = 1, 2, \ldots, r-1$. The next result is a consequence of the non-negativity assumption on J [Problem IV.9.1].

Proposition IV.3.2. (a) *For $h > 0$, $\bar{\omega}_+$ is the unique ground state.*

(b) *For $h < 0$, $\bar{\omega}_-$ is the unique ground state.*

(c) *For $h = 0$, the ground states include $\bar{\omega}_+$ and $\bar{\omega}_-$. These are the unique ground states if and only if J is irreducible on Λ.*

For $h > 0$, the measures $\{P_{\Lambda,\beta,h}; \beta > 0\}$ converge weakly to the unit point measure $\delta_{\bar{\omega}_+}$ as $\beta \to \infty$ [Problem IV.9.2]. That is, for $h > 0$ and 0 temperature the totally aligned ground state $\bar{\omega}_+$ is the only possible configuration. No randomness at all is left in the ferromagnet. A similar situation holds for $h < 0$.

The remarks in the previous paragraphs show that the finite-volume state $P_{\Lambda,\beta,h}$ defines a reasonable model for a ferromagnet. The form of $P_{\Lambda,\beta,h}$ can also be justified by means of an entropy principle. This principle

* $U_n(\omega) = \frac{1}{2}\sum_{j=1}^n y_j^2$ has the same form as $H_{\Lambda,0}(\omega)$ but with $J(i-j) = 0$ for all $i \neq j$.

will be generalized later in the chapter. Given a probability measure P on $\mathscr{B}(\Omega_\Lambda)$, we define the energy in P to be $U(\Lambda, h; P) = \int_{\Omega_\Lambda} H_{\Lambda,h}(\omega) P(d\omega)$. Let $U_{\min}$ and $U_{\max}$ denote the minimum and maximum, respectively, of $H_{\Lambda,h}(\omega)$ over Ω_Λ. Fix a number $U \in (U_{\min}, U_{\max})$. We prove that there exists a unique value β such that the finite-volume Gibbs state $P_{\Lambda,\beta,h}$ is the most random probability measure P on $\mathscr{B}(\Omega_\Lambda)$ which satisfies the energy constraint $U(\Lambda, h; P) = U$. The randomness in P, relative to the fixed measure $\pi_\Lambda P_\rho$, is measured by the negative of the relative entropy

$$(4.6) \qquad I^{(2)}_{\pi_\Lambda P_\rho}(P) = \int_{\Omega_\Lambda} \log \frac{dP}{d(\pi_\Lambda P_\rho)}(\omega) P(d\omega) = \sum_{\omega \in \Omega_\Lambda} \log \frac{P\{\omega\}}{\pi_\Lambda P_\rho\{\omega\}} \cdot P\{\omega\}.$$

Theorem IV.3.3.[6] *Let h real and $U \in (U_{\min}, U_{\max})$ be given. Then the following conclusions hold.*

(a) *There exists a unique value β such that $U(\Lambda, h; P_{\Lambda,\beta,h}) = U$.*

(b) *$I^{(2)}_{\pi_\Lambda P_\rho}(P)$ attains its infimum over the set $\{P \in \mathscr{M}(\Omega_\Lambda): U(\Lambda, h; P_{\Lambda,\beta,h}) = U\}$ at the unique measure $P = P_{\Lambda,\beta,h}$.*

Proof. (a) Define $c(t) = \log \int_{\Omega_\Lambda} \exp(tH_{\Lambda,h}(\omega)) \pi_\Lambda P_\rho(d\omega)$ for t real.* We have

$$c'(t) = \frac{\int_{\Omega_\Lambda} H_{\Lambda,h}(\omega) \exp[tH_{\Lambda,h}(\omega)] \pi_\Lambda P_\rho(d\omega)}{\int_{\Omega_\Lambda} \exp[tH_{\Lambda,h}(\omega)] \pi_\Lambda P_\rho(d\omega)},$$

$$c'(-\beta) = \int_{\Omega_\Lambda} H_{\Lambda,h}(\omega) P_{\Lambda,\beta,h}(d\omega) = U(P_{\Lambda,\beta,h}).$$

If ν denotes the $\pi_\Lambda P_\rho$-distribution of $H_{\Lambda,h}$, then $c(t) = \log \int_{\mathbb{R}} e^{tx} \nu(dx)$. This is the free energy function of ν, which was introduced in Section II.4. The support of ν is a finite set which equals the range of $H_{\Lambda,h}(\omega)$, $\omega \in \Omega_\Lambda$. By Theorem II.5.2(a), for $U \in (U_{\min}, U_{\max})$, there exists a unique value β such that $c'(-\beta) = U(\Lambda, h; P_{\Lambda,\beta,h}) = U$.

(b) $I^{(2)}_{\pi_\Lambda P_\rho}(P_{\Lambda,\beta,h}) = \int_{\Omega_\Lambda} [-\beta H_{\Lambda,h}(\omega) - \log Z] P_{\Lambda,\beta,h}(d\omega) = -\beta U - \log Z$, where $Z = Z(\Lambda, \beta, h)$. If $P \neq P_{\Lambda,\beta,h}$ is any other measure in $\mathscr{M}(\Omega_\Lambda)$ for which $U(\Lambda, h; P_{\Lambda,\beta,h}) = U$, then

$$\begin{aligned} I^{(2)}_{\pi_\Lambda P_\rho}(P) &= \int_{\Omega_\Lambda} \log \frac{dP}{d(\pi_\Lambda P_\rho)}(\omega) P(d\omega) \\ &= \int_{\Omega_\Lambda} \log \frac{dP}{dP_{\Lambda,\beta,h}}(\omega) P(d\omega) + \int_{\Omega_\Lambda} \log \frac{dP_{\Lambda,\beta,h}}{d(\pi_\Lambda P_\rho)}(\omega) P(d\omega) \\ &= I^{(2)}_{P_{\Lambda,\beta,h}}(P) + \int_{\Omega_\Lambda} [-\beta H_{\Lambda,h}(\omega) - \log Z] P(d\omega) \\ &= I^{(2)}_{P_{\Lambda,\beta,h}}(P) - \beta U - \log Z. \end{aligned}$$

* $c(t) = \log Z(\Lambda, -t, h)$.

Since $P \neq P_{\Lambda,\beta,h}$, $I^{(2)}_{P_{\Lambda,\beta,h}}(P)$ is positive [Proposition I.4.1(b)], and so $I^{(2)}_{\pi_\Lambda P_\rho}(P) > -\beta U - \log Z$. We conclude that

$$I^{(2)}_{\pi_\Lambda P_\rho}(P) \geq -\beta U - \log Z = I^{(2)}_{\pi_\Lambda P_\rho}(P_{\Lambda,\beta,h})$$

with equality if and only if $P = P_{\Lambda,\beta,h}$. □

Let $S_\Lambda(\omega)$ be the random sum $\sum_{j\in\Lambda} Y_j(\omega)$, called the (total) *spin in* Λ. The *magnetization* is defined as the expectation

$$\begin{aligned} M(\Lambda,\beta,h) &= \int_{\Omega_\Lambda} S_\Lambda(\omega) P_{\Lambda,\beta,h}(d\omega) \\ &= \sum_{j\in\Lambda} \sum_{\omega\in\Omega_\Lambda} \omega_j \exp[-\beta H_{\Lambda,h}(\omega)] 2^{-|\Lambda|} \frac{1}{Z(\Lambda,\beta,h)}. \end{aligned} \tag{4.7}$$

The next theorem states properties of $M(\Lambda,\beta,h)$ which will be proved in Theorem V.4.2.

Theorem IV.3.4. (a) *For each* $\beta > 0$, $M(\Lambda,\beta,0) = 0$; $M(\Lambda,\beta,h)$ *is a non-negative concave function of* $h \geq 0$ *and a nondecreasing function of* h *real. It satisfies* $M(\Lambda,\beta,-h) = -M(\Lambda,\beta,h)$ *and* $|M(\Lambda,\beta,h)| \leq |\Lambda|$.

(b) *For each* $h \geq 0$, $M(\Lambda,\beta,h)$ *is a non-negative, nondecreasing function of* $\beta > 0$.

By (4.7), $M(\Lambda,\beta,h)$ is a continuous function of h. Since $M(\Lambda,\beta,0) = 0$, $M(\Lambda,\beta,h)$ converges to 0 as $h \to 0$ for *any* value of $\beta > 0$.

We will study the magnetization per site in the limit as the symmetric intervals Λ expand to fill $\mathbb{Z}$. This limit is called the *infinite-volume limit* or the *thermodynamic limit* and is denoted by $\Lambda \uparrow \mathbb{Z}$. We define the *specific magnetization* as

$$m(\beta,h) = \lim_{\Lambda\uparrow\mathbb{Z}} \frac{1}{|\Lambda|} M(\Lambda,\beta,h). \tag{4.8}$$

Since $M(\Lambda,\beta,0)$ equals 0, $m(\beta,0)$ equals 0. We will see that if the interaction is summable, then $m(\beta,h)$ exists, $m(\beta,+) = \lim_{h\to 0^+} m(\beta,h)$ exists, and $m(\beta,+)$ is non-negative. The model is said to exhibit *spontaneous magnetization* at inverse temperature β if $m(\beta,+) > 0 = m(\beta,0)$. The quantities $m(\beta,h)$ and $m(\beta,+)$ are studied in the next two sections.

IV.4. Spontaneous Magnetization for the Curie–Weiss Model

In order to see how spontaneous magnetization can result from the microscopic alignment effects built into $P_{\Lambda,\beta,h}$, we first consider the Curie–Weiss model.[7] This model is ideal for doing exact calculations, and the analysis of it involves interesting applications of large deviation theory. The Curie–Weiss model is defined in Example IV.3.1(b). To ease the notation, we replace Λ by the set $\{1, 2, \ldots, n\}$, where n is a positive integer. All quantities are indexed by n instead of by Λ. Thus the finite-volume Gibbs state is given by the formula

(4.9) $$P_{n,\beta,h}\{\omega\} = \exp[-\beta H_{n,h}(\omega)]\pi_n P_\rho\{\omega\}\cdot\frac{1}{Z(n,\beta,h)}, \qquad \omega\in\Omega_n,$$

where $\Omega_n = \{1,-1\}^n$,

$$H_{n,h}(\omega) = -\tfrac{1}{2}(\mathscr{J}_0/n)\sum_{i,j=1}^{n}\omega_i\omega_j - h\sum_{j=1}^{n}\omega_j,$$

$$Z(n,\beta,h) = \int_{\Omega_n}\exp[-\beta H_{n,h}(\omega)]\pi_n P_\rho(d\omega).$$

We write the Hamiltonian as a function of the sum $\sum_{j=1}^n \omega_j/n$:

$$H_{n,h}(\omega) = -n\left[\frac{1}{2}\mathscr{J}_0\left(\frac{\sum_{j=1}^n\omega_j}{n}\right)^2 + h\frac{\sum_{j=1}^n\omega_j}{n}\right].$$

This simple form makes the model easy to handle.

By (4.7) and (4.8), $m(\beta,h)$ equals $\lim_{n\to\infty}\langle S_n/n\rangle_{n,\beta,h}$, where $\langle - \rangle_{n,\beta,h}$ denotes expectation with respect to $P_{n,\beta,h}$. We prove that $m(\beta,+) = \lim_{h\to 0^+} m(\beta,h)$ equals 0 for all $0<\beta\le\mathscr{J}_0^{-1}$ and is positive for all $\beta>\mathscr{J}_0^{-1}$. Thus spontaneous magnetization occurs at all $\beta>\mathscr{J}_0^{-1}$. The number $\mathscr{J}_0^{-1}$ is called the *critical inverse temperature* for the Curie–Weiss model and is denoted by β_c^{CW}.

We will determine the limit of $\langle f(S_n/n)\rangle_{n,\beta,h}$ as $n\to\infty$ for any function $f\in\mathscr{C}(\mathbb{R})$. By doing this we will not only prove spontaneous magnetization but also determine the distribution limit of S_n/n as $n\to\infty$. Let $Q_n^{(1)}(dz)$ denote the distribution of the sum $\sum_{j=1}^n\omega_j/n$ with respect to the product measure $\pi_n P_\rho$. Then

(4.10) $$\begin{aligned}\left\langle f\left(\frac{S_n}{n}\right)\right\rangle_{n,\beta,h} &= \int_{\Omega_n} f\left(\frac{\sum_{j=1}^n\omega_j}{n}\right)\exp[-\beta H_{n,h}(\omega)]\pi_n P_\rho(d\omega)\cdot\frac{1}{Z(n,\beta,h)}\\ &= \int_{\mathbb{R}} f(z)\exp[n(\tfrac{1}{2}\beta\mathscr{J}_0 z^2 + \beta h z)]Q_n^{(1)}(dz)\cdot\frac{1}{Z(n,\beta,h)},\end{aligned}$$

and the partition function $Z(n,\beta,h)$ can be written as

(4.11) $$Z(n,\beta,h) = \int_{\mathbb{R}}\exp[n(\tfrac{1}{2}\beta\mathscr{J}_0 z^2 + \beta h z)]Q_n^{(1)}(dz).$$

By Theorem II.4.1, the distributions $\{Q_n^{(1)}; n = 1,2,\ldots\}$ have a large deviation property with $a_n = n$ and entropy function $I_\rho^{(1)}(z) = \sup_{t\in\mathbb{R}}\{tz - c_\rho(t)\} = \sup_{t\in\mathbb{R}}\{tz - \log\cosh t\}$. A simple calculation shows that

(4.12) $$I_\rho^{(1)}(z) = \begin{cases}\dfrac{1-z}{2}\log(1-z) + \dfrac{1+z}{2}\log(1+z) & \text{for } |z|\le 1,\\ \infty & \text{for } |z|>1.\end{cases}$$

We define $i_{\beta,h}(z) = -(\frac{1}{2}\beta\mathscr{J}_0 z^2 + \beta h z) + I_\rho^{(1)}(z)$. For large n (4.10) and (4.11) suggest the heuristic formula

$$\text{(4.13)} \quad \left\langle f\left(\frac{S_n}{n}\right)\right\rangle_{n,\beta,h} \approx \int_{\mathbb{R}} f(z)\exp[-ni_{\beta,h}(z)]dz \cdot \frac{1}{\int_{\mathbb{R}}\exp[-ni_{\beta,h}(z)]dz}.$$

According to this formula, the limit of $\langle f(S_n/n)\rangle_{n,\beta,h}$ as $n \to \infty$ should be determined by the points z at which the function $i_{\beta,h}(z)$ attains its infimum. In fact, this statement is true because of the large deviation result stated in Theorem II.7.2. We will locate the minimum points, then deduce the limit of $\langle f(S_n/n)\rangle_{n,\beta,h}$.

Minimum points of $i_{\beta,h}(z)$ satisfy the equation

$$\text{(4.14)} \quad \frac{\partial i_{\beta,h}(z)}{\partial z} = 0 \qquad \text{or} \qquad \beta\mathscr{J}_0 z + \beta h = (I_\rho^{(1)})'(z) = \frac{1}{2}\log\frac{1+z}{1-z}.$$

$(I_\rho^{(1)})'(z)$ is an odd function of $z \in [-1, 1]$, is concave for $z \geq 0$, and has slope $(I_\rho^{(1)})''(0) = 1$ at $z = 0$. Also, $|(I_\rho^{(1)})'(z)| \to \infty$ as $|z| \to 1$. Since the slope of the affine function $z \to \beta\mathscr{J}_0 z + \beta h$ is $\beta\mathscr{J}_0$, the nature of the solutions of (4.14) depends on whether $0 < \beta\mathscr{J}_0 \leq 1$ or $\beta\mathscr{J}_0 > 1$. For $0 < \beta \leq \mathscr{J}_0^{-1}$ and any real h, (4.14) has a unique solution $z(\beta, h)$, and $z(\beta, h)$ is the unique minimum point of $i_{\beta,h}(z)$. As $h \to 0$, $z(\beta, h) \to z(\beta, 0) = 0$ [Figure IV.3(a), (b)]. For $\beta > \mathscr{J}_0^{-1}$, Figures IV.3(c), (d) show the solutions of (4.14) for small $h > 0$ and for $h = 0$. For $\beta > \mathscr{J}_0^{-1}$ and $h \neq 0$, (4.14) has a unique solution $z(\beta, h)$ that has the same sign as h, and $z(\beta, h)$ is the unique minimum point of $i_{\beta,h}(z)$. For $\beta > \mathscr{J}_0^{-1}$ and $h = 0$, the minimum points are the nonzero solutions $z(\beta, +)$ and $z(\beta, -) = -z(\beta, +)$ of (4.14). As $h \to 0^+$, $z(\beta, h) \to z(\beta, +) > 0$ and as $h \to 0^-$, $z(\beta, h) \to z(\beta, -) < 0$.

Part (a) of the next theorem gives the limit of $\langle f(S_n/n)\rangle_{n,\beta,h}$ as $n \to \infty$. Part (b) shows that spontaneous magnetization occurs at all $\beta > \beta_c^{\mathrm{CW}} = \mathscr{J}_0^{-1}$. Part (c) states that S_n/n converges exponentially to $m(\beta, h)$ for $\beta > 0$, $h \neq 0$ and $0 < \beta \leq \mathscr{J}_0^{-1}$, $h = 0$ but that exponential convergence to a constant fails for $\beta > \mathscr{J}_0^{-1}$ and $h = 0$.

Theorem IV.4.1. (a) *Let f be a bounded continuous function from $\mathbb{R}$ to $\mathbb{R}$. Then*

$$\lim_{n\to\infty} \langle f(S_n/n)\rangle_{n,\beta,h}$$

$$= \begin{cases} f(z(\beta,h)) & \text{for } \beta > 0,\ h \neq 0 \text{ and } 0 < \beta \leq \mathscr{J}_0^{-1},\ h = 0, \\ \frac{1}{2}f(z(\beta,+)) + \frac{1}{2}f(z(\beta,-)) & \text{for } \beta > \mathscr{J}_0^{-1},\ h = 0, \end{cases}$$

(4.15)

(b) *Let $m(\beta, h)$ be the specific magnetization for the Curie–Weiss model. Then $m(\beta, h)$ equals $z(\beta, h)$ for $\beta > 0$, $h \neq 0$ and $0 < \beta \leq \mathscr{J}_0^{-1}$, $h = 0$, and for each choice of sign*

$$\text{(4.16)} \quad m(\beta, \pm) = \lim_{h\to 0^\pm} m(\beta, h) = \begin{cases} z(\beta, 0) = 0 & \text{for } 0 < \beta \leq \mathscr{J}_0^{-1}, \\ z(\beta, \pm) \neq 0 & \text{for } \beta > \mathscr{J}_0^{-1}. \end{cases}$$

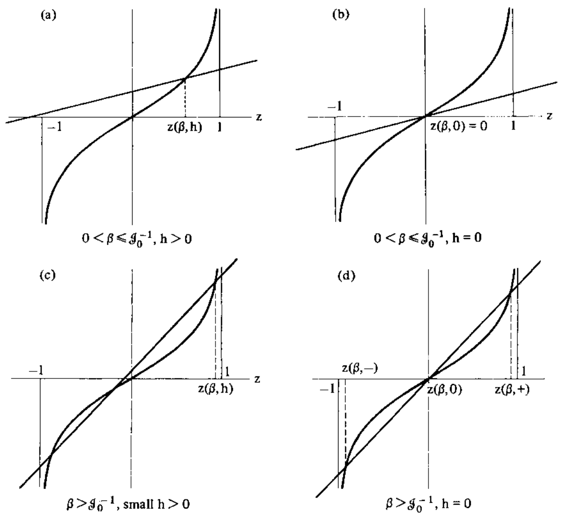


Figure IV.3. Solutions of the Curie–Weiss equation (4.14).

Values of β, h	Solutions of $\partial i_{\beta,h}(z)/\partial z = 0$	Minimum points of $i_{\beta,h}(z)$	Continuity properties
(a) $0 < \beta \le \mathscr{J}_0^{-1}$, $h \neq 0$	unique $z(\beta, h)$	$z(\beta, h)$	$z(\beta,h) \to z(\beta,0) = 0$ as $h \to 0$
(b) $0 < \beta \le \mathscr{J}_0^{-1}$, $h = 0$	unique $z(\beta, 0) = 0$	$z(\beta, 0) = 0$	
(c) $\beta > \mathscr{J}_0^{-1}$, $h \neq 0$	$z(\beta, h)$ (same sign as h) plus 2 roots of opposite sign for small h	$z(\beta, h)$	$z(\beta,h) \to \begin{cases} z(\beta,+) > 0 & \text{as } h \to 0^+ \\ z(\beta,-) < 0 & \text{as } h \to 0^- \end{cases}$
(d) $\beta > \mathscr{J}_0^{-1}$, $h = 0$	$z(\beta,+) > z(\beta,0) = 0 > z(\beta,-)$	$z(\beta,+)$, $z(\beta,-)$	

(c) $\frac{S_n}{n} \xrightarrow{\exp} m(\beta, h)$ *for* $\beta > 0, h \neq 0$ *and* $0 < \beta \leq \mathscr{J}_0^{-1}, h = 0.$

$\frac{S_n}{n} \xrightarrow{\mathscr{D}} \frac{1}{2}\delta_{m(\beta,+)}(dz) + \frac{1}{2}\delta_{m(\beta,-)}(dz)$ *for* $\beta > \mathscr{J}_0^{-1}, h = 0.$*

Proof. (a) Denote by $Q_{n,\beta,h}$ the $P_{n,\beta,h}$-distribution of S_n/n. According to (4.10)–(4.11), if A is a Borel subset of $\mathbb{R}$, then $Q_{n,\beta,h}\{A\}$ equals

$$\int_A \exp[n(\tfrac{1}{2}\beta\mathscr{J}_0 z^2 + \beta h z)]Q_n^{(1)}(dz) \cdot \frac{1}{\int_{\mathbb{R}} \exp[n(\frac{1}{2}\beta\mathscr{J}_0 z^2 + \beta h z)]Q_n^{(1)}(dz)}.$$

By Theorem II.7.2(a), $\{Q_{n,\beta,h}\}$ has a large deviation property with $a_n = n$ and entropy function

$$I_{\beta,h}(z) = i_{\beta,h}(z) - \inf_{z \in \mathbb{R}} i_{\beta,h}(z) \qquad \text{where } i_{\beta,h}(z) = I_\rho^{(1)}(z) - (\tfrac{1}{2}\beta\mathscr{J}_0 z^2 + \beta h z).$$
(4.17)

For $\beta > 0, h \neq 0$ and $0 < \beta \leq \mathscr{J}_0^{-1}, h = 0$, let K be any closed set which does not contain the unique minimum point $z(\beta, h)$ of $i_{\beta,h}$. By Theorem II.7.2(b), there exists a number $N = N(K) > 0$ such that

(4.18) $$Q_{n,\beta,h}\{K\} \leq e^{-nN} \qquad \text{for all sufficiently large } n.$$

This yields the first line of (4.15). For $\beta > \mathscr{J}_0^{-1}$ and $h = 0$, if K is the closed set

$$\{z \in \mathbb{R} : |z - z(\beta, +)| \geq \varepsilon \text{ and } |z - z(\beta, -)| \geq \varepsilon\} \qquad \text{where } 0 < \varepsilon < z(\beta, +),$$

then for all sufficiently large n, $Q_{n,\beta,0}\{K\} \leq e^{-nN}$ for some $N = N(K) > 0$. This yields the second line of (4.15) since the measures $\{Q_{n,\beta,0}\}$ are symmetric.

(b) The range of S_n/n is contained in the interval $[-1, 1]$. If f is a function in $\mathscr{C}(\mathbb{R})$ such that $f(x) = x$ for $-1 \leq x \leq 1$, then part (b) follows from part (a) and the behavior of $z(\beta, h)$ as $h \to 0^+$ and $h \to 0^-$.

(c) For $\beta > 0, h \neq 0$ and $0 < \beta \leq \mathscr{J}_0^{-1}, h = 0$, $m(\beta, h)$ equals $z(\beta, h)$. The bound (4.18) implies that S_n/n converges exponentially to $m(\beta, h)$. The convergence in distribution for $\beta > \mathscr{J}_0^{-1}$ and $h = 0$ follows from the second line of (4.15) since f is an arbitrary function in $\mathscr{C}(\mathbb{R})$. □

Remark IV.4.2. By (4.11) and Theorem II.7.3(a)

(4.19) $$\lim_{n \to \infty} \frac{1}{n} \log Z(n, \beta, h) = \sup_{z \in \mathbb{R}} \{\tfrac{1}{2}\beta\mathscr{J}_0 z^2 + \beta h z - I_\rho^{(1)}(z)\}.$$

The function $\psi(\beta, h) = -\beta^{-1} \lim_{n\to\infty} n^{-1} \log Z(n, \beta, h)$ is called the *specific Gibbs free energy* for the Curie–Weiss model. The limit (4.19) can be derived without using large deviations [see Problem IV.9.4]. In Ellis and Newman (1978b), the limit (4.15) is derived without using large deviations.

*The second line of part (c) means that S_n/n converges in distribution to a random variable distributed by $\frac{1}{2}\delta_{m(\beta,+)} + \frac{1}{2}\delta_{m(\beta,-)}$.

Let us consider the form of the entropy function $I_{\beta,h}(z)$ defined in (4.17). For $\beta > 0, h \neq 0$ and $0 < \beta \leq \mathscr{J}_0^{-1}, h = 0, I_{\beta,h}(z)$ is a convex function on $\mathbb{R}$ with a unique minimum point at $z(\beta, h)$ $(I_{\beta,h}(z(\beta, h)) = 0)$. However, for $\beta > \mathscr{J}_0^{-1}$ and $h = 0$, $I_{\beta,h}(z)$ is not a convex function. It has minimum points at $z(\beta, +)$ and $z(\beta, -)$ and is positive for all other values of z. Another example of a nonconvex entropy function was given in Example II.6.2.

This completes our discussion of spontaneous magnetization for the Curie–Weiss model. Other aspects of the model will be studied in Section V.9.

IV.5. Spontaneous Magnetization for General Ferromagnets on $\mathbb{Z}$

In the Curie–Weiss model the interaction $J(i-j)$ equals $\mathscr{J}_0/|\Lambda|$ for each i and j in Λ. Thus the interaction depends on the set Λ. We now consider other finite-volume Gibbs states $P_{\Lambda,\beta,h}$ on symmetric intervals Λ of $\mathbb{Z}$. The interactions J are assumed to satisfy the following hypotheses*: J is independent of Λ, J is nonnegative on $\mathbb{Z}$ (*ferromagnetic*) and is not identically zero, J is symmetric, and $\sum_{k\in\mathbb{Z}} J(k) < \infty$. This summability hypothesis restricts the interaction strength between distant spins. J is called a *summable ferromagnetic interaction* on $\mathbb{Z}$ and the corresponding spin models are called *general ferromagnets* on $\mathbb{Z}$. These models cannot be treated by the large deviation technique which was used in the Curie–Weiss case. Less direct methods are required. In this section, we illustrate one of the most powerful of these methods, which is convexity. The main fact about general ferromagnets on $\mathbb{Z}$ is that unless the interaction has infinite range, there is no spontaneous magnetization. The behavior of models on $\mathbb{Z}$ contrasts sharply with the behavior of models on $\mathbb{Z}^D$, $D \geq 2$. The latter exhibit spontaneous magnetization for all nontrivial interactions, regardless of whether they have finite range or infinite range [Theorem V.5.1].

A useful function for studying the magnetization is the *Gibbs free energy*. It is defined as

$$\Psi(\Lambda, \beta, h) = -\beta^{-1} \log Z(\Lambda, \beta, h), \tag{4.20}$$

where $Z(\Lambda, \beta, h) = \int_{\Omega_\Lambda} \exp[-\beta H_{\Lambda,h}(\omega)]\pi_\Lambda P_\rho(d\omega)$. $\Psi(\Lambda, \beta, h)$ has a simple relation to the magnetization:

$$\begin{aligned} -\frac{\partial\Psi(\Lambda,\beta,h)}{\partial h} &= -\beta^{-1}\frac{\partial Z(\Lambda,\beta,h)}{\partial h}\cdot\frac{1}{Z(\Lambda,\beta,h)} \\ &= \sum_{j\in\Lambda}\sum_{\omega\in\Omega_\Lambda} \omega_j \exp[-\beta H_{\Lambda,h}(\omega)]2^{-|\Lambda|}\frac{1}{Z(\Lambda,\beta,h)} = M(\Lambda,\beta,h). \end{aligned} \tag{4.21}$$

Our proofs of the existence and properties of the specific magnetization $m(\beta, h) = \lim_{\Lambda\uparrow\mathbb{Z}} |\Lambda|^{-1} M(\Lambda, \beta, h)$ will be based on the function

* More general interactions are discussed in Appendix C.3.

(4.22) $$\psi(\beta, h) = \lim_{\Lambda \uparrow \mathbb{Z}} \frac{1}{|\Lambda|} \Psi(\Lambda, \beta, h),$$

called the *specific Gibbs free energy*. The existence of the limit (4.22) for a summable interaction J is proved in Appendix D.1. Another proof based on level-3 large deviations is given in Theorem IV.7.3 below.

There is a degenerate case of the specific Gibbs free energy that is worth pointing out. If the interaction J is identically zero, then $Z(\Lambda, \beta, h) = \int_{\Omega_\Lambda} \exp(\beta h \sum_{j \in \Lambda} \omega_j) \pi_\Lambda P_\rho(d\omega)$ and $\psi(\beta, h) = -\beta^{-1} \log \int_{\{1,-1\}} e^{\beta h x} \rho(dx) = -\beta^{-1} \log \cosh \beta h$. Thus $\psi(\beta, h)$ equals $-\beta^{-1} c_\rho(\beta h)$, where $c_\rho(t)$ is the free energy function of the measure ρ. Properties of the specific Gibbs free energy are proved next.

Theorem IV.5.1. *Let J be a summable ferromagnetic interaction on $\mathbb{Z}$. Then the following conclusions hold.*

(a) *For each $\beta > 0$, the specific Gibbs free energy $\psi(\beta, h)$ is a concave, even function of h real and is a continuously differentiable function of $h \neq 0$.*

(b) *For $\beta > 0$ and h real, the specific magnetization $m(\beta, h) = \lim_{\Lambda \uparrow \mathbb{Z}} |\Lambda|^{-1} M(\Lambda, \beta, h)$ exists. For $\beta > 0$ and $h \neq 0$, $m(\beta, h) = -\partial\psi(\beta, h)/\partial h$.*

Proof. (a) Concavity is preserved under pointwise limits. Therefore the concavity of $\psi(\beta, h)$ will follow from the concavity of $\Psi(\Lambda, \beta, h)$. The latter is equivalent to the inequality

(4.23) $$Z(\Lambda, \beta, \lambda h_1 + (1-\lambda) h_2) \leq Z(\Lambda, \beta, h_1)^\lambda \cdot Z(\Lambda, \beta, h_2)^{1-\lambda}$$

for any h_1 and h_2 real and $0 < \lambda < 1$. The left-hand side equals

(4.24) $$\begin{aligned} &\int_{\Omega_\Lambda} \exp\left(\beta\lambda h_1 \sum_{j \in \Lambda} \omega_j\right) \exp\left(\beta(1-\lambda) h_2 \sum_{j \in \Lambda} \omega_j\right) \\ &\cdot \exp\left(\frac{\beta}{2} \sum_{i,j \in \Lambda} J(i-j)\omega_i\omega_j\right) \pi_\Lambda P_\rho(d\omega), \end{aligned}$$

and so (4.23) follows from Hölder's inequality* applied to the functions $\exp(\beta\lambda h_1 \sum_{j \in \Lambda} \omega_j)$ and $\exp(\beta(1-\lambda) h_2 \sum_{j \in \Lambda} \omega_j)$. The evenness of $\psi(\beta, \cdot)$ follows from the fact that $H_{\Lambda,-h}(\omega) = H_{\Lambda,h}(-\omega)$.

(b) The following proof is due to Preston (1974a). The Gibbs free energy is related to the magnetization by the formula

(4.25) $$\frac{1}{|\Lambda|}(\Psi(\Lambda, \beta, h) - \Psi(\Lambda, \beta, 0)) = -\int_0^h \frac{1}{|\Lambda|} M(\Lambda, \beta, s) ds,$$

which is equivalent to (4.21). We would like to prove the existence of the specific magnetization by passing to the limit $\Lambda \uparrow \mathbb{Z}$ in (4.25). The left-hand side becomes $\psi(\beta, h) - \psi(\beta, 0)$, but care is needed in handling the integral on the right. Consider $h \geq 0$ ($h < 0$ is handled similarly). Since $0 \leq M(\Lambda, \beta, h)$

*Corollary VI.4.2 with $p = 1/\lambda$, $q = 1/(1-\lambda)$.

$\leq |\Lambda|$, there exists an infinite subsequence $\{\Lambda'\}$ of $\{\Lambda\}$ such that $m(\beta, h) = \lim_{\Lambda' \uparrow \mathbb{Z}} |\Lambda'|^{-1} M(\Lambda', \beta, h)$ exists for every rational number $h \geq 0$. $M(\Lambda', \beta, 0)$ equals 0 and thus $m(\beta, 0)$ equals 0. Since $M(\Lambda', \beta, h)$ is a concave function of $h \geq 0$, a standard convexity result implies that $m(\beta, h)$ exists for all $h \geq 0$ [Theorem VI.3.3(a)]. The limit $m(\beta, h)$ is concave for $h \geq 0$ and hence is continuous for $h > 0$ [Theorem VI.3.1]. By the Lebesgue dominated convergence theorem, $m(\beta, h)$ satisfies

$$\psi(\beta, h) - \psi(\beta, 0) = -\int_0^h m(\beta, s)ds. \tag{4.26}$$

The preceding argument may be repeated for the functions $\{|\bar{\Lambda}|^{-1} M(\bar{\Lambda}, \beta, h)\}$, where $\{\bar{\Lambda}\}$ is an arbitrary infinite subsequence of $\{\Lambda\}$. This leads to a limit function, say $\bar{m}(\beta, h)$, which satisfies $\bar{m}(\beta, 0) = 0$ as well as equation (4.26). Therefore, $\bar{m}(\beta, h)$ equals $m(\beta, h)$. We conclude that the limit $\lim_{\Lambda \uparrow \mathbb{Z}} |\Lambda|^{-1} M(\Lambda, \beta, h)$ exists along the entire sequence $\{\Lambda\}$ and equals $m(\beta, h)$. Finally, (4.26) implies that $\psi(\beta, h)$ is a continuously differentiable function of $h \neq 0$ and $\partial\psi(\beta, h)/\partial h = -m(\beta, h)$. □

The specific Gibbs free energy $\psi(\beta, h)$ need not be differentiable at $h = 0$. The relationship between spontaneous magnetization and the existence of $\partial\psi(\beta, 0)/\partial h$ is made explicit in the following theorem. Recall that spontaneous magnetization is said to occur at inverse temperature β if $m(\beta, +) = \lim_{h \to 0^+} m(\beta, h)$ is positive.

Theorem IV.5.2.* *Let J be a summable ferromagnetic interaction on $\mathbb{Z}$. Then the following conclusions hold.*

(a) *For $\beta > 0$ and $h \neq 0$, $m(\beta, h) = -\partial\psi(\beta, h)/\partial h$.*

(b) *For each $\beta > 0$, $m(\beta, 0) = 0$; $m(\beta, h)$ is a non-negative, concave function of $h \geq 0$ and a nondecreasing function of h real. It satisfies $m(\beta, -h) = -m(\beta, h)$ and $|m(\beta, h)| \leq 1$.*

(c) *For each $h \geq 0$, $m(\beta, h)$ is a non-negative, nondecreasing function of $\beta > 0$.*

(d) *For each $\beta > 0$, the limits $m(\beta, +) = \lim_{h \to 0^+} m(\beta, h)$ and $m(\beta, -) = \lim_{h \to 0^-} m(\beta, h)$ exist and $m(\beta, -) = -m(\beta, +)$; $m(\beta, +)$ is a non-negative, nondecreasing function of $\beta > 0$.*

(e) *For each $\beta > 0$,*

$$m(\beta, +) = -\frac{\partial\psi(\beta, 0)}{\partial h^+} \geq m(\beta, 0) = 0 \geq m(\beta, -) = -\frac{\partial\psi(\beta, 0)}{\partial h^-}. \tag{4.27}$$

Thus spontaneous magnetization occurs at β if and only if $\psi(\beta, h)$ is not differentiable at $h = 0$.

These properties of $m(\beta, h)$ may be read from Figure IV.1. Part (a) of

*Other properties of $m(\beta, h)$ are derived in Problem V.13.1.

Theorem IV.5.2 was proved in Theorem IV.5.1. The properties listed in parts (b) and (c) were stated for $M(\Lambda, \beta, h)$ in Theorem IV.3.4 and are preserved under passage to the limit $\Lambda \uparrow \mathbb{Z}$. Since $\psi(\beta, h)$ is concave for h real and differentiable for $h \neq 0$, $\partial\psi/\partial h$ is nonincreasing for $h \neq 0$. Hence $m(\beta, +) = \lim_{h\to 0^+} m(\beta, h) = -\lim_{h \to 0^+} \partial\psi(\beta, h)/\partial h$ exists and equals the right-hand derivative of $-\psi$ at $h = 0$ $(-\partial\psi(\beta, 0)/\partial h^+)$; similarly for $m(\beta, -)$. Since for each $h \geq 0$, $m(\beta, h)$ is a non-negative, nondecreasing function of $\beta > 0$, the same properties hold as for $m(\beta, +)$. This proves part (d) as well as (4.27). Since $\psi(\beta, h)$ is differentiable at $h = 0$ if and only if $\partial\psi(\beta, 0)/\partial h^+ = \partial\psi(\beta, 0)/\partial h^-$, we obtain the last assertion in part (e).

According to part (e), spontaneous magnetization corresponds to a discontinuity in the first-order derivative $\partial\psi/\partial h$. Hence it is known as a *first-order phase transition.*

Spontaneous magnetization indicates a strong cooperation among the individual spins, and it occurs only if the alignment effects built into the finite-volume Gibbs states persist in the limit $\Lambda \uparrow \mathbb{Z}$. Thus, one might expect spontaneous magnetization to occur only if distant spins have a suitably strong interaction and the temperature is sufficiently low. The next theorem justifies this intuition.

Theorem IV.5.3. *Let J be a summable ferromagnetic interaction on $\mathbb{Z}$ and define the positive number $\mathscr{J}_0 = \sum_{k \in \mathbb{Z}} J(k)$. Define the critical inverse temperature*

$$\beta_c = \sup\{\beta > 0 : m(\beta, +) = 0\}.$$

Then the following conclusions hold.

(a) *β_c is well-defined and $\mathscr{J}_0^{-1} \leq \beta_c \leq \infty$; in particular $\beta_c > 0$.*

(b) *$\beta_c = \infty$ if $\sum_{k\in\mathbb{Z}} |k| J(k) < \infty$.*

(c) *$\beta_c < \infty$ if $J(k) = k^{-2}$ $(k \neq 0)$ or if $J > 0$ and $J(k) \sim |k|^{-\alpha}$, some $1 < \alpha < 2$.**

(d) *For $0 < \beta < \beta_c$, $\psi(\beta, h)$ is differentiable at $h = 0$ and $m(\beta, +) = 0$. For $\beta > \beta_c$, the differentiability fails and spontaneous magnetization occurs:*

$$(4.28) \qquad m(\beta, +) = -\frac{\partial\psi(\beta, 0)}{\partial h^+} > 0 > m(\beta, -) = -\frac{\partial\psi(\beta, 0)}{\partial h^-}.$$

Comments on proof. (a) Since $m(\beta, +)$ is a non-negative, nondecreasing function of $\beta > 0$, β_c is well-defined. Let $m^{\mathrm{CW}}(\beta, h)$ denote the specific magnetization for the Curie–Weiss model with interaction $\mathscr{J}_0/|\Lambda|$, where $\mathscr{J}_0$ equals $\sum_{k\in\mathbb{Z}} J(k)$. Pearce (1981) shows that for $\beta > 0$ and $h \geq 0$, $m(\beta, h) \leq m^{\mathrm{CW}}(\beta, h)$. Since $m^{\mathrm{CW}}(\beta, +) = \lim_{h\to 0^+} m^{\mathrm{CW}}(\beta, h)$ is positive for $\beta > \mathscr{J}_0^{-1}$ [Theorem IV.4.1(b)], it follows that $\beta_c \geq \mathscr{J}_0^{-1}$. Pearce's method is outlined in Problem IV.9.7.[8]

(b), (c) These parts are discussed below.

* $J(k) \sim |k|^{-\alpha}$ means $\lim_{|k|\to\infty} [\log J(k)/\log |k|] = -\alpha$.

(d) For $0 < \beta < \beta_c$, $m(\beta, +)$ equals 0 and so $\psi(\beta, h)$ is differentiable at $h = 0$. For $\beta > \beta_c$, $m(\beta, +)$ is positive and (4.28) follows from (4.27).[9]

We show a special case of part (b) by proving that β_c equals ∞ for the Ising model.

Example IV.5.4. For the Ising model,

(4.29) $$Z(\Lambda, \beta, h) = \int_{\Omega_\Lambda} \exp\left(\beta \mathscr{J} \sum_{j=-N}^{N-1} \omega_j \omega_{j+1} + \beta h \sum_{j=-N}^{N} \omega_j\right) \pi_\Lambda P_\rho(d\omega).$$

$Z(\Lambda, \beta, h)$ can be expressed in terms of a matrix product. For $\alpha_1, \alpha_2 \in \{1, -1\}$, define $B(\alpha_1, \alpha_2) = \frac{1}{2}\exp[\beta \mathscr{J} \alpha_1 \alpha_2 + \beta h(\alpha_1 + \alpha_2)/2]$ and let $B = B_{\beta,h}$ be the 2×2 symmetric matrix

$$\begin{pmatrix} B(1, 1) & B(1, -1) \\ B(-1, 1) & B(-1, -1) \end{pmatrix}.$$

B is called the *transfer matrix* of the model. Λ equals $\{j \in \mathbb{Z} : |j| \leq N\}$ and

$$\begin{aligned} Z(\Lambda, \beta, h) &= \frac{1}{2} \sum_{\{\omega_j = \pm 1;\, j \in \Lambda\}} a(\omega_{-N}) B(\omega_{-N}, \omega_{-N+1}) \cdots B(\omega_{N-1}, \omega_N) a(\omega_N) \\ &= \frac{1}{2} \sum_{\omega_N, \omega_{-N} = \pm 1} a(\omega_{-N}) B^{2N}(\omega_{-N}, \omega_N) a(\omega_N), \end{aligned}$$

(4.30)

where $a(x) = \exp(\frac{1}{2}\beta h x)$. The larger eigenvalue of $B_{\beta,h}$ is

$$\lambda(B_{\beta,h}) = \tfrac{1}{2} e^{\beta \mathscr{J}} [\cosh \beta h + (\sinh^2 \beta h + e^{-4\beta \mathscr{J}})^{1/2}].$$

By Lemma IX.4.1(d) (Perron–Frobenius)

(4.31) $$-\beta \psi(\beta, h) = \lim_{\Lambda \uparrow \mathbb{Z}} \frac{1}{|\Lambda|} \log Z(\Lambda, \beta, h) = \log \lambda(B_{\beta,h}).$$

For each $\beta > 0$, $\psi(\beta, h)$ is a real analytic function of real h. Hence $\partial \psi(\beta, 0)/\partial h$ exists for each $\beta > 0$ and β_c equals ∞. This calculation can be generalized to any finite-range interaction, and as with the Ising model, one finds that β_c equals ∞ [Ruelle (1969, Section 5.6)]. See Appendix C.6 for details.

According to part (b) of Theorem IV.5.3, β_c equals ∞ not just for interactions of finite range but whenever $\sum_{k \in \mathbb{Z}} |k| J(k)$ is finite. A method of proof is outlined in Problem IV.9.8. This leaves the infinite-range case where $\sum_{k \in \mathbb{Z}} J(k)$ is finite but $\sum_{k \in \mathbb{Z}} |k| J(k)$ diverges. It follows from Dyson (1969a) that β_c is finite if J is positive and $J(k) \sim |k|^{-\alpha}$ for some $1 < \alpha < 2$. Fröhlich and Spencer (1982a) proved that β_c is finite also in the borderline case $J(k) = k^{-2}$ $(k \neq 0)$. These proofs of spontaneous magnetization are difficult and are omitted.[10]

Finally, we study convergence properties of the random variables $S_\Lambda/|\Lambda|$, which define the spin per site in Λ. By (4.7), the expectation of $S_\Lambda/|\Lambda|$ with respect to the finite-volume Gibbs state $P_{\Lambda,\beta,h}$ gives the magnetization per site, $M(\Lambda,\beta,h)/|\Lambda|$. By analogy with the Curie–Weiss model, we expect that as $\Lambda \uparrow \mathbb{Z}$, $S_\Lambda/|\Lambda|$ converges exponentially to the specific magnetization $m(\beta,h)$ for all $\beta > 0$, $h \neq 0$, and $0 < \beta < \beta_c$, $h = 0$.

Theorem IV.5.5. *Let J be a summable ferromagnetic interaction on $\mathbb{Z}$. Then there exists a constant m such that*

$$S_\Lambda/|\Lambda| \xrightarrow{\text{exp}} m \qquad \text{with respect to } \{P_{\Lambda,\beta,h}\} \tag{4.32}$$

if and only if $\psi(\beta,h)$ is differentiable with respect to h. In this case m equals $m(\beta,h) = -\partial\psi(\beta,h)/\partial h$. Thus (4.32) *holds for all $\beta > 0$, $h \neq 0$ and for all $0 < \beta < \beta_c$, $h = 0$;* (4.32) *fails for $\beta > \beta_c$ and $h = 0$ [Theorems* IV.5.1 *and* IV.5.3].

Proof. According to Section II.6, the exponential convergence can be proved by considering the free energy function $c_{\beta,h}(t)$ of the sequence $\{S_\Lambda\}$. For t real, $c_{\beta,h}(t) = \lim_{\Lambda\uparrow\mathbb{Z}} c_{\Lambda,\beta,h}(t)$, where

$$\begin{aligned} c_{\Lambda,\beta,h}(t) &= \frac{1}{|\Lambda|}\log\int_{\Omega_\Lambda} \exp[tS_\Lambda(\omega)]P_{\Lambda,\beta,h}(d\omega) \\ &= \frac{1}{|\Lambda|}\log\int_{\Omega_\Lambda} \exp[-\beta H_{\Lambda,h}(\omega) + tS_\Lambda(\omega)]\pi_\Lambda P_\rho(d\omega)\cdot\frac{1}{Z(\Lambda,\beta,h)}. \end{aligned}$$

For each $\omega \in \Omega_\Lambda$,

$$\begin{aligned} -\beta H_{\Lambda,h}(\omega) + tS_\Lambda(\omega) &= \frac{\beta}{2}\sum_{i,j\in\Lambda} J(i-j)\omega_i\omega_j + (\beta h + t)\sum_{j\in\Lambda}\omega_j \\ &= -\beta H_{\Lambda,h+t/\beta}(\omega). \end{aligned}$$

Hence

$$\begin{aligned} c_{\Lambda,\beta,h}(t) &= \frac{1}{|\Lambda|}\log\frac{Z(\Lambda,\beta,h+t/\beta)}{Z(\Lambda,\beta,h)} \\ &= -\beta\frac{1}{|\Lambda|}[\Psi(\Lambda,\beta,h+t/\beta) - \Psi(\Lambda,\beta,h)] \end{aligned}$$

and $c_{\beta,h}(t) = -\beta[\psi(\beta,h+t/\beta) - \psi(\beta,h)]$. By Theorem II.6.3, there exists a constant m such that $S_\Lambda/|\Lambda| \xrightarrow{\text{exp}} m$ if and only if $c_{\beta,h}(t)$ is differentiable at $t = 0$. In this case, m equals $c'_{\beta,h}(0)$. But $c'_{\beta,h}(0)$ exists if and only if $\partial\psi(\beta,h)/\partial h$ exists, and then $c'_{\beta,h}(0) = -\partial\psi(\beta,h)/\partial h$. This completes the proof. □

Theorem IV.5.5 implies that spontaneous magnetization occurs at β if and only if $S_\Lambda/|\Lambda|$ fails to converge exponentially to $m(\beta,0) = 0$. For this reason, we are justified in calling spontaneous magnetization a *level-*1 *phase transition.* Let us interpret the level-1 phase transition in terms of entropy. According to the proof of Theorem IV.5.5, the free energy function of the sequence $\{S_\Lambda\}$ with respect to $\{P_{\Lambda,\beta,h}\}$ is

$$c_{\beta,h}(t) = -\beta[\psi(\beta, h + t/\beta) - \psi(\beta, h)]. \tag{4.33}$$

For $0 < \beta < \beta_c$ and any h real, $c_{\beta,h}(t)$ is differentiable for all t real. Hence by Theorem II.6.1, the $P_{\Lambda,\beta,h}$-distributions of $S_\Lambda/|\Lambda|$ have a large deviation property with $a_\Lambda = |\Lambda| = 2N + 1$ and entropy function

$$\begin{aligned} I_{\beta,h}(z) &= \sup_{t\in\mathbb{R}}\{tz - c_{\beta,h}(t)\} = \sup_{t\in\mathbb{R}}\{tz + \beta\psi(\beta, h + t/\beta)\} - \psi(\beta,h) \\ &= \sup_{x\in\mathbb{R}}\{\beta xz + \beta\psi(\beta,x)\} - [\beta hz + \beta\psi(\beta,h)]. \end{aligned} \tag{4.34}$$

$I_{\beta,h}(z)$ is a convex function and it attains its infimum of 0 at the unique point $c'_{\beta,h}(0) = m(\beta,h)$ [Theorem II.6.3]. The situation is different for $\beta > \beta_c$. Let us focus on the case $h = 0$. The function $c_{\beta,0}(t)$ is not differentiable at $t = 0$, and $I_{\beta,0}(z)$ does not attain its infimum at a unique point. The theory of Legendre–Fenchel transforms shows that $I_{\beta,0}(z)$ attains its infimum on the whole interval

$$\begin{aligned} [(c_{\beta,0})'_-(0), (c_{\beta,0})'_+(0)] &= [-\partial\psi(\beta,0)/\partial h^-, -\partial\psi(\beta,0)/\partial h^+] \\ &= [m(\beta,-), m(\beta,+)], \end{aligned}$$

where $m(\beta,+)$ is the spontaneous magnetization [Theorem VII.2.1(g)]. According to part (b) of Theorem II.6.1,

$$\limsup_{\Lambda\uparrow\mathbb{Z}} \frac{1}{|\Lambda|}\log P_{\Lambda,\beta,0}\{S_\Lambda/|\Lambda|\in K\} \le -\inf_{z\in K} I_{\beta,0}(z) \quad \text{for each closed set } K \text{ in } \mathbb{R}.$$

Thus, if K is disjoint from the interval $[m(\beta,-), m(\beta,+)]$, then the probabilities $P_{\Lambda,\beta,0}\{S_\Lambda/|\Lambda|\in K\}$ decay exponentially as $\Lambda\uparrow\mathbb{Z}$. However, since $c_{\beta,0}(t)$ fails to be differentiable at $t = 0$, we are unable to apply part (c) of Theorem II.6.1 to conclude that the $P_{\Lambda,\beta,0}$-distributions of $S_\Lambda/|\Lambda|$ have a large deviation property. If the large deviation property does hold with some entropy function $I(z)$, then $I_{\beta,0}(z)$ equals the closed convex hull of $I(z)$ [see Problem VII.8.2].

This completes our discussion of finite-volume Gibbs states. In the next section, we consider probability measures which describe the infinite-volume ferromagnet.

IV.6. Infinite-Volume Gibbs States and Phase Transitions

The finite-volume Gibbs states studied in the previous three sections are probability measures on the finite sets $\Omega_\Lambda = \{1, -1\}^\Lambda$. In this section we study states of general ferromagnets on $\mathbb{Z}$. The states are probability measures on the infinite-volume configuration space $\Omega = \{1, -1\}^{\mathbb{Z}}$. These measures are obtained from the finite-volume Gibbs states by weak limits ($\Lambda\uparrow\mathbb{Z}$). Equivalent notions of infinite-volume measures are discussed in Appendix C.

The set $\{1, -1\}$ is topologized by the discrete topology and the set Ω by the product topology. According to Tychonoff's theorem, Ω is compact. The σ-field generated by the open sets of the product topology is called the Borel σ-field of Ω and is denoted by $\mathscr{B}(\Omega)$. $\mathscr{B}(\Omega)$ coincides with the σ-field generated by the cylinder sets of Ω [Propositions A.3.2 and A.3.5(b)]. Let $\mathscr{M}(\Omega)$ be the set of probability measures on $\mathscr{B}(\Omega)$ and $\mathscr{M}_s(\Omega)$ the subset of $\mathscr{M}(\Omega)$ consisting of strictly stationary probability measures. Strict stationarity is a natural condition since most of the infinite-volume Gibbs states which we obtain are strictly stationary with respect to spatial translations, or (as it is usually said) are translation invariant. A translation invariant, infinite-volume Gibbs state is called a *phase*.

Throughout this section, we fix a summable ferromagnetic interaction J on $\mathbb{Z}$. The finite-volume Gibbs state $P_{\Lambda,\beta,h}$ defined in Section IV.3 will be modified by means of *external conditions*. The measure $P_{\Lambda,\beta,h}$ models a ferromagnet on the set $\Lambda = \{j\in\mathbb{Z}: |j| \le N\}$, where N is a non-negative integer. External conditions correspond physically to the situation where an experimenter prepares the complement of Λ, $\Lambda^c = \mathbb{Z}\backslash\Lambda$, by fixing a configuration on that set. Let $\tilde{\omega}$ be a point in the set $\Omega_{\Lambda^c} = \{1,-1\}^{\Lambda^c}$. The coordinates $\tilde{\omega}_j$, $j\in\Lambda^c$, denote the values of the fixed external spins at the sites of Λ^c. When we want to indicate the dependence of $\tilde{\omega}$ upon Λ, we will write $\tilde{\omega}(\Lambda)$. We define the Hamiltonian of a configuration $\omega\in\Omega_\Lambda = \{1,-1\}^\Lambda$ to be*

$$H_{\Lambda,h,\tilde{\omega}}(\omega) = -\frac{1}{2}\sum_{i,j\in\Lambda} J(i-j)\omega_i\omega_j - \sum_{i\in\Lambda}\left(h + \sum_{j\in\Lambda^c} J(i-j)\tilde{\omega}_j\right)\omega_i. \tag{4.35}$$

Thus each $\tilde{\omega}_j$, $j\in\Lambda^c$, interacts with the spins in Λ through the given interaction. Compare $H_{\Lambda,h,\tilde{\omega}}$ with the Hamiltonian $H_{\Lambda,h}$ in (4.3). The external condition $\tilde{\omega}$ changes $H_{\Lambda,h}$ by altering the external field acting at each site $i\in\Lambda$ from the value h to the value $h_i = h + \sum_{j\in\Lambda^c} J(i-j)\tilde{\omega}_j$. Since J is summable, h_i is well-defined. The finite-volume Gibbs state on Λ with external condition $\tilde{\omega}$ is defined to be the probability measure $P_{\Lambda,\beta,h,\tilde{\omega}}$ on $\mathscr{B}(\Omega_\Lambda)$ which assigns to each $\{\omega\}$, $\omega\in\Omega_\Lambda$, the probability

$$P_{\Lambda,\beta,h,\tilde{\omega}}\{\omega\} = \exp[-\beta H_{\Lambda,h,\tilde{\omega}}(\omega)]\pi_\Lambda P_\rho\{\omega\}\cdot\frac{1}{Z(\Lambda,\beta,h,\tilde{\omega})}. \tag{4.36}$$

In this formula, β is positive and $Z(\Lambda,\beta,h,\tilde{\omega})$ is the normalization $\int_{\Omega_\Lambda}\exp[-\beta H_{\Lambda,h,\tilde{\omega}}(\omega)]\pi_\Lambda P_\rho(d\omega)$. There are two important choices of $\tilde{\omega}$. If each $\tilde{\omega}_j = 1$ (resp., -1), then the external condition is called *plus* (resp., *minus*) and the measure is written as $P_{\Lambda,\beta,h,+}$ (resp., $P_{\Lambda,\beta,h,-}$). Expectation with respect to $P_{\Lambda,\beta,h,\tilde{\omega}}$ will be denoted by $\langle - \rangle_{\Lambda,\beta,h,\tilde{\omega}}$.

We have defined external conditions by means of points $\tilde{\omega}$ in Ω_{Λ^c}. One can allow other external conditions such as *free* (each $\tilde{\omega}_j = 0$ in (4.35)) or *periodic*

*We omit from (4.35) the interaction between $\tilde{\omega}_i$ and $\tilde{\omega}_j$ and between h and $\tilde{\omega}_i$. These interactions are constants independent of ω. If included in (4.35), they would cancel out in (4.36).

(one modifies the definition of $J(i-j)$). These external conditions are useful in certain applications. However, restricting external conditions to be points in Ω_{Λ^c} allows for a cleaner formulation of infinite-volume Gibbs states and facilitates the proof of the equivalence between these states and other notions of infinite-volume measures [Appendix C].

Let $\mathscr{C}(\Omega)$ be the space of bounded, continuous, real-valued functions on Ω with the supremum norm. We say that a sequence $\{P_n; n = 1, 2, \ldots\}$ in $\mathscr{M}(\Omega)$ *converges weakly* to $P \in \mathscr{M}(\Omega)$, and write $P_n \Rightarrow P$ or $P = w\text{-}\lim_{n\to\infty} P_n$, if $\int_\Omega f dP_n \to \int_\Omega f dP$ for every $f \in \mathscr{C}(\Omega)$. If a weak limit P exists, then it is unique. With respect to weak convergence, $\mathscr{M}(\Omega)$ is a compact metric space [Theorem A.11.2].

Let $\{P_n; n = 1, 2, \ldots\}$ be an arbitrary sequence in $\mathscr{M}(\Omega)$. We call a subset $\mathscr{S}$ of $\mathscr{C}(\Omega)$ a *convergence-determining class* if the existence of the limit $\lim_{n\to\infty} \int_\Omega f dP_n$ for each $f \in \mathscr{S}$ implies that $\{P_n; n = 1, 2, \ldots\}$ converges weakly to some probability measure P. Since Ω is compact, $\mathscr{C}(\Omega)$ itself is a convergence-determining class because of the Riesz representation theorem. Other well-known examples are the subset consisting of all product functions $f(\omega) = \prod_{i \in B} \omega_i$ for B a finite subset of $\mathbb{Z}$ (define $f(\omega) = 1$ if B is empty) and the subset consisting of all functions $f(\omega) = \chi_\Sigma(\omega)$ for Σ a cylinder set in Ω. These examples are discussed in Theorem A.11.3. Another useful convergence-determining class is given in the next lemma.

Lemma IV.6.1. *For B a nonempty finite subset of $\mathbb{Z}$, define $f_B(\omega) = \prod_{i \in B} [\frac{1}{2}(1 + \omega_i)]$. For B the empty set, define $f_B(\omega) = 1$. Then the subset of $\mathscr{C}(\Omega)$ consisting of all functions $f_B(\omega)$ is a convergence-determining class.*

Proof. The limit $\lim_{n\to\infty} \int_\Omega f_B dP_n$ exists for any finite set B if and only if the limit $\lim_{n\to\infty} \int_\Omega f dP_n$ exists for all functions $f(\omega) = \prod_{i \in B} \omega_i$, where $f(\omega) = 1$ if B is empty. Hence the lemma is a consequence of Theorem A.11.3(a). □

In order to define infinite-volume Gibbs states, we extend each finite-volume Gibbs state $P_{\Lambda,\beta,h,\tilde{\omega}}$ to a probability measure on $\mathscr{B}(\Omega)$. Let $B_{\Lambda,\tilde{\omega}}$ denote the set $\{\omega \in \Omega : \omega_j = \tilde{\omega}_j \text{ for each } j \in \Lambda^c\}$ and π_Λ the projection of Ω onto Ω_Λ defined by $(\pi_\Lambda \omega)_i = \omega_i$, $i \in \Lambda$. We define the extension $\bar{P}_{\Lambda,\beta,h,\tilde{\omega}}$ of the finite-volume Gibbs state by setting

$$\bar{P}_{\Lambda,\beta,h,\tilde{\omega}}\{A\} = P_{\Lambda,\beta,h,\tilde{\omega}}\{\pi_\Lambda(A \cap B_{\Lambda,\tilde{\omega}})\} = \sum_{\omega \in \pi_\Lambda(A \cap B_{\Lambda,\tilde{\omega}})} P_{\Lambda,\beta,h,\tilde{\omega}}\{\omega\} \tag{4.37}$$

for A a Borel subset of Ω. The right-hand side of (4.37) is given by (4.36). Since the support of $\bar{P}_{\Lambda,\beta,h,\tilde{\omega}}$ is the set $B_{\Lambda,\tilde{\omega}}$, the extension is compatible with the external condition. We note a useful property of $\bar{P}_{\Lambda,\beta,h,\tilde{\omega}}$. Let f be a function in $\mathscr{C}(\Omega)$ such that the value $f(\omega)$ depends on only finitely many coordinates of ω; let these coordinates be $\omega_{i_1}, \ldots, \omega_{i_r}$. If Λ is any symmetric interval containing the sites $i_1, \ldots, i_r$, then the restriction of f to the coordinates $\{\omega_i; i \in \Lambda\}$ defines a function in $\mathscr{C}(\Omega_\Lambda)$. We denote the restriction by the same symbol f; if $\omega = \pi_\Lambda \bar{\omega}$, $\omega \in \Omega_\Lambda$ and $\bar{\omega} \in \Omega$, then $f(\omega) = f(\bar{\omega})$. It is easy to check [Problem IV.9.9] that

$$\int_\Omega f(\omega) \bar{P}_{\Lambda,\beta,h,\tilde{\omega}}(d\omega) = \int_{\Omega_\Lambda} f(\omega) P_{\Lambda,\beta,h,\tilde{\omega}}(d\omega). \tag{4.38}$$

An example of such a function f is the function f_B in the previous lemma. From now on, we will denote the extension $\bar{P}_{\Lambda,\beta,h,\tilde{\omega}}$ by the same symbol $P_{\Lambda,\beta,h,\tilde{\omega}}$ used for the original measure. The extension will be called a finite-volume Gibbs state.

Since the space $\mathscr{M}(\Omega)$ is compact, any sequence of finite-volume Gibbs states $\{P_{\Lambda,\beta,h,\tilde{\omega}(\Lambda)}; \Lambda \uparrow \mathbb{Z}\}$ with arbitrary external conditions $\{\tilde{\omega}(\Lambda)\}$ has a convergent subsequence $\{P_{\Lambda',\beta,h,\tilde{\omega}(\Lambda')}\}$. Each of these weak limits is an infinite-volume state for the given interaction. It is natural to investigate the dependence of the limits upon the choice of external conditions. For different values of β and h several situations occur: there is a unique weak limit regardless of the choice of external conditions and the limit is translation invariant; the weak limit depends upon the choice of external conditions and is either translation invariant or not. In the second situation, a phase transition is said to occur. We shall call this a *level-3 phase transition* in order to distinguish it from the level-1 phase transition which is spontaneous magnetization.

Consider the set of limits

$$\mathscr{G}^0_{\beta,h} = \{P \in \mathscr{M}(\Omega) : P = w\text{-}\lim_{\Lambda' \uparrow \mathbb{Z}} P_{\Lambda',\beta,h,\tilde{\omega}(\Lambda')}\}, \tag{4.39}$$

where $\{\Lambda'\}$ is any increasing sequence of symmetric intervals whose union is $\mathbb{Z}$ and $\tilde{\omega}(\Lambda')$ is any external condition for Λ'. Define $\mathscr{G}_{\beta,h}$ to be the closed convex hull of $\mathscr{G}^0_{\beta,h}$. This is the intersection of all closed convex subsets of $\mathscr{M}(\Omega)$ containing $\mathscr{G}^0_{\beta,h}$. Equivalently, $\mathscr{G}_{\beta,h}$ is the closure, with respect to weak convergence, of the set of convex combinations

$$\left\{P \in \mathscr{M}(\Omega) : P = \sum_{j=1}^r \lambda_j P_j, \lambda_j > 0, \sum_{j=1}^r \lambda_j = 1, P_j \in \mathscr{G}^0_{\beta,h}\right\}. \tag{4.40}$$

Each measure $P \in \mathscr{G}_{\beta,h}$ is called an *infinite-volume Gibbs state*. A level-3 phase transition is said to occur if $\mathscr{G}_{\beta,h}$ consists of more than one measure.

The passage from $\mathscr{G}^0_{\beta,h}$ to $\mathscr{G}_{\beta,h}$ has an interesting physical interpretation. We denote the limit P in (4.39) by $P_{\beta,h,\{\omega(\Lambda')\}}$. This measure corresponds physically to the situation where an experimenter is sure that the external condition for each Λ' is $\omega(\Lambda')$. The experimenter may also be uncertain about the external conditions. Such an uncertainty is represented by the convex combination P in (4.40), where $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the probabilities of r different choices.

Before proving properties of infinite-volume Gibbs states, we extend the definition of Gibbs free energy to the state $P_{\Lambda,\beta,h,\tilde{\omega}}$ with external condition $\tilde{\omega}$. Recall that for the finite-volume Gibbs state $P_{\Lambda,\beta,h}$ without external condition, we defined $\Psi(\Lambda,\beta,h) = -\beta^{-1}\log Z(\Lambda,\beta,h)$, where $Z(\Lambda,\beta,h) = \int_{\Omega_\Lambda}\exp[-\beta H_{\beta,h}(\omega)]\pi_\Lambda P_\rho(d\omega)$. If $\sum_{k\in\mathbb{Z}} J(k) < \infty$, then the specific Gibbs free energy $\psi(\beta,h) = \lim_{\Lambda\uparrow\mathbb{Z}}|\Lambda|^{-1}\Psi(\Lambda,\beta,h)$ exists [Appendix D.1]. Given an external condition $\tilde{\omega} = \tilde{\omega}(\Lambda)$, we define the Gibbs free energy in the state $P_{\Lambda,\beta,h,\tilde{\omega}}$ by the formula

$$\begin{aligned}\Psi(\Lambda,\beta,h,\tilde{\omega}) &= -\beta^{-1}\log Z(\Lambda,\beta,h,\tilde{\omega})\\ &= -\beta^{-1}\log\int_{\Omega_\Lambda}\exp[-\beta H_{\Lambda,h,\tilde{\omega}}(\omega)]\pi_\Lambda P_\rho(d\omega).\end{aligned}$$

The next lemma shows that for any sequence of external conditions $\{\tilde{\omega}(\Lambda);\ \Lambda\uparrow\mathbb{Z}\}$ the limit $\lim_{\Lambda\uparrow\mathbb{Z}}|\Lambda|^{-1}\Psi(\Lambda,\beta,h,\tilde{\omega}(\Lambda))$ exists and is independent of the sequence chosen.

Lemma IV.6.2. *Let J be a summable ferromagnetic interaction on $\mathbb{Z}$. Then for $\beta > 0$ and h real,* $\lim_{\Lambda\uparrow\mathbb{Z}}|\Lambda|^{-1}\Psi(\Lambda,\beta,h,\tilde{\omega}(\Lambda))$ *exists and is independent of the choice of $\{\tilde{\omega}(\Lambda)\}$. The limit equals* $\lim_{\Lambda\uparrow\mathbb{Z}}|\Lambda|^{-1}\Psi(\Lambda,\beta,h) = \psi(\beta,h)$.

Proof. We use the comparison lemma. Lemma II.7.4. For any probability measure P on $\mathscr{B}(\Omega_\Lambda)$ and real-valued functions f and g on Ω_Λ

$$\left|\log\int_{\Omega_\Lambda} e^f dP - \log\int_{\Omega_\Lambda} e^g dP\right| \le \|f-g\|_\infty,$$

where $\|-\|_\infty$ denotes the maximum over Ω_Λ. It follows that

$$\frac{1}{|\Lambda|}|\Psi(\Lambda,\beta,h,\tilde{\omega}) - \Psi(\Lambda,\beta,h)| \le \frac{1}{|\Lambda|}\|H_{\Lambda,h,\tilde{\omega}} - H_{\Lambda,h}\|_\infty. \tag{4.41}$$

Since $\psi(\beta,h) = \lim_{\Lambda\uparrow\mathbb{Z}}|\Lambda|^{-1}\Psi(\Lambda,\beta,h)$ exists, it suffices to prove that $|\Lambda|^{-1}\|H_{\Lambda,h,\tilde{\omega}} - H_{\Lambda,h}\|_\infty \to 0$ as $\Lambda\uparrow\mathbb{Z}$. For any $\varepsilon > 0$, there exists a positive integer N_ε so that $\sum_{|k|>N_\varepsilon} J(k) < \varepsilon$. For any $\omega\in\Omega_\Lambda$

$$\frac{1}{|\Lambda|}|H_{\Lambda,h,\tilde{\omega}}(\omega) - H_{\Lambda,h}(\omega)| \le \frac{1}{|\Lambda|}\sum_{i\in\Lambda}\sum_{j\in\Lambda^c} J(i-j) \le \frac{1}{|\Lambda|}{\sum}' J(i-j) + \varepsilon, \tag{4.42}$$

where $\sum'$ denotes the sum over all $i\in\Lambda$ and $j\in\Lambda^c$ such that $|i-j|\le N_\varepsilon$. Since $\sum' J(i-j) \le \max_{k\in\mathbb{Z}}(J(k))\cdot 2N_\varepsilon^2$, the proof is complete. □

We recall from Theorems IV.5.1 and IV.5.3(d) that for all $\beta > 0$, $h \ne 0$ and $0 < \beta < \beta_c$, $h = 0$, $\partial\psi(\beta,h)/\partial h$ exists and equals minus the specific magnetization $m(\beta,h) = \lim_{\Lambda\uparrow\mathbb{Z}}|\Lambda|^{-1}M(\Lambda,\beta,h)$. $M(\Lambda,\beta,h)$ is the magnetization in the finite-volume Gibbs state $P_{\Lambda,\beta,h}$ without external condition. This relation will be extended to the states $\{P_{\Lambda,\beta,h,\tilde{\omega}}\}$ by means of the following useful convexity result.

Lemma IV.6.3. *Let $\{f_n; n = 1, 2, \ldots\}$ be a sequence of convex functions on an open interval A of $\mathbb{R}$ such that $f(t) = \lim_{n\to\infty} f_n(t)$ exists for every $t \in A$. If each f_n and f are differentiable at some point $t_0 \in A$, then $\lim_{n\to\infty} f_n'(t_0)$ exists and equals $f'(t_0)$.*

Proof. Define $g_n(t) = (f_n(t) - f_n(t_0))/(t - t_0)$ and $g(t) = (f(t) - f(t_0))/(t - t_0)$ for $t \in A$, $t \neq t_0$. Then $g_n(t) \to g(t)$ since by hypothesis $f_n \to f$ on A. The convexity of f_n implies that $g_n(s) \le f_n'(t_0) \le g_n(t)$ for $s < t_0 < t$ $(s, t \in A)$.* Taking $n \to \infty$, we see that

$$\text{(4.43)} \qquad \sup_{\{s \in A: s < t_0\}} g(s) \le \liminf_{n\to\infty} f_n'(t_0) \le \limsup_{n\to\infty} f_n'(t_0) \le \inf_{\{t \in A: t > t_0\}} g(t).$$

As the pointwise limit of convex functions, f is convex, and so $f'(t_0) = \sup_{\{s \in A: s < t_0\}} g(s) = \inf_{\{t \in A: t > t_0\}} g(t)$. Inserting this in (4.43) completes the proof. □

Given an external condition $\tilde{\omega} = \tilde{\omega}(\Lambda)$, define $M(\Lambda, \beta, h, \tilde{\omega}) = \sum_{i \in \Lambda} \langle \omega_i \rangle_{\Lambda, \beta, h, \tilde{\omega}}$. This sum is the magnetization in the finite-volume Gibbs state $P_{\Lambda, \beta, h, \tilde{\omega}}$. By the same calculation as in (4.21), $\partial \Psi(\Lambda, \beta, h, \tilde{\omega})/\partial h$ equals $-M(\Lambda, \beta, h, \tilde{\omega})$, and as in the proof of Theorem IV.5.1, $\Psi(\Lambda, \beta, h, \tilde{\omega})$ is a concave function of h real. Since $|\Lambda|^{-1}\Psi(\Lambda, \beta, h, \tilde{\omega}(\Lambda)) \to \psi(\beta, h)$ and $\partial \psi(\beta, h)/\partial h$ exists for $\beta > 0$, $h \neq 0$ and $0 < \beta < \beta_c$, $h = 0$, Lemma IV.6.3 yields the following important fact.

Lemma IV.6.4. *Let J be be a summable ferromagnetic interaction on $\mathbb{Z}$. Then for $\beta > 0$, $h \neq 0$ and $0 < \beta < \beta_c$, $h = 0$, $\lim_{\Lambda \uparrow \mathbb{Z}} |\Lambda|^{-1} M(\Lambda, \beta, h, \tilde{\omega}(\Lambda))$ exists and is independent of the choice of $\{\tilde{\omega}(\Lambda)\}$. For these values of β and h,*

$$\text{(4.44)} \qquad \lim_{\Lambda \uparrow \mathbb{Z}} \frac{1}{|\Lambda|} M(\Lambda, \beta, h, \tilde{\omega}(\Lambda)) = -\frac{\partial \psi(\beta, h)}{\partial h} = m(\beta, h).$$

Let T denote the shift mapping on Ω. A probability measure P on $\mathscr{B}(\Omega)$ is said to be *translation invariant* if for each Borel set A $P\{T^{-1}A\} = P\{A\}$. Let $\mathscr{M}_s(\Omega)$ denote the set of translation invariant probability measures on $\mathscr{B}(\Omega)$. A measure $P \in \mathscr{M}_s(\Omega)$ is called *ergodic* if $P\{A\}$ equals 0 or 1 for any Borel set A which satisfies $T^{-1}A = A$. We have denoted the set of infinite-volume Gibbs states by $\mathscr{G}_{\beta,h}$. A measure P in $\mathscr{G}_{\beta,h}$ is called a *phase* if P is translation invariant. Let $m(\beta, h)$ be the specific magnetization and let $m(\beta, +) = \lim_{h\to 0^+} m(\beta, h)$ and $m(\beta, -) = \lim_{h \to 0^-} m(\beta, h)$. According to Theorem IV.5.3 there exists a critical inverse temperature $\beta_c \in (0, \infty]$ such that spontaneous magnetization occurs at all $\beta > \beta_c$ but not at any $0 < \beta < \beta_c$; i.e.,

$$m(\beta, +) = 0 = m(\beta, -) \qquad \text{for } 0 < \beta < \beta_c,$$
$$m(\beta, +) > 0 > m(\beta, -) \qquad \text{for } \beta > \beta_c.$$

* See page 214.

The next theorem describes phases of the ferromagnet and relates the occurrence of a level-3 phase transition to that of spontaneous magnetization.

Theorem IV.6.5. *Let J be a summable ferromagnetic interaction on $\mathbb{Z}$. Then the following conclusions hold.*

(a) *For each $\beta > 0$ and h real, the weak limits*

$$(4.45) \qquad P_{\beta,h,+} = \operatorname*{w-lim}_{\Lambda \uparrow \mathbb{Z}} P_{\Lambda,\beta,h,+}, \qquad P_{\beta,h,-} = \operatorname*{w-lim}_{\Lambda \uparrow \mathbb{Z}} P_{\Lambda,\beta,h,-}$$

exist and are translation invariant. Thus $\mathscr{G}_{\beta,h}$ and $\mathscr{G}_{\beta,h} \cap \mathscr{M}_s(\Omega)$ are nonempty. The measures $P_{\beta,h,+}$ and $P_{\beta,h,-}$ are ergodic.

(b) *$P_{\beta,h,+}$ equals $P_{\beta,h,-}$ if and only if $\partial\psi(\beta,h)/\partial h$ exists. Thus, for $\beta > 0$, $h \neq 0$ and for $0 < \beta < \beta_c$, $h = 0$, $P_{\beta,h,+}$ equals $P_{\beta,h,-}$. For these values of β and h, define $P_{\beta,h} = P_{\beta,h,+} = P_{\beta,h,-}$.*

(c) *If $\partial\psi(\beta,h)/\partial h$ exists, then $P_{\beta,h}$ is the unique measure in $\mathscr{G}_{\beta,h}$ and thus in $\mathscr{G}_{\beta,h} \cap \mathscr{M}_s(\Omega)$. No level-3 phase transition occurs. The mean $\int_\Omega \omega_0 P_{\beta,h}(d\omega)$ equals the specific magnetization $m(\beta,h)$.*

(d) *For $\beta > \beta_c$, $P_{\beta,0,+} \neq P_{\beta,0,-}$. In fact,*

$$(4.46) \quad \int_\Omega \omega_0 P_{\beta,0,+}(d\omega) = m(\beta,+) > 0 > \int_\Omega \omega_0 P_{\beta,0,-}(d\omega) = m(\beta,-).$$

Thus for $\beta > \beta_c$ and $h = 0$, a level-3 phase transition occurs.

(e) *For $\beta > \beta_c$, $\mathscr{G}_{\beta,0} \cap \mathscr{M}_s(\Omega)$ contains (at least) all the measures $P_{\beta,0}^{(\lambda)} = \lambda P_{\beta,0,+} + (1-\lambda)P_{\beta,0,-}$, $0 \le \lambda \le 1$.*

The proof of this theorem requires several new results which we will present as lemmas at the end of the section. First, we interpret the contents of the theorem. See Note 11 for further comments on the structure of $\mathscr{G}_{\beta,h}$ and $\mathscr{G}_{\beta,h} \cap \mathscr{M}_s(\Omega)$.

Part (a). Let $S_\Lambda(\omega) = \sum_{j \in \Lambda} \omega_j$ be the spin in a symmetric interval Λ. The ergodic theorem [Theorem A.11.5] implies that if $P \in \mathscr{M}_s(\Omega)$ is ergodic, then

$$(4.47) \qquad \lim_{\Lambda \uparrow \mathbb{Z}} \frac{S_\Lambda(\omega)}{|\Lambda|} = \int_\Omega \omega_0 P(d\omega) \qquad P\text{-a.s.}$$

Let P be the ergodic infinite-volume Gibbs state $P_{\beta,h,+}$ or $P_{\beta,h,-}$. Then $\lim_{\Lambda \uparrow \mathbb{Z}} S_\Lambda(\omega)/|\Lambda|$, which is the limiting spin per site in a sample ω drawn from the magnet, is a constant independent of ω P-a.s. The measures $P_{\beta,h,+}$ and $P_{\beta,h,-}$ are called *pure phases* of the magnet. Stronger clustering properties of $P_{\beta,h,+}$ and $P_{\beta,h,-}$ are given in Corollary A.11.12.

Parts (b)–(c). According to Lemma IV.6.4, for $\beta > 0$, $h \neq 0$ and $0 < \beta < \beta_c$, $h = 0$, the limiting magnetization per site, $\lim_{\Lambda \uparrow \mathbb{Z}} |\Lambda|^{-1} M(\Lambda,\beta,h,\tilde{\omega}(\Lambda))$, exists and is independent of the choice of external conditions $\{\tilde{\omega}(\Lambda)\}$. It equals the specific magnetization $m(\beta, h)$. For these values of β and h, the independence of external conditions extends to the infinite-volume Gibbs states; that is, $P_{\beta,h} = P_{\beta,h,+} = P_{\beta,h,-}$ is the unique measure in $\mathscr{G}_{\beta,h}$ as well as in $\mathscr{G}_{\beta,h} \cap \mathscr{M}_s(\Omega)$. The ergodic limit (4.47) and the fact that*

$$\int_\Omega \omega_0 P_{\beta,h}(d\omega) = m(\beta, h) > 0 \qquad \text{for } h > 0 \tag{4.48}$$

imply that with respect to $P_{\beta,h}$, for $h > 0$ almost every $\omega \in \Omega$ has a majority of the spins $+1$. There is a similar statement for $h < 0$. This support property of $P_{\beta,h}$ is the infinite-volume analog of the alignment effect built into the finite-volume Gibbs states $\{P_{\Lambda,\beta,h}\}$.

Parts (d)–(e). For $\beta > \beta_c$, the distinct measures $P_{\beta,0,+}$ and $P_{\beta,0,-}$ are both ergodic, and so there exist disjoint subsets $A_{\beta,+}$ and $A_{\beta,-}$ of Ω such that $P_{\beta,0,+}\{A_{\beta,+}\} = 1$ and $P_{\beta,0,-}\{A_{\beta,-}\} = 1$ [Theorem A.11.7(b)]. The ergodic limit (4.47) and the fact that $\int_\Omega \omega_0 P_{\beta,0,+}(d\omega) = m(\beta, +) > 0$ imply that with respect to $P_{\beta,0,+}$, almost every $\omega \in A_{\beta,+}$ has a majority of the spins $+1$. Similarly, with respect to $P_{\beta,0,-}$, almost every $\omega \in A_{\beta,-}$ has a majority of the spins -1. The measures $P_{\beta,0,+}$ and $P_{\beta,0,-}$ are called a *pure plus phase* and a *pure minus phase*, respectively. These measures are simply related by the formula $P_{\beta,0,+}(d(-\omega)) = P_{\beta,0,-}(d\omega)$, which follows from the same relation for the finite-volume Gibbs states $P_{\Lambda,\beta,0,+}$ and $P_{\Lambda,\beta,0,-}$. For $0 < \lambda < 1$, the nonergodic measure $P_{\beta,0}^{(\lambda)} = \lambda P_{\beta,0,+} + (1-\lambda)P_{\beta,0,-}$ is called a *mixed phase*. With respect to this measure, the limiting spin per site, $\lim_{\Lambda \uparrow \mathbb{Z}} S_\Lambda(\omega)/|\Lambda|$, is not a constant but depends on the choice of ω [see Theorem IV.6.6(c)].

Level-3 phase transition. That $\mathscr{G}_{\beta,h}$ consists of a unique measure for $\beta > 0$, $h \neq 0$ and $0 < \beta < \beta_c$, $h = 0$, contrasts with the nonuniqueness of measures in $\mathscr{G}_{\beta,0}$ for $\beta > \beta_c$. We have called this nonuniqueness a level-3 phase transition. Formula (4.46) shows that spontaneous magnetization is the contraction of the level-3 phase transition onto level-1. The level-3 phase transition can also be looked at as a *symmetry-breaking transition*.[12] For $h = 0$, the microscopic interaction energy between each pair of spins ω_i, ω_j equals $-J(i-j)\omega_i\omega_j$. For all i and j, these energy terms are invariant with respect to sign changes $(\omega_i, \omega_j) \to (-\omega_i, -\omega_j)$ in the spins. However for $\beta > \beta_c$ neither of the states $P_{\beta,0,+}$, $P_{\beta,0,-}$ retains this invariance.

The next result refines the ergodic theorem by giving additional convergence properties of the microscopic sums $\{S_\Lambda/|\Lambda|\}$ as $\Lambda \uparrow \mathbb{Z}$. We see that exponential convergence to a constant distinguishes the values $\beta > 0$, $h \neq 0$ and $0 < \beta < \beta_c$, $h = 0$ from the values $\beta > \beta_c$. $h = 0$.

*The positivity of $m(\beta, h)$ for $h > 0$ is proved in a footnote on page 165. Also see Problem V.13.1(b).

Theorem IV.6.6. *Let J be a summable ferromagnetic interaction on $\mathbb{Z}$. Then the following conclusions hold.*

(a) *For $\beta > 0$, $h \neq 0$ and for $0 < \beta < \beta_c$, $h = 0$,*

$$S_\Lambda/|\Lambda| \xrightarrow{\text{a.s.}} m(\beta, h) \quad \text{and} \quad S_\Lambda/|\Lambda| \xrightarrow{\exp} m(\beta, h) \quad \text{w.r.t. } P_{\beta,h} \quad \text{as } \Lambda \uparrow \mathbb{Z}^D.$$

(b) *For $\beta > \beta_c$,*

$$\frac{S_\Lambda}{|\Lambda|} \xrightarrow{\text{a.s.}} m(\beta, +) \quad \text{w.r.t. } P_{\beta,0,+},$$

$$\frac{S_\Lambda}{|\Lambda|} \xrightarrow{\text{a.s.}} m(\beta, -) \quad \text{w.r.t. } P_{\beta,0,-} \quad \text{as } \Lambda \uparrow \mathbb{Z}.$$

In each case, exponential convergence fails.

(c) *For $\beta > \beta_c$ and each $0 < \lambda < 1$, there exists a random variable $Y_\beta^{(\lambda)}$ on Ω with distribution $\lambda\delta_{m(\beta,+)} + (1-\lambda)\delta_{m(\beta,-)}$ such that $S_\Lambda/|\Lambda| \xrightarrow{\text{a.s.}} Y_\beta^{(\lambda)}$ w.r.t. $P_{\beta,0}^{(\lambda)} = \lambda P_{\beta,0,+} + (1-\lambda)P_{\beta,0,-}$ as $\Lambda \uparrow \mathbb{Z}$.*

With respect to the various measures in this theorem, large deviation bounds for $S_\Lambda/|\Lambda|$ can be derived from Theorem II.6.1. We omit the formulas, which are easily worked out as in the discussion at the end of Section IV.5 (see Lemma IV.6.11 for the calculation of the free energy function).[13]

We now turn to the proofs of Theorems IV.6.5 and IV.6.6. The first step is to prove that the weak limits $P_{\beta,h,+} = \text{w-lim}_{\Lambda\uparrow\mathbb{Z}} P_{\Lambda,\beta,h,+}$ and $P_{\beta,h,-} = \text{w-lim}_{\Lambda\uparrow\mathbb{Z}} P_{\Lambda,\beta,h,-}$ exist. According to Lemma IV.6.1, it suffices to prove that for each finite set B the limits $\lim_{\Lambda\uparrow\mathbb{Z}} \int_\Omega f_B dP_{\Lambda,\beta,h,+}$ and $\lim_{\Lambda\uparrow\mathbb{Z}} \int_\Omega f_B dP_{\Lambda,\beta,h,-}$ exist, where $f_B(\omega) = \prod_{i\in B}[\frac{1}{2}(1+\omega_i)]$ for B nonempty. By the discussion after that lemma, each integral $\int_\Omega f_B dP_{\Lambda,\beta,h,+}$ or $\int_\Omega f_B dP_{\Lambda,\beta,h,-}$ equals $\int_{\Omega_\Lambda} f_B dP_{\Lambda,\beta,h,+}$ or $\int_{\Omega_\Lambda} f_B dP_{\Lambda,\beta,h,-}$, respectively, provided Λ contains B. The proof depends upon a powerful monotonicity result due to Fortuin, Kastelyn, and Ginibre (1971) and known as the FKG inequality.

The FKG inequality is valid for a more general measure than the finite-volume Gibbs state $P_{\Lambda,\beta,h,\tilde\omega}$. Let Λ be an arbitrary nonempty finite subset of $\mathbb{Z}$, $\{J_{ij}; i,j\in\Lambda\}$ a set of non-negative real numbers, and $\{h_i; i\in\Lambda\}$ a set of real numbers. Let P be the probability measure on $\mathscr{B}(\Omega_\Lambda)$ which assigns to each $\{\omega\}$, $\omega\in\Omega_\Lambda$, the probability

$$P\{\omega\} = \exp[-H(\omega)]\pi_\Lambda P_\rho\{\omega\}\cdot\frac{1}{Z}. \tag{4.49}$$

In this formula, $H(\omega)$ equals $-\frac{1}{2}\sum_{i,j\in\Lambda} J_{ij}\omega_i\omega_j - \sum_{i\in\Lambda} h_i\omega_i$ and Z equals $\int_{\Omega_\Lambda} \exp[-H(\omega)]\pi_\Lambda P_\rho(d\omega)$. Expectation with respect to P is denoted by $\langle - \rangle_{\Lambda,\{h_i\}}$ or by $\langle - \rangle$. The measure P reduces to $P_{\Lambda,\beta,h,\tilde\omega}$ if Λ is a symmetric interval, $J_{ij} = \beta J(i-j)$, and $h_i = h + \sum_{j\in\Lambda^c} J(i-j)\tilde\omega_j$.

Let ω and $\bar\omega$ be points in Ω_Λ. We write $\omega \le \bar\omega$ if $\omega_i \le \bar\omega_i$ for each $i\in\Lambda$. A real-valued function f on Ω_Λ is said to be *nondecreasing* if $f(\omega) \le f(\bar\omega)$ whenever $\omega \le \bar\omega$. For example, the function $Y(\omega) = \omega_i$ for $i\in\Lambda$ is nondecreasing as is $f_B(\omega)$ for any subset B of Λ. However, $f(\omega) = \omega_i\omega_j$ ($i \neq j$ in Λ) is not nondecreasing. The FKG inequality is stated in part (a) of the next theorem. A useful consequence of the inequality is stated in part (b).[14]

Theorem IV.6.7. *Let f and g be nondecreasing functions on Ω_Λ. Then the following conclusions hold.*

(a) *For any values of $\{h_i\}$, the covariance of f and g is non-negative:*

$$\langle fg\rangle_{\Lambda,\{h_i\}} - \langle f\rangle_{\Lambda,\{h_i\}}\langle g\rangle_{\Lambda,\{h_i\}} \geq 0.$$

(b) $\{h_i\} \leq \{\bar{h}_i\}$ *implies* $\langle f\rangle_{\Lambda,\{h_i\}} \leq \langle f\rangle_{\Lambda,\{\bar{h}_i\}}$.

Proof. (a) The following proof is due to Battle and Rosen (1980). The argument is by induction on the number of sites $|\Lambda|$ in Λ. We have

$$\langle fg\rangle - \langle f\rangle\langle g\rangle = \int_{\Omega_\Lambda}\int_{\Omega_\Lambda} [f(\omega) - f(\tilde{\omega})][g(\omega) - g(\tilde{\omega})]P(d\omega)P(d\tilde{\omega}). \tag{4.50}$$

If $|\Lambda| = 1$, then the integrand is non-negative since f and g are nondecreasing. Hence $\langle fg\rangle - \langle f\rangle\langle g\rangle \geq 0$. Assume now that the inequality has been proved for $|\Lambda| = 1, \ldots, n-1$, some $n \geq 2$, and consider the inequality for $|\Lambda| = n$. Fix any site α in Λ. We set $\omega = (\omega', \omega_\alpha)$, where ω' has the $n-1$ components $\{\omega_i; i \in \Lambda\backslash\{\alpha\}\}$, and rewrite $H(\omega)$ in the form

$$H(\omega', \omega_\alpha) = -\frac{1}{2}\sum_{i,j\in\Lambda\backslash\{\alpha\}} J_{ij}\omega_i\omega_j - \sum_{i\in\Lambda\backslash\{\alpha\}}\left(h_i + \frac{1}{2}(J_{i\alpha} + J_{\alpha i})\omega_\alpha\right)\omega_i$$
$$-\frac{1}{2}J_{\alpha\alpha} - h_\alpha\omega_\alpha.$$

For fixed $\omega_\alpha \in \{1, -1\}$, let $v_{\omega_\alpha}(d\omega')$ be the probability measure on $\Omega_{\Lambda\backslash\{\alpha\}}$ defined by

$$v_{\omega_\alpha}(d\omega') = \exp[-H(\omega', \omega_\alpha)]\pi_{\Lambda\backslash\{\alpha\}}P_\rho(d\omega')\cdot\frac{1}{Z(\omega_\alpha)},$$

where $Z(\omega_\alpha)$ equals $\int_{\Omega_{\Lambda\backslash\{\alpha\}}}\exp[-H(\omega', \omega_\alpha)]\pi_{\Lambda\backslash\{\alpha\}}P_\rho(d\omega')$. We now write

$$\langle fg\rangle = \int_{\Omega_\Lambda} f(\omega)g(\omega)P(d\omega)$$
$$= \int_{\{1,-1\}}\left[\int_{\Omega_{\Lambda\backslash\{\alpha\}}} f(\omega', \omega_\alpha)g(\omega', \omega_\alpha)v_{\omega_\alpha}(d\omega')\right]Z(\omega_\alpha)\rho(d\omega_\alpha)\cdot\frac{1}{Z}.$$

The inductive hypothesis clearly applies to the ω'-integral. It follows that

$$\langle fg\rangle \geq \int_{\{1,-1\}} \phi(\omega_\alpha)\gamma(\omega_\alpha)Z(\omega_\alpha)\rho(d\omega_\alpha)\cdot\frac{1}{Z}, \tag{4.51}$$

where $\phi(\omega_\alpha) = \int_{\Omega_{\Lambda\setminus\{\alpha\}}} f(\omega', \omega_\alpha)\nu_{\omega_\alpha}(d\omega')$ and $\gamma(\omega_\alpha) = \int_{\Omega_{\Lambda\setminus\{\alpha\}}} g(\omega', \omega_\alpha)\nu_{\omega_\alpha}(d\omega')$. We prove just below that ϕ and γ are nondecreasing. The result for $|\Lambda| = 1$ and (4.51) yield (4.50):

$$\begin{aligned}\langle fg\rangle &\geq \int_{\{1,-1\}} \phi(\omega_\alpha)Z(\omega_\alpha)\rho(d\omega_\alpha)\cdot\frac{1}{Z}\cdot\int_{\{1,-1\}} \gamma(\omega_\alpha)Z(\omega_\alpha)\rho(d\omega_\alpha)\cdot\frac{1}{Z}\\ &= \langle f\rangle\langle g\rangle.\end{aligned}$$

We prove that $\phi(-1) \leq \phi(1)$. The same proof shows that γ is nondecreasing. In the definition of $\nu_{\omega_\alpha}(d\omega')$, replace $\omega_\alpha \in \{1, -1\}$ by a real parameter t. We prove that the function $t \to \int_{\Omega_{\Lambda\setminus\{\alpha\}}} f(\omega', 1)\nu_t(d\omega')$ is nondecreasing. Since f is nondecreasing, it will then follow that

$$\begin{aligned}\phi(-1) &= \int_{\Omega_{\Lambda\setminus\{\alpha\}}} f(\omega', -1)\nu_{-1}(d\omega') \leq \int_{\Omega_{\Lambda\setminus\{\alpha\}}} f(\omega', 1)\nu_{-1}(d\omega')\\ &\leq \int_{\Omega_{\Lambda\setminus\{\alpha\}}} f(\omega', 1)\nu_1(d\omega') = \phi(1).\end{aligned}$$

We have

$$\begin{aligned}&\frac{d}{dt}\int_{\Omega_{\Lambda\setminus\{\alpha\}}} f(\omega', 1)\nu_t(d\omega')\\ &\qquad = \int_{\Omega_{\Lambda\setminus\{\alpha\}}} f(\omega', 1)\left[-\frac{\partial H(\omega', t)}{\partial t}\right]\nu_t(d\omega')\\ &\qquad\quad - \int_{\Omega_{\Lambda\setminus\{\alpha\}}} f(\omega', 1)\nu_t(d\omega')\cdot\int_{\Omega_{\Lambda\setminus\{\alpha\}}}\left[-\frac{\partial H(\omega', t)}{\partial t}\right]\nu_t(d\omega').\end{aligned}$$

The function $-\partial H(\omega', t)/\partial t$ equals $\sum_{i\in\Lambda\setminus\{\alpha\}} \frac{1}{2}(J_{i\alpha} + J_{\alpha i})\omega_i + h_\alpha$, and so it is a nondecreasing function on $\Omega_{\Lambda\setminus\{\alpha\}}$. By the inductive hypothesis, we conclude that $d\int_{\Omega_{\Lambda\setminus\{\alpha\}}} f(\omega', 1)\nu_t(d\omega')/dt \geq 0$. The FKG inequality is proved.

(b) Since $\partial H/\partial h_i = -\omega_i$ and

$$\frac{\partial Z}{\partial h}\cdot\frac{1}{Z} = \int_{\Omega_\Lambda} \omega_i \exp[-H(\omega)]\pi_\Lambda P_\rho(d\omega)\cdot\frac{1}{Z} = \langle\omega_i\rangle,$$

we have

$$\begin{aligned}\frac{\partial}{\partial h_i}\langle f\rangle &= \frac{\partial}{\partial h_i}\left\{\int_{\Omega_\Lambda} f(\omega)\exp[-H(\omega)]\pi_\Lambda P_\rho(d\omega)\cdot\frac{1}{Z}\right\}\\ &= \int_{\Omega_\Lambda} f(\omega)\cdot\omega_i\exp[-H(\omega)]\pi_\Lambda P_\rho(d\omega)\cdot\frac{1}{Z} - \langle f\rangle Z_{h_i}\cdot\frac{1}{Z} \qquad (4.52)\\ &= \langle f\cdot\omega_i\rangle - \langle f\rangle\langle\omega_i\rangle.\end{aligned}$$

This is non-negative as f and ω_i are nondecreasing. Thus $\langle f\rangle_{\Lambda,\{h_i\}}$ is a nondecreasing function of each h_i. Part (b) is proved. □

Let J be a non-negative, summable, symmetric function on $\mathbb{Z}$. Denote by $\langle - \rangle_{\Lambda,\beta,h,+}$ (resp., $\langle - \rangle_{\Lambda,\beta,h,-}$) expectation with respect to the measure P in (4.49) with $J_{ij} = \beta J(i-j)$ and $h_i = h + \sum_{j \in \Lambda^c} J(i-j)(+1)$ (resp., $h_i = h + \sum_{j \in \Lambda^c} J(i-j)(-1)$).

Corollary IV.6.8. *Let $\{\Lambda_n; n = 1, 2, \ldots\}$ be an increasing sequence of finite subsets of $\mathbb{Z}$ such that $\Lambda_n \uparrow \mathbb{Z}$ as $n \to \infty$. For B a nonempty finite set in $\mathbb{Z}$, pick an integer $n(B)$ so that B is contained in Λ_n for all $n \geq n(B)$; $f_B(\omega) = \prod_{i \in B} [(1 + \omega_i)/2]$ is a nondecreasing function on Ω_{Λ_n} whenever $n \geq n(B)$. For $\beta > 0$ and h real, the following conclusions hold.*

(a) *For $n \geq n(B)$, $\langle f_B \rangle_{\Lambda_n,\beta,h,+}$ is a nonincreasing sequence as $n \to \infty$; $\langle f_B \rangle_{\Lambda_n,\beta,h,-}$ is a nondecreasing sequence as $n \to \infty$.*

(b) *The limits $\langle f_B \rangle_{\beta,h,+} = \lim_{\Lambda_n \uparrow \mathbb{Z}} \langle f_B \rangle_{\Lambda_n,\beta,h,+}$ and $\langle f_B \rangle_{\beta,h,-} = \lim_{\Lambda_n \uparrow \mathbb{Z}} \langle f_B \rangle_{\Lambda_n,\beta,h,-}$ both exist.*

Proof. (a) The external field h can be chosen to be site-dependent, say $\tilde{h}_i$. If $\tilde{h}_i$ tends to ∞, then h_i tends to ∞. Choose $\bar{n} > n \geq n(B)$, so that $\Lambda_n \subset \Lambda_{\bar{n}}$. The FKG inequality implies that $\langle f_B \rangle_{\Lambda_{\bar{n}},\beta,h,+} \leq \langle f_B \rangle_{\Lambda_n,\beta,h,+}$ because the latter can be obtained from the former by taking $h_i \to \infty$ for each $i \in \Lambda_{\bar{n}} \backslash \Lambda_n$. A similar proof shows that if $\bar{n} > n \geq n(B)$, then $\langle f_B \rangle_{\Lambda_{\bar{n}},\beta,h,-} \geq \langle f_B \rangle_{\Lambda_n,\beta,h,+}$.

(b) This follows from part (a). □

We now prove two lemmas from which Theorem IV.6.5 will follow. Let Λ be a symmetric interval of $\mathbb{Z}$. For $\beta > 0$ and h real, let $P_{\Lambda,\beta,h,\tilde{\omega}}$ be the finite-volume Gibbs state on Λ with external condition $\tilde{\omega}$ corresponding to a summable ferromagnetic interaction J. We write $\langle - \rangle_{\Lambda,h,\tilde{\omega}}$ for expectation with respect to $P_{\Lambda,\beta,h,\tilde{\omega}}$.

Lemma IV.6.9. *Let $f_B(\omega) = \prod_{i \in B} [\frac{1}{2}(1 + \omega_i)]$ for B a nonempty finite subset of $\mathbb{Z}$. For $\beta > 0$ and h real, the following conclusions hold.*

(a) *The limits $\langle f_B \rangle_{h,+} = \lim_{\Lambda \uparrow \mathbb{Z}} \langle f_B \rangle_{\Lambda,h,+}$ and $\langle f_B \rangle_{h,-} = \lim_{\Lambda \uparrow \mathbb{Z}} \langle f_B \rangle_{\Lambda,h,-}$ exist and for any $k \in \mathbb{Z}$ $\langle f_{B+k} \rangle_{h,+} = \langle f_B \rangle_{h,+}$ and $\langle f_{B+k} \rangle_{h,-} = \langle f_B \rangle_{h,-}$.*

(b) *For any real number h_0, $\lim_{h \to h_0^+} \langle f_B \rangle_{h,+} = \langle f_B \rangle_{h_0,+}$ and $\lim_{h \to h_0^-} \langle f_B \rangle_{h,-} = \langle f_B \rangle_{h_0,-}$.*

(c) *For any symmetric interval Λ containing B and any external condition $\tilde{\omega}$, $\langle f_B \rangle_{\Lambda,h,-} \leq \langle f_B \rangle_{\Lambda,h,\tilde{\omega}} \leq \langle f_B \rangle_{\Lambda,h,+}$.*

(d) $0 \leq \langle f_B \rangle_{h,+} - \langle f_B \rangle_{h,-} \leq |B| \, (\langle \omega_0 \rangle_{h,+} - \langle \omega_0 \rangle_{h,-})$.

Proof. (a) By Corollary IV.6.8(b), the limits

$$\langle f_B \rangle_{h,+} = \lim_{\Lambda \uparrow \mathbb{Z}} \langle f_B \rangle_{\Lambda,h,+} \qquad \text{and} \qquad \langle f_{B+k} \rangle_{h,+} = \lim_{\Lambda \uparrow \mathbb{Z}} \langle f_{B+k} \rangle_{\Lambda,h,+}$$

both exist. The interaction strength $J(i-j)$ between each pair of sites i, j is translation invariant. Hence if Λ contains B, then $\langle f_{B+k} \rangle_{\Lambda+k,h,+} = \langle f_B \rangle_{\Lambda,h,+}$ and

$$\lim_{\Lambda\uparrow\mathbb{Z}} \langle f_{B+k}\rangle_{\Lambda+k,h,+} = \lim_{\Lambda\uparrow\mathbb{Z}} \langle f_B\rangle_{\Lambda,h,+} = \langle f_B\rangle_{h,+}.$$

We can find a sequence $\{\bar{\Lambda}_n; n = 1, 2, \ldots\}$ of symmetric intervals such that $\bar{\Lambda}_1 \subseteq \bar{\Lambda}_2 + k \subseteq \bar{\Lambda}_3 \subseteq \bar{\Lambda}_4 + k \subseteq \ldots$ and $\bigcup_{n=1}^{\infty} \bar{\Lambda}_n = \mathbb{Z}$. By Corollary IV.6.8(b), $\lim_{\Lambda_n\uparrow\mathbb{Z}} \langle f_{B+k}\rangle_{\Lambda_n,h,+}$ exists, where $\Lambda_n = \bar{\Lambda}_n$ for n odd, $\Lambda_n = \bar{\Lambda}_n + k$ for n even. The existence of this limit implies that

$$\lim_{\Lambda\uparrow\mathbb{Z}} \langle f_{B+k}\rangle_{\Lambda,h,+} = \lim_{\Lambda\uparrow\mathbb{Z}} \langle f_{B+k}\rangle_{\Lambda+k,h,+}.$$

Combining this with the previous two displays, we conclude that $\langle f_{B+k}\rangle_{h,+}$ equals $\langle f_B\rangle_{h,+}$. That $\langle f_{B+k}\rangle_{h,-}$ equals $\langle f_B\rangle_{h,-}$ is proved similarly.

(b) Let Λ contain B. Since $\langle f_B\rangle_{h,+} \le \langle f_B\rangle_{\Lambda,h,+}$,

$$\limsup_{h\to h_0^+} \langle f_B\rangle_{h,+} \le \lim_{h\to h_0^+} \langle f_B\rangle_{\Lambda,h,+} = \langle f_B\rangle_{\Lambda,h_0,+}. \tag{4.53}$$

Hence $\limsup_{h\to h_0^+} \langle f_B\rangle_{h,+} \le \lim_{\Lambda\uparrow\mathbb{Z}} \langle f_B\rangle_{\Lambda,h_0,+} = \langle f_B\rangle_{h_0,+}$. By the FKG inequality, if $h > h_0$, then $\langle f_B\rangle_{h_0,+} \le \langle f_B\rangle_{h,+}$, and so $\langle f_B\rangle_{h_0,+} \le \liminf_{h\to h_0^+} \langle f_B\rangle_{h,+}$. It follows that $\lim_{h\to h_0^+} \langle f_B\rangle_{h,+} = \langle f_B\rangle_{h_0,+}$. The second half of part (b) is proved similarly.

(c) For any external condition $\tilde{\omega}$, the field $h_i = h + \sum_{j\in\Lambda^c} J(i-j)\tilde{\omega}_j$ acting at site $i\in\Lambda$ lies between the field corresponding to the plus external condition and the field corresponding to the minus external condition. Hence the FKG inequality yields part (c).

(d) The function $f(\omega) = \sum_{i\in B}\omega_i - f_B(\omega)$ is nondecreasing. Hence by the FKG inequality, if Λ contains B, then

$$0 \le \langle f_B\rangle_{\Lambda,h,+} - \langle f_B\rangle_{\Lambda,h,-} \le \sum_{i\in B} [\langle\omega_i\rangle_{\Lambda,h,+} - \langle\omega_i\rangle_{\Lambda\cdot h,-}]. \tag{4.54}$$

Take $\Lambda\uparrow\mathbb{Z}$ and use the fact that $\langle\omega_i\rangle_{h,+} = \langle\omega_0\rangle_{h,+}$ and $\langle\omega_i\rangle_{h,-} = \langle\omega_0\rangle_{h,-}$ to complete the proof of part (d). □

Spontaneous magnetization occurs at β if and only if

$$m(\beta,+) = -\frac{\partial\psi(\beta,0)}{\partial h^+} = -\lim_{h\to 0^+} \frac{\partial\psi(\beta,h)}{\partial h}$$

is positive. The next lemma relates $m(\beta,h)$, $m(\beta,+)$, and $m(\beta,-)$ to the quantities $\langle\omega_0\rangle_{\beta,h,+}$ and $\langle\omega_0\rangle_{\beta,h,-}$, defined as

$$\langle\omega_0\rangle_{\beta,h,+} = \lim_{\Lambda\uparrow\mathbb{Z}} \int_\Omega \omega_0 P_{\Lambda,\beta,h,+}(d\omega),$$

$$\langle\omega_0\rangle_{\beta,h,-} = \lim_{\Lambda\uparrow\mathbb{Z}} \int_\Omega \omega_0 P_{\Lambda,\beta,h,-}(d\omega).$$

According to Corollary IV.6.8(b), the limits exist. Later, we will identify $\langle\omega_0\rangle_{\beta,h,+}$ and $\langle\omega_0\rangle_{\beta,h,-}$ as the means of infinite-volume Gibbs states $P_{\beta,h,+}$ and $P_{\beta,h,-}$, respectively.

Lemma IV.6.10. *Let J be a summable ferromagnetic interaction on $\mathbb{Z}$. Then the following conclusions hold.*

(a) *For $\beta > 0$ and $h \neq 0$,*

$$\langle \omega_0 \rangle_{\beta,h,+} = m(\beta,h) = -\frac{\partial \psi(\beta,h)}{\partial h} = \langle \omega_0 \rangle_{\beta,h,-}.$$

(b) *For $\beta > 0$ and $h = 0$,*

$$(4.55) \quad \begin{aligned} \langle \omega_0 \rangle_{\beta,0,+} &= m(\beta,+) = -\frac{\partial \psi(\beta,0)}{\partial h^+} \geq 0, \\ \langle \omega_0 \rangle_{\beta,0,-} &= m(\beta,-) = -\frac{\partial \psi(\beta,0)}{\partial h^-} \leq 0. \end{aligned}$$

Thus spontaneous magnetization occurs at β if and only if $\langle \omega_0 \rangle_{\beta,0,+} = m(\beta,+) > 0$.

Proof. (a) For any $\varepsilon > 0$, let Λ_ε be a symmetric interval such that $\langle \omega_0 \rangle_{\Lambda_\varepsilon,\beta,h,+} \leq \langle \omega_0 \rangle_{\beta,h,+} + \varepsilon$. Given another symmetric interval Λ which contains Λ_ε, define $B_\varepsilon(\Lambda) = \{i \in \Lambda: \Lambda_\varepsilon + i \subseteq \Lambda\}$. The sequence $\{\langle \omega_i \rangle_{\Lambda,\beta,h,+}\}$ is nonincreasing as $\Lambda \uparrow \mathbb{Z}$ [Corollary IV.6.8(a)], and so for any $i \in B_\varepsilon(\Lambda)$

$$(4.56) \quad \begin{aligned} \langle \omega_0 \rangle_{\beta,h,+} &= \langle \omega_i \rangle_{\beta,h,+} \leq \langle \omega_i \rangle_{\Lambda,\beta,h,+} \leq \langle \omega_i \rangle_{\Lambda_\varepsilon+i,\beta,h,+} \\ &= \langle \omega_0 \rangle_{\Lambda_\varepsilon,\beta,h,+} \leq \langle \omega_0 \rangle_{\beta,h,+} + \varepsilon. \end{aligned}$$

Thus for all symmetric intervals Λ containing Λ_ε

$$(4.57) \quad \langle \omega_0 \rangle_{\beta,h,+} \leq \frac{1}{|B_\varepsilon(\Lambda)|} \sum_{i \in B_\varepsilon(\Lambda)} \langle \omega_i \rangle_{\Lambda,\beta,h,+} \leq \langle \omega_0 \rangle_{\beta,h,+} + \varepsilon.$$

Recall the quantity $M(\Lambda,\beta,h,+) = \sum_{i \in \Lambda} \langle \omega_i \rangle_{\Lambda,\beta,h,+}$, which is the magnetization in the state $P_{\Lambda,\beta,h,+}$. Since $|\Lambda| \cdot |B_\varepsilon(\Lambda)|^{-1} \to 1$ as $\Lambda \uparrow \mathbb{Z}$, it follows from (4.57) that $\lim_{\Lambda \uparrow \mathbb{Z}} |\Lambda|^{-1} M(\Lambda,\beta,h,+) = \langle \omega_0 \rangle_{\beta,h,+}$ for any real h. But for $h \neq 0$ this limit also equals $m(\beta,h)$ [Lemma IV.6.4]. Thus for $h \neq 0$ $\langle \omega_0 \rangle_{\beta,h,+} = m(\beta,h) = -\partial \psi(\beta,h)/\partial h$. A similar proof shows that for $h \neq 0$ $\langle \omega_0 \rangle_{\beta,h,-} = m(\beta,h) = -\partial \psi(\beta,h)/\partial h$.

(b) In part (a) take $h \to 0^+$ and use Lemma IV.6.9(b) to obtain $\langle \omega_0 \rangle_{\beta,0,+} = m(\beta,+) = -\partial \psi(\beta,0)/\partial h^+$. This gives half of (4.55). The other half is proved similarly. □

Proof of Theorem IV.6.5. (a) The existence and translation invariance of the measures $P_{\beta,h,+} = w\text{-}\lim_{\Lambda \uparrow \mathbb{Z}} P_{\Lambda,\beta,h,+}$ and $P_{\beta,h,-} = w\text{-}\lim_{\Lambda \uparrow \mathbb{Z}} P_{\Lambda,\beta,h,-}$ follow from Lemmas IV.6.1 and IV.6.9(a). Thus, $P_{\beta,h,+}$ and $P_{\beta,h,-}$ belong to $\mathcal{G}_{\beta,h} \cap \mathcal{M}_s(\Omega)$. Let $\mathcal{D}$ be a nonempty subset of $\mathcal{M}(\Omega)$. A measure $P \in \mathcal{D}$ is called an *extremal point* of $\mathcal{D}$ if $P = \lambda P_1 + (1-\lambda)P_2$ for $0 < \lambda < 1$ and $P_1, P_2 \in \mathcal{D}$ implies $P_1 = P_2 = P$. The set of extremal points of $\mathcal{G}_{\beta,h} \cap \mathcal{M}_s(\Omega)$ is the set of ergodic measures in $\mathcal{G}_{\beta,h}$ [Theorem A.11.8(b)]. Hence in order to show that $P_{\beta,h,+}$ is ergodic, it suffices to prove that $P_{\beta,h,+}$ is extremal in

$\mathscr{G}_{\beta,h} \cap \mathscr{M}_s(\Omega)$.* We prove the stronger statement that $P_{\beta,h,+}$ is extremal in $\mathscr{G}_{\beta,h}$. Assume that $P_{\beta,h,+} = \lambda P_1 + (1-\lambda)P_2$ for $0 < \lambda < 1$ and $P_1, P_2 \in \mathscr{G}_{\beta,h}$. Lemma IV.6.9(c) implies that for any measure $P \in \mathscr{G}_{\beta,h}$, $\int_\Omega f_B dP \leq \int_\Omega f_B dP_{\beta,h,+}$ for all finite sets B in $\mathbb{Z}$ ($f_\phi(\omega) = 1$). The integrals $\{\int_\Omega f_B dP;\ B$ finite$\}$ determine P. Hence if either measure P_i in the decomposition of $P_{\beta,h,+}$ differs from $P_{\beta,h,+}$, then $\int_\Omega f_B dP_i < \int_\Omega f_B dP_{\beta,h,+}$ for some finite set B, and so

$$(4.58) \qquad \int_\Omega f_B dP_{\beta,h,+} = \lambda \int_\Omega f_B dP_1 + (1-\lambda)\int_\Omega f_B dP_2 < \int_\Omega f_B dP_{\beta,h,+}.$$

This contradiction proves that $P_{\beta,h,+}$ is extremal in $\mathscr{G}_{\beta,h}$ and thus is ergodic. A similar proof shows that $P_{\beta,h,-}$ is extremal in $\mathscr{G}_{\beta,h}$ and thus is ergodic.

(b) If $\partial\psi(\beta,h)/\partial h$ exists, then by Lemma IV.6.10(b) $\langle\omega_0\rangle_{\beta,h,+} = \langle\omega_0\rangle_{\beta,h,-}$. Lemma IV.6.9(d) implies that for any finite subset B in $\mathbb{Z}$, $\langle f_B\rangle_{\beta,h,+} = \langle f_B\rangle_{\beta,h,-}$. Thus, $P_{\beta,h,+} = P_{\beta,h,-}$. Conversely, suppose that $P_{\beta,h,+} = P_{\beta,h,-}$. If $h \neq 0$, then $\partial\psi(\beta,h)/\partial h$ exists by Theorem IV.5.1(a). If $h = 0$, then

$$\langle\omega_0\rangle_{\beta,0,+} = \lim_{\Lambda\uparrow\mathbb{Z}} \int_\Omega \omega_0 P_{\Lambda,\beta,0,+}(d\omega) = \int_\Omega \omega_0 P_{\beta,0,+}(d\omega) = \int_\Omega \omega_0 P_{\beta,0,-}(d\omega)$$
$$= \lim_{\Lambda\uparrow\mathbb{Z}} \int_\Omega \omega_0 P_{\Lambda,\beta,0,-}(d\omega) = \langle\omega_0\rangle_{\beta,0,-},$$

and $\partial\psi(\beta,0)/\partial h$ exists by Lemma IV.6.10(b).

(c) Lemma IV.6.9(c) implies that if $P_{\beta,h,+} = P_{\beta,h,-} = P_{\beta,h}$, then $\mathscr{G}_{\beta,h}$ and thus $\mathscr{G}_{\beta,h} \cap \mathscr{M}_s(\Omega)$ consist of the unique measure $P_{\beta,h}$. We have defined $\langle\omega_0\rangle_{\beta,h,+} = \lim_{\Lambda\uparrow\mathbb{Z}} \int_\Omega \omega_0 P_{\Lambda,\beta,h,+}(d\omega)$. But if $\partial\psi(\beta,h)/\partial h$ exists, then $P_{\Lambda,\beta,h,+} \Rightarrow P_{\beta,h}$, and so $\langle\omega_0\rangle_{\beta,h,+} = \int_\Omega \omega_0 P_{\beta,h}(d\omega)$. The equality $\int_\Omega \omega_0 P_{\beta,h}(d\omega) = m(\beta,h)$ follows from Lemma IV.6.10.

(d) For $\beta > \beta_c$, $\partial\psi(\beta,h)/\partial h$ does not exist at $h = 0$, and thus $\langle\omega_0\rangle_{\beta,h,+} \neq \langle\omega_0\rangle_{\beta,h,-}$. Therefore $P_{\beta,0,+} \neq P_{\beta,0,-}$. Formula (4.46) follows from Lemma IV.6.10(b) and the fact that $P_{\Lambda,\beta,0,+} \Rightarrow P_{\beta,0,+}$ and $P_{\Lambda,\beta,0,-} \Rightarrow P_{\beta,0,-}$.

(e) Since $\mathscr{G}_{\beta,0} \cap \mathscr{M}_s(\Omega)$ is convex, it must contain all the measures $\{P_{\beta,0}^{(\lambda)}; 0 \leq \lambda \leq 1\}$. This completes the proof of Theorem IV.6.5. □

The key step in proving Theorem IV.6.6 is to calculate the free energy functions of the sequence $\{S_\Lambda\}$ with respect to the infinite-volume Gibbs state $P_{\beta,h,+}$ and $P_{\beta,h,-}$, respectively.

Lemma IV.6.11. *Let J be a summable ferromagnetic interaction on $\mathbb{Z}$. For $\beta > 0$, h real, and t real, define*

$$c_{\beta,h,+}(t) = \lim_{\Lambda\uparrow\mathbb{Z}} \frac{1}{|\Lambda|} \log\langle\exp(tS_\Lambda)\rangle_{\beta,h,+},$$

$$c_{\beta,h,-}(t) = \lim_{\Lambda\uparrow\mathbb{Z}} \frac{1}{|\Lambda|} \log\langle\exp(tS_\Lambda)\rangle_{\beta,h,-}.$$

*The ergodicity of $P_{\beta,h,+}$ and $P_{\beta,h,-}$ also follows from Corollary A.11.12(a).

Then

$$c_{\beta,h,+}(t) = c_{\beta,h,-}(t) = -\beta[\psi(\beta, h + t/\beta) - \psi(\beta, h)]. \tag{4.59}$$

Proof. We first evaluate $c_{\beta,h,+}(t)$ for $t \geq 0$. Define

$$c_\Lambda(t) = \frac{1}{|\Lambda|}\log\langle\exp(tS_\Lambda)\rangle_{\beta,h,+}.$$

For Λ' a symmetric interval containing Λ, define $g_{\Lambda,\Lambda'}(t) = \langle\exp(tS_\Lambda)\rangle_{\Lambda',\beta,h,+}$. Since $P_{\Lambda',\beta,h,+} \Rightarrow P_{\beta,h,+}$ as $\Lambda' \uparrow \mathbb{Z}$, $|\Lambda|^{-1}\log g_{\Lambda,\Lambda'}(t) \to c_\Lambda(t)$ as $\Lambda' \uparrow \mathbb{Z}$ for each $t \geq 0$. We claim that

$$\langle\exp(tS_\Lambda)\rangle_{\Lambda,\beta,h,\tilde{\omega}} \leq g_{\Lambda,\Lambda'}(t) \leq \langle\exp(tS_\Lambda)\rangle_{\Lambda,\beta,h,+} \qquad \text{for } t \geq 0, \tag{4.60}$$

where $\tilde{\omega}$ is the external condition $\tilde{\omega}_j = 1$ for $j \in (\Lambda')^c$, $\tilde{\omega}_j = -1$ for $j \in \Lambda'\backslash\Lambda$. Indeed for $t \geq 0$, the function $\exp(t\sum_{j\in\Lambda}\omega_j)$ is nondecreasing on Ω_Λ, and the right-hand (resp., left-hand) side of (4.60) can be obtained from the middle term by letting the external field $h_i \to \infty$ (resp., $h_i \to -\infty$) for each $i \in \Lambda'\backslash\Lambda$. Hence (4.60) follows from the FKG inequality. Another application of FKG gives $\langle\exp(tS_\Lambda)\rangle_{\Lambda,\beta,h,-} \leq \langle\exp(tS_\Lambda)\rangle_{\Lambda,\beta,h,\tilde{\omega}}$. As in the proof of Theorem IV.5.5,

$$\begin{aligned}
\frac{1}{|\Lambda|}\log\langle\exp(tS_\Lambda)\rangle_{\Lambda,\beta,h,-} &= -\beta\frac{1}{|\Lambda|}[\Psi(\Lambda,\beta,h+t/\beta,-) - \Psi(\Lambda,\beta,h,-)],\\
\frac{1}{|\Lambda|}\log\langle\exp(tS_\Lambda)\rangle_{\Lambda,\beta,h,+} &= -\beta\frac{1}{|\Lambda|}[\Psi(\Lambda,\beta,h+t/\beta,+) - \Psi(\Lambda,\beta,h,+)].
\end{aligned} \tag{4.61}$$

Thus for $t \geq 0$

$$\begin{aligned}
&-\beta\frac{1}{|\Lambda|}[\Psi(\Lambda,\beta,h+t/\beta,-) - \Psi(\Lambda,\beta,h,-)]\\
&\qquad \leq c_\Lambda(t) \leq -\beta\frac{1}{|\Lambda|}[\Psi(\Lambda,\beta,h+t/\beta,+) - \Psi(\Lambda,\beta,h,+)].
\end{aligned} \tag{4.62}$$

By Lemma IV.6.2 (independence of the specific Gibbs free energy of the choice of external conditions), we conclude that

$$c_{\beta,h,+}(t) = \lim_{\Lambda\uparrow\mathbb{Z}} c_\Lambda(t) = -\beta[\psi(\beta,h+t/\beta) - \psi(\beta,h)].$$

A similar proof yields (4.59) for $c_{\beta,h,-}(t)$, $t \geq 0$. For $t < 0$, the function $-\exp(t\sum_{j\in\Lambda}\omega_j)$ is nondecreasing on Ω_Λ, and so (4.60) holds with the senses of the inequalities reversed. As above, we obtain (4.59) for $c_{\beta,h,+}(t)$, $t < 0$. A similar proof yields (4.59) for $c_{\beta,h,-}(t)$, $t < 0$.

Proof of Theorem IV.6.6. (a) For $\beta > 0$, $h \neq 0$ and $0 < \beta < \beta_c$, $h = 0$, $\partial\psi(\beta,h)/\partial h$ exists. Hence by the previous lemma and Theorem II.6.3,

$S_\Lambda/|\Lambda| \xrightarrow{\exp} -\partial\psi(\beta,h)/\partial h = m(\beta,h)$. The almost sure convergence follows from the ergodic theorem or Theorem II.6.4.

(b) By the ergodic theorem, $S_\Lambda/|\Lambda| \to \langle\omega_0\rangle_{\beta,0,+} = m(\beta,+)$ $P_{\beta,0,+}$-a.s. and $S_\Lambda/|\Lambda| \to \langle\omega_0\rangle_{\beta,0,-} = m(\beta,-)$ $P_{\beta,0,-}$-a.s. In each case, the almost sure convergence cannot be strengthened to exponential convergence since by Theorem II.6.3 exponential convergence is equivalent to the existence of $(c_{\beta,0,+})'(0)$ or $(c_{\beta,0,-})'(0)$, respectively. But $\partial\psi(\beta,0)/\partial h$ does not exist for $\beta > \beta_c$, and so by the previous lemma $(c_{\beta,0,+})'(0)$ and $(c_{\beta,0,-})'(0)$ do not exist for $\beta > \beta_c$.

(c) This follows from the ergodic theorem and Theorem A.11.7(b). □

We have now completed the proof of the existence of infinite-volume Gibbs states and studied convergence properties of the spin per site $S_\Lambda/|\Lambda|$ with respect to these measures. In the next section, we show how to characterize the set of translation invariant infinite-volume Gibbs states in terms of a variational principle.

IV.7. The Gibbs Variational Formula and Principle

Let J be a summable ferromagnetic interaction on $\mathbb{Z}$. The set $\mathscr{G}_{\beta,h}$ of infinite-volume Gibbs states was defined as the closed convex hull of the set of weak limits of finite-volume Gibbs states. The Gibbs variational principle is another approach to studying the infinite-volume ferromagnet. It characterizes the set of translation invariant infinite-volume Gibbs states directly, eliminating the need to consider weak limits at all. The Gibbs variational principle expresses the specific Gibbs free energy $\psi(\beta,h)$ as the supremum of an energy functional minus an entropy functional over $\mathscr{M}_s(\Omega)$. The set of measures at which the supremum is attained is exactly the set $\mathscr{G}_{\beta,h} \cap \mathscr{M}_s(\Omega)$ [Theorem IV.7.3]. We recall that in the Gibbs variational principle for the discrete ideal gas [Theorem III.8.2], there was a unique solution for each value of β. By contrast, for $\beta > \beta_c$, $h = 0$, the Gibbs variational principle for ferromagnets has nonunique solutions since the set $\mathscr{G}_{\beta,0} \cap \mathscr{M}_s(\Omega)$ contains the distinct measures $P_{\beta,0,+}$, $P_{\beta,0,-}$, and all convex combinations.

Before stating the Gibbs variational principle, we consider an analog which characterizes the finite-volume Gibbs state $P_{\Lambda,\beta,h,\tilde{\omega}}$. Let Λ be a symmetric interval of $\mathbb{Z}$ and $\tilde{\omega}$ an external condition. Given a probability measure P on $\mathscr{B}(\Omega_\Lambda)$, we define the energy in P to be

$$U(\Lambda,h,\tilde{\omega};P) = \int_{\Omega_\Lambda} H_{\Lambda,h,\tilde{\omega}}(\omega)P(d\omega),$$

where $H_{\Lambda,h,\tilde{\omega}}$ is the Hamiltonian defined in (4.35). $I^{(2)}_{\pi_\Lambda P_\rho}(P)$ denotes the relative entropy of P with respect to $\pi_\Lambda P_\rho$,

$$\int_{\Omega_\Lambda} \log \frac{dP}{d(\pi_\Lambda P_\rho)}(\omega) P(d\omega) = \sum_{\omega \in \Omega_\Lambda} \log \frac{P\{\omega\}}{\pi_\Lambda P_\rho\{\omega\}} \cdot P\{\omega\}.$$

$Z(\Lambda, \beta, h, \tilde{\omega})$ is the partition function $\int_{\Omega_\Lambda} \exp[-\beta H_{\Lambda,h,\tilde{\omega}}(\omega)] \pi_\Lambda P_\rho(d\omega)$.

Proposition IV.7.1. *For any $\beta > 0$, h real, and external condition $\tilde{\omega}$,*

$$\log Z(\Lambda, \beta, h, \tilde{\omega}) = \sup_{P \in \mathscr{M}(\Omega_\Lambda)} \{-\beta U(\Lambda, h, \tilde{\omega}; P) - I^{(2)}_{\pi_\Lambda P_\rho}(P)\},$$

and the supremum is attained at the unique measure $P = P_{\Lambda,\beta,h,\tilde{\omega}}$.

Proof. For any probability measure P on Ω_Λ,

$$\beta U(\Lambda, h, \tilde{\omega}; P) + I^{(2)}_{\pi_\Lambda P_\rho}(P) + \log Z(\Lambda, \beta, h, \tilde{\omega})$$
$$= \sum_{\omega \in \Omega_\Lambda} \log\left(\frac{P\{\omega\}}{\exp[-\beta H_{\Lambda,h,\tilde{\omega}}(\omega)] \cdot \pi_\Lambda P_\rho\{\omega\}/Z(\Lambda, \beta, h, \tilde{\omega})}\right) \cdot P\{\omega\}.$$

The sum equals the relative entropy of P with respect to $P_{\Lambda,\beta,h,\tilde{\omega}}$. Hence

$$\beta U(\Lambda, h, \tilde{\omega}; P) + I^{(2)}_{\pi_\Lambda P_\rho}(P) + \log Z(\Lambda, \beta, h, \tilde{\omega}) \geq 0$$

and equality holds if and only if $P = P_{\Lambda,\beta,h,\tilde{\omega}}$ [Proposition I.4.1(b)]. □

We now introduce the functionals which appear in the Gibbs variational principle. Let P be a translation invariant probability measure on $\mathscr{B}(\Omega)$, Λ a symmetric interval, and π_Λ the projection of Ω onto Ω_Λ defined by $(\pi_\Lambda \omega)_i = \omega_i$, $i \in \Lambda$. Define a probability measure $\pi_\Lambda P$ on $\mathscr{B}(\Omega_\Lambda)$ by requiring $\pi_\Lambda P\{F\} = P\{\pi_\Lambda^{-1} F\}$ for subsets F of Ω_Λ and consider the functional

$$U(\Lambda, h, \tilde{\omega}(\Lambda); \pi_\Lambda P) = \int_{\Omega_\Lambda} H_{\Lambda,h,\tilde{\omega}(\Lambda)}(\omega) \pi_\Lambda P(d\omega),$$

where $\tilde{\omega}(\Lambda)$ is an external condition. $I^{(2)}_{\pi_\Lambda P_\rho}(\pi_\Lambda P)$ denotes the relative entropy of $\pi_\Lambda P$ with respect to $\pi_\Lambda P_\rho$.

Lemma IV.7.2. *Let J be a summable ferromagnetic interaction on $\mathbb{Z}$. Then for $\beta > 0$ and h real, the following conclusions hold.*

(a) $\lim_{\Lambda \uparrow \mathbb{Z}} |\Lambda|^{-1} \log Z(\Lambda, \beta, h, \tilde{\omega}(\Lambda))$ *exists and is independent of the choice of $\{\tilde{\omega}(\Lambda)\}$. The limit equals $-\beta\psi(\beta, h)$, where $\psi(\beta, h)$ is the specific Gibbs free energy.*

(b) *For any $P \in \mathscr{M}_s(\Omega)$, $u(h, P) = \lim_{\Lambda \uparrow \mathbb{Z}} |\Lambda|^{-1} U(\Lambda, h, \tilde{\omega}(\Lambda); \pi_\Lambda P)$ exists and is independent of the choice of $\{\tilde{\omega}(\Lambda)\}$. The limit is given by*

$$u(h; P) = -\frac{1}{2} \sum_{k \in \mathbb{Z}} J(k) \int_\Omega \omega_0 \omega_k P(d\omega) - h \int_\Omega \omega_0 P(d\omega) \tag{4.63}$$

and is a bounded, affine, continuous functional of $P \in \mathscr{M}_s(\Omega)$. The functional $u(h; P)$ is called the specific energy in P.

(c) *For any* $P \in \mathcal{M}_s(\Omega)$, $\lim_{\Lambda \uparrow \mathbb{Z}} |\Lambda|^{-1} I^{(2)}_{\pi_\Lambda P_\rho}(\pi_\Lambda P)$ *exists and equals* $I_\rho^{(3)}(P)$, *the mean relative entropy of* P *with respect to* P_ρ. $I_\rho^{(3)}(P)$ *is an affine, lower semicontinuous function of* $P \in \mathcal{M}_s(\Omega)$.

Proof. (a) Lemma IV.6.2.

(b) Since P is translation invariant, $U(\Lambda, h, \tilde{\omega}(\Lambda); \pi_\Lambda P)$ equals

$$-\frac{1}{2}\sum_{k \in \mathbb{Z}} J(k) N(\Lambda, k) \int_\Omega \omega_0 \omega_k P(d\omega) - \sum_{i \in \Lambda} \sum_{j \in \Lambda^c} J(i-j)\tilde{\omega}_j \int_\Omega \omega_0 P(d\omega) \tag{4.64}$$

$$- h|\Lambda| \int_\Omega \omega_0 P(d\omega),$$

where $N(\Lambda, k)$ is the number of ordered pairs i, j in Λ for which $i - j = k$. As in the proof of Lemma IV.6.2, the $\tilde{\omega}$-term is $o(|\Lambda|)$ as $\Lambda \uparrow \mathbb{Z}$ [see (4.42)]. Since for each k $N(\Lambda, k) \le |\Lambda|$ and $|\Lambda|^{-1} N(\Lambda, k) \to 1$ as $\Lambda \uparrow \mathbb{Z}$, (4.63) follows. The continuous functionals $u_n(h, P) = -\frac{1}{2}\sum_{|k| \le n} J(k) \int_\Omega \omega_0 \omega_k P(d\omega) - h \int_\Omega \omega_0 P(d\omega)$ converge uniformly over $\mathcal{M}_s(\Omega)$ to $u(h; P)$ as $n \to \infty$. Hence $u(h; P)$ is continuous. The boundedness and affineness of $u(h; P)$ are obvious.

(c) This is proved in Section IX.2. □

The next theorem is due to Ruelle (1967) and Lanford and Ruelle (1969). Part (a) is called the *Gibbs variational formula*. Part (b), which characterizes the translation invariant infinite-volume Gibbs states as solutions of this formula, is called the *Gibbs variational principle*. Heuristically, the theorem follows from Proposition IV.7.1 by dividing each term in the latter by $|\Lambda|$ and taking $\Lambda \uparrow \mathbb{Z}$.

Theorem IV.7.3. *Let* J *be a summable ferromagnetic interaction on* $\mathbb{Z}$. *Then for* $\beta > 0$ *and* h *real, the following conclusions hold.*

(a) $-\beta\psi(\beta, h) = \sup_{P \in \mathcal{M}_s(\Omega)} \{-\beta u(h; P) - I_\rho^{(3)}(P)\}$.

(b) *The set of* $P \in \mathcal{M}_s(\Omega)$ *at which the supremum in part* (a) *is attained equals* $\mathcal{G}_{\beta,h} \cap \mathcal{M}_s(\Omega)$, *the set of translation invariant infinite-volume Gibbs states.*

First, we will check the consistency of the theorem with Theorem IV.6.5, which analyzed the structure of the set $\mathcal{G}_{\beta,h} \cap \mathcal{M}_s(\Omega)$. Then, we will prove the Gibbs variational formula using level-3 large deviations. In Appendix C.5 we sketch a proof of the Gibbs variational formula and principle for a much larger class of models than we are now considering. The proof is due to Föllmer (1973) and Preston (1976). In Appendix C.6 we solve the Gibbs variational formula for finite-range interactions, using techniques to be developed in Chapter IX.

The Gibbs variational principle makes explicit an energy-entropy competition which underlies the ferromagnetic phase transition. First consider $\beta = 0$. Then in the Gibbs variational formula the energy term is absent, and $\sup_{P \in \mathcal{M}_s(\Omega)} \{-I_\rho^{(3)}(P)\} = -\inf_{P \in \mathcal{M}_s(\Omega)} I_\rho^{(3)}(P)$ is attained at the unique measure P_ρ. This is consistent with Theorem IV.6.5 since for small β

$\mathscr{G}_{\beta,h} \cap \mathscr{M}_s(\Omega)$ consists of the unique measure $P_{\beta,h}$ and $P_{\beta,h} \Rightarrow P_\rho$ as $\beta \to 0^+$ [Problem V.13.1(d)]. Now consider large β. Then in the Gibbs variational formula the energy term dominates. For $h > 0$

$$\sup_{P \in \mathscr{M}_s(\Omega)} \{-u(h;P)\} = \sup_{P \in \mathscr{M}_s(\Omega)} \left\{\frac{1}{2}\sum_{k\in\mathbb{Z}} J(k)\int_\Omega \omega_0\omega_k dP + h\int_\Omega \omega_0 dP\right\}$$

is attained at the unique measure P_{δ_1}, which is the infinite product measure with identical one-dimensional marginals δ_1. This is consistent with Theorem IV.6.5 since for all $h > 0$ $\mathscr{G}_{\beta,h} \cap \mathscr{M}_s(\Omega)$ consists of the unique measure $P_{\beta,h}$ and $P_{\beta,h} \Rightarrow P_{\delta_1}$ as $\beta \to \infty$ [Problem V.13.1(d)]. P_{δ_1} is supported on the totally aligned plus-ground state $\bar{\omega}_+$ ($\bar{\omega}_{+,j} = 1$ for all $j \in \mathbb{Z}$). On the other hand, if $h = 0$, then $\sup_{P\in\mathscr{M}_s(\Omega)}\{-u(0;P)\}$ is attained at all the measures $P^{(\lambda)} = \lambda P_{\delta_1} + (1-\lambda)P_{\delta_{-1}}$, $0 \le \lambda \le 1$.* This would be consistent with Theorem IV.6.5(d)–(e) if whenever β_c is finite, $P_{\beta,0,+}$ (resp., $P_{\beta,0,-}$) converged weakly to P_{δ_1} (resp., $P_{\delta_{-1}}$) as $\beta \to \infty$. A proof of this statement for models on $\mathbb{Z}^D$, $D \ge 2$, is sketched in Problem V.13.1(d).

We now derive the Gibbs variational formula using level-3 large deviations. The interval Λ consists of the $2N+1$ integers j with $|j| \le N$. In order to ease the notation, we consider $h = 0$ and write $Z(\Lambda, \beta, 0)$ as

$$Z(n,\beta) = \int_\Omega \exp\left[\frac{\beta}{2}\sum_{i,j=1}^{n} J(i-j)\omega_i\omega_j\right] P_\rho(d\omega),$$

where $n = 2N+1$. A similar proof is valid for $h \ne 0$. A relatively easy case is the Ising model on $\mathbb{Z}$, for which

$$Z(n,\beta) = \int_\Omega \exp\left[\beta\mathscr{J}\sum_{j=1}^{n-1}\omega_j\omega_{j+1}\right] P_\rho(d\omega), \qquad \mathscr{J} > 0.$$

This equals the quantity $Z_n^{(3)}$ in Theorem II.7.3(c) with $G(\omega_j, \omega_{j+1}) = \beta\mathscr{J}\omega_j\omega_{j+1}$. Hence by (2.42)

$$-\beta\psi(\beta,0) = \lim_{n\to\infty}\frac{1}{n}\log Z(n,\beta) = \sup_{P\in\mathscr{M}_s(\Omega)}\left\{\beta\mathscr{J}\int_\Omega \omega_1\omega_2 P(d\omega) - I_\rho^{(3)}(P)\right\}.$$

This gives the Gibbs variational formula since $\int_\Omega \omega_1\omega_2 dP = \int_\Omega \omega_0\omega_1\, dP$.[15] We now prove the Gibbs variational formula for any finite-range interaction J. We write

$$\begin{aligned}\frac{1}{2}\sum_{i,j=1}^{n} J(i-j)\omega_i\omega_j &= \frac{n}{2}J(0) + \sum_{1\le j<i\le n} J(i-j)\omega_i\omega_j \\ &= \frac{n}{2}J(0) + \sum_{k=1}^{n-1} J(k)\sum_{j=1}^{n-k}\omega_j\omega_{k+j}.\end{aligned}$$

*The set of maximizing measures for $\sup_{P\in\mathscr{M}_s(\Omega)}\{-u(0;P)\}$ may contain other measures besides $\{P^{(\lambda)}, 0 \le \lambda \le 1\}$ (depending on J).

Thus if J has range L and $n > L$ then

$$Z(n,\beta) = \int_\Omega \exp\left\{\beta\left[\frac{n}{2}J(0) + \sum_{k=1}^{L} J(k)\sum_{j=1}^{n-k}\omega_j\omega_{k+j}\right]\right\}P_\rho(d\omega).$$

Let $R_n(\omega,\cdot)$ denote the empirical process $n^{-1}\sum_{k=0}^{n-1}\delta_{T^kY(n,\omega)}(\cdot)$, where T is the shift mapping on Ω and $Y(n,\omega)$ is the periodic point in Ω obtained by repeating $(Y_1(\omega), Y_2(\omega), \ldots, Y_n(\omega))$ periodically ($Y_j(\omega) = \omega_j$). We have

$$\int_\Omega \bar{\omega}_1\bar{\omega}_{k+1}R_n(\omega,d\bar{\omega}) = \frac{1}{n}\sum_{j=1}^{n-k}\omega_j\omega_{k+j} + \frac{1}{n}\times k \text{ cyclic terms},$$

where the k cyclic terms are $\omega_j\omega_{k+j-n}, j = n-k+1, \ldots, n$. Thus uniformly in ω

$$\left| n\sum_{k=1}^{L} J(k)\int_\Omega \bar{\omega}_1\bar{\omega}_{k+1}R_n(\omega,d\bar{\omega}) - \sum_{k=1}^{L} J(k)\sum_{j=1}^{n-k}\omega_j\omega_{k+j}\right| = O(L).$$

By the comparison lemma, Lemma II.7.4, $Z(n,\beta)$ has the same leading order asymptotic behavior as

$$\bar{Z}(n,\beta) = \int_\Omega \exp\left\{n\beta\left[\frac{1}{2}J(0) + \sum_{k=1}^{L} J(k)\int_\Omega \bar{\omega}_1\bar{\omega}_{k+1}R_n(\omega,d\bar{\omega})\right]\right\} P_\rho(d\omega)$$
$$= \int_{\mathscr{M}_s(\Omega)} \exp[-n\beta u(0;P)]\,Q_n^{(3)}(dP),$$

where $Q_n^{(3)}$ is the distribution of $R_n(\omega,\cdot)$ on $\mathscr{M}_s(\Omega)$. By Theorem IX.1.1, $\{Q_n^{(3)}\}$ has a large deviation property with $a_n = n$ and entropy function $I_\rho^{(3)}$. Varadhan's theorem, Theorem II.7.1, yields the Gibbs variational formula:

$$-\beta\psi(\beta,0) = \lim_{n\to\infty}\frac{1}{n}\log\bar{Z}(n,\beta) = \sup_{P\in\mathscr{M}_s(\Omega)}\{-\beta u(0;P) - I_\rho^{(3)}(P)\}.$$

We finally prove the Gibbs variational formula for an infinite-range, summable interaction J. Take $L > 0$ and define $J_L(k) = J(k)$ for $|k| \le L$ and $J_L(k) = 0$ for $|k| > L$. We use the result just proved for the finite-range interaction J_L. By taking L sufficiently large, we can make the quantities $-\beta\psi(\beta,0)$ and $\sup_{P\in\mathscr{M}_s(\Omega)}\{-\beta u(0,P) - I_\rho^{(3)}(P)\}$ corresponding to J_L arbitrarily close to the respective quantities corresponding to J. This completes the proof of Theorem IV.7.3(a).

For the general ferromagnet, the translation invariant infinite-volume Gibbs states are also characterized by an entropy principle which is equivalent to the Gibbs variational principle. The entropy principle involves an energy constraint, expressed in terms of the specific energy $u(h;P)$ in the state $P\in\mathscr{M}_s(\Omega)$. For each $\beta > 0$ and h real, define the set

$$A_{\beta,h} = \{u\in\mathbb{R}: u = u(h;P) \text{ for some } P\in\mathscr{G}_{\beta,h}\cap\mathscr{M}_s(\Omega)\}.$$

For $\beta > 0$, $h \neq 0$ and $0 < \beta < \beta_c$, $h = 0$, $\mathscr{G}_{\beta,h}\cap\mathscr{M}_s(\Omega)$ consists of the unique measure $P_{\beta,h}$, and so $A_{\beta,h}$ is the single point $\{u(h;P_{\beta,h})\}$. The situation for $\beta > \beta_c$, $h = 0$ can be different since it is possible for $\mathscr{G}_{\beta,0} \cap \mathscr{M}_s(\Omega)$ to contain two measures P_1 and P_2 for which $u(0;P_1) \neq u(0,P_2)$. Since $u(h;P)$ is a continuous functional of P, $A_{\beta,0}$ is a compact interval of $\mathbb{R}$. The next theorem generalizes the entropy principle in Theorem IV.3.3, which characterized the finite-volume Gibbs state $P_{\Lambda,\beta,h}$.[16]

Theorem IV.7.4. *Let J be a summable ferromagnetic interaction on $\mathbb{Z}$. For $\beta > 0$, h real, and $u \in A_{\beta,h}$ let $\mathscr{I}_{\beta,h,u}$ denote the subset of $\{P \in \mathscr{M}_s(\Omega): u(h;P) = u\}$ at which $\inf\{I_\rho^{(3)}(P): P \in \mathscr{M}_s(\Omega), u(h;P) = u\}$ is attained. Then the following conclusions hold.*

(a) $\bigcup_{u \in A_{\beta,h}} \mathscr{I}_{\beta,h,u} = \mathscr{G}_{\beta,h} \cap \mathscr{M}_s(\Omega)$.

(b) *For $\beta > \beta_c$, $h = 0$, and $u = u(0;P_{\beta,0,+})$, the set $\mathscr{I}_{\beta,0,u}$ contains (at least) all the infinite-volume Gibbs states $P_{\beta,0}^{(\lambda)} = \lambda P_{\beta,0,+} + (1-\lambda)P_{\beta,0,-}$, $0 \le \lambda \le 1$. Thus $\inf\{I_\rho^{(3)}(P): P \in \mathscr{M}_s(\Omega), u(0;P) = u(0;P_{\beta,0,+})\}$ is not attained at a unique measure.*

Proof. (a) Given $\bar{P} \in \mathscr{G}_{\beta,h} \cap \mathscr{M}_s(\Omega)$, set $u = u(h;\bar{P})$. Then by Theorem IV.7.3

$$-\beta\psi(\beta,h) = -\beta u - I_\rho^{(3)}(\bar{P}) \le -\beta u - \inf\{I_\rho^{(3)}(P): P \in \mathscr{M}_s(\Omega), u(h;P) = u\}$$

$$\le \sup_{P \in \mathscr{M}_s(\Omega)} \{-\beta u(h;P) - I_\rho^{(3)}(P)\} = -\beta\psi(\beta,h). \tag{4.65}$$

This implies that $\bar{P} \in \mathscr{I}_{\beta,h,u}$. Now assume that $\bar{P} \in \mathscr{I}_{\beta,h,u}$, some $u \in A_{\beta,h,}$ does not belong to $\mathscr{G}_{\beta,h} \cap \mathscr{M}_s(\Omega)$. Then for any $P_0 \in \mathscr{G}_{\beta,h} \cap \mathscr{M}_s(\Omega)$ with $u(h;P_0) = u$

$$-\beta u - I_\rho^{(3)}(\bar{P}) = -\beta u(h;\bar{P}) - I_\rho^{(3)}(\bar{P})$$

$$< -\beta\psi(\beta,h) = -\beta u(h;P_0) - I_\rho^{(3)}(P_0) = -\beta u - I_\rho^{(3)}(P_0).$$

Thus $I_\rho^{(3)}(P_0) < I_\rho^{(3)}(\bar{P})$. This contradicts the hypothesis that $\bar{P}$ belongs to $\mathscr{I}_{\beta,h,u}$. Part (a) is proved.

(b) By symmetry $u(0;P_{\beta,0,+})$ equals $u(0;P_{\beta,0,-})$. Hence for all $0 \le \lambda \le 1$, $u(0;P_{\beta,0}^{(\lambda)})$ equals $u(0;P_{\beta,0,+})$. Since $P_{\beta,0}^{(\lambda)}$ belongs to $\mathscr{G}_{\beta,h} \cap \mathscr{M}_s(\Omega)$, the proof of part (a) shows that $P_{\beta,0}^{(\lambda)}$ belongs to $\mathscr{I}_{\beta,h,u(0;P_{\beta,0,+})}$. □

The nonuniqueness property of $I_\rho^{(3)}$ expressed in part (b) of Theorem IV.7.4 is a level-3 analog of a nonuniqueness property of the function $I_{\beta,0}(z)$, $\beta > \beta_c$, discussed at the end of Section IV.5. The set of means $\{\int_\Omega \omega_0 P_{\beta,0}^{(\lambda)}(d\omega); 0 \le \lambda \le 1\}$ of the measures $\{P_{\beta,0}^{(\lambda)}; 0 \le \lambda \le 1\}$ is exactly the interval $[m(\beta,-), m(\beta,+)]$ on which $I_{\beta,0}(z)$ attains its infimum of 0.

One of the main results in this chapter is that if β_c if finite, then for $\beta > \beta_c$ and $h = 0$ there exist nonunique, translation invariant infinite-volume Gibbs states $P_{\beta,0,+}$ and $P_{\beta,0,+}$. In the next chapter we will study ferromagnetic models on the lattices $\mathbb{Z}^D$, $D \in \{2, 3, \dots\}$. These models share many features with the models on $\mathbb{Z}$. An important contrast is the fact that β_c is finite for any nontrivial interaction.

IV.8. Notes

Many of these notes apply with little or no change to ferromagnetic models on $\mathbb{Z}^D$, $D \in \{1, 2, \dots\}$. These models will be treated in the next chapter. The notes which do apply are so indicated.

1 (page 88) $(\mathbb{Z}^D, D \geq 1)$. The similarities between the phase transitions for liquid–gas systems and ferromagnetic systems are discussed by Stanley (1971, Chapters 1 and 2). A model that is based on analogies between the two kinds of systems is the lattice–gas model [Lee and Yang (1952), Stanley (1971, Appendix A)].

2 (page 89) $(\mathbb{Z}^D, D \geq 1)$ (a) Introductions to ferromagnetic models and related lattice systems can be found in Griffiths (1971, 1972), Spitzer (1971), Georgii (1972), Thompson (1972), Kindermann and Snell (1980), and Gross (1982). Ruelle (1969, 1978), Lanford (1973), Preston (1974b, 1976), Israel (1979), and Simon (1985) are advanced references. Wightman (1979) is a beautiful overview of the thermodynamics of phase transitions (based on Gibbs's geometric approach through convexity) and the mathematics of lattice systems. Lebowitz (1975) is a useful review of properties of ferromagnetic models. Also see Gallavotti (1972a).

(b) A rough sketch of the Ising model first appeared in a 1920 paper of Lenz, but the model was named after his student, E. Ising. Ising (1925) concluded that there is no phase transition for $D = 1$ but erroneously tried to generalize his argument to $D = 2$. Brush (1967) discusses the history of the model.

3 (page 93) $(\mathbb{Z}^D, D \geq 1)$ The discussion of correlations in Section IV.2 is based upon unpublished lecture notes of A. Sokal and upon Wilson (1979).

4 (page 94) $(\mathbb{Z}^D, D \geq 1)$ The divergence of the specific magnetic susceptibility $\chi(\beta, 0) = \partial m(\beta, 0)/\partial h$ at $\beta = \beta_c$ is related, in a liquid-gas system, to the phenomenon of critical opalescence, which is the strong scattering of light by the system at the critical point [Stanley (1971, Chapter 1)]. The strong scattering is caused by abnormally large density fluctuations in the system.

5 (page 95) $(\mathbb{Z}^D, D \geq 1)$ (a) A useful generalization of the finite-volume Gibbs state $P_{\Lambda,\beta,h}$ in (4.4) is to allow many-body interactions [see Appendix C.3]. A second generalization is to replace the measure $\rho = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$ by a nondegenerate symmetric probability measure ρ on $\mathbb{R}$ for which $\int_{\mathbb{R}} e^{\sigma x^2} \rho(dx)$ is finite for all $\sigma > 0$. The measure ρ is called a *single-site distribution*. For example, see Lebowitz and Presutti (1976), Newman (1976a, 1976c), Ruelle (1976), Sylvester (1976b), Cassandro et al. (1978), and van Beijeren and Sylvester (1978). A third generalization is to allow vector-valued spins. For example, the d-vector spin model corresponds to spins taking values in the surface of the unit sphere in $\mathbb{R}^d$, $d \in \{2, 3, \dots\}$; the single-site distribution is $\rho(dx) = \delta_1(\|x\|)$, $x \in \mathbb{R}^d$. The Heisenberg model is the case $d = 3$.

(b) Ferromagnetic models have been applied in a number of different areas. A useful technique for studying quantum fields is first to study the fields on a lattice, then to let the lattice spacing shrink to zero. The lattice approximations are ferromagnetic models whose single-site distributions have the form const $\cdot \exp(-P(x))dx$, $P(x)$ an even polynomial [Simon (1974), Guerra, Rosen, and Simon (1975), Rosen (1977), Glimm and Jaffe (1981)]. Stochastic models closely related to spin systems are studied in percolation theory [Kesten (1982), Aizenman and Newman (1984), Durrett (1984a, 1984b)] and in the theory of interacting particle systems [Spitzer (1970), Holley and Stroock (1976a, 1976b, 1977), Liggett (1977, 1985), Griffeath (1978), Durrett (1981)].

6 (page 97) ($\mathbb{Z}^D, D \geq 1$) In order to avoid the nonphysical values $\beta \leq 0$ in Theorem IV.3.3, restrict U to the interval $(U_{\min}, U_0)$, where $U_0 = \lim_{\beta \to 0^+} U(\Lambda, h; P_{\Lambda,\beta,h}) = \int_{\Omega_\Lambda} H_{\Lambda,h}(\omega)\pi_\Lambda P_\rho(d\omega) = -\frac{1}{2}J(0)|\Lambda|$.

7 (page 98) ($\mathbb{Z}^D, D \geq 1$) The Curie–Weiss model is discussed in Kac (1968) and in Thompson (1972). It is also known as the Husimi–Temperley model [Husimi (1953), Temperley (1954)].

8 (page 106) ($\mathbb{Z}^D, D \geq 1$) The first Curie–Weiss bounds on β_c and $m(\beta, h)$ were found by Fisher (1967a), Griffiths (1967c), and Thompson (1971). For subsequent work on such bounds, see Cassandro et al. (1978), Simon (1980b), Pearce (1981), Sokal (1982a), Slawny (1983), and the references listed in these papers. Newman (1981b) proves the bound $m(\beta, h) \leq m^{\mathrm{CW}}(\beta, h)$ for $\beta > 0$ and $h \geq 0$ by using a connection between the Curie–Weiss model and Burger's equation.

9 (page 107) ($\mathbb{Z}^D, D \geq 1$) Whether $m(\beta_c, +)$ equals 0 or is positive depends on the model [Lebowitz and Martin-Löf (1972, page 282)]. The first holds for the Ising model on $\mathbb{Z}^2$ [see (5.18)] while the second is believed to hold for the model on $\mathbb{Z}$ with $J(k) = k^{-2}, k \neq 0$ [see Note 10c]. In general, the value of $m(\beta_c, +)$ is not known.

10 (page 107) ($\mathbb{Z}$ only) (a) Simon and Sokal (1981) have an entropy-energy proof of the fact that $\sum_{k \in \mathbb{Z}} |k| J(k) < \infty$ implies that β_c is infinite. Dobrushin (1968c), Ruelle (1968), and Bricmont, Lebowitz, and Pfister (1979) show by different methods that if $\sum_{k \in \mathbb{Z}} |k| J(k)$ is finite, then there exists a unique infinite-volume Gibbs state for all $\beta > 0$ and h real. This implies that β_c is infinite [Theorem IV.6.5]. The latter three papers prove analogous results for interactions J of arbitrary sign.

(b) We show that β_c is finite if $J > 0$ and $J(k) \sim |k|^{-\alpha}$, some $1 < \alpha < 2$ [Theorem IV.5.3(c)]. There exist $b > 0$ and $1 < \gamma < 2$ such that $J(k) \geq b|k|^{-\gamma}$ for all $k \neq 0$. By Theorem V.4.3(f), β_c corresponding to J is less than or equal to β_c corresponding to the interaction $J(k) = b|k|^{-\gamma}$, $k \neq 0$. For the latter interaction, Dyson (1969a) proves that

$$\liminf_{|i-j| \to \infty} \lim_{\Lambda \uparrow \mathbb{Z}} \int_{\Omega_\Lambda} \omega_i \omega_j P_{\Lambda,\beta,0}(d\omega) > 0$$

for all sufficiently large β (long-range order).

Proposition IV.8.1. *If long-range order holds, then β_c is finite.*

Proof. By the GKS-2 inequality [see Remark V.4.1(a)], $\int_{\Omega_\Lambda} \omega_i\omega_j P_{\Lambda,\beta,0}(d\omega) \le \int_{\Omega_\Lambda} \omega_i\omega_j P_{\Lambda,\beta,0,+}(d\omega)$, where $P_{\Lambda,\beta,0,+}$ is the finite-volume Gibbs state with the plus external condition. By Theorem IV.6.5(a), Lemma IV.6.10(b), and Corollary A.11.12(b),

$$\lim_{|i-j|\to\infty} \lim_{\Lambda\uparrow\mathbb{Z}} \int_{\Omega_\Lambda} \omega_i\omega_j P_{\Lambda,\beta,0,+}(d\omega) = [m(\beta, +)]^2.$$

Thus $0 < \liminf_{|i-j|\to\infty} \lim_{\Lambda\uparrow\mathbb{Z}} \int_{\Omega_\Lambda} \omega_i\omega_j P_{\Lambda,\beta,0}(d\omega) \le [m(\beta, +)]^2$ for all sufficiently large β. It follows that β_c is finite. □

(c) The phase transition for the interaction $J(k) = k^{-2}$ ($k \neq 0$) [Fröhlich and Spencer (1982a)] is believed to be an unusual kind. Namely, $m(\beta, +)$ is a discontinuous function of β at $\beta = \beta_c$: $m(\beta, +) \ge \text{const} > 0$ for $\beta \ge \beta_c$ and $m(\beta, +) = 0$ for $0 < \beta < \beta_c$. This was first discussed by Thouless (1969) and proved in a related hierarchical model by Dyson (1971). Also see Simon and Sokal (1981) and Sokal (1982b).

(d) There is a large literature concerning models on $\mathbb{Z}$. For example, see Dyson (1969b, 1972), Dobrushin (1973), Kolomytsev and Rokhlenko (1978, 1979), Cassandro and Olivieri (1981), Rogers and Thompson (1981), Simon (1981), and Imbrie (1982).

11 (page 115) (a) ($\mathbb{Z}^D, D \ge 1$). The proof that the measures $P_{\beta,h,+}$ and $P_{\beta,h,-}$ are ergodic follows Slawny (1974, page 300). The latter paper uses the GKS-2 inequality instead of the FKG inequality. The rest of Theorem IV.6.5 and Lemmas IV.6.9–IV.6.10 are due to Lebowitz and Martin-Löf (1972).

(b) ($\mathbb{Z}$ only). Fannes, Vanheuverzwijn, and Verbeure (1982) prove that if $J(k)$ is monotonically decreasing for k sufficiently large (e.g., $J(k) = |k|^{-\alpha}$, $k \neq 0$, for $1 < \alpha \le 2$), then every infinite-volume Gibbs state is translation invariant. The proof is based on energy-entropy estimates.

(c) ($\mathbb{Z}^D, D \ge 1$). The fact that $\mathscr{G}_{\beta,h}$ consists of a unique measure for all sufficiently small β [Theorem IV.6.5(c)] also follows from a general uniqueness theorem of Dobrushin (1968a). See Lanford (1973, Section C2) and Simon (1979b).

(d) ($\mathbb{Z}^D, D \ge 1$). Part (e) of Theorem IV.6.5 can be strengthened. The following theorem is due to Lebowitz (1977).

Theorem IV.8.2. *Let J be a summable ferromagnetic interaction which is irreducible on* $\mathbb{Z}$ [see page 96]. *Pick $\beta \ge \beta_c$ such that $\partial(\beta\psi(\beta, 0))/\partial\beta$ exists. Then $\mathscr{G}_{\beta,0} \cap \mathscr{M}_s(\Omega)$ consists precisely of all the measures $P_{\beta,0}^{(\lambda)} = \lambda P_{\beta,0,+} + (1-\lambda)P_{\beta,0,-}$, $0 \le \lambda \le 1$; $\partial(\beta\psi(\beta, 0))/\partial\beta$ exists for all but at most countably many values of $\beta \ge \beta_c$.*

The quantity $\partial(\beta\psi(\beta, 0))/\partial\beta$ is called the *specific energy* [Problem IV.9.11].

12 (page 116) ($\mathbb{Z}^D, D \ge 1$) Symmetry breaking is discussed further in Glimm and Jaffe (1981, Section 5.3).

13 (page 117) Here is an interesting open problem. According to Theorem II.6.1(b), for $\beta > \beta_c$ and any $\varepsilon > 0$, $P_{\beta,0,+}\{S_\Lambda/|\Lambda| \geq m(\beta, +) + \varepsilon\}$ converges to 0 exponentially fast as $\Lambda \uparrow \mathbb{Z}$. By Theorem IV.6.6(b), for $\beta > \beta_c$ and $0 < \varepsilon < 2\mathrm{m}(\beta, +)$, $P_{\beta,0,+}\{S_\Lambda/|\Lambda| \leq m(\beta, +) - \varepsilon\}$ converges to 0 but not exponentially fast. What is the decay rate of these probabilities?

14 (page 118) The FKG inequality originated in work on percolation models [Harris (1960)] and has been generalized and applied in many ways. See Battle and Rosen (1980), Newman (1980, 1984), Eaton (1982), Graham (1983), and the references listed in these papers.

15 (page 128) Let $\Gamma = \{1, -1\}$ and define $\mathscr{M}_s(\Gamma^2)$ to be the subset of $\mathscr{M}(\Gamma^2)$ consisting of probability measures τ with equal one-dimensional marginals. Clearly, if P belongs to $\mathscr{M}_s(\Omega)$, then $\tau = \pi_2 P$ belongs to $\mathscr{M}_s(\Gamma^2)$. For the Ising model on $\mathbb{Z}$, we have

$$\begin{aligned} -\beta\psi(\beta,0) &= \sup_{P\in\mathscr{M}_s(\Omega)}\left\{\beta\mathscr{J}\int_\Omega \omega_1\omega_2 P(d\omega) - I_\rho^{(3)}(P)\right\} \\ &= \sup_{\tau\in\mathscr{M}_s(\Gamma^2)}\left\{\beta\mathscr{J}\int_{\Gamma^2}\omega_1\omega_2\tau(d\omega) \right. \\ &\qquad \left. - \inf\{I_\rho^{(3)}(P): P\in\mathscr{M}_s(\Omega), \pi_2 P = \tau\}\right\}. \end{aligned}$$

By Theorem IX.3.3, $\inf\{I_\rho^{(3)}(P): P\in\mathscr{M}_s(\Omega), \pi_2 P = \tau\}$ equals the function $I_{\rho,2}^{(3)}(\tau)$ defined in (9.6). Hence,

$$-\beta\psi(\beta,0) = \sup_{\tau\in\mathscr{M}_s(\Gamma^2)}\left\{\beta\mathscr{J}\int_{\Gamma^2}\omega_1\omega_2\tau(d\omega) - I_{\rho,2}^{(3)}(\tau)\right\}. \tag{4.66}$$

The latter can be derived directly if one expresses the partition function in terms of the empirical pair measure and uses Theorem IX.4.3 for $\alpha = 2$. Equation (4.31) in Section IV.5 gives another formula for $-\beta\psi(\beta,0)$ in terms of the larger eigenvalue of a 2×2 positive matrix. Theorem IX.4.4 shows the equality of the expressions for $-\beta\psi(\beta,0)$ given in (4.31) and (4.66). Any finite-range interaction on $\mathbb{Z}$ can be handled like the Ising model on $\mathbb{Z}$ [Appendix C.6].

16 (page 130) ($\mathbb{Z}^D, D \geq 1$) There is a contraction principle related to the entropy principle in Theorem IV.7.4. The function

$$\bar{I}_\rho^{(1)}(h;u) = \inf\{I_\rho^{(3)}(P): P\in\mathscr{M}_s(\Omega), u(h;P) = u\}$$

is the entropy function of the $\pi_\Lambda P_\rho$-distributions of $\{H_{\Lambda,h}/|\Lambda|\}$. The function $-\bar{I}_\rho^{(1)}(h;u)$ is called the *specific microcanonical entropy*. See Lanford (1973, Chapter B), Aizenman and Lieb (1981), and Simon (1985).

IV.9. Problems

Many of these problems extend with little or no change to ferromagnetic models on $\mathbb{Z}^D$, $D\in\{1,2,\ldots\}$. These models will be treated in the next chapter. The problems which do extend are so indicated.

IV.9.1. ($\mathbb{Z}^D, D \geq 1$). Prove Proposition IV.3.2.

IV.9.2. ($\mathbb{Z}^D, D \geq 1$). Prove that if $h > 0$, then as $\beta \to \infty$ the finite-volume Gibbs states $\{P_{\Lambda,\beta,0}; \beta > 0\}$ converge weakly to the unit point measure $\delta_{\bar{\omega}_+}$; $\bar{\omega}_+$ is the configuration in Ω_Λ defined by $\bar{\omega}_{+,j} = 1$ for each $j \in \Lambda$.

The next four problems concern the Curie–Weiss model [Section IV.4].

IV.9.3. Verify formula (4.12) and the table accompanying Figure IV.3.

IV.9.4 [Kac (1968, page 247)]. This problem shows how to derive the limit (4.19) without using large deviations.

(a) By substituting in (4.11) the identity

$$\exp(\tfrac{1}{2}y^2) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(ty - \tfrac{1}{2}t^2)\,dt, \qquad y = \sqrt{n\beta\mathscr{J}_0}\,z, \tag{4.66}$$

and carrying out the $Q_n^{(1)}(dz)$ integration, prove that

$$\lim_{n\to\infty} \frac{1}{n} \log Z(n,\beta,h) = \sup_{t\in\mathbb{R}} \left\{ \log \cosh t - \frac{(t-\beta h)^2}{2\beta\mathscr{J}_0} \right\}.$$

(b) Using Problem VI.7.14, prove that

$$\sup_{t\in\mathbb{R}} \{\log\cosh t - \tfrac{1}{2}(\beta\mathscr{J}_0)^{-1}(t-\beta h)^2\} = \sup_{z\in\mathbb{R}} \{\tfrac{1}{2}\beta\mathscr{J}_0 z^2 + \beta h z - I_\rho^{(1)}(z)\}.$$

IV.9.5. Equation (4.19) shows that for the Curie–Weiss model, the specific Gibbs free energy $\psi(\beta,h)$ equals $-\beta^{-1}\sup_{z\in\mathbb{R}}\{\frac{1}{2}\beta\mathscr{J}_0 z^2 + \beta h z - I_\rho^{(1)}(z)\}$.

(a) For $\beta > 0$ and $h \neq 0$, prove that $\partial\psi(\beta,h)/\partial h$ exists and $-\partial\psi(\beta,h)/\partial h = z(\beta,h)$, where $z(\beta,h)$ is specified in the table accompanying Figure IV.3.

(b) For $\beta > 0$ and $h > 0$, prove that $(I_\rho^{(1)})''(z(\beta,h)) > \beta\mathscr{J}_0$.

(c) For $\beta > 0$ and $h > 0$, prove that $\partial z(\beta,h)/\partial h > 0$, $\partial z(\beta,h)/\partial\beta > 0$, and $\partial^2 z(\beta,h)/\partial h^2 < 0$.

(d) Verify the conclusions of Theorems IV.5.1 and IV.5.2 for $\psi(\beta,h)$ and for the Curie–Weiss specific magnetization.

IV.9.6. [Ellis (1981)]. Let $c_{\beta,h}(t) = \lim_{n\to\infty} n^{-1}\log\int_{\Omega_n}\exp[tS_n(\omega)]P_{n,\beta,h}(d\omega)$, where $P_{n,\beta,h}$ is defined in (4.9). The function $c_{\beta,h}$ is the free energy function of the sequence $\{S_n; n = 1,2,\ldots\}$ for the Curie–Weiss model.

(a) Prove that $c_{\beta,h}(t) = \sup_{z\in\mathbb{R}}\{tz - i_{\beta,h}(z)\} + \inf_{z\in\mathbb{R}} i_{\beta,h}(z)$, where $i_{\beta,h}(z) = I_\rho^{(1)}(z) - (\frac{1}{2}\beta\mathscr{J}_0 z^2 + \beta h z)$.

(b) $c'_{\beta,h}(0)$ exists for $\beta > 0$, $h \neq 0$ and $0 < \beta \leq \mathscr{J}_0^{-1}$, $h = 0$, but $c'_{\beta,h}(0)$ does not exist for $\beta > \mathscr{J}_0^{-1}$, $h = 0$. Prove this statement first by explicit calculation and then by applying Theorems II.6.3 and IV.4.1.

(c) Evaluate the Legendre–Fenchel transform of $c_{\beta,h}$. What is the relationship between the Legendre–Fenchel transform and the function $I_{\beta,h}$ in (4.17)? [*Hint:* Theorem VI.5.8.]

The remaining problems concern general ferromagnets on $\mathbb{Z}$.

IV.9.7 ($\mathbb{Z}^D, D \geq 1$) [Pearce (1981)]. Let J be a summable ferromagnetic interaction on $\mathbb{Z}$ and set $\mathscr{J}_0 = \sum_{k \in \mathbb{Z}} J(k)$. Let $m(\beta, h)$ be the specific magnetization corresponding to J and $m^{\text{CW}}(\beta, h)$ the specific magnetization for the Curie–Weiss model with interaction $\mathscr{J}_0/|\Lambda|$. For $h = 0$, $m^{\text{CW}}(\beta, h)$ equals 0 and for $h > 0$, $m^{\text{CW}}(\beta, h)$ equals the unique positive root $z(\beta, h)$ of (4.14). The present problem shows that for $\beta > 0$ and $h \geq 0$, $m^{\text{CW}}(\beta, h) \geq m(\beta, h)$ and thus that $\beta_c \geq \mathscr{J}_0^{-1}$ [Theorem IV.5.3(a)].

Let Λ be a symmetric interval. For each $i, j \in \Lambda$, define $J_\Lambda(i - j) = \sum_{k \in \mathbb{Z}} J(i - j - k|\Lambda|)$. Fix $\beta > 0$ and $h \geq 0$. Let $P_{\Lambda,\beta,h}$ be the finite-volume Gibbs state corresponding to J and $P_{\Lambda,\beta,h,p}$ the finite-volume Gibbs state corresponding to J_Λ (p stands for the periodic boundary condition). For $i \in \Lambda$, define

$$\langle \omega_i \rangle_\Lambda = \int_{\Omega_\Lambda} \omega_i P_{\Lambda,\beta,h}(d\omega) \text{ and } \langle \omega_i \rangle_{\Lambda,p} = \int_{\Omega_\Lambda} \omega_i P_{\Lambda,\beta,h,p}(d\omega).$$

Since $h \geq 0$ and $J_{ij}^{(\Lambda)} \geq J(i - j)$, Table V.1(c) implies $\langle \omega_i \rangle_{\Lambda,p} \geq \langle \omega_i \rangle_\Lambda$ [page 147].

(a) Show that the sum $\sum_{j \in \Lambda} J_\Lambda(i - j)$ is independent of i and Λ and equals $\mathscr{J}_0 = \sum_{k \in \mathbb{Z}} J(k)$.

(b) Denote by $\langle - \rangle_0$ expectation with respect to $\exp(\beta(\mathscr{J}_0 z + h)\sum_{j \in \Lambda} \omega_j) \cdot \pi_\Lambda P_\rho(d\omega) \cdot (1/Z_0)$, where Z_0 is a normalization. Prove that

$$z - \langle \omega_i \rangle_{\Lambda,p} = \left\langle (z - \omega_i) \exp\left\{\frac{\beta}{2} \sum_{i,j \in \Lambda} J_\Lambda(i - j)(z - \omega_i)(z - \omega_j)\right\}\right\rangle_0 \cdot \frac{1}{Z},$$

where $z = z(\beta, h)$ and $Z = \langle \exp\{\frac{\beta}{2} \sum_{i,j \in \Lambda} J_\Lambda(i - j)(z - \omega_i)(z - \omega_j)\}\rangle_0$.

(c) Prove that for each positive integer r and site $j \in \Lambda$

$$\int_{\{1,-1\}} (z - \omega_j)^r \exp[\beta(\mathscr{J}_0 z + h)\omega_j] \rho(d\omega_j)$$
$$= \tfrac{1}{2}(z + 1)^r \exp[\beta(\mathscr{J}_0 z + h)]\{a(\beta, h) + (-1)^r a(\beta, h)^r\}$$

is positive, where $z = z(\beta, h)$ and $a(\beta, h) = \exp[-2\beta(\mathscr{J}_0 z + h)]$. [*Hint:* Use (4.14).]

(d) By expanding the exponential in part (b) and using part (c), prove that $z(\beta, h) > \langle \omega_i \rangle_{\Lambda,p} \geq \langle \omega_i \rangle_\Lambda$. Deduce that

$$m^{\text{CW}}(\beta, h) \geq m(\beta, h) = \lim_{\Lambda \uparrow \mathbb{Z}} \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \langle \omega_i \rangle_\Lambda \qquad \text{and} \qquad \beta_c \geq \mathscr{J}_0^{-1}.$$

IV.9.8 ($\mathbb{Z}$ only) [Sakai (1976), Bricmont, Lebowitz, and Pfister (1979)]. Let J be a summable ferromagnetic interaction on $\mathbb{Z}$ for which $\sum_{k \in \mathbb{Z}} |k| J(k)$ is finite. This problem shows that for all $\beta > 0$ and $h = 0$ there exists a unique translation invariant infinite-volume Gibbs state $P_{\beta,0}$. Theorem IV.6.5 implies that β_c equals ∞, thus proving Theorem IV.5.3(b).

By Lemma IV.6.9(c), it suffices to prove that $P_{\beta,0,+}$ equals $P_{\beta,0,-}$ for all $\beta > 0$.

(a) Let Σ be any cylinder set in Ω. Prove that for all sufficiently large symmetric intervals Λ

$$P_{\Lambda,\beta,0,-}\{\Sigma\} \leq \exp[4\beta \sum_{k\in\mathbb{Z}} |k|J(k)] \cdot P_{\Lambda,\beta,0,+}\{\Sigma\}.$$

[*Hint:* First prove the bound for cylinder sets of the form $\{\omega\in\Omega: \omega_i = \bar{\omega}_i$ for each $i\in\bar{\Lambda}\}$, where $\bar{\Lambda}$ is a symmetric interval and $\bar{\omega}$ is a configuration in $\Omega_{\bar{\Lambda}}$.]

(b) By Theorem IV.6.5(a), $P_{\Lambda,\beta,0,+} \Rightarrow P_{\beta,0,+}$ and $P_{\Lambda,\beta,0,-} \Rightarrow P_{\beta,0,-}$ as $\Lambda \uparrow \mathbb{Z}$. Deduce a contradiction to part (a) if $P_{\beta,0,+} \neq P_{\beta,0,-}$ [*Hint:* Theorem A.11.7(b)].

IV.9.9. ($\mathbb{Z}^D, D \geq 1$) Prove (4.38). [*Hint:* Suppose that $f\in\mathscr{C}(\Omega)$ depends only on the coordinates $\omega_{i_1}, \ldots, \omega_{i_r}$. If Λ contains the sites $i_1, \ldots, i_r$, then f can be written in the form $\sum_{i=1}^{s} a_i\chi_{\Sigma_i}$, where s is a positive integer, $a_1, \ldots, a_s$ are real numbers, and $\Sigma_1, \ldots, \Sigma_s$ are cylinder sets such that $\pi_\Lambda(\Sigma_i \cap B_{\Lambda,\tilde{\omega}}) = \pi_\Lambda\Sigma_i$.]

IV.9.10. ($\mathbb{Z}$ only). Fill in the details in the proof of the Gibbs variational formula for an infinite-range, summable interaction on $\mathbb{Z}$ and for an arbitrary value of h.

IV.9.11. ($\mathbb{Z}^D, D \geq 1$). Let J be a summable ferromagnetic interaction on $\mathbb{Z}$. $H_{\Lambda,h}$ denotes the Hamiltonian defined in (4.3); $P_{\Lambda,\beta,h}$ the corresponding finite-volume Gibbs state defined in (4.4); $Z(\Lambda,\beta,h)$ the partition function $\int_{\Omega_\Lambda} \exp[-\beta H_{\Lambda,h}(\omega)]\pi_\Lambda P_\rho(d\omega)$; $\Psi(\Lambda,\beta,h)$ the Gibbs free energy $-\beta^{-1}\log Z(\Lambda,\beta,h)$, and $\psi(\beta,h)$ the specific Gibbs free energy $\lim_{\Lambda\uparrow\mathbb{Z}} |\Lambda|^{-1}\Psi(\Lambda,\beta,h)$.

(a) Prove that $\beta\Psi(\Lambda,\beta,h)$ is a concave function of $\beta > 0$ and that

$$\frac{\partial(\beta\Psi(\Lambda,\beta,h))}{\partial\beta} = \int_{\Omega_\Lambda} H_{\Lambda,h}(\omega)P_{\Lambda,\beta,h}(d\omega).$$

(b) Prove that if $\partial(\beta\psi(\beta,h))/\partial\beta$ exists, then

$$\lim_{\Lambda\uparrow\mathbb{Z}} \frac{1}{|\Lambda|}\int_{\Omega_\Lambda} H_{\Lambda,h}(\omega)P_{\Lambda,\beta,h}(d\omega) = \frac{\partial(\beta\psi(\beta,h))}{\partial\beta}.$$

The limit is called the *specific energy*.