Benson Proper Efficiency in Set-Valued Optimization on Real Linear Spaces*

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Summary. In this work, a notion of cone-subconvexlikeness of set-valued maps on linear spaces is given and several characterizations are obtained. An alternative theorem is also established for this kind of set-valued maps. Using the notion of vector closure introduced recently by Adán and Novo, we also provide, in this framework, an adaptation of the proper efficiency in the sense of Benson for set-valued maps. The previous results are then applied to obtain different optimality conditions for this Benson-vectorial proper efficiency by using scalarization and multiplier rules.

1 Introduction

In the last decades, there has been an increasing interest in vector optimization problems with set-valued objectives or constraints. See, for instance, [7, 8, 10, 12, 13, 14, 15, 16] and references therein. This kind of optimization problems with set-valued maps are closely related to stochastic programming, control theory and economic theory.

In this work, we introduce a new concept of proper efficiency in the sense of Benson for an optimization problem with set-valued maps on real linear spaces, and we characterize this concept of proper efficiency. We introduce this Benson vectorial proper efficiency by using concepts and results given by Adán and Novo [1, 2, 3, 4, 5]. We extend the notion of cone-subconvexlikeness of set-valued maps on linear spaces and give several characterizations. We establish separation theorems and an alternative theorem for solid cones. We also analyze the behaviour of a cone-subconvexlike set-valued map via a positive linear operator. We prove scalarization theorems and characterize the

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Benson-vectorial proper efficiency in optimization problems of set-valued maps with cone-subconvexlikeness. Lastly, using a new generalized Slater constraint qualification, we obtain a Lagrange multiplier rule of algebraic type for vector optimization problems with set-valued maps.

2 Notations and Preliminaries

Throughout this work, we will assume always, unless stated specifically otherwise, that Y is a real linear space partially ordered by a convex cone $K \subset Y$ and A is a nonempty subset of Y. Let $\mathrm{cone}(A)$, $\mathrm{co}(A)$, $\mathrm{aff}(A)$, $\mathrm{span}(A)$ and $L(A) = \mathrm{span}(A - A)$ denote the generated cone, convex hull, affine hull, linear hull and associated linear subspace of A, respectively. In this section, we recall some algebraic concepts and known results.

The core (algebraic interior) and the intrinsic core (relative algebraic interior) of A are defined, respectively, as follows:

$\mathrm{cor}(A) = \{y \in A : \ \forall v \in Y, \ \exists t > 0, \ \forall \alpha \in [0,t], \ y + \alpha v \in A\}$,
$\mathrm{icr}(A) = \{y \in A : \ \forall v \in L(A), \ \exists t > 0, \ \forall \alpha \in [0,t], \ y + \alpha v \in A\}$.

We say that A is solid (relatively solid) if $\mathrm{cor}(A) \neq \emptyset$ ($\mathrm{icr}(A) \neq \emptyset$). It is clear that if $\mathrm{cor}(A) \neq \emptyset$ then $\mathrm{cor}(A) = \mathrm{icr}(A)$ because $L(A) = Y$.

It is well-known that for finite dimensional spaces there exist sets which are not solid but they are relatively solid, for example, any segment, ray or line in $\mathbb{R}^2$. At the end of this section we show an example in infinite dimension (see Example 1).

The algebraic closure of a set A is defined by

$$\mathrm{lin}(A) = A \cup \{y \in Y : \ \exists a \in A, \ [a,y) \subset A\}.$$

Except for solid convex sets, this concept is not satisfactory as a substitute for topological closure. In order to solve this problem, Adán and Novo [4] have introduced a weaker closure of algebraic type, which was called vector closure. This vector closure coincides with the algebraic closure for convex sets, and coincides with the topological closure for solid convex sets.

Definition 1. *Let A be a nonempty subset of Y. The vector closure of A is the set* $\mathrm{vcl}(A) = \{y \in Y : \ \exists v \in Y, \ \forall t > 0, \ \exists \alpha \in (0,t], \ y + \alpha v \in A\}$.

It is clear that $y \in \mathrm{vcl}(A)$ if and only if there exist $v \in Y$ and a sequence $\lambda_n \to 0^+$ such that $y + \lambda_n v \in A$ for all n. The set A is called vectorially closed if $A = \mathrm{vcl}(A)$.

We say that a cone K is pointed if $K \cap (-K) = \{0\}$. It is well-known that for a convex cone K, whose relative algebraic interior is non-empty, the following conditions hold:

(i) $\mathrm{icr}(K) \cup \{0\}$ is a convex cone,
(ii) $\mathrm{icr}(K) + K = \mathrm{icr}(K)$,

(*iii*) $\mathrm{icr}(\mathrm{icr}(K)) = \mathrm{icr}(\mathrm{icr}(K) \cup \{0\}) = \mathrm{icr}(K)$.

Denote by Y' the algebraic dual of Y and by A^+ the positive dual cone of A, that is,

$$A^+ = \{\varphi \in Y' \colon \varphi(a) \geq 0,\ \forall a \in A\}$$

and $A^{+s} = \{\varphi \in Y' \colon \varphi(a) > 0,\ \forall a \in A \setminus \{0\}\}$ is the strict positive dual of A. A^+ is a vectorially closed convex cone and

$$[\mathrm{cone}(A)]^+ = \mathrm{cone}(A^+) = [\mathrm{conv}(A)]^+ = \mathrm{conv}(A^+) = A^+.$$

Other properties that will be used and appear in [4, 5] are the following:

(*i*) $A, B \subset Y$, $A \subset B \Rightarrow \mathrm{vcl}(A) \subset \mathrm{vcl}(B)$,
(*ii*) $[\mathrm{vcl}(\mathrm{cone}(\mathrm{conv}(A)))]^+ = A^+ = \mathrm{vcl}(A^+)$.

If Y is a topological vector space, the interior and closure of a set A are denoted by $\mathrm{int}(A)$ and $\mathrm{cl}(A)$, respectively. It is easy to check the following inclusions

$$A \subset \mathrm{lin}(A) \subset \mathrm{vcl}(A) \subset \mathrm{cl}(A).$$

To illustrate the notions above we give an example in infinite dimension.

Example 1. Let Y be the vector space of all sequences of real numbers, let S be the subspace of Y of all convergent sequences:

$$S = \{a = (a_n) \in Y :\ \exists \lim a_n = \alpha \in \mathbb{R}\},$$

and let K be the subset of S of all sequences with nonnegative limit:

$$K = \{(a_n) \in S :\ \lim a_n \geq 0\}.$$

It is clear that K is a nonpointed convex cone with $K \cap (-K) = c_0$, the linear space of the sequences converging to zero. Furthermore, the vector space generated by K is S, i.e., $L(K) = K - K = S$, and it is easy to check that K is vectorially closed. Its intrinsic core is

$$\mathrm{icr}(K) = \{a \in K :\ \lim a_n > 0\}.$$

Indeed, let $a \in K$ such that $\alpha = \lim a_n > 0$, and let us see that $a \in \mathrm{icr}(K)$, i.e., that $\forall v \in S = L(K)$, $\exists t_0 > 0$ such that $a + tv \in K$, $\forall t \in (0, t_0]$. As the sequence $v = (v_n) \in S$ there exists $\lim v_n = \beta$. Then $\lim(a_n + tv_n) = \alpha + t\beta \geq 0$ for all $t \in (0, t_0]$ if we choose

$$t_0 = \begin{cases} 1 & \text{if } \beta \geq 0 \\ -\alpha/\beta & \text{if } \beta < 0. \end{cases}$$

Now pick $a \in \mathrm{icr}(K)$. As $\mathrm{icr}(K) \subset K$ we have $\lim a_n \geq 0$. Suppose that $\lim a_n = 0$. Let $v = (v_n)$ defined by $v_n = 1$ for all $n \in \mathbb{N}$. Since $-v \in S$ and $a \in \mathrm{icr}(K)$ there exists $t_0 > 0$ such that $a + t(-v) \in K$, $\forall t \in (0, t_0]$. This implies that $-t_0 = \lim(a_n + t_0(-v_n)) \geq 0$, which is a contradiction.

The following cone separation theorem is due to Adán and Novo [5, Theorem 2.2].

Theorem 1. *Let M, K be two vectorially closed and relatively solid convex cones in Y. Let K^+ be solid. If $M \cap K = \{0\}$ then there exists a linear functional $\varphi \in Y' \backslash \{0\}$ such that $\forall k \in K$, $m \in M$, $\varphi(k) \geq 0 \geq \varphi(m)$ and furthermore $\forall k \in K \backslash \{0\}$, $\varphi(k) > 0$.*

Throughout this work, we assume that, unless indicated otherwise, X is a set, Y and Z are linear spaces, $K \subset Y$ and $D \subset Z$ are pointed relatively solid convex cones, and $F\colon X \longrightarrow 2^Y$ and $G\colon X \longrightarrow 2^Z$ are set-valued maps with domain X. The image of a subset A of X under F is denoted by $F(A) = \cup_{x \in A} F(x)$.

Consider the following unconstrained (P) and constrained (CP) vector optimization problems with set-valued maps:

$$(P) \quad \begin{cases} K - \text{Min } F(x) \\ \text{subject to } x \in X, \end{cases}$$

$$(CP) \quad \begin{cases} K - \text{Min } F(x) \\ \text{subject to } x \in X, \ G(x) \cap (-D) \neq \emptyset. \end{cases}$$

The feasible set of (CP) is defined by

$$\Omega = \{x \in X : \ G(x) \cap (-D) \neq \emptyset\}. \tag{1}$$

In [5], Adán and Novo have introduced the following concept of proper efficient point of a set $S \subset Y$ in the framework of vector optimization problems in partially ordered real linear spaces.

Definition 2. *The set of Benson-vectorial (BeV) proper efficient points of $S \subset Y$ is defined by*

$$BeV(S) = \{y \in S : \ \text{vcl}(\text{cone}(S - y + K)) \cap (-K) = \{0\}\}.$$

If we assume that Y is a topological linear space, and in this definition we replace the vector closure by the topological closure, we obtain the usual Benson (Be) proper efficiency defined in [6]. Because of $\text{vcl}(S) \subset \text{cl}(S)$, it is clear that $\text{Be}(S) \subset \text{BeV}(S)$.

For a vector optimization problem with set-valued maps, we introduce the following concept of proper efficient solution.

Definition 3. *A point $x \in X$ is called a Benson-vectorial (BeV) proper efficient solution of problem (P) if there exists*

$$y \in F(x) \cap BeV(F(X)).$$

The pair (x, y) is called a Benson-vectorial proper minimizer of (P).

3 Cone-Subconvexlike Set-Valued Maps

It is well-known that convexity plays an important role in optimization theory. In this section, we propose the following notion of cone-subconvexlikeness for set-valued maps on linear spaces. As we shall see presently, this concept is weaker than other concepts of cone-subconvexlikeness for set-valued maps.

Let X be a set, let $F\colon X \longrightarrow 2^Y$ be a set-valued map with $\mathrm{dom}(F) = X$ and let $K \subset Y$ be a relatively solid convex cone.

Definition 4. *F is said to be K-subconvexlike on X if* $\exists k_0 \in \mathrm{icr}(K)$ *such that* $\forall x, x' \in X$, $\forall \alpha \in (0,1)$, $\forall \varepsilon > 0$,

$$\varepsilon k_0 + \alpha F(x) + (1-\alpha)F(x') \subset F(X) + K.$$

Proposition 1. *The following statements are equivalent:*
(a) F is K-subconvexlike on X.
(b) $\forall k \in \mathrm{icr}(K)$, $\forall x, x' \in X$, $\forall \alpha \in (0,1)$,

$$k + \alpha F(x) + (1-\alpha)F(x') \subset F(X) + \mathrm{icr}(K).$$

(c) $\forall x, x' \in X$, $\forall \alpha \in (0,1)$, $\exists k \in K$ *such that* $\forall \varepsilon > 0$

$$\varepsilon k + \alpha F(x) + (1-\alpha)F(x') \subset F(X) + K. \tag{2}$$

(d) $F(X) + \mathrm{icr}(K)$ *is a convex set.*

Proof. The implications (b) $\Rightarrow$ (a) $\Rightarrow$ (c) are clear. Let us see (c) $\Rightarrow$ (b). Let $k \in \mathrm{icr}(K)$, $x, x' \in X$, $\alpha \in (0,1)$. Then, by assumption, $\exists k' \in K$ such that $\forall \varepsilon > 0$ condition (2) holds (with k' instead of k). As $k \in \mathrm{icr}(K) = \mathrm{icr}(\mathrm{icr}(K))$, for $-k' \in L(K) = L(\mathrm{icr}(K) = \mathrm{span}(K - K)$ there exists $\varepsilon_0 > 0$ such that $k_0 = k + \varepsilon_0(-k') \in \mathrm{icr}(K)$. So,

$$\begin{aligned} k + \alpha F(x) + (1-\alpha)F(x') &= [\varepsilon_0 k' + \alpha F(x) + (1-\alpha)F(x')] + k_0 \\ &\subset F(X) + K + k_0 \subset F(X) + \mathrm{icr}(K) \end{aligned}$$

(the last inclusion is true because $K + \mathrm{icr}(K) \subset \mathrm{icr}(K)$).
(b) $\Rightarrow$ *(d)* Let $u, u' \in F(X) + \mathrm{icr}(K)$, $\alpha \in (0,1)$. Then, $u = y + k$, $u' = y' + k'$ with $y \in F(x)$, $y' \in F(x')$, $k, k' \in \mathrm{icr}(K)$, $x, x' \in X$. Therefore

$$\alpha u + (1-\alpha)u' = \alpha k + (1-\alpha)k' + \alpha y + (1-\alpha)y'.$$

As $\mathrm{icr}(K)$ is a convex set, $k_0 = \alpha k + (1-\alpha)k' \in \mathrm{icr}(K)$. So,

$$\alpha u + (1-\alpha)u' \in k_0 + \alpha F(x) + (1-\alpha)F(x') \subset F(X) + \mathrm{icr}(K).$$

(d) $\Rightarrow$ *(b)* Let $k \in \mathrm{icr}(K)$, $x, x' \in X$, $\alpha \in (0,1)$, $y \in F(x)$, $y' \in F(x')$, then

$$k + \alpha y + (1-\alpha)y' = \alpha(y + k) + (1-\alpha)(y' + k) \in F(X) + \mathrm{icr}(K)$$

because $F(X) + \mathrm{icr}(K)$ is a convex set by assumption, and $y + k, y' + k \in F(X) + \mathrm{icr}(K)$. □

Remark 1. Of course, we may define that F is K-subconvexlike on X in the sense of Li ([12], given for a solid cone), if $\exists k_0 \in \operatorname{icr}(K), \forall x, x' \in X, \forall \alpha \in (0,1)$, $\forall \varepsilon > 0$, $\exists x'' \in X$ such that

$$\varepsilon k_0 + \alpha F(x) + (1-\alpha)F(x') \subset F(x'') + K.$$

However, this notion is more restrictive than Definition 4 (see Example 2). When $\operatorname{cor}(K) \neq \emptyset$, K-subconvexlikeness in the sense of Li becomes exactly Definition 2.2 in Li [12] (see also [14, Definition 1.2]).

Example 2. Consider $X = [-2, 0]$, $F : X \longrightarrow 2^{\mathbb{R}^2}$ defined by $F(x) = [(-2,1), (0, 2+x)]$ and $K = \mathbb{R}_+ \times \{0\}$. It follows that $F(X) + \operatorname{icr}(K)$ is a convex set so F is K-subconvexlike on X. However, F is not K-subconvexlike on X in the sense of Li. Indeed, given $k_0 \in \operatorname{icr}(K)$ if $x = 0$, $x' = -2$, $\alpha = \frac{1}{5}$ and $\varepsilon = 1$ then $\forall x'' \in X$

$$\varepsilon k_0 + \alpha F(x) + (1-\alpha)F(x') \not\subset F(x'') + K$$

as can easily be checked.

The previous proposition can be considered an extension of Lemmas 3.1 and 3.2 in Li [13] which are valid in a topological linear space Y provided with a convex cone K whose interior is nonempty.

In order to simplify the notations we introduce a new definition.

Definition 5. *A set-valued map $F : X \longrightarrow 2^Y$ is said to be relatively solid K-subconvexlike on X if the following conditions hold:*

(i) F is K-subconvexlike on X,

(ii) $\operatorname{icr}(F(X) + \operatorname{icr}(K)) \neq \emptyset$.

Remark 2. If Y is finite dimensional, condition (ii) is always true whenever F is K-subconvexlike because $F(X) + \operatorname{icr}(K)$ is a convex set.

Example 3. Let Y and K be the sets of Example 1. Let A be the convex cone

$$A = \{(a_n) \in Y : \; a_n \geq 0, \; \forall n \in \mathbb{N}\}$$

and $C = A + \operatorname{icr}(K) = \{(a_n) \in Y : \; \liminf a_n > 0\}$. We have that C is a convex set and $\operatorname{icr}(C) = \emptyset$. So $F : A \longrightarrow 2^Y$, given by $F(x) = x + K$, is K-subconvexlike on A but is not relatively solid K-subconvexlike on A. However, if we consider $F : K \longrightarrow 2^Y$ is relatively solid K-subconvexlike on K.

In Theorem 3 we establish an alternative theorem for K-subconvexlike set-valued maps with K solid. Previously, in Theorem 2 we establish a partial result of alternative type when K is only a relatively solid cone.

Theorem 2. *Let $K \subset Y$ be a relatively solid convex cone. Assume that $F : X \rightarrow 2^Y$ is relatively solid K-subconvexlike on X. If there is no $x \in X$ such that*

$$F(x) \cap (-\operatorname{icr}(K)) \neq \emptyset, \tag{3}$$

then $\exists \varphi \in K^+ \setminus \{0\}$ such that $\varphi(y) \geq 0 \quad \forall y \in F(X)$.

Proof. The set $F(X) + \text{icr}(K)$ is convex by Proposition 1. From (3) it follows that $0 \notin F(X) + \text{icr}(K)$. So, $0 \notin \text{icr}(F(X) + \text{icr}(K))$. Using the support theorem [9, Theorem 6.C], there exists $\varphi \in Y' \setminus \{0\}$ such that

$$\varphi(y + k) \geq 0 \quad \forall y \in F(X),\ \forall k \in \text{icr}(K) \tag{4}$$

(and φ is strictly positive on $\text{icr}(F(X) + \text{icr}(K))$). With standard reasonings, from (4) it follows that $\varphi(k) \geq 0\ \forall k \in K$, i.e., $\varphi \in K^+$. If $\exists y \in F(X)$ such that $\varphi(y) < 0$, choosing $k \in \text{icr}(K)$ with $\varphi(k)$ small enough we obtain that $\varphi(y + k) < 0$, which contradicts (4). Hence, $\varphi(y) \geq 0$ for all $y \in F(X)$. □

Theorem 3. *Let K be a solid convex cone. If F is K-subconvexlike on X, then exactly one of the following systems is consistent:*
(i) $\exists x \in X$ such that $F(x) \cap (-\text{cor}(K)) \neq \emptyset$.
(ii) $\exists \varphi \in K^+ \setminus \{0\}$ such that $\forall y \in F(X)$, $\varphi(y) \geq 0$.

Proof. By [4, Proposition 6.(iii)], $\text{cor}(F(X) + \text{cor}(K)) = F(X) + \text{cor}(K)$, and consequently, condition (ii) in Definition 5 is satisfied. Therefore, by Theorem 2, not (i) $\Rightarrow$ (ii). If we assume that both (i) and (ii) are satisfied, then there exist $x \in X$, $y \in F(x) \cap (-\text{cor}(K))$ and $\varphi \in K^+ \setminus \{0\}$ such that $\varphi(y) \geq 0$. But, since $y \in -\text{cor}(K)$ and $\varphi \in K^+ \setminus \{0\}$, we deduce that $\varphi(y) < 0$ and by Theorem 2.2 in [12] this is a contradiction. □

Remark 3. This theorem is slightly more general than Theorem 2.1 of Li [14] because the notion of K-subconvexlikeness of this author is more restrictive than our notion, even when $\text{cor}(K) \neq \emptyset$ (see Remark 1). If we consider that Y is a topological vector space then Theorem 3 collapses into Lemma 3.3 in [13]. Indeed, when Y is a topological vector space and $\text{int}(K) \neq \emptyset$, then $\text{int}(K) = \text{cor}(K)$ and the linear functional φ satisfying condition (ii) is continuous because we can apply Theorem 3.7 in [17] since the open set $\text{int}(K)$ is contained in the set $\{y \in Y : \varphi(y) > 0\}$ [12, Lemma 2.2] as $\varphi \in K^+ \setminus \{0\}$. Let us note that if $\text{cor}(K) = \emptyset$ and $\text{icr}(K) \neq \emptyset$, then both (i) (with $\text{icr}(K)$ instead of $\text{cor}(K)$) and (ii) can be true. For instance, in $\mathbb{R}^2$, $K = \mathbb{R}_+ \times \{0\}$, $X = \{(x, 0) : x \in (0, 1]\}$, $F(x, 0) = (x, 0) - K$ and $\varphi(x, y) = y$.

Lemma 1. *Let S_1 be a relatively solid convex set of Y and $S_2 \subset Y$. If $S_1 \subset S_2$ and $\text{vcl}(S_1) = \text{vcl}(S_2)$, then $\text{icr}(S_1) = \text{icr}(S_2)$.*

Proof. One has $\text{aff}(S_1) = \text{aff}(S_2)$ because by assumption $\text{vcl}(S_1) = \text{vcl}(S_2)$ and for any set $S \subset Y$, $\text{aff}(S) = \text{aff}(\text{vcl}(S))$. Hence, as $S_1 \subset S_2$ we deduce that $\text{icr}(S_1) \subset \text{icr}(S_2)$. On the other hand, $S_2 \subset \text{vcl}(S_2) = \text{vcl}(S_1)$ and as S_2 and $\text{vcl}(S_1)$ have the same affine hull, we get that $\text{icr}(S_2) \subset \text{icr}(\text{vcl}(S_1)) = \text{icr}(S_1)$. The last equality is true by Proposition 4(i) in [4]. Consequently, the conclusion follows. □

Proposition 2. *Let S be a relative solid convex subset of Y and $\varphi : Y \to Z$ a linear map. Then $\varphi(\text{icr}(S)) = \text{icr}(\varphi(S))$.*

Proof. Firstly let us see that

$$\varphi(\mathrm{icr}(S)) \subset \mathrm{icr}(\varphi(S)). \tag{5}$$

(as a consequence, $\varphi(S)$ is relatively solid). It is obvious that $\varphi(L(S)) = L(\varphi(S))$. Take $a \in \mathrm{icr}(S)$ and let us prove that $\varphi(a) \in \mathrm{icr}(\varphi(S))$. Given $w \in L(\varphi(S))$, there exists $v \in L(S)$ satisfying $\varphi(v) = w$. As $a \in \mathrm{icr}(S)$, for $v \in L(S)$ there exists $t_0 > 0$ such that $a + tv \in S$ $\forall t \in (0, t_0]$. From here,

$$\varphi(a) + tw = \varphi(a) + t\varphi(v) \in \varphi(S) \quad \forall t \in (0, t_0],$$

and therefore, $\varphi(a) \in \mathrm{icr}(\varphi(S))$. Now, the reverse inclusion: $\mathrm{icr}(\varphi(S)) \subset \varphi(\mathrm{icr}(S))$. For this aim, let us see that $\varphi(S)$ and $\varphi(\mathrm{icr}(S))$ have the same vector closure. We have that

$$\varphi(\mathrm{vcl}(S)) \subset \mathrm{vcl}(\varphi(S)). \tag{6}$$

Indeed, choose $b \in \mathrm{vcl}(S)$, then there exists $v \in Y$ such that $\forall \alpha' > 0$ $\exists \alpha \in (0, \alpha']$ such that $b + \alpha v \in S$. Hence, $\varphi(b) + \alpha\varphi(v) \in \varphi(S)$. This means that $\varphi(b) \in \mathrm{vcl}(\varphi(S))$. The following inclusions are clear taking into account (6):

$$\varphi(S) \subset \varphi(\mathrm{vcl}(S)) = \varphi(\mathrm{vcl}(\mathrm{icr}(S))) \subset \mathrm{vcl}(\varphi(\mathrm{icr}(S))) \subset \mathrm{vcl}(\varphi(S)).$$

From this chain, we select the following:

$$\varphi(S) \subset \mathrm{vcl}(\varphi(\mathrm{icr}(S))) \subset \mathrm{vcl}(\varphi(S)).$$

Taking vector closure and using that $\mathrm{vcl}(\mathrm{vcl}(B)) = \mathrm{vcl}(B)$, if B is a relative solid convex set, by [4, Proposition 3(iii)] (as $\varphi(\mathrm{icr}(S)) = \varphi(\mathrm{icr}(\mathrm{icr}(S))) \subset \mathrm{icr}(\varphi(\mathrm{icr}(S)))$, by condition (5) and as S is a relative solid, $\varphi(\mathrm{icr}(S))$ is a relative solid too) we have that:

$$\mathrm{vcl}(\varphi(S)) \subset \mathrm{vcl}(\varphi(\mathrm{icr}(S))) \subset \mathrm{vcl}(\varphi(S)).$$

Therefore, $\mathrm{vcl}(\varphi(S)) = \mathrm{vcl}(\varphi(\mathrm{icr}(S)))$, and by Lemma 1,

$$\mathrm{icr}(\varphi(S)) = \mathrm{icr}(\varphi(\mathrm{icr}(S))) \subset \varphi(\mathrm{icr}(S)).$$

Using (5), we have the conclusion. □

Next we analyze the postcomposition of a K-subconvexlike set-valued map with a positive linear map.

Let $\mathcal{L}(Y, Z)$ be the set of all linear maps φ from Y to Z, and let $\mathcal{L}_+(Y, Z)$ be the subset of positive linear maps, i.e.,

$$\mathcal{L}_+(Y, Z) = \{\varphi \in \mathcal{L}(Y, Z) : \ \varphi(K) \subset D\}.$$

Proposition 3. *Let $F : X \longrightarrow 2^Y$ be K-subconvexlike on X. If $\varphi \in \mathcal{L}_+(Y, Z)$, then $\varphi \circ F$ is D-subconvexlike on X.*

Proof. By Proposition 1(c), $\forall x, x' \in X$, $\forall \alpha \in (0,1)$, $\exists k \in K$ such that $\forall \varepsilon > 0$ we have

$$\varepsilon k + \alpha F(x) + (1-\alpha)F(x') \subset F(X) + K,$$

and therefore,

$$\varepsilon\varphi(k) + \alpha(\varphi \circ F)(x) + (1-\alpha)(\varphi \circ F)(x') \subset (\varphi \circ F)(X) + \varphi(K) \subset (\varphi \circ F)(X) + D.$$

As $\varphi(k) \in D$, statement (c) of Proposition 1 is satisfied for $\varphi \circ F$ and consequently, $\varphi \circ F$ is D-subconvexlike on X. □

Corollary 1. *Let $(F, G) : X \longrightarrow 2^{Y \times Z}$ be $K \times D$-subconvexlike on X.*
(i) If $\varphi \in K^+$ then $(\varphi \circ F, G)$ is $\mathbb{R}_+ \times D$-subconvexlike on X.
(ii) If $\psi \in \mathcal{L}_+(Z, Y)$ then $F + \psi \circ G$ is K-subconvexlike on X.

Proof. It is enough to apply Proposition 3 to (F, G) and the positive linear function $(y, z) \in Y \times Z \mapsto (\varphi(y), z)$ in part (i), and to the positive linear function $(y, z) \in Y \times Z \mapsto y + \psi(z)$ in part (ii). □

4 Benson-Vectorial Proper Efficiency

In this section we analyze different optimality conditions for Benson-vectorial proper efficiency, by using a pointed relatively solid convex cone and K-subconvexlike set-valued maps. Firstly, we establish a necessary condition and a sufficient condition through scalarization. Then, we obtain optimality conditions by using multiplier rules of algebraic type.

Now X is a set, Y is a linear space and $K \subset Y$ is a pointed relatively solid convex cone.

Let $\varphi \in \mathcal{L}(Y, \mathbb{R})$. We can associate to problem (P) the following scalar optimization problem with a set-valued map:

$$(SP_\varphi) \quad \begin{cases} \text{Min } (\varphi \circ F)(x) \\ \text{subject to } x \in X. \end{cases}$$

Definition 6. *If $x_0 \in X$, $y_0 \in F(x_0)$ and*

$$\varphi(y_0) \leq \varphi(y) \quad \forall y \in F(x),\ \forall x \in X,$$

then x_0 is called a minimal solution of problem (SP_φ), and (x_0, y_0) is called a minimizer of problem (SP_φ).

Theorem 4. *Let $\varphi \in K^{+s}$. If (x_0, y_0) is a minimizer of (SP_φ) then (x_0, y_0) is a Benson-vectorial proper minimizer of (P).*

Proof. Assume that (x_0, y_0) is not a Benson-vectorial proper minimizer. Then there exists

$$y \in \text{vcl}[\text{cone}(F(X) - y_0 + K)] \cap (-K) \quad \text{with} \quad y \neq 0.$$

Then $y \in -K$ and, since $\varphi \in K^{+s}$, we have that

$$\varphi(y) < 0. \tag{7}$$

On the other hand, as $y \in \text{vcl}[\text{cone}(F(X)-y_0+K)]$, due to the definition of vcl, there exist $v \in Y$ and a sequence $\lambda_n \to 0^+$ such that $y + \lambda_n v \in \text{cone}(F(X) - y_0 + K)$ for all n. So, there exist sequences $\{\alpha_n\} \subset \mathbb{R}^+$, $\{y_n\} \subset F(X)$ and $\{k_n\} \subset K$ such that $y + \lambda_n v = \alpha_n(y_n - y_0 + k_n)$. Since φ is linear, we deduce

$$\varphi(y) + \lambda_n\varphi(v) = \alpha_n(\varphi(y_n) - \varphi(y_0) + \varphi(k_n)). \tag{8}$$

By hypothesis (x_0, y_0) is a minimizer of (SP_φ) and $\varphi \in K^{+s}$ so we have that $\varphi(y) \geq \varphi(y_0)$ for all $y \in F(X)$ and $\varphi(k_n) \geq 0$ for all n. From this and (8) it follows that for all n

$$\varphi(y) + \lambda_n\varphi(v) \geq 0.$$

As $\lambda_n \to 0^+$, we get $\varphi(y) \geq 0$, which contradicts (7). Therefore (x_0, y_0) is a Benson-vectorial proper minimizer of (P). □

As a consequence of the previous result, if we consider a topological linear space Y and we replace the vector closure by the topological closure and the relative algebraic interior by the topological interior, the previous proof is valid too. Therefore, the result above is an extension of Theorem 4.1 in Li [13].

To establish sufficient conditions we need some convexity properties and the following lemma.

Lemma 2. *Let S be a relatively solid convex set of Y. Then*

$$\text{icr}(S) \subset \text{icr}(\text{cone}(S)). \tag{9}$$

Proof. Firstly, let us prove that

$$L(\text{cone}(S)) = \text{aff}(S \cup \{0\}) = \begin{cases} L(S) & \text{if } 0 \in \text{aff}(S) \\ L(S) + \mathbb{R}s_0 & \text{if } 0 \notin \text{aff}(S), \end{cases} \tag{10}$$

where s_0 is an arbitrary element of S and $\mathbb{R}s_0$ is the linear subspace generated by s_0. Indeed, the statement is obvious when $0 \in \text{aff}(S)$. Thus, assume that $0 \notin \text{aff}(S)$. The linear subspace $L(S) + \mathbb{R}s_0$ is the smallest affine variety which contains $S \cup \{0\}$ because:

1) $S \subset L(S) + s_0 \subset L(S) + \mathbb{R}s_0$ and $\{0\} \subset L(S) + \mathbb{R}s_0$.

2) If V is an affine variety containing $S \cup \{0\}$, then $\text{aff}(S) = L(S) + s_0 \subset V$ and V is a linear subspace of Y. So, $L(S) \subset V - s_0 = V$ and $\mathbb{R}s_0 \subset V$ since $s_0 \in S \subset V$. Therefore, $L(S) + \mathbb{R}s_0 \subset V$.

Secondly, let us see equation (9). Let $a \in \text{icr}(S)$, we have to prove that $\forall u \in L(\text{cone}(S))$,

$$\exists t_0 > 0 \text{ such that } a + tu \in \text{cone}(S) \quad \forall t \in (0, t_0]. \tag{11}$$

Taking into account equation (10), it is enough to prove (11) in the following cases: (i) $u \in L(S)$, (ii) $u = s_0$ and (iii) $u = -s_0$.

(i) Let $u \in L(S)$. As $a \in \text{icr}(S)$, then there is $t_0 > 0$ such that $a + tu \in S \subset \text{cone}(S)$ $\forall t \in (0, t_0]$, i.e., (11) is satisfied.

(ii) Now, $u = s_0$. Then, as $a, s_0 \in \text{cone}(S)$ we have $a + ts_0 \in \text{cone}(S)$ $\forall t \geq 0$ since $\text{cone}(S)$ is a convex cone.

(iii) Finally, $u = -s_0$. As $a \in \text{icr}(S) \subset S$ and $s_0 \in S$ (so $a - s_0 \in L(S)$), there exists $\gamma > 0$ such that

$$s_1 := s_0 + (1 + \gamma)(a - s_0) = a + \gamma(a - s_0) \in S.$$

The equation $a + t(-s_0) = \rho s_1$ in the unknown (t, ρ) has solution (t_0, ρ_0) where $t_0 = \gamma/(1+\gamma) > 0$ and $\rho_0 = 1/(1+\gamma) > 0$. Hence $a + t_0(-s_0) = \rho_0 s_1 \in \text{cone}(S)$, and therefore $[a, a + t_0(-s_0)] \subset \text{cone}(S)$ (i.e., (11) is true). □

Theorem 5. *Assume that K is vectorially closed and $\text{cor}(K^+) \neq \emptyset$. Let F be relatively solid K-subconvexlike on X. If (x_0, y_0) is a Benson-vectorial proper minimizer of (P) then there exists $\varphi \in K^{+s}$ such that (x_0, y_0) is a minimizer of (SP_φ).*

Proof. Since (x_0, y_0) is a Benson-vectorial proper minimizer then

$$-\text{vcl}[\text{cone}(F(X) - y_0 + K)] \cap K = \{0\}. \tag{12}$$

As $\text{vcl}[\text{cone}(F(X) - y_0 + \text{icr}(K))] \subset \text{vcl}[\text{cone}(F(X) - y_0 + K)]$, then

$$-\text{vcl}[\text{cone}(F(X) - y_0 + \text{icr}(K))] \cap K = \{0\}. \tag{13}$$

Let us see that $\text{vcl}[\text{cone}(F(X) - y_0 + \text{icr}(K))]$ is a vectorially closed relatively solid convex cone. It is clear that $\text{vcl}[\text{cone}(F(X) - y_0 + \text{icr}(K))]$ is a cone. Because of F is relatively solid K-subconvexlike on X, $\text{icr}[F(X) + \text{icr}(K)] \neq \emptyset$ and $F(X) + \text{icr}(K)$ is a convex set, then $\text{icr}[F(X) - y_0 + \text{icr}(K)] \neq \emptyset$ [5, Proposition 2.1(ii)] and $F(X) - y_0 + \text{icr}(K)$ is convex too. Therefore, $\text{cone}(F(X) - y_0 + \text{icr}(K))$ is convex and applying Lemma 2 we obtain that

$$\text{icr}[\text{cone}(F(X) - y_0 + \text{icr}(K))] \neq \emptyset.$$

Applying Proposition 3(iii)-(iv) in [4], we obtain that $\text{vcl}[\text{cone}(F(X) - y_0 + \text{icr}(K))]$ is vectorially closed and convex. On the other hand, by Proposition 4(i) in [4], $\text{vcl}[\text{cone}(F(X) - y_0 + \text{icr}(K))]$ is a relatively solid set. Under these conditions we can apply the separation Theorem 1, so taking into account condition (13), there exists $\varphi \in K^{+s} \setminus \{0\}$ such that

$$\varphi(v) \geq 0 \quad \text{for all } v \in \text{vcl}[\text{cone}(F(X) - y_0 + \text{icr}(K))].$$

Since $F(X) - y_0 + \text{icr}(K) \subset \text{vcl}[\text{cone}(F(X) - y_0 + \text{icr}(K))]$ we have

$$\varphi(y) - \varphi(y_0) + \varphi(k) \geq 0 \quad \text{for all } y \in F(X) \text{ and } k \in \text{icr}(K).$$

Due to $\varphi \in K^{+s}$ and $\lambda k \in \text{icr}(K)$ for all $\lambda > 0$, it follows that

$$\varphi(y) - \varphi(y_0) \geq 0 \quad \text{for all } y \in F(X).$$

Therefore (x_0, y_0) is a minimizer of (SP_φ). □

From the theorems above we obtain the following corollary, which gives us a characterization of Benson-vectorial proper minimizers under K-subconvexlikeness.

Corollary 2. *Let K^+ be solid and K be vectorially closed. Let F be relatively solid K-subconvexlike on X. Then (x_0, y_0) is a Benson-vectorial proper minimizer of (P) if and only if (x_0, y_0) is a minimizer of (SP_φ) for some $\varphi \in K^{+s}$.*

Therefore if we consider a topological linear space Y and $\text{int}(K) \neq \emptyset$ then Theorem 5 and Corollary 2 can be considered extensions of Theorem 4.2 and Corollary 4.1 in Li [13].

Finally, we give a generalized Slater constraint qualification in order to obtain a Lagrange multiplier rule of algebraic type for constrained vector optimization problems with set-valued maps.

Definition 7. *We say that the optimization problem (CP) satisfies the generalized Slater constraint qualification if there exists $x \in X$ such that $G(x) \cap -\text{icr}(D) \neq \emptyset$.*

Theorem 6. *Let $cor(K^+) \neq \emptyset$. Suppose that (F, G) is relatively solid $K \times D$-subconvexlike on X, F is relatively solid K-subconvexlike on Ω and $\text{aff}(\text{icr}(D)) = \text{aff}(\text{icr}[G(X) + \text{icr}(D)])$. If (CP) satisfies the generalized Slater constraint qualification and (x_0, y_0) is a Benson-vectorial proper minimizer of (CP) then there exists $T \in \mathcal{L}_+(Z, Y)$ such that $0 \in T(G(x_0))$ and (x_0, y_0) is a Benson-vectorial proper minimizer of the unconstrained problem*

$$\begin{aligned} &K - Min\ (F + T \circ G)(x) \\ &subject\ to\ x \in X. \end{aligned}$$

Proof. Since F is relatively solid K-subconvexlike on Ω, we can apply Theorem 5 to problem (CP), then there exists a linear functional $\varphi \in K^{+s}$ such that (x_0, y_0) is a minimizer of the scalar problem

$$\text{Min}\{\varphi[F(x)] : \ x \in \Omega\},$$

i.e.

$$\varphi(y) \geq \varphi(y_0) \quad \text{for all } y \in F(\Omega). \tag{14}$$

Let $H : X \longrightarrow 2^{\mathbb{R} \times Z}$ be the set-valued map defined by

$$H(x) = [\varphi(F(x)) - \varphi(y_0)] \times G(x) = \varphi(F(x)) \times G(x) - (\varphi(y_0), 0).$$

As a consequence of (14) we have

$$H(X) \cap -\text{icr}(\mathbb{R}_+ \times D) = \emptyset. \tag{15}$$

Since (F, G) is $K \times D$-subconvexlike on X then, by Corollary 1(i), we have that $H = (\varphi \circ (F - y_0), G) = (\varphi \circ F - \varphi(y_0), G)$ is $\mathbb{R}_+ \times D$-subconvexlike on X. From here and by $\text{icr}[(F, G)(X) + \text{icr}(K \times D)] \neq \emptyset$, applying Proposition 2 we obtain that

$$\text{icr}[(\varphi \circ (F - y_0), G)(X) + \text{icr}(\mathbb{R}_+ \times D)] \neq \emptyset.$$

Thus, H is relatively solid $\mathbb{R}_+ \times D$-subconvexlike on X. Together with (15), by Theorem 2 applied to H, we obtain that there exists $(r, \psi) \in \mathbb{R}_+ \times D^+ \backslash \{(0, 0)\}$ such that

$$r[\varphi(F(x) - y_0)] + \psi[G(x)] \geq 0 \quad \text{for all } x \in X \tag{16}$$

and (see the proof of Theorem 2)

$$(r, \psi)(y', z') > 0 \text{ for all } (y', z') \in \text{icr}[(\varphi \circ (F - y_0), G)(X) + \text{icr}(\mathbb{R}_+ \times D)]. \tag{17}$$

We note that $r > 0$. Otherwise, if $r = 0$ then from condition (17) it results

$$\psi(\text{icr}[G(X) + \text{icr}(D)]) > 0. \tag{18}$$

As a consequence of the generalized Slater constraint qualification, $0 \in G(X) + \text{icr}(D)$ so $\text{icr}(D) \subset G(X) + \text{icr}(D)$. On the other hand, by hypothesis, $\text{aff}(\text{icr}(D)) = \text{aff}(\text{icr}[G(X) + \text{icr}(D)])$, therefore

$$\text{icr}(D) = \text{icr}(\text{icr}(D)) \subset \text{icr}[G(X) + \text{icr}(D)]$$

and by (18) we obtain that

$$\psi(\text{icr}(D)) > 0. \tag{19}$$

Again, because of the generalized Slater constraint qualification, there exists some $x' \in X$ and $z' \in G(x') \cap -\text{icr}(D) \neq \emptyset$ and, consequently, by (19), $\psi(z') < 0$ and by (16), $\psi(z') \geq 0$, which is a contradiction. Thus, $r > 0$. Since $x_0 \in \Omega$ and $\psi \in D^+$ then there exists $z' \in G(x_0) \cap -D$ such that $\psi(z') \leq 0$. Taking $x = x_0$ and $y_0 \in F(x_0)$ in (16) we have that $\psi(z') \geq 0$, so $\psi(z') = 0$. Hence,

$$0 \in \psi[G(x_0)]. \tag{20}$$

As $r \neq 0$ and $\varphi \in K^{+s}$, we can choose $k \in K$ such that $r\varphi(k) = 1$. We define the operator $T \colon Z \longrightarrow Y$ by

$$T(z) = \psi(z)k. \tag{21}$$

It is clear that $T(D) \subset K$, i.e., $T \in \mathcal{L}_+(Z,Y)$. By (20), $0 \in T(G(x_0))$ and consequently

$$y_0 \in F(x_0) \subset F(x_0) + T(G(x_0)).$$

Now, from (16) and (21) we have that for all $x \in X$

$$r\varphi[F(x)+T(G(x))] = r\varphi[F(x)]+\psi[G(x)]r\varphi(k) = r\varphi[F(x)]+\psi[G(x)] \geq r\varphi(y_0)$$

If we divide this inequality by $r > 0$ we obtain that (x_0, y_0) is a minimizer of the scalar problem

$$K - \text{Min}\{[\varphi \circ (F + T \circ G)](x) : \ x \in X\}.$$

According to Theorem 4, (x_0, y_0) is a Benson-vectorial proper minimizer of the unconstrained optimization problem

$$K - \text{Min}\{(F + T \circ G)(x) : \ x \in X\}. \qquad \square$$

Remark 4. It is easy to check that the condition $\text{aff}(\text{icr}(D)) = \text{aff}(\text{icr}(G(X) + \text{icr}(D))$ is weaker than $\text{cor}(D) \neq \emptyset$. Indeed if $\text{cor}(D) \neq \emptyset$ then

$$\text{aff}(\text{cor}(D)) = \text{aff}[\text{cor}(G(X) + \text{cor}(D))] = Z.$$

Theorem 7. *Consider problem (CP). Assume $\text{cor}(K^+) \neq \emptyset$. Let (F, G) be a $K \times D$-subconvexlike set-valued map on X. If there exists a positive linear operator $T \in \mathcal{L}_+(Z,Y)$ and a pair (x_0, y_0) with $x_0 \in \Omega$ and $y_0 \in F(x_0)$ such that:*

(i) (x_0, y_0) is a Benson-vectorial proper minimizer of the problem

$$K - Min \ (F + T \circ G)(x) \quad \textit{subject to } x \in X,$$

(ii) $0 \in T(G(x_0))$ and

(iii) $\text{icr}[(F + T \circ G)(X) + \text{icr}(K)] \neq \emptyset$.

Then (x_0, y_0) is a Benson-vectorial proper minimizer of problem (CP).

Proof. Since (F, G) is $K \times D$-subconvexlike on X by Corollary 1(ii) $F + T \circ G$ is K-subconvexlike on X. Moreover, by assumption (iii), $F + T \circ G$ is relatively solid K-subconvexlike on X. So, applying Theorem 5 there exists $\varphi \in K^{+s}$ such that for all $x \in X$

$$\varphi(F(x) + T(G(x))) \geq \varphi(y_0)$$

Hence,

$$\varphi(F(x)) + \varphi(T(G(x))) \geq \varphi(y_0) \text{ for all } x \in X \tag{22}$$

Therefore, if $x \in \Omega$, there exists $z \in G(x)$ such that $z \in -D$. On the other hand, as $T \in \mathcal{L}_+(Z, Y)$, $T(z) \in -K$ and $\varphi \in K^{+s}$, we obtain $\varphi(T(z)) \leq 0$. From this, according to (22) and taking $z \in G(x)$, for each $y \in F(x)$ we obtain

$$\varphi(y) \geq \varphi(y) + \varphi(T(z)) \geq \varphi(y_0).$$

Hence, for all $y \in F(\Omega)$, one has $\varphi(y) \geq \varphi(y_0)$. As $y_0 \in F(x_0) \subset F(\Omega)$, applying Theorem 4, (x_0, y_0) is a Benson-vectorial proper minimizer of the problem (CP). $\square$

Once again our results extend Theorems 5.1 and 5.2 in Li [13] which are given in the framework of topological linear spaces with solid cones.

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