Second-Order Conditions in $C^{1,1}$ Vector Optimization with Inequality and Equality Constraints

Ivan Ginchev[1], Angelo Guerraggio[2], and Matteo Rocca[3]

[1] Technical University of Varna, Department of Mathematics, Studentska Str. 1 9010 Varna, Bulgaria. ginchev@ms3.tu-varna.acad.bg

[2] University of Insubria, Department of Economics, Via Ravasi 2 21100 Varese, Italy. aguerraggio@eco.uninsubria.it

[3] University of Insubria, Department of Economics, Via Ravasi 2 21100 Varese, Italy. mrocca@eco.uninsubria.it

Summary. The present paper studies the following constrained vector optimization problem: $\min_C f(x)$, $g(x) \in -K$, $h(x) = 0$, where $f : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^n \to \mathbb{R}^p$ are $C^{1,1}$ functions, $h : \mathbb{R}^n \to \mathbb{R}^q$ is C^2 function, and $C \subset \mathbb{R}^m$ and $K \subset \mathbb{R}^p$ are closed convex cones with nonempty interiors. Two type of solutions are important for the consideration, namely w-minimizers (weakly efficient points) and i-minimizers (isolated minimizers). In terms of the second-order Dini directional derivative second-order necessary conditions a point x^0 to be a w-minimizer and second-order sufficient conditions x^0 to be an i-minimizer of order two are formulated and proved. The effectiveness of the obtained conditions is shown on examples.

1 Introduction

In this paper we deal with the constrained vector optimization problem

$$\min_C f(x)\,, \quad g(x) \in -K\,, \quad h(x) = 0\,, \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^n \to \mathbb{R}^p$ and $h : \mathbb{R}^n \to \mathbb{R}^q$ are given functions, and $C \subset \mathbb{R}^m$ and $K \subset \mathbb{R}^p$ are closed convex cones with nonempty interiors. The inclusion $g(x) \in -K$ generalizes constraints of inequality type (in fact it is equivalent to $\langle \eta, g(x) \rangle \le 0$, $\eta \in K'$). This remark explains why the word *inequality* appears in the title of the paper. In the case when f and g are $C^{1,1}$ functions and h is C^2 function we derive second-order optimality conditions for a point x^0 to be a solution of this problem. The paper is thought as a continuation of the investigation initiated by the authors in [8], [9] and [10], where either unconstrained problems or problems with only inequality constraints are studied. Recall that a function is said to be $C^{k,1}$ if it is k-times Fréchet differentiable with locally Lipschitz k-th derivative. The $C^{0,1}$ functions are the locally Lipschitz functions. The $C^{1,1}$ functions have been introduced in Hiriart-Urruty, Strodiot, Hien Nguen [16] and since then have found various application in optimization. In particular second-order conditions for $C^{1,1}$ scalar problems are studied in [16, 6, 19, 28, 27]. Second-order optimality conditions in vector optimization are investigated in [1, 4, 18, 24, 26], and what concerns $C^{1,1}$ vector optimization in [12, 13, 21, 22, 23]. The given in the present paper approach and results generalize that of [23].

The assumption that f and g are defined on the whole space $\mathbb{R}^n$ is taken for convenience. Since we deal only with local solutions of problem (1), evidently our results generalize straightforward for functions f and g being defined on an open subset of $\mathbb{R}^n$. Usually the solutions of (1) are called points of efficiency. We prefer, like in the scalar optimization, to call them minimizers. In Section 2 we introduce different type of minimizers. Among them in our considerations an important role play the w-minimizers (weakly efficient points) and the i-minimizers (isolated minimizers). When we say first or second-order conditions we mean as usual conditions expressed in suitable first or second-order derivatives of the given functions. Here we deal with the Dini directional derivatives. In Section 2 we define the second-order Dini derivative. In Section 3 we recall after [10] second-order optimality conditions for problems with only inequality constraints. In Section 4 we prove second-order sufficient conditions for $C^{1,1}$ problems with both inequality and equality constraints. Section 5 indicates necessary optimality conditions. Section 6 points out directions for further investigations.

2 Preliminaries

For the norm and the dual parity in the considered finite-dimensional spaces we write $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$. From the context it should be clear to exactly what spaces these notations are applied.

For the cone $M \subset \mathbb{R}^k$ its positive polar cone is $M' = \{\zeta \in \mathbb{R}^k \mid \langle\zeta, \phi\rangle \geq 0 \text{ for all } \phi \in M\}$. The cone M' is closed and convex, and $M'' := (M')' = \mathrm{clco}M$, see Rockafellar [25, Theorem 14.1, page 121].

If $\phi \in \mathrm{clconv}M$ we set $M'[\phi] = \{\zeta \in M' \mid \langle\zeta, \phi\rangle = 0\}$. Then $M'[\phi]$ is a closed convex cone and $M'[\phi] \subset M'$. Consequently its positive polar cone $M[\phi] := (M'[\phi])'$ is a closed convex cone, $M \subset M[\phi]$ and $(M[\phi])' = M'[\phi]$. In this paper we apply the notation $M[\phi]$ for $M = K$ and $\phi = -g(x^0)$.

Given a set $A \subset \mathbb{R}^k$, then the distance from $y \in \mathbb{R}^k$ to A is $d(y, A) = \inf\{\|a-y\| \mid a \in A\}$. The oriented distance from y to A is defined by $D(y, A) = d(y, A) - d(y, \mathbb{R}^k \setminus A)$. The function D is introduced in Hiriart-Urruty [14, 15]. In the case of a convex set A, Ginchev, Hoffmann [11] show that $D(y, A) = \sup_{\|\xi\|=1} (\langle\xi, y\rangle - \sup_{a\in A}\langle\xi, a\rangle)$, which for $A = -C$ and C a closed convex cone gives $D(y, -C) = \sup\{\langle\xi, y\rangle \mid \xi \in C', \|\xi\| = 1\}$.

In terms of the distance function we have

$$K[-g(x^0)] = \{w \in \mathbb{R}^p \mid \limsup_{t\to 0^+} \frac{1}{t}\, d(-g(x^0) + tw, K) = 0\},$$

that is $K[-g(x^0)]$ is the contingent cone [3] of K at $-g(x^0)$.

We call the solutions of problem (1) minimizers. The solutions are understood in a local sense. In any case a solution is a feasible point x^0, that is a point satisfying the constraints $g(x^0) \in -K$, $h(x^0) = 0$.

The feasible point x^0 is said to be a w-minimizer (weakly efficient point) for (1) if there exists a neighbourhood U of x^0, such that $f(x) \notin f(x^0) - \mathrm{int}C$ for all feasible points $x \in U$. The feasible point x^0 is said to be an e-minimizer (efficient point) for (1) if there exists a neighbourhood U of x^0, such that $f(x) \notin f(x^0) - (C \setminus \{0\})$ for all feasible points $x \in U$. We say that the feasible point x^0 is a s-minimizer (strong minimizer) for (1) if there exists a neighbourhood U of x^0, such that $f(x) \notin f(x^0) - C$ for all feasible points $x \in U \setminus \{x^0\}$.

As in [8] it can be proved that the feasible point x^0 is a w-minimizer (s-minimizer) for the vector problem (1) if and only if x^0 is a minimizer (strong minimizer) for the scalar problem

$$\min D(f(x) - f(x^0), -C)\,, \quad g(x) \in -K\,, \quad h(x) = 0.$$

This observation motivates the following definition. We say that the feasible point x^0 is an isolated minimizer (for short *i-minimizer*) of order k, $k > 0$, for (1) if there exists a neighbourhood U of x^0 and a constant $A > 0$ such that

$$D(f(x) - f(x^0), -C) \geq A\, \|x - x^0\|^k \quad \text{for all feasible} \quad x \in U. \tag{2}$$

Since any two norms in a finite dimensional real space are equivalent, the notion of an i-minimizer is norm-independent.

Obviously, each i-minimizer is a s-minimizer. Further each s-minimizer is an e-minimizer and each e-minimizer is a w-minimizer (under the assumption $C \neq \mathbb{R}^m$).

The concept of an isolated minimizer for scalar problems is introduced in Auslender [2]. For vector problems it has been extended in Ginchev [7], Ginchev, Guerraggio, Rocca [8], and under the name of strict minimizers in Jiménez [17] and Jiménez, Novo [18]. We prefer the name *isolated minimizer* given originally by A. Auslender.

In the sequel we establish optimality conditions for problem (1) in terms of the second-order Dini derivative (for short *Dini derivative*). For a given $C^{1,1}$ function $\Phi : \mathbb{R}^n \to \mathbb{R}^k$ we define the second-order Dini derivative $\Phi''_u(x^0)$ of Φ at x^0 in direction $u \in \mathbb{R}^n$ by

$$\Phi''_u(x^0) = \operatorname*{Limsup}_{t \to 0^+} \frac{2}{t^2} \left(\Phi(x^0 + tu) - \Phi(x^0) - t\,\Phi'(x^0)u\right) .$$

If Φ is twice Fréchet differentiable at x^0 then the Dini derivative is a singleton and can be expressed in terms of the Hessian $\Phi''_u(x^0) = \Phi''(x^0)(u,u)$.

We deal often with the Dini derivative of the function $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^{m+p}$, $\Phi(x) = (f(x), g(x))$. Then we use the notation $\Phi''_u(x^0) = (f(x^0), g(x^0))''_u$. Let us turn attention that always $(f(x^0), g(x^0))''_u \subset f''_u(x^0) \times g''_u(x^0)$, but in general these two sets do not coincide. The following lemma gives some useful properties of the differential quotient.

Lemma 1 ([10]). *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^k$ be a $C^{1,1}$ function and Φ' be Lipschitz with constant L on the ball $\{x \mid \|x - x^0\| \le r\}$, where $x^0 \in \mathbb{R}^n$ and $r > 0$. Then, for $u, v \in \mathbb{R}^m$ and $0 < t < r/\max(\|u\|, \|v\|)$ we have*

$$\| \frac{2}{t^2} \left(\Phi(x^0 + tv) - \Phi(x^0) - t\Phi'(x^0)v\right) - \frac{2}{t^2} \left(\Phi(x^0 + tu) - \Phi(x^0) - t\Phi'(x^0)u\right) \| \le L\,(\|u\| + \|v\|)\, \|v - u\|\,.$$

In particular, for $v = 0$ we get

$$\| \frac{2}{t^2} \left(\Phi(x^0 + tu) - \Phi(x^0) - t\Phi'(x^0)u\right) \ \| \le L\, \|u\|^2.$$

3 Inequality Constraints, Sufficient Conditions

Here we consider the problem with only inequality constraints

$$\min\nolimits_C f(x)\,, \quad g(x) \in -K. \tag{3}$$

After [10] we recall a result establishing second-order sufficient optimality conditions. In the next section it will be applied to treat the problem with both equality and inequality constraints. We put

$$\Delta_I(x^0) = \big\{(\xi, \eta) \in C' \times K'[-g(x^0)] \setminus \{(0, 0)\} \mid \langle \xi, f'(x^0)\rangle + \langle \eta, g'(x^0)\rangle = 0\big\}$$

$$= \big\{(\xi, \eta) \in C' \times K' \mid (\xi, \eta) \neq 0,\ \langle \eta, g(x^0)\rangle = 0,\ \langle \xi, f'(x^0)\rangle + \langle \eta, g'(x^0)\rangle = 0\big\}$$

using the subscript I to underline that Δ_I is a set associated to the problem with only inequality constraints.

Theorem 1 ([10]). *Consider problem (3) with f and g being $C^{1,1}$ functions, and C and K closed convex cones with nonempty interiors. Let x^0 be a feasible point. Suppose that for each $u \in \mathbb{R}^n \setminus \{0\}$ one of the following two conditions is satisfied:*

$$\mathbb{S}'_i : \quad (f'(x^0)u,\, g'(x^0)u) \notin -(C \times K[-g(x^0)]),$$

$$\mathbb{S}''_i : \quad (f'(x^0)u,\, g'(x^0)u) \in -(C \times K[-g(x^0)] \setminus \mathrm{int}C \times \mathrm{int}K[-g(x^0)]) \ \text{and}\ \forall\, (y^0, z^0) \in (f(x^0), g(x^0))''_u : \exists\, (\xi^0, \eta^0) \in \Delta_I(x^0) : \langle \xi^0, y^0\rangle + \langle \eta^0, z^0\rangle > 0.$$

Then x^0 is an i-minimizer of order two for problem (3).

Theorem 1 generalizes Theorem 4.2 from Liu, Neittaanmäki, Křížek [23] in the following aspects. Theorem 1 in opposite to [23] concerns arbitrary and not only polyhedral cones C and K. In Theorem 1 the conclusion is that x^0 is an i-minimizer of order two, while in [23] the weaker conclusion is proved that the reference point x^0 is only an e-minimizer.

4 Inequality and Equality Constraints, Sufficient Conditions

In Theorem 2 we establish sufficient conditions for the general problem (1) with both inequality and equality constraints. If the functions f, g, h are at least C^1, we put

$$\Delta(x^0) = \{(\xi,\eta,\zeta) \in C' \times K' \times \mathbb{R}^q \mid (\xi,\eta,\zeta) \neq (0,0,0),\ \langle \eta,\, g(x^0)\rangle = 0,$$
$$\langle \xi,\, f'(x^0)u\rangle + \langle \eta,\, g'(x^0)u\rangle + \langle \zeta,\, h'(x^0)u\rangle = 0 \text{ for } u \in \ker h'(x^0)\}\,.$$

Theorem 2. *Consider problem (1) with f, $g \in C^{1,1}$ and $h \in C^2$, and C and K closed convex cones with nonempty interiors. Let x^0 be a feasible point and let the vectors $h_1'(x^0)$, ..., $h_q'(x^0)$, which are the components of $h'(x^0)$, be linearly independent. Let the vectors $\bar{u}^j \in \mathbb{R}^n$, $j = 1,\dots,q$, be determined by*

$$h_k'(x^0)\bar{u}^j = 0 \quad \text{for} \quad k \neq j, \quad \text{and} \quad h_j'(x^0)\bar{u}^j = 1\,. \tag{4}$$

Suppose that for each $u \in \ker h'(x^0) \setminus \{0\}$ one of the following two conditions is satisfied.

$$\mathbb{S}' : \qquad (f'(x^0)u,\, g'(x^0)u) \notin -(C \times K[-g(x^0)])\,,$$

$$\begin{aligned}\mathbb{S}'' : \quad & (f'(x^0)u,\, g'(x^0)u) \in -(C \times K[-g(x^0)] \setminus \mathrm{int}C \times \mathrm{int}K[-g(x^0)]) \\ & \text{and } \forall\, (y^0,z^0) \in (f(x^0),g(x^0))''_u : \exists\, (\xi^0,\eta^0,\zeta^0) \in \Delta(x^0) : \\ & \langle \xi^0,\, y^0\rangle + \langle \eta^0,\, z^0\rangle + \langle \zeta^0,\, h''(x^0)(u,u)\rangle > 0 \\ & \text{with } \zeta^0 = (\zeta_j^0)_{j=1}^q \text{ satisfying (5), where}\end{aligned}$$

$$\zeta_j^0 = -\langle \xi^0,\, f'(x^0)\bar{u}^j\rangle - \langle \eta^0,\, g'(x^0)\bar{u}^j\rangle\,, \quad j = 1,\dots,q\,. \tag{5}$$

Then x^0 is an i-minimizer of order two for problem (1).

Before going on with the proof we transform our problem (1) to a problem with only inequality constraints. Determine $\bar{u}^1,\dots,\bar{u}^q \in \mathbb{R}^n$ by (1). For each $j = 1,\dots,q$, equalities (1) constitute a system of linear equations with respect to the components of $\bar{u}^j$, which due to the linear independence of $h_1'(x^0)$, ..., $h_q'(x^0)$ has a solution. Moreover, the vectors $\bar{u}^1,\dots,\bar{u}^q$ solving this system are linearly independent and $\mathbb{R}^n$ is decomposed into a direct sum $\mathbb{R}^n = L \oplus L'$, where $L = \ker h'(x^0)$ and $L' = \mathrm{lin}\{\bar{u}^1,\dots,\bar{u}^q\}$. Let $u^1,\dots,u^{n-q}$ be $n-q$ linearly independent vectors in $L = \ker h'(x^0)$. We consider the system of equations:

$$h_k(x^0 + \sum_{i=1}^{n-q} \tau_i u^i + \sum_{j=1}^{q} \sigma_j \bar{u}^j) = 0\,, \quad k = 1, \ldots, q\,. \tag{6}$$

Taking $\tau_1, \ldots, \tau_{n-q}$ as independent variables and $\sigma_1, \ldots, \sigma_q$ as dependent variables, we see that this system satisfies the requirements of the implicit function theorem at the point $\tau_1 = \cdots = \tau_{n-q} = 0$, $\sigma_1 = \cdots = \sigma_q = 0$ (at this point h_k take values $h_k(x^0) = 0$ because x^0 is feasible, and the Jacobian is the unit matrix and hence nondegenerate). The implicit function theorem gives that in a neighbourhood of x^0 given by $|\tau_i| < \bar{\tau}$, $i = 1, \ldots, n-q$, $|\sigma_j| < \bar{\sigma}$, $j = 1, \cdots, q$, this system possesses a unique solution $\sigma_j = \sigma_j(\tau_1, \ldots, \tau_{n-q})$, $j = 1, \ldots, q$. The functions $\sigma_j = \sigma_j(\tau_1, \ldots, \tau_{n-q})$ are C^2 and $\sigma_j(0, \ldots, 0) = 0$.

Lemma 2. *Consider problem (1) with $h \in C^1$, for which $h'_1(x^0), \cdots, h'_q(x^0)$, are linearly independent, and C and K are closed convex cones. Then x^0 is a w-minimizer or i-minimizer of order k for (1) if and only if $\tau^0 = 0$ is respectively a w-minimizer or i-minimizer of order k for the problem*

$$\min_C \bar{f}(\tau_1, \ldots, \tau_{n-q}), \quad \bar{g}(\tau_1, \ldots, \tau_{n-q}) \in -K\,, \tag{7}$$

where

$$\begin{aligned} \bar{f}(\tau_1, \ldots, \tau_{n-q}) &= f(x^0 + \sum_{i=1}^{n-q} \tau_i u^i + \sum_{j=1}^{q} \sigma_j(\tau_1, \ldots, \tau_{n-q})\, \bar{u}^j)\,, \\ \bar{g}(\tau_1, \ldots, \tau_{n-q}) &= g(x^0 + \sum_{i=1}^{n-q} \tau_i u^i + \sum_{j=1}^{q} \sigma_j(\tau_1, \ldots, \tau_{n-q})\, \bar{u}^j)\,. \end{aligned} \tag{8}$$

Proof. From the implicit function theorem every feasible point x sufficiently close to x^0 admits a representation

$$x = x^0 + \sum_{i=1}^{n-q} \tau_i u^i + \sum_{j=1}^{q} \sigma_j(\tau_1, \ldots, \tau_{n-q})\, \bar{u}^j \tag{9}$$

with $\tau = (\tau_1, \ldots, \tau_{n-q})$ close to $\tau^0 = 0$ and $\sigma_j(\tau_1, \ldots, \tau_{n-q})$ the unique C^1 solution of (6) with value $\sigma^0 = 0$ at $\tau^0 = 0$. Therefore it is obvious that x^0 is a w-minimizer for (1) if and only if τ^0 is a w-minimizer for (7). Suppose now that x^0 is an i-minimizer of order k. Then for some neighbourhood U of x^0 and some $A > 0$ inequality (2) has place. Replacing here x with (9) we get for all τ being close to τ^0 and feasible for (7) the inequality

$$D(\bar{f}(\tau) - \bar{f}(\tau^0), -C) \geq A\, \|x(\tau) - x^0\|^k\,. \tag{10}$$

Expressing $x = x(\tau)$ by (9) and applying the Taylor formula for $\sigma_j(\tau_1, \ldots, \tau_{n-q})$ and the forthcoming expressions for the derivatives we get

$$x(\tau) - x^0 = \sum_{i=1}^{n-q} \tau_i u^i + o(\|\tau\|)\,.$$

With the account that by choice $u^1, \dots, u^{n-q}$ are linearly independent, we see that close to $\tau^0 = 0$ there exist positive constants A' and A'', such that

$$A' \|\tau - \tau^0\| \leq \|x(\tau) - x^0\| \leq A'' \|\tau - \tau^0\| .$$

These inequalities, together with (10) show that x^0 is an i-minimizer of order k for (1) if and only if $\tau^0 = 0$ is an i-minimizer of order k for (7). □

Now we calculate the derivatives of $\sigma_j(\tau_1, \dots, \tau_{n-q})$ at $\tau^0 = (0, \dots, 0)$. We have

$$\sigma_j\big|_{\tau^0} = \sigma_j(0, \dots, 0) = 0, \quad j = 1 \dots, q . \tag{11}$$

For the first-order derivatives differentiating (6) with respect to τ_i we get

$$h_k'(x^0 + \sum_{i=1}^{n-q} \tau_i u^i + \sum_{j=1}^{q} \sigma_j \bar{u}^j)(u^i + \sum_{j=1}^{q} \frac{\partial \sigma_j}{\partial \tau_i} \bar{u}^j) = 0 .$$

For $\tau = \tau^0 = 0$ we get

$$h_k'(x^0)(u^i + \sum_{j=1}^{q} \frac{\partial \sigma_j}{\partial \tau_i}\bigg|_{\tau^0} \bar{u}^j) = 0 ,$$

whence with account of $u^i \in \ker h'(x^0)$ and (1) we obtain

$$\frac{\partial \sigma_j}{\partial \tau_i}\bigg|_{\tau^0} = 0, \quad j = 1, \dots, q, \quad i = 1, \dots, n - q . \tag{12}$$

Now we calculate the second-order derivatives:

$$h_k''(x^0 + \sum_{i=1}^{n-q} \tau_i u^i + \sum_{j=1}^{q} \sigma_j \bar{u}^j)(u^{i'} + \sum_{j=1}^{q} \frac{\partial \sigma_j}{\partial \tau_{i'}} \bar{u}^j ,\, u^{i''} + \sum_{j=1}^{q} \frac{\partial \sigma_j}{\partial \tau_{i''}} \bar{u}^j)$$
$$+ h_k'(x^0 + \sum_{i=1}^{n-q} \tau_i u^i + \sum_{j=1}^{q} \sigma_j \bar{u}^j) \sum_{j=1}^{q} \frac{\partial^2 \sigma_j}{\partial \tau_{i'} \partial \tau_{i''}} \bar{u}^j = 0 .$$

For $\tau = \tau^0 = 0$ with account of $u^i \in \ker h'(x^0)$ and (1) we get

$$h_k''(x^0)(u^{i'} ,\, u^{i''}) + \sum_{j=1}^{q} \frac{\partial^2 \sigma_j}{\partial \tau_{i'} \partial \tau_{i''}}\bigg|_{\tau^0} h_k'(x^0)\, \bar{u}^j = 0 .$$

After all, substituting k with j, we obtain

$$\frac{\partial^2 \sigma_j}{\partial \tau_{i'} \partial \tau_{i''}}\bigg|_{\tau^0} = -h_j''(x^0)(u^{i'} ,\, u^{i''}), \quad j = 1, \dots, q, \quad i', i'' = 1, \dots, n - q . \tag{13}$$

Proof of Theorem 2. According to Lemma 2 to show that x^0 is an i-minimizer of order two for (1) we must show that $\tau^0 = 0$ is an i-minimizer of order two for the problem with only inequality constraints (7). For this purpose it is enough to check that the sufficient conditions of Theorem 1 applied to problem (7) are satisfied. Since $\bar{g}(\tau^0) = g(x^0)$ we see that x^0 feasible for (1) implies τ^0 feasible for (7). Similarly $K[-\bar{g}(\tau^0)] = K[-g(x^0)]$. Theorem 1 reformulated for problem (7) gives:

Suppose that for each $\tau \in \mathbb{R}^{n-q} \setminus \{0\}$ one of the following two conditions holds:

$$\bar{\mathbb{S}}' : \qquad (\bar{f}'(\tau^0)\tau,\, \bar{g}'(\tau^0)\tau) \notin -(C \times K[-\bar{g}(\tau^0)])\,,$$

$$\begin{aligned}\bar{\mathbb{S}}'' : \quad & (\bar{f}'(\tau^0)\tau,\, \bar{g}'(\tau^0)\tau) \in -(C \times K[-\bar{g}(\tau^0)] \setminus intC \times intK[-\bar{g}(\tau^0)]) \\ & and\ \forall\, (\bar{y}^0, \bar{z}^0) \in (\bar{f}(\tau^0), \bar{g}(\tau^0))''_\tau : \exists\, (\xi^0, \eta^0) \in \bar{\Delta}(0) : \\ & \langle \xi^0,\, \bar{y}^0 \rangle + \langle \eta^0,\, \bar{z}^0 \rangle > 0\,.\end{aligned}$$

Then τ^0 is an i-minimizer of order two for problem (7). Here

$$\bar{\Delta}(\tau^0) =$$

$$\{(\xi, \eta) \in C' \times K' \mid (\xi, \eta) \neq (0,0),\ \langle \eta, \bar{g}(\tau^0)\rangle = 0,\ \langle \xi, \bar{f}'(\tau^0)\rangle + \langle \eta, \bar{g}'(\tau^0)\rangle = 0\}.$$

We prove the theorem by showing that conditions $\mathbb{S}'$ and $\mathbb{S}''$ imply respectively $\bar{\mathbb{S}}'$ and $\bar{\mathbb{S}}''$. To show that $\mathbb{S}'$ implies $\bar{\mathbb{S}}'$ we get consecutively:

$$\begin{aligned}\frac{\partial}{\partial \tau_i}\, \bar{f}(\tau_1, \ldots, \tau_{n-q}) &= \frac{\partial}{\partial \tau_i} f(x^0 + \sum_{i=1}^{n-q} \tau_i u^i + \sum_{j=1}^{q} \sigma_j(\tau_1, \ldots, \tau_{n-q})\, \bar{u}^j) \\ &= f'(x^0 + \sum_{i=1}^{n-q} \tau_i u^i + \sum_{j=1}^{q} \sigma_j\, \bar{u}^j)(u^i + \sum_{j=1}^{q} \frac{\partial \sigma_j}{\partial \tau_i}\, \bar{u}^j),\end{aligned}$$

$$\frac{\partial}{\partial \tau_i}\, \bar{f}(0) = f'(x^0)u^i = (f_1'(x^0)u^i,\, \ldots,\, f_m'(x^0)u^i),$$

$$\bar{f}'(0)\, \tau = \sum_{i=1}^{n-q} \frac{\partial}{\partial \tau_i}\, \bar{f}(0)\, \tau_i = f'(x^0) \sum_{i=1}^{n-q} \tau_i u^i. \tag{14}$$

Similarly

$$\frac{\partial}{\partial \tau_i}\, \bar{g}(\tau_1, \ldots, \tau_{n-q}) = g'(x^0 + \sum_{i=1}^{n-q} \tau_i u^i + \sum_{j=1}^{q} \sigma_j\, \bar{u}^j)(u^i + \sum_{j=1}^{q} \frac{\partial \sigma_j}{\partial \tau_i}\, \bar{u}^j),$$

$$\frac{\partial}{\partial \tau_i}\, \bar{g}(0) = (g_1'(x^0)u^i,\, \ldots,\, g_p'(x^0)u^i),$$

$$\bar{g}'(0)\, \tau = \sum_{i=1}^{n-q} \frac{\partial}{\partial \tau_i}\, \bar{g}(0)\, \tau_i = g'(x^0) \sum_{i=1}^{n-q} \tau_i u^i.$$

Putting

$$u = \sum_{i=1}^{n-q} \tau_i u^i \in \mathrm{ker} h'(x^0) \tag{15}$$

we see that while τ varies in $\mathbb{R}^{n-q} \setminus \{0\}$ the vector u takes all values from $\mathrm{ker}h'(x^0) \setminus \{0\}$. Consequently condition $\bar{\mathbb{S}}'$ is equivalent to $\mathbb{S}'$, that is to

$$(f'(x^0)u,\, g'(x^0)u) \notin -(C \times K[-g(x^0)]) \quad \text{for} \quad u \in \mathrm{ker}h'(x^0) \setminus \{0\}\,. \tag{16}$$

Next we show that $\mathbb{S}''$ implies $\bar{\mathbb{S}}''$. The above calculations show the equivalence of the first parts of $\mathbb{S}''$ and $\bar{\mathbb{S}}''$, where only first order derivatives appear. Now we compare the second parts of $\mathbb{S}''$ and $\bar{\mathbb{S}}''$. For this purpose we must find first a relation between the Dini derivatives of $(f(x^0),\, g(x^0))''_u$ and $(\bar{f}(\tau^0),\, \bar{g}(\tau^0))''_\tau$. Initially we will consider the case of $f,\, g \in C^2$. Then

$$(f(x^0),\, g(x^0))''_u = (f''_u(x^0), g''_u(x^0)) = (f''(x^0)(u,u),\, g''(x^0)(u,u))$$

is a singleton. Similarly $\bar{f},\, \bar{g} \in C^2$ and

$$(\bar{f}(0),\, \bar{g}(0))''_\tau = (\bar{f}''(0)(\tau,\tau),\, \bar{g}''(0)(\tau,\tau))$$

is a singleton. We have consecutively

$$\begin{aligned} \frac{\partial^2}{\partial \tau_{i'} \partial \tau_{i''}} \bar{f}(\tau_1, \ldots, \tau_{n-q}) &= \frac{\partial^2}{\partial \tau_{i'} \partial \tau_{i''}} f(x^0 + \sum_{i=1}^{n-q} \tau_i u^i + \sum_{j=1}^{q} \sigma_j \bar{u}^j) \\ = f''(x^0 + \sum_{i=1}^{n-q} \tau_i u^i + \sum_{j=1}^{q} \sigma_j \bar{u}^j)&(u^{i'} + \sum_{j=1}^{q} \frac{\partial \sigma_j}{\partial \tau_{i'}} \bar{u}^j,\, u^{i''} + \sum_{j=1}^{q} \frac{\partial \sigma_j}{\partial \tau_{i''}} \bar{u}^j) \\ &+ f'(x^0 + \sum_{i=1}^{n-q} \tau_i u^i + \sum_{j=1}^{q} \sigma_j \bar{u}^j) \sum_{j=1}^{q} \frac{\partial^2 \sigma_j}{\partial \tau_{i'} \partial \tau_{i''}} \bar{u}^j, \end{aligned}$$

$$\frac{\partial^2}{\partial \tau_{i'} \partial \tau_{i''}} \bar{f}(0) = f''(x^0)(u^{i'},\, u^{i''}) - \sum_{j=1}^{q} h''_j(x^0)(u^{i'},\, u^{i''})\, f'(x^0)\bar{u}^j. \tag{17}$$

Therefore for u given by (4) we have

$$\bar{f}''(0)(\tau,\tau) = f''(x^0)(u,u) - \sum_{j=1}^{q} h''_j(x^0)(u,u)\, f'(x^0)\bar{u}^j. \tag{18}$$

Similarly

$$\bar{g}''(0)(\tau,\tau) = g''(x^0)(u,u) - \sum_{j=1}^{q} h''_j(x^0)(u,u)\, g'(x^0)\bar{u}^j. \tag{19}$$

Now we show that when the assumptions on f and g are relaxed from C^2 to $C^{1,1}$ still there exist formulas similar to (18) and (19). In fact the only reason to consider in advance the case of $f, g \in C^2$ was to elaborate some heuristics. In the relaxed case we show the following result. Let f, $g \in C^{1,1}$ and $h \in C^2$ be such that $h'_1(x^0), \dots, h'_q(x^0)$ are linearly independent. Suppose that $(\bar{y}^0, \bar{z}^0) \in (\bar{f}(0), \bar{g}(0))''_\tau$ and

$$\begin{aligned} \bar{y}^0 &= \lim_k \frac{2}{t_k^2} \left(\bar{f}(t_k\tau) - \bar{f}(0) - t_k \bar{f}'(0)\tau\right), \\ \bar{z}^0 &= \lim_k \frac{2}{t_k^2} \left(\bar{g}(t_k\tau) - \bar{g}(0) - t_k \bar{g}'(0)\tau\right). \end{aligned} \tag{20}$$

Let $u = u(\tau)$ be determined by (4). We will prove that the following limits exist

$$\begin{aligned} y^0 &= \lim_k \frac{2}{t_k^2} \left(f(x^0 + t_k u) - f(x^0) - t_k f'(x^0)u\right), \\ z^0 &= \lim_k \frac{2}{t_k^2} \left(\bar{g}(x^0 + t_k u) - g(x^0) - t_k g'(x^0)u\right), \end{aligned} \tag{21}$$

and satisfy (similarly to (18)–(19)) the relations

$$\begin{aligned} \bar{y}^0 &= y^0 - \sum_{j=1}^q h''_j(x^0)(u,u)\, f'(x^0)\bar{u}^j, \\ \bar{z}^0 &= z^0 - \sum_{j=1}^q h''_j(x^0)(u,u)\, g'(x^0)\bar{u}^j. \end{aligned} \tag{22}$$

Fix τ. Let now t be a positive real variable and put for brevity $\hat{u} = u + (1/t) \sum_{j=1}^q \sigma_j(t\tau)\, \bar{u}^j$. Then

$$\frac{2}{t^2} \left(\bar{f}(t\tau) - \bar{f}(0) - t\,\bar{f}'(0)\tau\right)$$

$$= \frac{2}{t^2} \left(f(x^0 + t\hat{u}) - f(x^0) - t\, f'(0)\hat{u}\right) + \frac{2}{t^2}\, f'(x^0) \sum_{j=1}^q \sigma_j(t\tau)\, \bar{u}^j.$$

The Taylor formula with regard to (4), (1) and (13) gives

$$\sigma_j(t\tau) = \frac{1}{2}\sigma''_j(\tau^0)(t\tau, t\tau) + o(t^2) = -\frac{1}{2}\, t^2\, h''_j(x^0)(u,u) + o(t^2),$$

whence

$$\frac{2}{t^2} \left(\bar{f}(t\tau) - \bar{f}(0) - t\,\bar{f}'(0)\tau\right) = \frac{2}{t^2} \left(f(x^0 + t\hat{u}) - f(x^0) - t\, f'(0)\hat{u}\right)$$

$$- \sum_{j=1}^q h''_j(x^0)(u,u)\, f'(x^0)\bar{u}^j.$$

A similar representation exists for f replaced by g. From these representations and (20) it follows that there exist the limits

$$\hat{y}^0 = \lim_k \frac{2}{t_k^2} \left(f(x^0 + t_k\hat{u}) - f(x^0) - t_k\, f'(x^0)\hat{u}\right),$$
$$\hat{z}^0 = \lim_k \frac{2}{t_k^2} \left(g(x^0 + t_k\hat{u}) - g(x^0) - t_k\, g'(x^0)\hat{u}\right),$$

and

$$\begin{aligned}\bar{y}^0 &= \hat{y}^0 - \sum_{j=1}^{q} h_j''(x^0)(u,u)\, f'(x^0)\bar{u}^j,\\ \bar{z}^0 &= \hat{z}^0 - \sum_{j=1}^{q} h_j''(x^0)(u,u)\, g'(x^0)\bar{u}^j.\end{aligned} \tag{23}$$

Applying now Lemma 1 we get

$$\begin{aligned}&\| \frac{2}{t_k^2} \left(f(x^0 + t_k\hat{u}) - f(x^0) - t_k\, f'(x^0)\hat{u}\right)\\ &- \frac{2}{t_k^2} \left(f(x^0 + t_k u) - f(x^0) - t_k\, f'(x^0)u\right) \|\\ \le L\left(\|\hat{u}\| + \|u\|\right)\|\hat{u} - u\| &= L\left(\|\hat{u}\| + \|u\|\right) \frac{1}{t_k} \| \sum_{j=1}^{q} \sigma_j(t_k\tau)\bar{u}^j \| = o(1)\,.\end{aligned}$$

A similar estimation exists for f replaced by g. In consequence, these inequalities show that there exist the limits (21) and it holds $y^0 = \hat{y}^0$, $z^0 = \hat{z}^0$. These equalities and formulas (10) imply (22).

Now we prove the second part of $\bar{\mathbb{S}}''$ as a consequence of $\mathbb{S}''$. Take $(\bar{y}^0,\, \bar{z}^0) \in (\bar{f}(\tau^0),\, \bar{g}(\tau^0))''_\tau$ with $\tau \in \mathbb{R}^{n-q} \setminus \{0\}$ and let (20) be satisfied. Then the limits (21) exist and define $(y^0,\, z^0) \in (f(x^0),\, g(x^0))''_u$, where u and τ are related by (4). The latter gives $u \in \ker h'(x^0) \setminus \{0\}$. Since $\mathbb{S}''$ holds, therefore there exists $(\xi^0,\, \eta^0,\, \zeta^0) \in \Delta(x^0)$ such that ζ^0 satisfies (5) and $\langle \xi^0,\, y^0 \rangle + \langle \eta^0,\, z^0 \rangle + \langle \zeta^0,\, h''(x^0)(u,u) \rangle > 0$. Substituting ζ^0 with (5) and applying (22) we get

$$\begin{aligned}0 &< \langle \xi^0,\, y^0 \rangle + \langle \eta^0,\, z^0 \rangle + \langle \zeta^0,\, h''(x^0)(u,u) \rangle\\ &= \langle \xi^0,\, y^0 - \sum_{j=1}^{q} h_j''(x^0)(u,u)\, f'(x^0)\bar{u}^j \rangle + \langle \eta^0,\, z^0 - \sum_{j=1}^{q} h_j''(x^0)(u,u)\, g'(x^0)\bar{u}^j \rangle\\ &= \langle \xi^0,\, \bar{y}^0 \rangle + \langle \eta^0,\, \bar{z}^0 \rangle\,.\end{aligned}$$

To demonstrate that the second part of $\bar{\mathbb{S}}''$ is satisfied it remains to show that $(\xi^0, \eta^0) \in \bar{\Delta}(\tau^0)$. This follows from the following observations. We have $(\xi^0, \eta^0) \neq (0,\, 0)$, since otherwise (5) would give $(\xi^0, \eta^0, \zeta^0) = (0,\, 0,\, 0)$. It holds $\langle \eta^0,\, \bar{g}(\tau^0) \rangle = \langle \eta^0,\, g(x^0) \rangle = 0$. Finally, for $\tau \in \mathbb{R}^{n-q}$ and u determined by (4) we have

$$\langle \xi^0,\, \bar{f}'(\tau^0)\tau \rangle + \langle \eta^0,\, \bar{g}'(\tau^0)\tau \rangle = \langle \xi^0,\, f'(x^0)u \rangle + \langle \eta^0,\, g'(x^0)u \rangle = 0. \qquad \square$$

The next example shows that the optimality in particular vector optimization problems can be checked effectively on the base of Theorem 2 and known calculus rules.

Example 1. Consider problem (1), for which $n = 3$, $m = 2$, $p = 1$, $q = 2$, the cones are $C = \mathbb{R}^2_+$ and $K = \mathbb{R}_+$, and the functions f, g, h, are given by

$$f(x_1, x_2, x_3) = (-2x_1^2 - 2x_2^2 + x_3,\, x_1^2 + x_2^2 - x_3),$$

$$g(x_1, x_2, x_3) = x_1|x_1| + x_2|x_2| - x_3,$$

$$h(x_1, x_2, x_3) = (x_1 + x_2,\, x_3).$$

Then the point $x^0 = (0,\, 0,\, 0)$ is an i-minimizer of order 2, which can be established on the base of Theorem 2, as it is shown below.

The problem is $C^{1,1}$ and not C^2 because of the function g. We have

$$f(x^0) = (0, 0), \quad g(x^0) = 0, \quad h(x^0) = (0, 0).$$

The point x^0 is feasible and it holds $C' = \mathbb{R}^2_+$, $K' = \mathbb{R}_+$, $K[-g(x^0)] = \mathbb{R}_+$,

$$f'(x)u = (-4x_1u_1 - 4x_2u_2 + u_3,\, 2x_1u_1 + 2x_2u_2 - u_3),$$

$$g'(x)u = 2u_1|x_1| + 2u_2|x_2| - u_3,$$

$$f'(x^0)u = (u_3,\, -u_3)\,, \quad g'(x^0)u = -u_3,$$

$$h_1'(x^0) = (1,\, 1,\, 0)\,, \quad h_2'(x^0) = (0,\, 0,\, 1).$$

Obviously $h_1'(x^0)$ and $h_2'(x^0)$ are linearly independent, and

$$\ker h'(x^0) = \{u \in \mathbb{R}^3 \mid u_1 + u_2 = 0,\, u_3 = 0\}\,.$$

$$(f'(x^0)u,\, g'(x^0)u) = (0,\, 0) \in \mathbb{R}^2 \times \mathbb{R} \quad \text{for} \quad u \in \ker h'(x^0),$$

$$\Delta(x^0) = C' \times K' \times \mathbb{R}^2 \setminus \{(0,\, 0,\, 0)\}.$$

For each $u \in \ker h'(x^0) \setminus \{0\}$ condition $\mathbb{S}'$ is not satisfied. We prove that for such u condition $\mathbb{S}''$ holds. We have

$$(f'(x^0)u,\, g'(x^0)u) \in -(C \times K[-g(x^0)] \setminus \operatorname{int}C \times \operatorname{int}K[-g(x^0)])\,.$$

The second-order derivatives at x^0 are

$$f_u''(x^0) = f''(x^0)(u, u) = (-4u_1^2 - 4u_2^2,\, 2u_1^2 + 2u_2^2)\,,$$

$$g_u''(x^0) = 2u_1|u_1| + 2u_2|u_2|\,, \quad h''(x^0)(u, u) = (0,\, 0)\,.$$

Turn attention that $(f(x^0),\, g(x^0))_u'' = (f_u''(x^0),\, g_u''(x^0))$ is single-valued. The assumption $u \in \ker h'(x^0) \setminus \{0\}$ means $u_1 + u_2 = 0$, $u_3 = 0$. The vectors $\bar{u}^1$, $\bar{u}^2$ satisfying (1) can be chosen as $\bar{u}^1 = (1/2,\, 1/2,\, 0)$, $\bar{u}^2 = (0,\, 0,\, 1)$. According to (5) the vector $\zeta^0 = (\zeta_1^0, \zeta_2^0)$ is expressed by $\xi^0 = (\xi_1^0,\, \xi_2^0)$ and η_0 as $\zeta^0 = (0,\, -\xi_1^0 + \xi_2^0 + \eta^0)$. Now for $y^0 = f''(x^0)(u, u)$, $z^0 = g_u''(x^0)$, $\xi^0 = (0,\, 1) \in C'$, $\eta^0 = 0 \in K'[-g(x^0]$ and $u \in \ker h'(x^0) \setminus \{0\}$ we get

$$\langle \xi^0,\, y^0 \rangle + \langle \eta^0,\, z^0 \rangle + \langle \zeta^0,\, h''(x^0)(u, u) \rangle$$

$$= -4\xi_1^0(u_1^2 + u_2^2) + 2\xi_2^0(u_1^2 + u_2^2) + \eta^0(2u_1|u_1| + 2u_2|u_2|) = 2(u_1^2 + u_2^2) > 0\,,$$

which shows that condition $\mathbb{S}''$ holds.

5 Necessary Conditions

The following Theorem 3 gives second-order necessary conditions for the problem (3) with only inequality constraints.

Theorem 3 ([10]). *Consider problem (3) with f and g being $C^{1,1}$ functions, and C and K closed convex cones with nonempty interiors. Let x^0 be a w-minimizer for (3). Then for each $u \in \mathbb{R}^n$ the following two conditions hold:*

$$\mathbb{N}'_i: \quad (f'(x^0)u,\, g'(x^0)u) \notin -(\mathrm{int}C \times \mathrm{int}K[-g(x^0)])\,,$$

$$\mathbb{N}''_i: \quad \text{if } (f'(x^0)u,\, g'(x^0)u) \in -(C \times K[-g(x^0)] \setminus \mathrm{int}C \times \mathrm{int}K[-g(x^0)]) \\ \text{then } \forall\, (y^0, z^0) \in (f(x^0), g(x^0))''_u : \exists\, (\xi^0, \eta^0) \in \Delta_I(x^0) : \\ \langle \xi^0,\, y^0\rangle + \langle \eta^0,\, z^0\rangle \geq 0\,.$$

Here $\Delta_I(x^0)$ has the same meaning as in Theorem 1. Theorem 3 generalizes Theorem 3.1 in Liu, Neittaanmäki, Křířek [23], which states the same thesis under the stronger assumption that C and K are polyhedral cones and C is acute.

The same elimination procedure as in Theorem 2 reduces problem (1) with both equality and inequality constraints to a problem with only inequality constraints to which we can apply Theorem 3. In such a way we obtain the following result:

Theorem 4. *Consider problem (1) with $f, g \in C^{1,1}$ and $h \in C^2$, and C and K closed convex cones with nonempty interiors. Let the vectors $h'_1(x^0), \ldots, h'_q(x^0)$, which are the components of $h'(x^0)$, be linearly independent and let the vectors $\bar{u}^j \in \mathbb{R}^n$ be determined by (1). Suppose that x^0 is a w-minimizer for (1). Then for each $u \in \ker h'(x^0)$ the following two conditions hold:*

$$\mathbb{N}': \quad (f'(x^0)u,\, g'(x^0)u) \notin -(\mathrm{int}C \times \mathrm{int}K[-g(x^0)])\,,$$

$$\mathbb{N}'': \quad \text{if } (f'(x^0)u,\, g'(x^0)u) \in -(C \times K[-g(x^0)] \setminus \mathrm{int}C \times \mathrm{int}K[-g(x^0)]) \\ \text{then } \forall\, (y^0, z^0) \in (f(x^0), g(x^0))''_u : \exists\, (\xi^0, \eta^0, \zeta^0) \in \Delta(x^0) : \\ \langle \xi^0,\, y^0\rangle + \langle \eta^0,\, z^0\rangle + \langle \zeta^0,\, h''(x^0)(u,u)\rangle \geq 0 \\ \text{and } \zeta^0 = (\zeta^0_j)_{j=1}^q \text{ satisfies (5).}$$

Here $\Delta(x^0)$ has the same meaning as in Theorem 2. The next example shows that the finding of the solutions of particular vector optimization problems can be effectively based on Theorem 4 and known calculus rules.

Example 2. Consider problem (1), for which $n = 3$, $m = 2$, $p = 1$, $q = 2$, the cones are $C = \mathbb{R}^2_+$ and $K = \mathbb{R}_+$, and the functions f, g, h, are given by

$$f(x_1, x_2, x_3) = (-2x_1^2 - 2x_2^2 + x_3,\; x_1^2 + x_2^2 - x_3)\,,$$

$$g(x_1, x_2, x_3) = x_1|x_1| + x_2|x_2| - x_3\,,$$

$$h(x_1, x_2, x_3) = (x_1 + x_2,\; 3x_1^2 + 3x_2^2 - 2x_3)\,.$$

Then the point $x^0 = (0,\, 0,\, 0)$ is not a w-minimizer, which can be established on the base of Theorem 4, as it is shown below.

Like in Example 1 we have $f(x^0) = (0,0)$, $g(x^0) = 0$, $h(x^0) = (0,0)$, $C' = \mathbb{R}^2_+$, $K' = \mathbb{R}_+$, $K[-g(x^0)] = \mathbb{R}_+$, $f'(x^0)u = (u_3, -u_3)$, $g'(x^0)u = -u_3$, $h'_1(x^0) = (1, 1, 0)$, $h'_2(x^0) = (0, 0, -2)$. Obviously $h'_1(x^0)$ and $h'_2(x^0)$ are linearly independent, and

$$\mathrm{ker} h'(x^0) = \{u \in \mathbb{R}^3 \mid u_1 + u_2 = 0,\, u_3 = 0\}.$$

$$(f'(x^0)u,\, g'(x^0)u) = (0,\, 0) \in \mathbb{R}^2 \times \mathbb{R} \quad \text{for} \quad u \in \mathrm{ker} h'(x^0)\,,$$
$$\Delta(x^0) = C' \times K' \times \mathbb{R}^2 \setminus \{(0,\, 0,\, 0)\}\,.$$

For each $u \in \mathrm{ker} h'(x^0)$ condition $\mathbb{N}'$ is satisfied. We prove that for some $u \in \mathrm{ker} h'(x^0)$ condition $\mathbb{N}''$ with ζ^0 distinguished by (5) does not hold. Observe that for any such u the statement in the first part of $\mathbb{N}''$ is true

$$(f'(x^0)u,\, g'(x^0)u) \in -(C \times K[-g(x^0)] \setminus \mathrm{int} C \times \mathrm{int} K[-g(x^0)])\,.$$

The second-order derivatives at x^0 are

$$f''_u(x^0) = f''(x^0)(u,u) = (-4u_1^2 - 4u_2^2,\, 2u_1^2 + 2u_2^2)\,,$$

$$g''_u(x^0) = 2u_1|u_1| + 2u_2|u_2|\,, \quad h''(x^0)(u,u) = (0,\, 6u_1^2 + 6u_2^2)\,.$$

Here $(f(x^0),\, g(x^0))''_u = (f''_u(x^0),\, g''_u(x^0))$ is single-valued. The vectors $\bar{u}^1$, $\bar{u}^2$ satisfying (1) can be chosen as $\bar{u}^1 = (1/2,\, 1/2,\, 0)$, $\bar{u}^2 = (0,\, 0,\, -1/2)$. According to (5) the vector $\zeta^0 = (\zeta_1^0,\, \zeta_2^0)$ is expressed by $\xi^0 = (\xi_1^0,\, \xi_2^0)$ and η^0 as $\zeta^0 = (0,\, (1/2)\xi_1^0 - (1/2)\xi_2^0 - (1/2)\eta^0)$. Now for $y^0 = f''(x^0)(u,u)$, $z^0 = g''_u(x^0)$, $\xi^0 \in C'$, $\eta' \in K'[g(x^0)]$, $(\xi^0, \eta^0) \neq (0,0)$ and $u \in \mathrm{ker} h'(x^0) \setminus \{0\}$ we get

$$\langle \xi^0,\, y^0 \rangle + \langle \eta^0,\, z^0 \rangle + \langle \zeta^0,\, h''(x^0)(u,u) \rangle$$
$$= -4\xi_1^0(u_1^2 + u_2^2) + 2\xi_2^0(u_1^2 + u_2^2) + \eta^0(2u_1|u_1| + 2u_2|u_2|) + 6\zeta_2^0(u_1^2 + u_2^2)$$
$$= -\xi_1^0(u_1^2 + u_2^2) - \xi_2^0(u_1^2 + u_2^2) + \eta^0(2u_1|u_1| + 2u_2|u_2| - 3u_1^2 - 3u_2^2)$$
$$\leq -(\xi_1^0 + \xi_2^0 + \eta^0)(u_1^2 + u_2^2) < 0\,,$$

which shows that for any $u \in \mathrm{ker} h'(x^0) \setminus \{0\}$ condition $\mathbb{N}''$ with ζ^0 distinguished by (5) does not hold. Thus, in spite that condition $\mathbb{N}'$ is satisfied for any $u \in \mathrm{ker} h'(x^0)$, there are u for which $\mathbb{N}''$ fails. According to Theorem 4 the point x^0 is not a w-minimizer.

6 Final Comments

A natural question is, whether it is possible to relax the smoothness assumptions for the function h from C^2 to $C^{1,1}$. This problem is reasonable for the sake of the uniformity of the assumptions for all function data in the considered constrained problem (1). Having in mind the formulations of Theorems

2 and 4 it is not difficult to predict the anticipated result for the case of h being only $C^{1,1}$. It is clear by analogy, that the eventual proof should be based on an implicit function theorem for $C^{1,1}$ functions. Implicit function theorems in nonsmooth analysis are investigated by many authors and in many settings. Some variant with application to $C^{1,1}$ optimization gives Kummer [20]. However for our consideration the variant for directionally differentiable functions developed in Demyanov, Rubinov [5, Chapter VI, Section 1] seems to be more suitable. Still, there is a need for some adjustment. For instance, it is important to have calculation rules for the second-order Dini directional derivatives of the implicit function. Therefore, an attempt to move in this direction demands a development of new ideas and will overburden in some sense the present paper. For this reason we postpone the discussion on the possible relaxation of the smoothness assumptions for h.

References

1. B. Aghezzaf. Second-order necessary conditions of the Kuhn-Tucker type in multiobjective programming problems. Control Cybernet. 28(2):213–224, 1999.
2. A. Auslender. Stability in mathematical programming with nondifferentiable data. SIAM J. Control Optim., 22: 239–254, 1984.
3. J.-P. Aubin and H. Frankowska. Set-Valued Analysis. Birkhäuser, Boston, 1990.
4. S. Bolintinéanu and M. El Maghri. Second-order efficiency conditions and sensitivity of efficient points. J. Optim. Theory Appl., 98(3):569–592, 1998.
5. V. F. Demyanov and A. M. Rubinov. Constructive Nonsmooth Analysis. Peter Lang, Frankfurt am Main, 1995.
6. P. G. Georgiev and N. Zlateva. Second-order subdifferentials of $C^{1,1}$ functions and optimality conditions. Set-Valued Anal. 4(2): 101–117, 1996.
7. I. Ginchev. Higher order optimality conditions in nonsmooth vector optimization. In: A. Cambini, B. K. Dass, L. Martein (Eds.), "Generalized Convexity, Generalized Monotonicity, Optimality Conditions and Duality in Scalar and Vector Optimization", J. Stat. Manag. Syst. 5(1–3): 321–339, 2002.
8. I. Ginchev, A. Guerraggio, and M. Rocca. First-order conditions for $C^{0,1}$ constrained vector optimization. In: F. Giannessi, A. Maugeri (eds.), Variational analysis and applications, Proc. Erice, June 20– July 1, 2003, Kluwer Acad. Publ. & Springer, Dordrecht–Berlin, 2005, to appear.
9. I. Ginchev, A. Guerraggio, and M. Rocca. From scalar to vector optimization. Appl. Math., to appear.
10. I. Ginchev, A. Guerraggio, and M. Rocca. Second-order conditions in $C^{1,1}$ constrained vector optimization. In: J.-B Hiriart-Urruty, C. Lemarechal, B. Mordukhovich, Jie Sun, Roger J.-B. Wets (eds.), Variational Analysis, Optimization, and their Applications, Math. Program., Series B, to appear.
11. I. Ginchev and A. Hoffmann. Approximation of set-valued functions by single-valued one. Discuss. Math. Differ. Incl. Control Optim., 22: 33–66, 2002.
12. A. Guerraggio and D. T. Luc. Optimality conditions for $C^{1,1}$ vector optimization problems. J. Optim. Theory Appl., 109(3):615–629, 2001.
13. A. Guerraggio and D. T. Luc. Optimality conditions for $C^{1,1}$ constrained multiobjective problems. J. Optim. Theory Appl., 116(1):117–129, 2003.

14. J.-B. Hiriart-Urruty. New concepts in nondifferentiable programming. Analyse non convexe, Bull. Soc. Math. France 60:57–85, 1979.
15. J.-B. Hiriart-Urruty. Tangent cones, generalized gradients and mathematical programming in Banach spaces. Math. Oper. Res. 4:79–97, 1979.
16. J.-B. Hiriart-Urruty, J.-J Strodiot, and V. Hien Nguen: Generalized Hessian matrix and second order optimality conditions for problems with $C^{1,1}$ data. Appl. Math. Optim. 11:169–180, 1984.
17. B. Jiménez. Strict efficiency in vector optimization. J. Math. Anal. Appl. 265: 264–284, 2002.
18. B. Jiménez and V. Novo. First and second order conditions for strict minimality in nonsmooth vector optimization. J. Math. Anal. Appl. 284:496–510, 2003.
19. D. Klatte and K. Tammer. On the second order sufficient conditions to perturbed $C^{1,1}$ optimization problems. Optimization 19:169–179, 1988.
20. B. Kummer. An implicit function theorem for $C^{0,1}$ equations and parametric $C^{1,1}$ optimization. J. Math. Anal. Appl. 158:35–46, 1991.
21. L. Liu. The second-order conditions of nondominated solutions for $C^{1,1}$ generalized multiobjective mathematical programming. J. Syst. Sci. Math. Sci. 4(2):128–138, 1991.
22. L. Liu and M. Křířek. The second-order optimality conditions for nonlinear mathematical programming with $C^{1,1}$ data. Appl. Math. 42:311–320, 1997.
23. L. Liu, P. Neittaanmäki, and M. Křířek. Second-order optimality conditions for nondominated solutions of multiobjective programming with $C^{1,1}$ data. Appl. Math. 45:381–397, 2000.
24. C. Malivert. First and second order optimality conditions in vector optimization. Ann. Sci. Math. Québec 14:65–79, 1990.
25. R. T. Rockafellar. Convex Analysis. Princeton University Press, Princeton, 1970.
26. S. Wang. Second-order necessary and sufficient conditions in multiobjective programming. Numer. Funct. Anal. Opim. 12:237–252, 1991.
27. X. Q. Yang. Second-order conditions in $C^{1,1}$ optimization with applications. Numer. Funct. Anal. Optim. 14:621–632, 1993.
28. X. Q. Yang and V. Jeyakumar. Generalized second-order directional derivatives and optimization with $C^{1,1}$ functions. Optimization 26:165–185, 1992.