Some Characterizations of Convex Games *

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Summary. Several characterizations of convexity for totally balanced games are presented. As a preliminary result, it is first shown that the core of any subgame of a nonnegative totally balanced game can be easily obtained from the maximum average value (MAV) function of the game. This result is then used to get a characterization of convex games in terms of MAV functions. It is also proved that a game is convex if and only if all of its marginal games are totally balanced.

1 Introduction

This paper contains some characterizations of convexity for totally balanced games. Totally balancedness was defined by Shapley and Shubik [8] as the property of having all subgames with nonempty core. These authors proved that totally balanced games coincide with market games generated by exchange economies whose traders have continuous concave utility functions. Another characterization of totally balanced games, namely, as flow games, was provided by Kalai and Zemel [3]. A flow game arises from a directed network each of whose arcs has a given capacity and belongs to a unique player; the worth of a coalition is the maximum flow that can be sent from the source to the sink by using only the arcs owned by its members. The totally balanced character of flow games is a consequence of the max flow-min cut theorem of Ford and Fulkerson [2], according to which the maximum source to sink flow equals the minimum capacity of a cut (i.e., of a set of arcs such that, when removed from the network, nothing can be sent from the source to the sink). Nonnegative totally balanced games are also known to be equivalent to linear production games in the sense of Owen [6]. Indeed, to any nonnegative

* This work has been supported by the Ministerio de Ciencia y Tecnología (Spain) and the FEDER, project BEC2002-00642, and by the Departament d'Universitats, Recerca i Societat de la Informació, Direcció General de Recerca de la Generalitat de Catalunya, project 2001SGR-00162. The author thanks the support of the Barcelona Economics Program of CREA.

game one can associate the linear production game in which the resources are the players, each of which owns only one unit of himself, the goods are the nonempty coalitions, each of which can be sold at a price equal to its worth, and to produce one unit of a given coalition one requires one unit of each of its members. One can easily show that the linear production game so defined is precisely the totally balanced cover of the initial game (i.e., its smallest totally balanced majorant). Note that this linear production representation of a nonnegative totally balanced game needs n resources (n being the number of players) and $2^n - 1$ goods. An alternative linear production representation, requiring just one good and at most $2^n - 1$ resources, can be deduced from the observation, due to Kalai and Zemel [3], that the class of totally balanced games is the span of the additive games by the minimum operation.

Section 2 deals with nonnegative totally balanced games. For these games, a duality theory has been proposed by Martínez-Legaz [5], relating them to a special class of convex functions. To each nontrivial nonnegative game, one associates its maximum average value (MAV) function, which is convex and contains all the information on the game provided that it is totally balanced. Since totally balanced games have all subgames with nonempty core, the natural question arises how to compute these cores from the MAV function of the game. A simple answer to this question is given in Section 3, where it is shown that the computation of the core of a subgame reduces to minimizing the MAV function of the game subject to a simple linear constraint. In sections 4 and 5 we consider a very special class of totally balanced games, namely, that of convex games. Both sections have in common that they analyze convexity from the point of view of totally balancedness. In Section 4, this analysis is made by means of MAV functions: We characterize convex games in terms of the optimal solutions to the optimization problems that arise in the computation of the cores of the subgames. Section 5 analyzes convex games by means of their marginal games; the main result in this section establishes that convex games are precisely those games all of whose marginal games are totally balanced.

We shall use some basic notions of convex analysis (in particular, the concept of subdifferential), for which we refer to the classical book by Rockafellar [7].

2 The MAV Function of a TU Game

A TU game is a pair $\Gamma = (N, v)$, where N is a finite set of players, and $v : 2^N \longrightarrow \mathbb{R}$ is a function, called the characteristic function of the game, defined on the power set of N and satisfying the condition $v(\emptyset) = 0$. In this section we will only consider nontrivial nonnegative games, i.e., those whose characteristic function satisfies $v(S) \geq 0$ for all $S \in 2^N$ and is not identically zero. As is well known, there is no loss of generality in assuming that a totally balanced game is nonnegative, since one can replace the original game by another strategically equivalent 0-normalized game, which is totally balanced and nonnegative. For such games, the following duality theory has been developed by Martínez-Legaz [5]. One defines $\mu : \mathbb{R}_+^N \setminus \{0\} \longrightarrow \mathbb{R}_{++} \cup \{+\infty\}$, the maximum average value (MAV) function of Γ, by

$$\mu(w) = \max_{S \subset N} \frac{v(S)}{w(S)} \qquad (w = (w_i)_{i \in N} \in \mathbb{R}_+^N \setminus \{0\})$$

(with the conventions $\frac{\alpha}{0} = +\infty$ for any $\alpha > 0$ and $\frac{0}{0} = 0$), where $w(S) = \sum_{i \in S} w_i$. This function admits the following economic interpretation: if the components of w represent the salaries demanded by the players and $v(S)$ is the total amount of output produced to an employer by a set S of players when they use his resources, then $\mu(w)$ is the maximum amount of output per unit of money spent that the employer can obtain by hiring a coalition. In order to make this paper self-contained, we restate here the main results (Theorem 2.1 and Corollary 2.2) in Martínez-Legaz [5]:

Theorem 1. *The MAV function $\mu : \mathbb{R}_+^N \setminus \{0\} \longrightarrow \mathbb{R}_{++} \cup \{+\infty\}$ of any nontrivial nonnegative TU game $\Gamma = (N, v)$ is a positively homogeneous of degree -1 continuous convex function, finite valued on $\mathbb{R}_{++}^N$, such that, at each point where the gradient exists, all of its nonzero components are the same. Conversely, if $\mu : \mathbb{R}_+^N \setminus \{0\} \longrightarrow \mathbb{R}_{++} \cup \{+\infty\}$ satisfies these conditions then there exists a unique nontrivial totally balanced nonnegative TU game $\Gamma = (N, v)$ having μ as its MAV function; its characteristic function v is given by*

$$v(S) = \min_{w \in \mathbb{R}_+^N \setminus \{0\}} \mu(w) w(S) \qquad \forall S \subset N \tag{1}$$

(with the convention $(+\infty) \cdot 0 = +\infty$).

Corollary 1. *Let $\Gamma = (N, v)$ be a nontrivial nonnegative TU game with MAV function μ and let $\tilde{v} : 2^N \longrightarrow \mathbb{R}$ be defined by*

$$\tilde{v}(S) = \min_{w \in \mathbb{R}_+^N \setminus \{0\}} \mu(w) w(S) \qquad \forall S \subset N. \tag{2}$$

Then $\tilde{\Gamma} = (N, \tilde{v})$ is the totally balanced cover of Γ, i.e., $\tilde{v}$ is the smallest majorant of v that defines a totally balanced game.

Corollary 2. *The MAV function of any nontrivial nonnegative TU game coincides with that of its totally balanced cover.*

Proof. According to Theorem 1, for any nontrivial nonnegative n-person TU game there is a unique totally balanced game with the same MAV function; by Corollary 1, this totally balanced game is precisely the totally balanced cover of the initial game. □

To illustrate Corollary 1, consider the game (N, v) with $N = \{1, 2, 3\}$ and v defined by

$$v(S) = \begin{cases} 0 & \text{if } |S| \leq 1, \\ 1 & \text{if } |S| \geq 2. \end{cases}$$

One can easily check that the MAV function μ of this game is given by

$$\mu(w_1, w_2, w_3) = \frac{1}{\min\{w_1 + w_2, w_1 + w_3, w_2 + w_3\}}. \tag{3}$$

Thus, according to (2), the characteristic function of the totally balanced cover $(N, \tilde{v})$ of (N, v) is given by

$$\tilde{v}(S) = \begin{cases} \min_{w \in \mathbb{R}^3_+ \setminus \{0\}} \mu(w_1, w_2, w_3) w_i = 0 & \text{if } S = \{i\}, \\ \min_{w \in \mathbb{R}^3_+ \setminus \{0\}} \mu(w_1, w_2, w_3)(w_i + w_j) = 1 & \text{if } S = \{i, j\}, \text{ with } i \neq j, \\ \min_{w \in \mathbb{R}^3_+ \setminus \{0\}} \mu(w_1, w_2, w_3)(w_1 + w_2 + w_3) = 3/2 & \text{if } S = N. \end{cases}$$

Indeed, the minima in this formula are attained, e.g., at the points (w_1, w_2, w_3) given by $w_i = 0$ and $w_j = 1$ for $j \neq i$ in the first case, and at $(1, 1, 1)$ in the other two cases. Notice also that, by Corollary 2, the MAV function of $(N, \tilde{v})$ is μ.

3 Computing the Core of a Subgame

From Theorem 1, it follows that the characteristic function of a nontrivial nonnegative totally balanced game Γ can be recovered from its MAV function by means of (1). It turns out that, in this case, μ contains all the information on the game. Therefore, it is in principle possible to compute the cores of the subgames of Γ (which are nonempty as Γ is totally balanced) directly from μ. A way for doing it is suggested by the following theorem.

Theorem 2. *Let $\Gamma = (N, v)$ be a nontrivial nonnegative totally balanced TU game with MAV function μ and let $T \subset N$ be such that $v(T) > 0$. For any $x \in \mathbb{R}^T_+ \setminus \{0\}$, the following statements are equivalent:*

(1) The point x belongs to the core of the subgame $\Gamma_T = (T, v_{|2^T})$.
(2) There exists $\overline{w} \in \mathbb{R}^N_+ \setminus \{0\}$ such that $x = \overline{w}_T := (\overline{w}_i)_{i \in T}$ and $\mu(\overline{w}) = 1$; for every $\overline{w} \in \mathbb{R}^N_+ \setminus \{0\}$ satisfying these conditions, $\frac{\overline{w}}{x(T)}$ is an optimal solution of

$$(\mathcal{P}_T) \quad \begin{array}{l} \text{minimize } \mu(w) \\ \text{subject to } w(T) = 1. \end{array}$$

(3) There exists $\overline{w} \in \mathbb{R}^N_+ \setminus \{0\}$ such that $x = \overline{w}_T$, $\mu(\overline{w}) = 1$ and $\frac{\overline{w}}{x(T)}$ is an optimal solution of $(\mathcal{P}_T)$.

Proof. To prove the implication (1) $\Rightarrow$ (2), let x be a core element of Γ_T and take any $\overline{w} \in \mathbb{R}^N_+ \setminus \{0\}$ such that $\overline{w}_T = x$ and $v(S) \leq \overline{w}(S)$ for all $S \not\subset T$ (this condition can be achieved by giving sufficiently high values to $\overline{w}_i$ for $i \notin T$). Since we also have $v(S) \leq \overline{w}(S)$ for all $S \subset T$ (as $\overline{w}_T = x$ is in the core of Γ_T), it follows that $\mu(\overline{w}) \leq 1$. But we actually have $\mu(\overline{w}) = 1$, as a consequence of

$$\mu(\overline{w}) \geq \frac{v(T)}{\overline{w}(T)} = \frac{v(T)}{x(T)} = 1.$$

Let $\overline{w} \in \mathbb{R}^N_+ \setminus \{0\}$ be any point satisfying $x = \overline{w}_T$ and $\mu(\overline{w}) = 1$. By $\overline{w}_T = x$, the point $\overline{w}/x(T)$ is a feasible solution to problem $(\mathcal{P}_T)$. To show that it is optimal, it suffices to observe that, for each feasible $w \in \mathbb{R}^N_+ \setminus \{0\}$, one has

$$\mu(w) \geq \frac{v(T)}{w(T)} = v(T) = x(T) = \mu(\overline{w})x(T) = \mu\left(\frac{\overline{w}}{x(T)}\right).$$

Implication (2) $\Longrightarrow$ (3) is obvious. Let us now prove (3) $\Longrightarrow$ (1). Given $\overline{w}$ as in (3) and any $S \subset T$, we have

$$v(S) \leq \mu(\overline{w})\overline{w}(S) = \overline{w}(S) = x(S).$$

Take $w \in \mathbb{R}^N_+ \setminus \{0\}$ such that $\mu(w)w(T) = v(T)$ (the existence of w follows from Corollary 1). From the optimality of $\overline{w}/x(T)$, we deduce that

$$x(T) = \mu(\overline{w})x(T) = \mu\left(\frac{\overline{w}}{x(T)}\right) \leq \mu\left(\frac{w}{w(T)}\right) = \mu(w)w(T) = v(T);$$

hence $x(T) \leq v(T)$. Since the opposite inequality also holds, we conclude that x belongs to the core of Γ_T. □

As a particular case of Theorem 2, the next result characterizes the core of the game itself.

Corollary 3. *Let Γ be as in Theorem 2 with $v(N) > 0$. For any $x \in \mathbb{R}^N_+ \setminus \{0\}$, the following statements are equivalent:*

(1) x belongs to the core of Γ.
(2) $\mu(x) = 1$ and $\frac{x}{x(N)}$ is an optimal solution of $(\mathcal{P}_N)$.

Theorem 2 shows that each point belonging to the core of a subgame Γ_T induces an optimal solution of the associated optimization problem $(\mathcal{P}_T)$. In the opposite direction, we have

Corollary 4. *Let Γ and T be as in Theorem 2. For any $w \in \mathbb{R}^N_+ \setminus \{0\}$, the following statements are equivalent:*

(1) w is an optimal solution of $(\mathcal{P}_T)$.
(2) $w(T) = 1$ and $\mu(w)w_T$ belongs to the core of Γ_T.

Proof. Let $x = \mu(w)w_T$. If (1) holds then $\overline{w} := \mu(w)w$ satisfies (3) of Theorem 2, hence (2) follows from the implication (3) $\Rightarrow$ (1) in that theorem. Conversely, if (2) holds then $\overline{w} := \mu(w)w$ satisfies $x = \overline{w}_T$ and $\mu(\overline{w}) = 1$; hence, by (1) $\Rightarrow$ (2) in Theorem 2, we obtain (1). □

The preceding results allow us to interpret Problem $(\mathcal{P}_T)$ in economic terms as a mathematical formulation of the following question: Given the total amount $w(T) = 1$ of the salaries received by the members of T, which amount of output $\mu(w)$ per unit of money spent the employer will obtain in the worst case (i.e., under the least favorable distribution of those salaries)? In other words, which is the guaranteed return per unit of money spent to the employer of an investment of which one money unit is assigned to paying salaries to the members of T? By Corollary 4, an optimal solution w of $(\mathcal{P}_T)$ satisfies $\mu(w) = \mu(w)w(T) = \mu(w)w_T(T) = v(T)$, so that the optimal value of $(\mathcal{P}_T)$ (i.e., the guaranteed return considered above) is precisely $v(T)$. Following (2) of Corollary 4, the optimal solution w gives us the weights according to which the payoff $\mu(w) = v(T)$ should be distributed among the members of T.

In view of Theorem 2 and Corollary 4, to compute the core of a (nontrivial) subgame Γ_T one can apply the following method: find all optimal solutions $\overline{w}$ to the problem $(\mathcal{P}_T)$; the elements in the core of Γ_T are just those of the form $\mu(\overline{w})\overline{w}_T$. Indeed, by Corollary 4, each $\mu(\overline{w})\overline{w}_T$ belongs to the core of Γ_T. Conversely, each element x in the core of Γ_T can be obtained in this way. To see this, take $\overline{w}$ as in (3) of Theorem 2. Then $\overline{w}/x(T)$ is an optimal solution of $(\mathcal{P}_T)$ and, as $\mu(\overline{w}) = 1$, one has

$$x = \overline{w}_T = \mu(\overline{w})\overline{w}_T = \mu\left(\frac{\overline{w}}{x(T)}\right)\frac{\overline{w}_T}{x(T)}.$$

One can illustrate this method by computing the core of the unanimity game $\Gamma^P = (N, v^P)$ associated to a nonempty coalition $P \subset N$, whose characteristic function is given by

$$v^P(S) = \begin{cases} 1 \text{ if } S \supset P \\ 0 \text{ otherwise.} \end{cases}$$

As shown in Martínez-Legaz [5], the MAV function μ^P of Γ^P is simply $\mu^P(w) = \dfrac{1}{w(P)}$. Therefore, the minimizers of $\mu^P(w)$ under the constraint $w(N) = 1$ are those $\overline{w} \in \mathbb{R}^n_+ \setminus \{0\}$ such that $\overline{w}(P) = 1$ and $\overline{w}_{N \setminus P} = 0$. Since these points satisfy $\mu^P(\overline{w}) = 1$, it follows that they are the core elements of Γ^P.

As a second example, consider the game $\Gamma = (N, v)$ with $N = \{1, 2, 3\}$ and v defined by

$$v(S) = \begin{cases} 0 & \text{if } S = \{i\} \\ 1 & \text{if } S = \{i, j\}, \text{ with } i \neq j, \\ \frac{3}{2} & \text{if } S = N. \end{cases}$$

As shown above, the MAV function μ of this game is as in (3). To find the core elements of Γ one therefore has to look for the minimizers of (3) under the constraints $w_1 + w_2 + w_3 = 1,\ w_i \geq 0 \quad (i = 1, 2, 3)$. Since, by the first constraint, the right hand side of (3) is equal to $\frac{1}{1-\max\{w_1,w_2,w_3\}}$, this is equivalent to minimizing $\max\{w_1, w_2, w_3\}$ under the same constraints. This problem has a unique optimal solution, namely, the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. As $\mu(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \frac{3}{2}$, it turns out that the core of Γ is $\left\{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right\}$.

To summarize, our results show that the computation of the core of any subgame of a nonnegative totally balanced game reduces to the minimization of a convex function (of nonnegative variables) under one linear constraint. Although this can be regarded as an easy problem, one should keep in mind that to use this method requires first computing the MAV function of the game, which is, in general, a hard task. So we do not claim that our method has any advantage upon the standard one consisting in solving the inequality system that defines the core (except when the MAV function is known or easy to compute); however, it allows one to express easily the core of a nontrivial nonnegative totally balanced game directly in terms of its MAV function. The importance of this fact lies in that the MAV function provides an alternative representation of the game, but such representation would not be of much use if one could not express standard concepts, like the core, in terms of it in an easy way.

4 Characterizing Convex Games in Terms of Their MAV Functions

A very important class of totally balanced games is that of convex games. One says that $\Gamma = (N, v)$ is convex if for every two coalitions S and T one has

$$v\,(S) + v(T) \leq v(S \cup T) + v(S \cap T).$$

The term "convex" is due to the property of "increasing returns" enjoyed by these games. Indeed, it is well-known that Γ is convex if and only if it satisfies

$$v\,(S \cup \{i\}) - v\,(S) \leq v\,(T \cup \{i\}) - v(T)$$

for each $i \in N$ and every coalitions S, T such that $S \subset T \subset N \backslash \{i\}$. An example of convex games is provided by unanimity games (see Section 3).

In this section we give a necessary and sufficient condition for a nonnegative totally balanced game to be convex, in terms of its MAV function. This condition will be based upon the following characterization of convex games, due to Einy and Shitovitz [1, Props. 3.8 and 4.2]:

Proposition 1. *Let Γ be a totally balanced TU game. The following statements are equivalent:*

(1) Γ is convex.
(2) For every $S,T \subset N$ with $S \subset T$ and every core element x of Γ_S there is a core element w of Γ_T such that $w_S = x$.

Theorem 3. *Let Γ be as in Theorem 2. The following statements are equivalent:*

(1) Γ is convex.
(2) For every $S,T \subset N$ with $S \subset T$ and $v(S) > 0$ and every optimal solution $\overline{w}$ of $(\mathcal{P}_S)$ there is an optimal solution $\overline{\overline{w}}$ of $(\mathcal{P}_T)$ and $\lambda > 0$ such that $\overline{\overline{w}}_S = \lambda \overline{w}_S$ and $\overline{\overline{w}}/\overline{\overline{w}}(S)$ is an optimal solution of $(\mathcal{P}_S)$.

Proof. Let us first recall that a totally balanced game is nonnegative if and only if it is monotonic (see Martínez-Legaz [5, Prop. 2.3]). Hence, if S, T and N are as in (2) then $v(T) > 0$.

(1) $\Longrightarrow$ (2). If $S \subset T \subset N$, $v(S) > 0$ and $\overline{w}$ is an optimal solution of $(\mathcal{P}_S)$ then, by Corollary 4, $\overline{w}(S) = 1$ and $\mu(\overline{w})\overline{w}_S$ belongs to the core of Γ_S. According to Proposition 1, there exists a core element w' of Γ_T such that $w'_S = \mu(\overline{w})\overline{w}_S$. By Theorem 2, there exists $w'' \in \mathbb{R}^N_+ \setminus \{0\}$ such that $w''_T = w'$, $\mu(w'') = 1$ and $w''/w'(T)$ is an optimal solution of $(\mathcal{P}_T)$. Then, for $\overline{\overline{w}} := w''/w'(T)$ and $\lambda := \mu(\overline{w})/w'(T)$, one has

$$\overline{\overline{w}}_S = \frac{w''_S}{w'(T)} = \frac{(w''_T)_S}{w'(T)} = \frac{w'_S}{w'(T)} = \frac{\mu(\overline{w})\overline{w}_S}{w'(T)} = \lambda \overline{w}_S.$$

Moreover $\overline{\overline{w}}/\overline{\overline{w}}(S)$ is an optimal solution of $(\mathcal{P}_S)$, since it is a feasible point and has the same objective function value as the optimal solution $\overline{w}$:

$$\mu\left(\frac{\overline{\overline{w}}}{\overline{\overline{w}}(S)}\right) = \mu\left(\frac{w''}{w''(S)}\right) = w''(S)\,\mu(w'') = w''(S) = w''_T(S) = w'(S)$$
$$= w'_S(S) = \mu(\overline{w})\overline{w}_S(S) = \mu(\overline{w})\overline{w}(S) = \mu(\overline{w}).$$

(2) $\Longrightarrow$ (1). We shall prove that condition (2) above implies condition (2) of Proposition 1. Let $S \subset T \subset N$ and x be a core element of Γ_S. If $v(S) = 0$, from the monotonicity of v it follows that $x = 0$ and v vanishes at each subcoalition of S. Therefore one can easily check that, taking any core element $y = (y_i)_{i \in T\setminus S}$ of $\Gamma_{T\setminus S}$, the vector $w = (w_i)_{i\in T}$ defined by

$$w_i := 0 \text{ if } i \in S, \quad w_i := y_i + \frac{v(T) - v(T\setminus S)}{|T| - |S|} \text{ if } i \in T\setminus S,$$

belongs to the core of Γ_T and satisfies $w_S = 0 = x$. If $v(S) > 0$ then, by Theorem 2, there exists $\overline{w} \in \mathbb{R}^N_+ \setminus \{0\}$ such that $x = \overline{w}_S$, $\mu(\overline{w}) = 1$ and $\overline{w}/x(S)$ is an optimal solution of $(\mathcal{P}_S)$. According to condition (2), there are an optimal solution $\overline{\overline{w}}$ of $(\mathcal{P}_T)$ and $\lambda > 0$ such that $\overline{\overline{w}}_S = \lambda \overline{w}_S / x(S)$ and $\overline{\overline{w}}/\overline{\overline{w}}(S)$ is an optimal solution of $(\mathcal{P}_S)$; by $\overline{w}_S = x$, one has $\lambda = \overline{\overline{w}}_S(S) = \overline{\overline{w}}(S)$, so that

$$\overline{\overline{w}}_S = \frac{\overline{\overline{w}}(S)}{x(S)} \overline{w}_S = \frac{\overline{\overline{w}}(S)}{x(S)} x.$$

Let $w' := \frac{x(S)}{\overline{\overline{w}}(S)} \overline{\overline{w}}$. Since both $\frac{\overline{\overline{w}}}{\overline{\overline{w}}(S)}$ and $\frac{\overline{w}}{x(S)}$ are optimal solutions of $(\mathcal{P}_S)$, we have

$$\begin{aligned} \mu(w') &= \mu\left(\frac{x(S)}{\overline{\overline{w}}(S)} \overline{\overline{w}}\right) = \frac{1}{x(S)} \mu\left(\frac{\overline{\overline{w}}}{\overline{\overline{w}}(S)}\right) \\ &= \frac{1}{x(S)} \mu\left(\frac{\overline{w}}{x(S)}\right) = \mu(\overline{w}) = 1. \end{aligned}$$

On the other hand,

$$\frac{w'}{w'(T)} = \frac{\overline{\overline{w}}}{\overline{\overline{w}}(T)} = \overline{\overline{w}}$$

is an optimal solution of $(\mathcal{P}_T)$. Therefore, by Theorem 2, $w := w'_T$ belongs to the core of Γ_T; moreover, it satisfies

$$w_S = (w'_T)_S = w'_S = \frac{x(S)}{\overline{\overline{w}}(S)} \overline{\overline{w}}_S = x. \quad \square$$

5 Characterizing Convex Games in Terms of Their Marginals

Since totally balancedness is not a sufficient condition for a game $\Gamma = (N, v)$ to be convex, a natural question to ask is which additional conditions imposed on a totally balanced game ensure its convexity. The answer is given by the following theorem, which says that the required conditions are the totally balancedness of the marginal games as well. By the marginal game relative to coalition $T \subset N$, we mean the game $\Gamma'_T = (N \backslash T, v'_T)$ whose characteristic function is defined by $v'_T(S) = v(T \cup S) - v(T)$.

Theorem 4. *Let $\Gamma = (N, v)$ be a TU game. The following statements are equivalent:*

(1) Γ is convex.
(2) Γ'_T is convex for every $T \subset N$.
(3) Γ'_T is totally balanced for every $T \subset N$.
(4) Γ'_T is superadditive for every $T \subset N$.

Proof. To prove (1) $\Longrightarrow$ (2), let $T \subset N$ and $S_1, S_2 \subset N \backslash T$. Since Γ is convex, we have

$$\begin{aligned} v'_T(S_1) + v'_T(S_2) &= v(T \cup S_1) + v(T \cup S_2) - 2v(T) \leq \\ &\leq v(T \cup S_1 \cup S_2) + v(T \cup (S_1 \cap S_2)) - 2v(T) = \\ &= v'_T(S_1 \cup S_2) + v'_T(S_1 \cap S_2), \end{aligned}$$

which shows that v'_T is convex. Implications (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) follow from the well-known facts that all convex games are totally balanced and that the latter are superadditive. So, it only remains to prove (4) $\Longrightarrow$ (1); to this aim, it suffices to observe that, for each $S_1, S_2 \subset N$, one has

$$\begin{aligned} v(S_1) + v(S_2) &= v'_{S_1 \cap S_2}(S_1 \backslash S_2) + v'_{S_1 \cap S_2}(S_2 \backslash S_1) + 2v(S_1 \cap S_2) \leq \\ &\leq v'_{S_1 \cap S_2}((S_1 \cup S_2) \setminus (S_1 \cap S_2)) + 2v(S_1 \cap S_2) = \\ &= v(S_1 \cup S_2) + v(S_1 \cap S_2), \end{aligned}$$

where the inequality follows from the superadditivity of $v'_{S_1 \cap S_2}$. □

The equivalence between statements (1) and (4) of the preceding theorem was implicitly used in Martínez-Legaz [4] to prove Proposition 20 on a characterization of convex games in terms of indirect functions. Based on the equivalence (1) $\Longleftrightarrow$ (2), we will next present an alternative characterization of convex games, similar to that of totally balanced games in terms of balanced sets of coalitions (cf., e.g., Shapley and Shubik [8]). To this aim, we need to introduce the following notion:

Definition 1. *A collection $\mathcal{B}$ of subsets of $P \subset N$ is marginally $P-$balanced if $\bigcap_{S \in \mathcal{B}} S \notin \mathcal{B}$ and there exist positive weights $\{\gamma_S\}_{S \in \mathcal{B}}$ such that for each $i \in P \backslash \left(\bigcap_{S \in \mathcal{B}} S \right)$ one has $\sum_{\substack{S \in \mathcal{B} \\ S \ni i}} \gamma_S = 1$.*

Corollary 5. *A TU game $\Gamma = (N, v)$ is convex if and only if*

$$v(P) \geq \sum_{S \in \mathcal{B}} \gamma_S v(S) - \left(\sum_{S \in \mathcal{B}} \gamma_S - 1 \right) v \left(\bigcap_{S \in \mathcal{B}} S \right)$$

for every $P \subset N$ and every marginally P-balanced collection $\mathcal{B}$ with weights $\{\gamma_S\}_{S \in \mathcal{B}}$.

Proof. The "only if" part follows from the totally balancedness of v'_T, with $T = \cap_{S \in \mathcal{B}} S$, and the fact that the marginal P-balancedness of $\mathcal{B}$ is equivalent to the balancedness of $\{S \backslash T\}_{S \in \mathcal{B}}$ as a collection of subsets of $P \backslash T$, associating to each $S \backslash T$ the weight γ_S. To prove the converse, given $S, T \subset N$ with $S \not\subset T$ and $T \not\subset S$, let $P = S \cup T$. Then $\{S, T\}$ is marginally P-balanced with $\gamma_S = \gamma_T = 1$. Thus, the assumed inequality reduces to

$$v(S \cup T) \geq v(S) + v(T) - v(S \cap T). \quad \square$$

The interest of Corollary 5 lies in that it allows for an easy comparison between convex games and totally balanced games. Notice that the condition stated in Corollary 5 reduces to that of totally balancedness when restricted to collections $\mathcal{B}$ having an empty intersection. Moreover, it admits the following interpretation. If a fraction γ_S of coalition S forms (in the sense, e.g., that coalition S works during γ_S units of time), thus yielding an output $\gamma_S v(S)$, the total output that P can obtain is at least the sum of all these outputs minus that paid, by their extra effort, to the subcoalition consisting of those players who contributed $\sum_{S \in \mathcal{B}} \gamma_S$ (greater than 1) units of themselves (i.e., those players who worked during more than one unit of time). This payment is the output they would be able to obtain by themselves with this extra effort. Note that, as $\mathcal{B}$ is marginally P-balanced, the other players contribute exactly one unit of themselves.

Acknowledgments. I am grateful to Carles Rafels for his stimulating comments on convex games and for some useful information about the existing literature, and to Michael Maschler for suggestions on improvements and extensions of an earlier version. I also thank two anonymous referees for their careful reading and helpful corrections.

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