Sufficiency Conditions for Infinite Horizon Optimal Control Problems

Sabine Pickenhain[1] and Valeriya Lykina[2]

[1] Brandenburg University of Technology Cottbus, Germany.
sabine@math.tu-cottbus.de
[2] Brandenburg University of Technology Cottbus, Germany.
lykina@math.tu-cottbus.de

Summary. In this paper we formulate and use the duality concept of Klötzler (1977) for infinite horizon optimal control problems. The main idea is choosing weighted Sobolev and weighted L_p spaces as the state and control spaces, respectively. Different criteria of optimality are known for specific problems, e.g. the overtaking criterion of von Weizsäcker (1965), the catching up criterion of Gale (1967) and the sporadically catching up criterion of Halkin (1974). Corresponding to these criteria we develop the duality theory and prove sufficient conditions for local optimality. Here we use some remarkable properties of weighted spaces. An example is presented where the solution is obtained in the framework of these weighted spaces, but which does not belong to standard Sobolev spaces.

1 Introduction

It is well known that in problems of economic growth we have to deal with infinite horizon optimal control problems. The range of applications of such type of problems is large, starting with famous Ramsey accumulation model up to diverse problems in continuum mechanics. Numerous advertising models and renewable resources models go back to control problems with infinite horizon as well [8, 16]. Applications of infinite horizon problems in continuum mechanics were studied by Leizarowitz and Mizel [12], and Zaslavski [19].
The usual maximum principle cannot easily be adjusted to the case of infinite horizon problems as it was first demonstrated in an example of Halkin [9]. Since the usual transversality condition does not hold anymore, some authors have investigated particular situations where ad-hoc transversality conditions are necessary for optimality. Such transversality conditions were obtained by Aseev and Kryazhimskiy [1], Michel [14] and Smirnov [17]. The simplest way to solve optimal control problems with infinite horizon is to find a solution on a finite interval and try to extend the solution onto the whole half-axis. But there is no guarantee for the extended solution to be optimal on an infinite interval. For that reason the proof of optimality is very important and is usually

based on sufficient conditions. A lot of work has been done in the last decades to prove necessary conditions for problems in the calculus of variations, see e.g. [4, 5], and optimal control, see e.g. [6]. Results concerning sufficiency conditions were derived via Fenchel-Rockafellar duality by Rockafellar [15], Aubin and Clarke [2], Magill [13], and Benveniste and Scheinkman [3]. In our paper we use the duality concept of Klötzler [10] and a special choice of state and control spaces to obtain sufficiency conditions. Considering the exponential factor $e^{-\rho t}$ as a density function we propose to choose weighted Sobolev and weighted L_p-spaces as state and control spaces, respectively, defined in the second section. Here we include a brief review of important aspects concerning differences between Lebesgue and improper Riemann integrals, which can influence optimality on an infinite interval. According to [8] and [6], there are several optimality criteria for considered class of problems and they are introduced in section 3. The fourth section is devoted to the development of the duality theory taking some properties of weighted spaces into account. A localized problem and the corresponding dual problem are formulated in section 5. The last section includes sufficiency conditions, which are proved via linear approach in the dual problem. An example illustrating existence of optimal solution with respect to weighted spaces-while no solution in usual Sobolev spaces exists- is presented as well.

2 Problem Formulation

We deal with problems of the following type: Minimize the functional

$$J(x,u) = \int_0^\infty f(t, x(t), u(t))\nu(t)\, dt \tag{1}$$

with respect to all

$$(x,u) \in W^{1,n}_{p,\nu}(0,\infty) \times L^r_{p,\nu}(0,\infty) \tag{2}$$

fulfilling the

State equation	$\dot{x}(t) = g(t, x(t), u(t))$ a.e. on $(0,\infty)$,	(3)
Control restriction	$u(t) \in U$ a.e. on $(0,\infty)$,	(4)
Initial condition	$x(0) = x_0$.	(5)

Here U is a nonempty compact set in $\mathbb{R}^r$. The spaces $W^{1,n}_{p,\nu}(0,\infty)$ and $L^r_{p,\nu}(0,\infty)$ will be defined below.

2.1 Weighted Sobolev Spaces

We consider weighted Sobolev spaces $W^{1,n}_{p,\nu}(\Omega)$ as subspaces of weighted $L^n_{p,\nu}(\Omega)$ spaces of those absolutely continuous functions x for which both x and its derivative $\dot{x}$ lie in $L^n_{p,\nu}(\Omega)$, see [11].

Let $\Omega = [0,\infty)$ and let $\mathcal{M}^n = \mathcal{M}(\Omega;\mathbb{R}^n)$ denote the space of Lebesgue measurable functions defined on Ω with values in $\mathbb{R}^n$. The function $\nu : \Omega \to \mathbb{R}_+\backslash\{0\}$ is a density function if $\nu \in \mathcal{M}$ and

$$\int_\Omega \nu(t)dt < \infty.$$

Let $\nu \in C(\Omega)$, $0 < \nu(t) < \infty$ be given, then we define the space $L^n_{p,\nu}(\Omega)$ by

$$L^n_{p,\nu}(\Omega) = \{x \in \mathcal{M}^n |\, \|x\|^p_p := \int_\Omega |x(t)|^p \nu(t)\, dt < \infty\,\}, \quad (\text{when } p \geq 2)$$
$$L^n_{\infty,\nu}(\Omega) = \{x \in \mathcal{M}^n |\, \|x\|_\infty := \operatorname*{ess\,sup}_{t\in\Omega} |x(t)|\nu(t) < \infty\,\} \quad (\text{when } p = \infty)$$

and the weighted Sobolev space by

$$W^{1,n}_{p,\nu}(\Omega) = \{x \in \mathcal{M}^n |\, x \in L^n_{p,\nu}(\Omega), \dot{x} \in L^n_{p,\nu}(\Omega)\,\} \quad (p = \infty).$$

Here $\dot{x}$ is the distributional derivative of x in the sense of [18, p.49]. This space, equipped with the norm

$$\|x\|^p_{W^{1,n}_{p,\nu}} = \int_\Omega \{|x(t)| + |\dot{x}(t)|\}^p \nu(t) dt,$$

is a Banach space. For later use we also define the space

$$L^{n\times n}_{\infty,\nu^{-1}}(\Omega) = \left\{ Q \in \mathcal{M}^{n\times n} \;\middle|\; \|Q\|_\infty := \max_{i,j} \left(\operatorname*{ess\,sup}_{t\in\Omega} \frac{|Q_{i,j}(t)|}{\nu(t)} \right) < \infty \right\}.$$

For $x \in L^n_{p,\nu}(\Omega)$ and $y \in L^n_{q,\nu^{1-q}}(\Omega)$ the scalar product $\ll x, y \gg$ in $L^n_2(\Omega)$ defines a continuous bilinear form, since

$$\begin{aligned} |\ll x, y \gg| &\leq \int_0^\infty |x(t)|\nu^{1/p}(t)|y(t)|\nu^{-1+1/q}(t)dt \\ &\leq \|x\|_{L^n_{p,\nu}(\Omega)} \|y\|_{L^n_{q,\nu^{1-q}}(\Omega)} \end{aligned}$$

holds true. For the special case $p = 2$ one has $[L^n_{2,\nu}(\Omega)]^* = L^n_{2,\nu}(\Omega)$ due to the Riesz representation theorem. Therefore, we obtain the following relation between the scalar products in $L^n_{2,\nu}(\Omega)$ and $L^n_2(\Omega)$: For $x \in L^n_{2,\nu}(\Omega)$ and $y \in L^n_{2,\nu^{-1}}(\Omega)$ there exists $\hat{y} \in L^n_{2,\nu}(\Omega)$ such that

$$\begin{aligned} \langle x, \hat{y} \rangle_{L^n_{2,\nu}(\Omega)} &= \ll x, y \gg_{L^n_2(\Omega)} \\ \hat{y} &= y/\nu. \end{aligned} \tag{6}$$

Equation (6) is essentially used to formulate the duality theory in the sense of Klötzler in the following sections.

Remark. It is well known, see [7], that the inclusion $L^n_{p,\nu}(\Omega) \subseteq L^n_{q,\nu}(\Omega)$ holds true for all $p \geq q$, i.e. there is a $C \in \mathbb{R}_+$ such that

$$\|x\|_{L^n_{q,\nu}} \leq C \, \|x\|_{L^n_{p,\nu}} . \tag{7}$$

Note that here and in the proofs of other sections we abbreviate $L^n_{p,\nu}(\Omega)$ by $L^n_{p,\nu}$ in the indices.

Now some aspects concerning the integral in (1) should be mentioned. We assume that the function f in (1) is continuously differentiable and allow both Lebesgue and improper Riemann integrals to appear in (1). The main difference between the Lebesgue and improper Riemann integrals is that one of them may not exist while the other one is convergent. In the case

$$\int\limits_0^\infty |f(t,x(t),u(t))| \, \nu(t)dt < \infty, \tag{8}$$

both Lebesgue and Riemann improper integrals exist and coincide [7] and we have

$$\begin{aligned} \int\limits_0^\infty f(t,x(t),u(t))\nu(t)dt &= \lim_{T\to\infty} \int\limits_0^T f(t,x(t),u(t))d\nu(t)dt \\ &= \lim_{T\to\infty} J_T(x(t),u(t)). \end{aligned} \tag{9}$$

But it can happen, that the integral in (8), i.e. Lebesgue integral, does not exist and at the same time the Riemann integral is conditionally convergent.

3 Global Optimality Criteria

In the case of infinite horizon optimal control problems the standard optimality notion should be newly defined. Namely, there are several new optimality criteria [8], which are also suitable in the case of a divergent integral in (1). We introduce global optimality criteria for the case when the integral in (1) is understood in the Lebesgue sense.

Definition 1. *Suppose that the integral in (1) exists. Furthermore, denote the problem (1)-(5) by (P_∞). Let (x^*,u^*) be an admissible pair of (P_∞). For any other arbitrary admissible pair (x,u) and for $T \geq 0$, let*

$$\Delta(T) = \int_0^T f(t,x(t),u(t))\nu(t)\,dt - \int_0^T f(t,x^*(t),u^*(t))\nu(t)\,dt.$$

Then the pair (x^*,u^*) is called optimal for (P_∞) in the sense of

1. *criterion L1, if for any admissible pair* (x,u) *we have* $\lim_{T\to\infty} \Delta(T) \geq 0$*;*
2. *criterion L2, if for any admissible pair* (x,u) *there exists a moment* τ *such that for all* $T \geq \tau$ *we have* $\Delta(T) \geq 0$ *(overtaking criterion of von Weizsäcker (1965)).*

Optimality in the sense of L1 coincides with usual optimality, while L2-optimality is stronger than the first one. The definition of local optimality will be introduced later.

Remark. In the case of Riemann improper integral in (1) there are some other optimality criteria which are defined in [8].

4 Duality Theory

Before formulating the duality theory for infinite horizon optimal control problems, we prove:

Lemma 1. *Let* (x^*,u^*) *be an admissible pair of* (P_∞) *and* $S : \Omega \times \mathbb{R}^n \to \mathbb{R}$ *be a function of the form*

$$S(t,\xi) = a(t) + y(t)^T(\xi - x^*(t)) + 1/2(\xi - x^*(t))^T Q(t)(\xi - x^*(t)), \quad (10)$$

with $a \in W_1^1(\Omega)$, $y \in W^{1,n}_{q,\nu^{1-q}}(\Omega)$, *and* $Q \in W^{1,n\times n}_{\infty,\nu^{-1}}(\Omega)$ *symmetric. Assume also that* $p \geq q$*. Then, for any* $x \in W^{1,n}_{p,\nu}(\Omega)$ *with* $x(0) = x_0$*, one has:*

$$\lim_{T\to\infty} S(T, x(T)) = 0, \quad (11)$$

$$\int_0^\infty \frac{d}{dt} S(t,x(t))dt = -S(0,x_0), \quad (12)$$

Proof. Observe that

$$\tilde{S} := \int_0^\infty |S(t,x(t))|\, dt \leq \int_0^\infty |a(t)|\, dt + \int_0^\infty \left|y(t)^T(x(t) - x^*(t))\right| dt$$
$$+ \frac{1}{2}\int_0^\infty \left|(x(t) - x^*(t))^T Q(t)(x(t) - x^*(t))\right| dt.$$

Applying Hölder's inequality we obtain

$$\tilde{S} \leq \|a\|_{W_1^1} + \left(\int_0^\infty |y(t)|^q \nu^{1-q}(t)dt\right)^{1/q} \cdot \left(\int_0^\infty |x(t) - x^*(t)|^p \nu(t)dt\right)^{1/p}$$

$$+\frac{1}{2}\left(\int_0^\infty |(x(t)-x^*(t))|^p\,\nu(t)dt\right)^{1/p}\cdot\left(\int_0^\infty |Q(t)(x(t)-x^*(t))|^q\,\nu^{1-q}(t)dt\right)^{1/q}.$$

This yields

$$\begin{aligned}\tilde{S} &\le \|a\|_{W_1^1} + \|y\|_{L^n_{q,\nu^{1-q}}}\|x-x^*\|_{L^n_{p,\nu}} + \frac{1}{2}\|x-x^*\|_{L^n_{p,\nu}}\|Q(x-x^*)\|_{L^n_{q,\nu^{1-q}}}\\ &\le \|a\|_{W_1^1} + \|y\|_{L^n_{q,\nu^{1-q}}}\|x-x^*\|_{L^n_{p,\nu}} + C\|x-x^*\|^2_{L^n_{p,\nu}}\|Q\|_{L^{n\times n}_{\infty,\nu^{-1}}} < \infty.\end{aligned}$$

The last estimate is true because of

$$\begin{aligned}\|Q(x-x^*)\|^q_{L^n_{q,\nu^{1-q}}} &= \int_0^\infty |Q(t)(x-x^*)(t)|^q\,\nu^{1-q}(t)dt\\ &\le 2^q\int_0^\infty\left(\max_{i,j}\ \operatorname*{ess\,sup}_{t\ge 0}\frac{|Q_{i,j}(t)|}{\nu(t)}\right)^q |(x-x^*)(t)|^q\,\nu(t)dt\\ &\le 2^q\|Q\|^q_{L^{n\times n}_{\infty,\nu^{-1}}}\cdot\|(x-x^*)\|^q_{L^n_{q,\nu}}\\ &\le \left(2C\|Q\|_{L^{n\times n}_{\infty,\nu^{-1}}}\cdot\|(x-x^*)\|_{L^n_{p,\nu}}\right)^q.\end{aligned}$$

To estimate $\|(x-x^*)\|_{L^n_{q,\nu}}$ we applied (7). The convergence of $\int_0^\infty |S(t,x(t))|\,dt$ yields (11), since

$$\begin{aligned}\lim_{T\to\infty}\int_0^T S(t,x(t))dt &= \lim_{T\to\infty}\left(\int_0^{T-1} S(t,x(t))dt + \int_{T-1}^T S(t,x(t))dt\right)\\ &= \lim_{T\to\infty}\int_0^T S(t,x(t))dt + \lim_{\tau\to\infty} S(\tau,x(\tau)),\end{aligned}$$

where τ is an element in $[T-1,T]$. Condition (12) can now easily be derived applying (11). □

We introduce the Hamiltonian as

$$\mathcal{H}(t,\xi,\eta) = \sup_{v\in U} H(t,\xi,v,\eta), \tag{13}$$

where

$$H(t,\xi,v,\eta) = -f(t,\xi,v) + \frac{1}{\nu(t)} < \eta, g(t,\xi,v) >$$

represents the Pontrjagin function. Furthermore, we define the set

$$Y = \left\{ S : \Omega \times \mathbb{R}^n \to \mathbb{R} \;\middle|\; \begin{array}{c} S(t,\xi) = a(t) + y(t)^T(\xi - x^*(t)) \\ \quad + \frac{1}{2}(\xi - x^*(t))^T Q(t)(\xi - x^*(t)) \\ \\ a \in W_1^1,\ y \in W^{1,n}_{\infty,\nu^{1-q}},\ Q \in W^{1,n\times n}_{\infty,\nu^{-1}} \\ Q - \text{symmetric} \\ \\ \frac{1}{\nu(t)}\partial_t S(t,\xi) + \mathcal{H}(t,\xi,\partial_\xi S(t,\xi)) \le 0 \\ \forall (t,\xi) \in \Omega \times \mathbb{R}^n \end{array} \right\}.$$

Using the dual problem formalism described in [10] we construct a problem (D_∞) and prove:

Theorem 1. *Let a problem (P_∞) be given. Then for the problem*

$$(D_\infty) \quad \textit{maximize } g_\infty(S) := -S(0,x_0) \textit{ with respect to } S \in Y,$$

the weak duality relation

$$\inf(P_\infty) \ge \sup(D_\infty) \tag{14}$$

holds.

Proof. Let (x,u) be admissible for (P_∞) and S be admissible for (D_∞), i.e. $S \in Y$. Then we have

$$\begin{aligned} J(x,u) &= \int_0^\infty f(t,x(t),u(t))\nu(t)dt \\ &= \int_0^\infty \left(-H(t,x(t),u(t),\partial_\xi S(t,x(t)))\right)\nu(t)dt \\ &+ \int_0^\infty \left(\frac{\partial_\xi S(t,x(t))}{\nu(t)} g(t,x(t),u(t))\right)\nu(t)dt \\ &= \int_0^\infty \left(-H(t,x(t),u(t),\partial_\xi S(t,x(t))) - \frac{\partial_t S(t,x(t))}{\nu(t)}\right)\nu(t)dt \\ &+ \int_0^\infty \left(\frac{\partial_t S(t,x(t))}{\nu(t)} + \frac{\partial_\xi S(t,x(t))}{\nu(t)}\dot{x}(t)\right)\nu(t)dt \\ &\ge -\int_0^\infty \left(\mathcal{H}(t,x(t),\partial_\xi S(t,x(t))) + \frac{\partial_t S(t,x(t))}{\nu(t)}\right)\nu(t)dt \\ &+ \int_0^\infty \left(\partial_t S(t,x(t)) + \partial_\xi S(t,x(t))\dot{x}(t)\right)dt \\ &\ge -\int_0^\infty \sup_{\xi\in\mathbb{R}^n}\left\{\left(\mathcal{H}(t,\xi,\partial_\xi S(t,\xi)) + \frac{\partial_t S(t,\xi)}{\nu(t)}\right)\right\}\nu(t)dt \\ &+ \int_0^\infty \left(\partial_t S(t,x(t)) + \partial_\xi S(t,x(t))\dot{x}(t)\right)dt. \end{aligned}$$

This shows that

$$J(x,u) \geq \int_0^\infty \frac{d}{dt} S(t,x(t))dt = \lim_{T\to\infty} \int_0^T \frac{d}{dt} S(t,x(t))dt$$
$$= \lim_{T\to\infty} S(T,x(T)) - S(0,x(0)) = -S(0,x_0),$$

completing the proof in this way. □

Remark. As we can see, the proper decision variable in the dual problem (D_∞) is (a,y,Q), but we use $S \in Y$ for simplicity.

The next two corollaries provide sufficiency conditions for global optimality in the sense of criterion L1 and criterion L2, respectively.

Corollary 1. *An admissible pair (x^*,u^*) is a global minimizer of (P_∞) in the sense of criterion L1, if there exists an admissible S^* for (D_∞), such that the following conditions are fulfilled for almost all $t>0$:*

$$(M)\ \ \mathcal{H}(t,x^*(t),\partial_\xi S^*(t,x^*(t))) = H(t,x^*(t),u^*(t),\partial_\xi S^*(t,x^*(t))),$$

$$(HJ) \qquad \tfrac{1}{\nu(t)}\partial_t S^*(t,x^*(t)) + \mathcal{H}(t,x^*(t),\partial_\xi S(t,x^*(t))) = 0.$$

Proof. This follows immediately from Theorem 1. □

Remark. The boundary condition

$$(B) \qquad \lim_{T\to\infty} S^*(T,x^*(T)) = 0 \tag{15}$$

is automatically satisfied due to Lemma 1.

Corollary 2. *An admissible pair (x^*,u^*) is a global minimizer of (P_∞) in the sense of criterion L2, if there exists a family $\{(S_T^*)\}_{T\geq\tau} \subset Y$, for a sufficiently large τ, such that the following conditions are fulfilled for almost all $t \in (0,T)$:*

$$(M_T)\ \mathcal{H}(t,x^*(t),\partial_\xi(S_T^*)(t,x^*(t))) = H(t,x^*(t),u^*(t),\partial_\xi(S_T^*)(t,x^*(t))),$$

$$(HJ_T) \quad \tfrac{1}{\nu(t)}\partial_t(S_T^*)(t,x^*(t)) + \mathcal{H}(t,x^*(t),\partial_\xi(S_T^*)(t,x^*(t))) = 0,$$

$$(B_T) \qquad \inf_{\xi\in\mathbb{R}^n} S_T^*(T,\xi) = S_T^*(T,x^*(T)). \tag{16}$$

Proof: According to criterion L2, we obtain the following inequalities for all $T \geq \tau$ and $S_T^* \in Y$:

$$\begin{aligned}
J_T(x,u) &= \int_0^T f(t,x(t),u(t))\nu(t)dt \\
&= \int_0^T \left(-H(t,x(t),u(t),\partial_\xi S_T^*(t,x(t))) - \frac{\partial_t S_T^*(t,x(t))}{\nu(t)}\right)\nu(t)dt \\
&+ \int_0^T \left(\frac{\partial_t S_T^*(t,x(t))}{\nu(t)} + \frac{\partial_\xi S_T^*(t,x(t))}{\nu(t)}\dot{x}(t)\right)\nu(t)dt \\
&\geq -\int_0^T \left(\mathcal{H}(t,x(t),\partial_\xi S_T^*(t,x(t))) + \frac{\partial_t S_T^*(t,x(t))}{\nu(t)}\right)\nu(t)dt \qquad (17)\\
&+ \int_0^T \left(\partial_t S_T^*(t,x(t)) + \partial_\xi S_T^*(t,x(t))\dot{x}(t)\right)dt \\
&\geq -\int_0^T \sup_{\xi\in\mathbb{R}^n}\left\{\left(\mathcal{H}(t,\xi,\partial_\xi S_T^*(t,\xi)) + \frac{\partial_t S_T^*(t,\xi)}{\nu(t)}\right)\right\}\nu(t)dt \\
&+ S_T^*(T,x(T)) - S_T^*(0,x(0)) \\
&\geq \inf_{\xi\in\mathbb{R}^n} S_T^*(T,\xi) - S_T^*(0,x_0).
\end{aligned}$$

All inequalities in (17) become equalities if the conditions (M_T), (HJ_T) and (B_T) are satisfied for the pair (x^*,u^*). This means that for all $T \geq \tau$ the strong duality relation for problems with finite horizon, see [10],

$$J_T(x^*,u^*) = \inf_{\xi\in\mathbb{R}^n}\{S_T^*(T,\xi) - S_T^*(0,x_0)\} \qquad (18)$$

holds. Having in mind the definition of criterion L2, we can easily see that the pair is the optimal solution of the problem (P_∞) in the sense of criterion L2. □

Remark. It follows from (16) that the transversality condition $y_T(T) = 0$ has to be satisfied for all $T \geq \tau$.

5 Formulation of the Local Problem and Local Optimality Criteria

In this section we discuss local optimality. Evidently every function from $W^{1,n}_{p,\nu}$ is absolutely continuous. For that reason the imbedding of the weighted Sobolev space into the space of continuous functions allows us to formulate the notion of strong local optimality as follows.

Definition 2. *An admissible pair (x^*,u^*) of (P_∞) is strong local optimal in the sense of criterion L1, if there is a function $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ such that $J(x^*,u^*) \leq J(x,u)$ for any admissible pair (x,u) of (P_∞) satisfying $|(x(t) - x^*(t))\nu(t)| < \delta(t)$ for all $t > 0$.*

In this paper we concentrate only on L1 strong local optimality while definition of L2 strong local optimality will be omitted. The problem (P_∞) can now be localized by writing (2) in the form

$$[x\,,\,u] \in W^{1,n}_{p,\nu}(\Omega) \times L^r_{p,\nu}(\Omega),\;\; x(t) \in \mathcal{K}_{\delta,\nu}(x^*(t)),$$

where

$$\mathcal{K}_{\delta,\nu}(x^*(t)) := \{\xi \in \mathbb{R}^n \,|\, |(\xi - x^*(t))\nu(t)| < \delta(t)\,\}.$$

The localized version of problem (P_∞) will be denoted by $(P_{\infty,\mathrm{loc}})$. Now we define the set

$$Y_{\mathrm{loc}} = \left\{ S \in Y \;\middle|\; \begin{array}{c} \frac{1}{\nu(t)}\partial_t S(t,\xi) + \mathcal{H}(t,\xi,\partial_\xi S(t,\xi)) \le 0 \\ \\ \text{on } \{(t,\xi)\,|t \in \Omega,\; \xi \in \mathcal{K}_{\delta,\nu}(x^*(t))\,\} \end{array} \right\}.$$

Using this notation we now formulate an equivalent version of Theorem 1 for the localized case.

Theorem 2. *Let us consider the problem $(P_{\infty,\mathrm{loc}})$. Then, for problem*

$$(D_{\infty,\mathrm{loc}}) \quad \textit{maximize } \; g_\infty(S) := -S(0,x_0) \; \textit{with respect to } S \in Y_{\mathrm{loc}},$$

the weak duality relation $\inf(P_{\infty,\mathrm{loc}}) \ge \sup(D_{\infty,\mathrm{loc}})$ holds.

Remark. The corresponding versions of Corollaries 1 and 2 hold true for $(P_{\infty,\mathrm{loc}})$ if S is local admissible in the dual problem, i.e. admissible in $(D_{\infty,\mathrm{loc}})$.

6 Sufficiency Conditions for Local Optimality

6.1 Auxiliary Result

It is important to ascertain that the adjoint variable belongs to the desirable space $W^{1,n}_{q,\nu^{1-q}}(\Omega)$. For this aim we prove

Lemma 2. *Consider an admissible pair (x^*, u^*) of (P_∞). Assume that, for some constant $C \in \mathbb{R}_+$, we have:*

$$|\partial_\xi g(t,x^*(t),u^*(t))| \le C(\| x^* \|_{L^n_{p,\nu}(\Omega)} + \|u^*\|_{L^r_{p,\nu}(\Omega)}),$$
$$\partial_\xi f(t,x^*(t),u^*(t)) \in L^n_{q,\nu}(\Omega); \;\; \omega(t,0) \in L^n_{q,\nu^{1-q}}(\Omega),$$

$$t \to \Phi(t) = \int_0^t \omega(t,s)\partial_\xi f(s,x^*(s),u^*(s))\nu(s)ds \; \textit{is in} \; L^n_{q,\nu^{1-q}}(\Omega).$$

Here $\omega(t,s)$ denotes the Green matrix defined for $t \ge s$ as a solution of the system

$$\frac{d\omega(t,s)}{dt} = -\partial_\xi g(t,x^*(t),u^*(t))\omega(t,s)\;, \quad \omega(s,s) = I. \tag{19}$$

Then the solution y of the adjoint equation

$$\dot{y}(t) = -y(t)^T\partial_\xi g(t,x^*(t),u^*(t)) + \partial_\xi f(t,x^*(t),u^*(t))\nu(t) \tag{20}$$

is an element of the weighted Sobolev space $W^{1,n}_{q,\nu^{1-q}}(\Omega)$.

Proof. The solution of (20) can be written as

$$y(t) = \omega(t,0)y(0) + \Phi(t),$$

where $\omega(t,s)$ is a solution of the system (19). Noticing

$$\|y\|_{W^{1,n}_{q,\nu^{1-q}}} \leq C\left(\|y\|_{L^n_{q,\nu^{1-q}}} + \|\dot{y}\|_{L^n_{q,\nu^{1-q}}}\right),$$

we estimate first the function y itself

$$\begin{aligned}
\|y\|^q_{L^n_{q,\nu^{1-q}}} &= \int_0^\infty |\omega(t,0)y(0) + \Phi(t)|^q\, \nu^{1-q}(t)dt \\
&\leq 2^q \int_0^\infty |\omega(t,0)y(0)|^q\, \nu^{1-q}(t)dt + 2^q \int_0^\infty |\Phi(t)|^q\, \nu^{1-q}(t)dt \\
&= 2^q\, \|\omega(t,0)y(0)\|^q_{L^n_{q,\nu^{1-q}}} + 2^q\, \|\Phi\|^q_{L^n_{q,\nu^{1-q}}},
\end{aligned}$$

and then its distributional derivative

$$\begin{aligned}
\|\dot{y}\|^q_{L^n_{q,\nu^{1-q}}} &= \int_0^\infty \left|-y(t)^T \partial_\xi g(t,x^*(t),u^*(t)) + \partial_\xi f(t,x^*(t),u^*(t))\nu(t)\right|^q \nu^{1-q}dt \\
&\leq (2C)^q \left(\|\, x^* \,\|_{L^n_{p,\nu}} + \|u^*\|_{L^n_{p,\nu}}\right)^q \int_0^\infty |y(t)|^q\, \nu^{1-q}(t)dt \\
&+ 2^q \int_0^\infty |\partial_\xi f(t,x^*(t),u^*(t))\nu(t)|^q\, \nu^{1-q}(t)dt \\
&= \left(2\tilde{C}\, \|y\|_{L^n_{q,\nu^{1-q}}}\right)^q + 2^q\, \|\partial_\xi f\|^q_{L^n_{q,\nu}} \\
&\leq \left(4\tilde{C}\right)^q \left(\|\omega(t,0)y(0)\|^q_{L^n_{q,\nu^{1-q}}} + \|\Phi\|^q_{L^n_{q,\nu^{1-q}}}\right) + 2^q\, \|\partial_\xi f\|^q_{L^n_{q,\nu}}\,.
\end{aligned}$$

Under the assumptions of this lemma, both $\|y\|_{L^n_{q,\nu^{1-q}}}$ and $\|\dot{y}\|_{L^n_{q,\nu^{1-q}}}$ are finite and we conclude $y \in W^{1,n}_{q,\nu^{1-q}}(\Omega)$. □

6.2 The Main Result on Sufficiency Conditions

We now present the main result of this paper and prove sufficiency conditions for local optimality. We have developed the duality theory via quadratic approach in the dual problem, but we now formulate the following theorem applying the linear approach. To derive analogous sufficiency conditions by means of the quadratic approach we need some a priori assumptions which guarantee that $Q \in L^{n\times n}_{\infty,\nu^{-1}}(\Omega)$ holds. This will be a task of further studies.

Theorem 3. *Let the assumptions of Lemma 2 be satisfied for an admissible pair (x^*, u^*) of $(P_{\infty,\mathrm{loc}})$. Suppose that y solves (20) and fulfills the conditions*

$$\partial^2_{vv} H(t, x^*(t), u^*(t), y(t)) < 0, \tag{21}$$

$$\mathcal{H}(t, x^*(t), y(t)) = H(t, x^*(t), u^*(t), y(t)), \tag{22}$$

$$\partial^2_{\xi\xi}\mathcal{H}(t, x^*(t), y(t)) \; is\, negative - definite, \tag{23}$$

almost everywhere on Ω. Then the pair (x^*, u^*) is a strong local minimizer of $(P_{\infty,\mathrm{loc}})$ in the sense of criterion L1.

Proof. In order to verify whether an S defined in (10) is admissible for the problem $(D_{\infty,\mathrm{loc}})$ we define the defect of the Hamilton-Jacobi differential equation as

$$\begin{aligned} \Lambda(t,\xi) &= \frac{1}{\nu(t)}\partial_t S(t,\xi) + \mathcal{H}(t,\xi,\partial_\xi S(t,\xi)) \\ &= \frac{1}{\nu(t)}\left(\dot a(t) + \dot y(t)^T(\xi - x^*(t)) - y(t)^T \dot x^*(t)\right) + \mathcal{H}(t,\xi,\partial_\xi S(t,\xi)). \end{aligned}$$

Choosing $a(t)$ from the Hamilton-Jacobi differential equation

$$\Lambda(t, x^*(t)) = 0 \tag{24}$$

we obtain

$$\dot a(t) = (y(t)^T \dot x^*(t) - \mathcal{H}(t, x^*(t), y(t)))\nu(t). \tag{25}$$

Substitution of (25) into the expression for $\Lambda(t,\xi)$ yields

$$\Lambda(t,\xi) = \frac{1}{\nu(t)}\dot y(t)^T(\xi - x^*(t)) + \mathcal{H}(t,\xi,\partial_\xi S(t,\xi)) - \mathcal{H}(t, x^*(t), y(t)).$$

It can easily be seen that S belongs to Y_{loc} if $x^*(t)$ maximizes $\Lambda(t,\xi)$ in a $\delta(t)$-neighborhood of $x^*(t)$ for all $t > 0$. For that reason we consider a parametric optimization problem

$$(P_t) \quad \text{maximize } \Lambda(t,\xi) \text{ with respect to } \xi \in \mathcal{K}_{\delta,\nu}(x^*(t)).$$

The first order necessary condition

$$\partial_\xi \Lambda(t, x^*(t)) = \frac{1}{\nu(t)}\dot y(t) + \partial_\xi \mathcal{H}(t, x^*(t), y(t)) = 0 \tag{26}$$

together with the second order sufficiency condition represented by (23) guarantee that $x^*(t)$ solves the problem (P_t) for all $t > 0$. Due to condition (21), $\partial_\xi \mathcal{H}(t, x^*(t), y(t))$ consists of only one point and the canonical equation (26) can be rewritten in form (20). Since the conditions of Lemma 2 are satisfied we obtain $y \in W^{1,n}_{q,\nu^{1-q}}(\Omega)$. It means that S has the form (10) with $Q(t) \equiv 0$ and Lemma 1 can be applied to get the condition (15) of the generalized maximum principle for the criterion L1 satisfied. The maximum condition stated in (22) and two other conditions of Corollary 1 are satisfied, what allows us to deduce that the pair (x^*, u^*) is a strong local minimizer of $(P_{\infty,\mathrm{loc}})$ in the sense of criterion L1. □

6.3 An Example

We consider the Production-Inventory Model [16]

$$\min_{u\geq 0}\left\{ J(x,u)=\int_0^{\infty} e^{-\rho t}\left(\frac{h}{2}(x-\hat{x})^2+\frac{c}{2}(u-\hat{u})^2\right)dt\right\} \tag{27}$$

$$\dot{x}(t)=u(t)-v_0,\ \ x(0)=x_0, \tag{28}$$

where $h>0$, $c>0$, $\rho\geq 0$, $\hat{x}$, $\hat{u}$ denote the inventory holding cost coefficient, the production cost coefficient, the constant discount rate, the inventory goal level and the production goal level, respectively. The state equation expresses that the inventory x at time t is increased by the production rate $u(t)$ and decreased by the constant sales rate v_0.

We find the Pontrjagin function

$$H(t,\xi,v,\eta)=-\frac{h}{2}(\xi-\hat{x})^2-\frac{c}{2}(v-\hat{u})^2+\frac{\eta}{e^{-\rho t}}\,(v-v_0) \tag{29}$$

and verify the condition (21) of Theorem 3

$$\partial^2_{vv}H(t,\xi,v,\eta)=-c<0\Longrightarrow \partial^2_{vv}H(t,x^*(t),u^*(t),y(t))<0,$$

which is evidently satisfied. Furthermore, obtaining the control from maximum condition (22)

$$u^*(t)=\max\left\{\hat{u}+\frac{\eta}{c}e^{\rho t},0\right\},$$

we assume $\hat{u}$ to be large enough that the production rate always gives a nonnegative value:

$$u^*(t)=\hat{u}+\frac{\eta}{c}e^{\rho t}. \tag{30}$$

Substitution of u^* into (29) yields the Hamilton function

$$\mathcal{H}(t,\xi,\eta)=-\frac{h}{2}(\xi-\hat{x})^2+\frac{\eta^2}{2c}e^{2\rho t}+\eta e^{\rho t}(\hat{u}-v_0).$$

Relation (20) together with the state equation define the canonical system

$$\begin{cases} \dot{y}(t)=-y(t)^T\underbrace{\partial_\xi g(t,x^*(t),u^*(t))}_{=0}+\partial_\xi f(t,x^*(t),u^*(t))\nu(t)\\ \dot{x}^*(t)=\hat{u}+\frac{y(t)}{c}e^{\rho t}-v_0,\ \ x^*(0)=x_0,\end{cases}$$

which can be rewritten as follows

$$\begin{cases} \dot{y}(t)=h(x^*(t)-\hat{x})e^{-\rho t}\\ \dot{x}^*(t)=\hat{u}+\frac{y(t)}{c}e^{\rho t}-v_0,\ \ x^*(0)=x_0.\end{cases} \tag{31}$$

By differentiating the first equation and replacing the expression for $\dot{y}(t)$ by $h(x^*(t)-\hat{x})e^{-\rho t}$, one gets

$$\begin{aligned}\ddot{y}(t) &= h\left(\dot{x}^*(t)e^{-\rho t} - \rho(x^*(t)-\hat{x})e^{-\rho t}\right)\\ &= h\left(\hat{u} + \frac{y(t)}{c}e^{\rho t} - v_0\right)e^{-\rho t} - \rho\dot{y}(t).\end{aligned}$$

The equation for the adjoint variable becomes

$$\ddot{y}(t) + \rho\dot{y}(t) - \frac{h}{c}y = h(\hat{u}-v_0)e^{-\rho t}.$$

By using the notation

$$k_1 = \frac{-\rho + \sqrt{\rho^2 + 4h/c}}{2}, \quad k_2 = \frac{-\rho - \sqrt{\rho^2+4h/c}}{2},$$

one can write $y(t) = c_1 e^{k_1 t} + c_2 e^{k_2 t} + (v_0 - \hat{u})ce^{-\rho t}$. Then we obtain the state function from the first equation of (31):

$$x^*(t) = \frac{1}{h}\left\{c_1 k_1 e^{-k_2 t} + c_2 k_2 e^{-k_1 t}\right\} - (v_0 - \hat{u})\frac{c\rho}{h} + \hat{x}.$$

The initial condition from (28) yields

$$c_2 = \frac{(x_0 - \hat{x})h - c_1 k_1 - c(\hat{u} - v)\rho}{k_2}.$$

In order to satisfy $y \in W^{1,n}_{q,\nu^{1-q}}$ it is necessary to set $\lim_{t\to\infty} y(t) = 0$. This can occur only if $c_1 = 0$ holds true, otherwise $y(t)$ tends to infinity. The complete solution of (31) is stated below:

$$\begin{aligned}y(t) &= c_2 e^{k_2 t} + (v_0 - \hat{u})ce^{-\rho t},\\ x^*(t) &= \frac{1}{h}\left\{c_2 k_2 e^{-k_1 t}\right\} - (v_0 - \hat{u})\frac{c\rho}{h} + \hat{x}.\end{aligned}$$

Using (30) we derive the control function

$$u^*(t) = \frac{c_2}{c}e^{-k_1 t} + v_0.$$

We now investigate the question concerning the spaces the solution belongs to. The function $x^*(t)$ does not belong to any usual Sobolev space W_p, since the constant $|(v_0 - \hat{u})c\rho|^p$ is not integrable over the infinite interval. The same holds true for the control function u^* as it includes a constant as well. We try to figure out whether these functions belong to some weighted Sobolev space $W^{1,n}_{p,\nu}$ and weighted $L^r_{p,\nu}$ space, respectively. Moreover, we will show that for all ω and p satisfying

$$\frac{\omega}{p} < \rho \tag{32}$$

we have $x^* \in W^{1,n}_{p,\nu}$, $y \in W^{1,n}_{q,\nu^{1-q}}$, and $u^* \in L^r_{p,\nu}$, where $\nu(t) = e^{-\omega t}$, $\omega > 0$. For that purpose we estimate as follows

$$\|x^*\|^p_{L^n_{p,\nu}} = \int_0^\infty \left| \frac{1}{h}\left(c_2 k_2 e^{-k_1 t} + (\hat{u} - v_0)c\right) + \hat{x} \right|^p e^{-\omega t} dt$$

$$\leq 2^p \int_0^\infty \left| \frac{c_2 k_2}{h} \right|^p e^{(-pk_1 - \omega)t} dt + 2^p \int_0^\infty \left| \frac{(\hat{u} - v_0)c}{h} + \hat{x} \right|^p e^{-\omega t} dt < \infty.$$

The first integral on the right-hand-side converges due to the positivity of k_1, and it allows us to say $x^* \in L^n_{p,\nu}$. Almost the same estimate for $\|\dot{x}^*\|^p_{L^n_{p,\nu}}$ implies $x^* \in W^{1,n}_{p,\nu}$. We repeat the whole procedure for $\|u^*\|^p_{L^r_{p,\nu}}$ and obtain

$$\|u^*\|^p_{L^r_{p,\nu}} = \int_0^\infty \left| \frac{c_2}{c} e^{-k_1 t} + v_0 \right|^p e^{-\omega t} dt$$

$$\leq 2 \left| \frac{c_2}{c} \right|^p \int_0^\infty e^{(-pk_1 - \omega)t} dt + (2v_0)^p \int_0^\infty e^{-\omega t} dt < \infty.$$

By (32), one derives $u^* \in L^r_{p,\nu}$. Setting $q = \frac{p}{p-1}$ and $1 - q = -\frac{1}{p-1}$, one gets

$$\|y\|^q_{L^n_{q,\nu^{1-q}}} = \int_0^\infty \left| c_2 e^{k_2 t} + (\hat{u} - v_0) c e^{-\rho t} \right|^{\frac{p}{p-1}} e^{\frac{\omega}{p-1} t} dt$$

$$\leq |2c_2|^{\frac{p}{p-1}} \int_0^\infty e^{(\frac{p}{p-1} k_2 + \frac{\omega}{p-1})t} dt + |2(v_0 - \hat{u})c|^{\frac{p}{p-1}} \int_0^\infty e^{\frac{-\rho p + \omega}{p-1} t} dt < \infty.$$

Repeating the same procedure for $\|\dot{y}\|^q_{L^n_{q,\nu^{1-q}}}$ we prove $y \in W^{1,n}_{q,\nu^{1-q}}$. The inclusion $y \in W^{1,n}_{q,\nu^{1-q}}$ is necessary in order to justify the application of Theorem 3. Now it remains to verify (23), but this can be easily done because $\partial^2_{\xi\xi}\mathcal{H}(t, x^*(t), y(t)) = -h < 0$. As a consequence, all the conditions of Theorem 3 are satisfied and we can conclude that the pair (x^*, u^*) is a strong local minimizer of the problem (27)-(28) in the sense of the criterion L1.

References

1. S.M. Aseev and A.V. Kryazhimskiy. The Pontryagin maximum Principle and transversality conditions for a class of optimal control problems with infinite time horizons. Interim Report IR-03-013, IIASI, Laxenburg, Austria, 2003.

2. J.P. Aubin and F.H. Clarke. Shadow prices and duality for a class of optimal control problems. SIAM J. Conrol Optim., 17(5):567-586, 1979.
3. L.M. Benveniste and J.A. Scheinkman. Duality theory for dynamic optimization models of economics: The continuous time case. J. Econ. Theory 27:1-29, 1982.
4. J. Blot and P. Michel. First-order conditions for infinite-horizon variational problems. J. Optim. Theory Appl., 88(2):339-364, 1996.
5. J. Blot and N. Hayek. Second-Order Necessary Conditions for the Infinite-Horizon Variational Problems. Math. Oper. Res., 21(4):979-990, 1996.
6. D.A. Carlson, A.B. Haurie, and A. Leizarowitz. Infinite Horizon Optimal Control. Springer-Verlag, New York, 1991.
7. J. Elstrodt. Maß und Integrationstheorie. Springer-Verlag, Heidelberg, 1996.
8. G. Feichtinger and R.F. Hartl. Optimale Kontrolle Ökonomischer Prozesse. Walter de Gruyter, Berlin-New York, 1986.
9. H. Halkin. Necessary conditions for optimal control problems with infinite horizons. Econometrica, 42(2):267-272, 1974.
10. R. Klötzler. On a general conception of duality in optimal control. Proceed. Equadiff 4, Prague, 1977.
11. A. Kufner. Weighted Sobolev Spaces. John Wiley and Sons, New York,1985.
12. A. Leizarowitz and V.J. Mizel. One dimensional infinite-horizon variational problems arising in continuum mechanics. Archive for Rational Mechanics and Analysis, 106(2):161-194, 1988.
13. M.J.P. Magill. Pricing optimal horizon problems. J. Math. Anal. Appl., 88:398-421, 1982.
14. P. Michel. On the transversality condition in infinite horizon optimal problems, Econometrica, 50(4):975-985, 1982.
15. R.T. Rockafellar. Convex processes and hamilton dynamical systems. In: Convex Analysis and Mathematical Economics. (Proc. Sympos., Univ. of Tilburg, 1978). Lecture Notes in Economics and Mathematical Systems, 168, Springer-Verlag, Berlin 1979, pp. 122-136.
16. S.P. Sethi and G.L. Thompson. Optimal Control Theory, Kluwer Academic Publishers, Boston, 2000.
17. G.V. Smirnov. Transversality condition for infinite horizon problems, J. Optim. Theory Appl., 88(3):671-688, 1996.
18. K. Yosida. Functional Analysis, Springer-Verlag, New York, 1974.
19. A.J. Zaslavski. The existence of periodic minimal energy configurations for one-dimensional infinite horizon variational problems arising in continuum mechanics. J. Math. Anal. Appl. 194:459-476, 1995.