L^1–Optimal Boundary Control of a String to Rest in Finite Time

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Summary. In this paper, the problem to control a finite string to the zero state in finite time from a given initial state by controlling the state at the two boundary points is considered. The corresponding optimal control problem where the objective function is the L^1-norm of the controls is solved in the sense that the controls that are successful and minimize at the same time the objective function are determined as functions of the initial state.

1 Introduction

We consider a string of finite length that is governed by the wave equation. The string is controlled through the boundary values at both ends of the string (two–point Dirichlet control). The boundary control of the wave equation has been studied by many authors and results about exact controllability are well–known. The method of moments is an important tool to analyze this system (see e.g. [1, 7, 8, 10, 12] and the references therein). Also the controllability of the discretized problems and the relation between the optimal controls for the continuous and the discrete case have been the subject of recent investigations, see [14]. A related problem of one–point time optimal control has been solved in [11], where the control functions are assumed to have a second derivative whose norm is constrained. In [13], exact controllability is studied for a string with elastic fixing at one end.

In this paper, our main interest is to study the structure of the optimal controls and to give an explicit representation of the optimal controls in terms of the given initial data. This yields valuable test examples for numerical algorithms.

From a given initial state where the position and the integral of the velocity are given by a Lebesgue–integrable function the system is controlled to the zero state in a given finite time.

To guarantee that this control problem is solvable for all initial states, the control time has to be greater than or equal to the time that a wave needs to travel from one end of the string to the other (the characteristic time). In Theorem 1 we give an exact controllability result where the initial states that can be steered to zero with boundary controls from the spaces L^p ($p \in [1, \infty]$) are characterized: These are the initial states where the initial position and the integral of the initial velocity are functions in the spaces L^p on the space interval.

The requirements that the target state is reached in the given terminal time do not determine a unique solution. So we can choose from the set of successful controls a point that minimizes our objective function which is the L^1–norm of the controls. In general, this optimization problem does not have a unique solution. In Theorem 2 the solutions are given explicitly in terms of the initial data.

In [2], [4] and [6], we have studied the related problem to steer the system from the zero state to a given terminal state in such a way that the L^p-norm ($p \in [2, \infty]$) of the control functions is minimized. In these papers, the method of moments and Fourier–series have been used in the proofs. In the present paper we use the method of characteristics for our proofs. Note that in contrast to the L^1–case, for $p \in (1, \infty)$ the corresponding optimal controls are uniquely determined.

This paper has the following structure: We define the optimal control problem and some important auxiliary variables, for example the characteristic time and the defect. Then the problem is transformed and reformulated in terms of the Riemann invariants. For this purpose, we use the d'Alembert solution of the wave equation. After the introduction of auxiliary functions as variables in the optimization problem, the exact controllability result Theorem 1 can be proved. Then the objective function is also written in terms of the auxiliary functions, which allows to reformulate the optimization problem such that it decouples to time–parametric finite dimensional problems that can be solved explicitly. (These auxiliary problems also do not have a unique solution.) This allows to solve the optimal control problem. In Theorem 2, the solutions of the L^1–optimal control problem are given in terms of the initial state. Finally we present some examples.

2 The Problem

Let $L^1(0, T)$ denote the space of Lebesgue–integrable functions on the interval $(0, T)$, and let

$$\|(u_1, u_2)\|_{1,(0,T)} = \int_0^T |u_1(t)| + |u_2(t)|\, dt.$$

Let the length $L > 0$, the time $T > 0$ and the wave velocity $c > 0$ be given. Let $y_0 \in L^1(0, L)$ and y_1 be given such that the function $x \mapsto \int_0^x y_1(s)\, ds$ is in $L^1(0, L)$.

We consider the problem

$$\mathcal{P}: \qquad \text{minimize } \|(u_1,\, u_2)\|_{1,(0,T)} \text{ subject to } u_1, u_2 \in L^1(0,T) \text{ and} \tag{1}$$

$$y(x,0) = y_0(x),\; y_t(x,0) = y_1(x),\; x \in (0,L) \tag{2}$$

$$y(0,t) = u_1(t),\; y(L,t) = u_2(t),\; t \in (0,T) \tag{3}$$

$$y_{tt}(x,t) = c^2 y_{xx}(x,t),\; (x,t) \in (0,L) \times (0,T) \tag{4}$$

$$y(x,T) = 0,\; y_t(x,T) = 0,\; x \in (0,L). \tag{5}$$

3 Definition of the Characteristic Time

Define the characteristic time $t_0 = L/c$ that a characteristic curve needs to travel from one end of the string to the other. In the sequel we assume that

$$T \geq t_0.$$

For the solution of the problem, we need to know how often the characteristic time t_0 fits into the time interval $[0,T]$. Define the natural number

$$k = \max\{j \in I\!N : j\, t_0 \leq T\} \tag{6}$$

and the defect

$$\Delta = T - k\, t_0 \geq 0. \tag{7}$$

The definition of Δ implies the equation $T = k\, t_0 + \Delta$.

4 Transformation of the Problem

In order to come closer to a solution of Problem $\mathcal{P}$, we transform it to a form that we can solve. For this purpose, we write the solution of the wave equation in the form

$$y(x,t) = [\alpha(x+ct) + \beta(x-ct)]/2 \tag{8}$$

which means that we describe our solution in terms of the Riemann invariants or in other words, as the sum of travelling waves. For an introduction to linear hyperbolic systems see [9].

The *end conditions* (5) yield the equations

$$\alpha(x+cT) + \beta(x-cT) = 0,\; \alpha'(x+cT) - \beta'(x-cT) = 0,\; x \in (0,L) \tag{9}$$

where the derivatives are in the sense of distributions. This is equivalent to

$$\alpha(x) = -\beta(x-2cT),\; \alpha'(x) = \beta'(x-2cT),\; x \in (cT, cT+L). \tag{10}$$

Differentiation of the first equation in (10) yields

$$\alpha'(x) = -\beta'(x - 2cT),\ x \in (cT, cT + L)$$

hence we have $\alpha'(x) = -\alpha'(x)$ and thus

$$\alpha'(x) = 0,\ x \in (cT, cT + L);\ \ \beta'(x) = 0,\ x \in (-cT, -cT + L). \tag{11}$$

So the first equation in (10) implies that there exists a real constant r such that

$$\alpha(x) = r,\ x \in (cT, cT + L);\ \ \beta(x) = -r,\ x \in (-cT, -cT + L). \tag{12}$$

We have shown that if (8) satisfies the end conditions (5), then (12) holds. The reverse statement is obviously true.

The *initial conditions* (2) yield the equations

$$y_0(x) = (1/2)\,[\alpha(x) + \beta(x)],\ y_1(x) = (c/2)\,[\alpha'(x) - \beta'(x)],\ x \in (0, L). \tag{13}$$

Hence we have

$$y_0(x) + (1/c)\int_0^x y_1(s)\,ds = \alpha(x) - k_1,\ x \in (0, L) \tag{14}$$

$$y_0(x) - (1/c)\int_0^x y_1(s)\,ds = \beta(x) + k_1,\ \ x \in (0, L) \tag{15}$$

for a real constant k_1 that we can choose as zero, which implies

$$\alpha(x) = y_0(x) + (1/c)\int_0^x y_1(s)\,ds,\ x \in (0, L), \tag{16}$$

$$\beta(x) = y_0(x) - (1/c)\int_0^x y_1(s)\,ds,\ x \in (0, L). \tag{17}$$

We have shown that if (8) satisfies the initial conditions (2), then (16), (17) hold. The converse also holds: If α, β satisfy (16), (17), the initial conditions (2) are valid for y given by (8).

5 Exact Controllability

The considerations in the last section imply the following exact controllability result:

Theorem 1. *Let $T \geq L/c$ and $p \in [1, \infty]$ be given. The initial boundary–value problem (2)–(4) has a travelling waves solution in the sense (8) that satisfies the end conditions (5) with u_1, $u_2 \in L^p(0, T)$, if and only if the initial states y_0, y_1 satisfy the following conditions: $y_0 \in L^p(0, L)$ and $Y_1 \in L^p(0, L)$, where $Y_1(x) = \int_0^x y_1(s)\,ds$, that is $y_1 \in W^{-1,p}(0, L)$.*

This implies that Problem $\mathcal{P}$ is solvable if and only if y_0 and Y_1 are in $L^1(0, L)$.

Proof of one direction. Assume that y_0 and $Y_1 \in L^p(0, L)$. Define

$$u_1(t) = y(0,t) = [\alpha(ct) + \beta(-ct)]/2$$

$$u_2(t) = y(L,t) = [\alpha(L+ct) + \beta(L-ct)]/2$$

where the functions $\alpha \in L^p(0, L+ct)$, $\beta \in L^p(-cT, L)$ are chosen such that (12) and (16), (17) hold, for example with $r = 0$ and $\alpha(x) = 0$ for $x \in (L, cT)$ and $\beta(x) = 0$ for $x \in (L - cT, 0)$. Then the solution y given by (8) satisfies the initial conditions (2) and the end conditions (5). Moreover, u_1 and u_2 are in $L^p(0,T)$. The proof of the converse is given in the next section. □

Remark 1: For the case $p \in [2, \infty]$, Theorem 1 is already proved in [6] using Fourier series. Note however, that in [6] the initial state is the zero state which is controlled in the time T to the target state (y_0, y_1).

6 Definition of Auxiliary Functions and Completion of the Proof of Theorem 1

For $j \in \{0, 1, ..., k\}$ and $t \in (0, t_0)$ define the functions

$$\alpha_j(t) = \alpha(ct + jL),\ \beta_j(t) = \beta(-ct - (j-1)L) \tag{18}$$

and for $t \in (0, \Delta)$ define

$$\alpha_{k+1}(t) = \alpha(ct + (k+1)L),\ \beta_{k+1}(t) = \beta(-ct - kL). \tag{19}$$

The functions α_j, β_j are useful as decision variables in the transformed optimization problem. We will state the constraints in terms of the functions α_j, β_j: Since

$$[cT, cT + L] = [kL + c\Delta, (k+1)L + c\Delta]$$
$$= [kL + c\Delta, (k+1)L] \cup [(k+1)L, (k+1)L + c\Delta]$$

and

$$[-cT, -cT + L] = [-kL - c\Delta, -(k-1)L - c\Delta]$$
$$= [-kL - c\Delta, -kL] \cup [-kL, -(k-1)L - c\Delta]$$

the constraints (12) are equivalent to the conditions

$$\alpha_k(t) = r,\ t \in (\Delta, t_0),\ \alpha_{k+1}(t) = r,\ t \in (0, \Delta), \tag{20}$$

$$\beta_k(t) = -r,\ t \in (\Delta, t_0),\ \beta_{k+1}(t) = -r,\ t \in (0, \Delta). \tag{21}$$

This means that the functions α_{k+1}, β_{k+1} are constant on $(0, \Delta)$ and the functions α_k, β_k are constant on (Δ, t_0) with the same absolute values but with opposite signs.

Conditions (16) and (17) are respectively equivalent to

$$\alpha_0(t) = y_0(ct) + (1/c) \int_0^{ct} y_1(s)\, ds,\ t \in (0, t_0), \tag{22}$$

$$\beta_0(t) = y_0(L - ct) - (1/c) \int_0^{L-ct} y_1(s)\, ds,\ t \in (0, t_0), \tag{23}$$

so the values of the functions α_0, β_0 are prescribed by the initial conditions.

We can represent the control functions u_1, u_2 in terms of α_j, β_j in the following way. Define the intervals

$$I_j^1 = [jt_0, jt_0 + \Delta],\ j \in \{0, 1, 2, ..., k\}, \tag{24}$$

$$I_j^2 = [jt_0 + \Delta, (j+1)t_0],\ j \in \{0, 1, 2, ..., k-1\}. \tag{25}$$

Then for $t \in I_j^1$ or $t \in I_j^2$ we have

$$u_1(t) = [\,\alpha_j(t - jt_0) + \beta_{j+1}(t - jt_0)\,]/2, \tag{26}$$

$$u_2(t) = [\,\alpha_{j+1}(t - jt_0) + \beta_j(t - jt_0)\,]/2. \tag{27}$$

Now we complete the proof of Theorem 1. Assume that controls u_1, $u_2 \in L^p(0,T)$ are given such that the travelling waves solution (8) satisfies the initial conditions (2) and the end conditions (5). The end conditions (20), (21) imply that the functions α_{k+1}, β_{k+1} are in $L^p(0,\Delta)$. Then (26) and the fact that u_1 is in $L^p(0,T)$ imply that α_k is also in $L^p(0,\Delta)$. Equation (27) and the fact that $u_2 \in L^p(0,T)$ imply that β_k is also in $L^p(0,\Delta)$. Analogous arguments show that α_{k-1}, β_{k-1} are in $L^p(0,\Delta)$ and repeating the argument shows that α_0, β_0 is in $L^p(0,\Delta)$.

The end conditions (20), (21) imply that the functions α_k, β_k are in $L^p(\Delta, t_0)$. Then (26) and $u_1 \in L^p(0,T)$ imply that α_{k-1} is also in $L^p(\Delta, t_0)$. Equation (27) and the fact that $u_2 \in L^p(0,T)$ imply that β_{k-1} is also in $L^p(\Delta, t_0)$. Repeating the argument implies that α_0, β_0 are in $L^p(\Delta, t_0)$.

Thus we have shown that α_0, β_0 are in $L^p(0, t_0)$. Equations (22), (23) imply that y_0 is in $L^p(0,L)$ and that Y_1 is in $L^p(0,L)$.

7 Reformulation of the Optimization Problem in terms of α_j, β_j

We start by transforming our objective function

$$J(u_1, u_2) = \int_0^T |u_1(t)| + |u_2(t)|\, dt. \tag{28}$$

We have

$$J(u_1, u_2) = \sum_{j=0}^{k} \int_{jt_0}^{jt_0+\Delta} |u_1(t)| + |u_2(t)|\, dt + \sum_{j=0}^{k-1} \int_{jt_0+\Delta}^{(j+1)t_0} |u_1(t)| + |u_2(t)|\, dt$$

$$= \sum_{j=0}^{k} \int_0^{\Delta} |u_1(t+jt_0)| + |u_2(t+jt_0)| \, dt + \sum_{j=0}^{k-1} \int_{\Delta}^{t_0} |u_1(t+jt_0)| + |u_2(t+jt_0)| \, dt$$

$$= \int_0^{\Delta} \sum_{j=0}^{k} |u_1(t+jt_0)| + |u_2(t+jt_0)| \, dt + \int_{\Delta}^{t_0} \sum_{j=0}^{k-1} |u_1(t+jt_0)| + |u_2(t+jt_0)| \, dt$$

$$= \int_0^{\Delta} \sum_{j=0}^{k} \left[\frac{1}{2} |\alpha_j(t) + \beta_{j+1}(t)| + \frac{1}{2} |\alpha_{j+1}(t) + \beta_j(t)| \right] dt$$

$$+ \int_{\Delta}^{t_0} \sum_{j=0}^{k-1} \left[\frac{1}{2} |\alpha_j(t) + \beta_{j+1}(t)| + \frac{1}{2} |\alpha_{j+1}(t) + \beta_j(t)| \right] dt$$

$$=: F(\alpha_j|_{(0,\Delta)}, \beta_j|_{(0,\Delta)}, j \in \{1, ..., k\}; \;\; \alpha_j|_{(\Delta,t_0)}, \beta_j|_{(\Delta,t_0)}, j \in \{1, ..., k-1\}). \tag{29}$$

Now we write down our optimization problem in terms of the unknown functions α_j, β_j. If we have determined a solution pair α_j, β_j, we obtain the corresponding controls u_1, u_2 from (26), (27). In this sense Problem $\mathcal{P}$ is equivalent to the problem:

$$\text{minimize the objective function } F \text{ given in (29)} \tag{30}$$

over the functions

$$\alpha_j|_{(0,\Delta)}, \beta_j|_{(0,\Delta)} \in L^1(0, \Delta), \; j \in \{1, ..., k\},$$

$$\alpha_j|_{(\Delta,t_0)}, \beta_j|_{(\Delta,t_0)} \in L^1(\Delta, t_0), \; j \in \{1, ..., k-1\}$$

where α_0, β_0 are given in (22), (23) and $\alpha_k|_{(\Delta,t_0)}$, $\beta_k|_{(\Delta,t_0)}$, $\alpha_{k+1}|_{(0,\Delta)}$, $\beta_{k+1}|_{(0,\Delta)}$ are given by (20), (21).

7.1 Definition of a Time–Parametric Optimization Problem

For $t \in (0, t_0)$ and a natural number m consider the optimization problem

$$H(t,m): \;\; \min \sum_{j=0}^{m-1} \frac{1}{2} |\alpha_j(t) + \beta_{j+1}(t)| + \frac{1}{2} |\alpha_{j+1}(t) + \beta_j(t)| \tag{31}$$

where the numbers $\alpha_0(t)$, $\beta_0(t)$ and $\alpha_m(t)$, $\beta_m(t)$ are given and the decision variables are $\alpha_1(t)$,...,$\alpha_{m-1}(t)$, $\beta_1(t)$,...,$\beta_{m-1}(t)$. If $m = 1$ there are no decision variables. The objective function of $H(t,m)$ is the integrand of the function F given in (29) at a single point $t \in (0, t_0)$, so the idea of $H(t,m)$ is to minimize the integrand of Problem (30) at a single point in time.

We obtain solutions of Problem (30) by solving the optimization problems $H(t, k+1)$ for $t \in (0, \Delta)$ and $H(t, k)$ for $t \in (\Delta, t_0)$ almost everywhere, that is we minimize the integrand in the objective function J pointwise a.e.. Consider solutions $\alpha_j(t)$, $\beta_j(t)$ of these optimization problems as functions of t. If these functions are Lebesgue integrable, they are condidates for a solution of the optimization problem (30) and thus yield solutions of the optimal control problem $\mathcal{P}$. In fact, the solutions $\alpha_j(t)$, $\beta_j(t)$ are coupled by the real parameter r from (20), (21). So we reduce the original infinite–dimensional problem to the problem to find the value of the real number r for which the objective function evaluated at the corresponding solutions $\alpha_j(t)$, $\beta_j(t)$ has minimal value.

7.2 Solution of a Time–Parametric Optimization Problem

Consider problem $H(t,m)$ for a fixed time $t \in (0, t_0)$. Since t is fixed, we call the decision variables $\alpha_1,\ldots,\alpha_{m-1}$, $\beta_1,\ldots,\beta_{m-1}$ and omit the parameter t. We introduce new variables:

$$\begin{aligned} \gamma_j &= \alpha_j + \beta_{j+1}\ ,\ \delta_j = \alpha_{j+1} + \beta_j \quad \text{for } j \text{ even},\\ \gamma_j &= \alpha_{j+1} + \beta_j\ ,\ \delta_j = \alpha_j + \beta_{j+1} \quad \text{for } j \text{ odd}. \end{aligned}$$

We have

$$\sum_{j=0}^{m-1} (-1)^j \gamma_j = \begin{cases} \alpha_0 - \alpha_m & \text{if } m \text{ is even},\\ \alpha_0 + \beta_m & \text{if } m \text{ is odd}. \end{cases}$$

If $\alpha_m = r = -\beta_m$ as in (20), (21), this yields for all m the equation

$$\sum_{j=0}^{m-1} (-1)^j \gamma_j = \alpha_0 - r = c_1.$$

Similarly,

$$\sum_{j=0}^{m-1} (-1)^j \delta_j = \begin{cases} \beta_0 - \beta_m & \text{if } m \text{ is even},\\ \beta_0 + \alpha_m & \text{if } m \text{ is odd}. \end{cases}$$

If $\alpha_m = r = -\beta_m$ as in (20), (21), this yields for all m the equation

$$\sum_{j=0}^{m-1} (-1)^j \delta_j = \beta_0 + r = c_2.$$

We also have

$$\sum_{j=0}^{m-1} |\alpha_j(t) + \beta_{j+1}(t)| + |\alpha_{j+1}(t) + \beta_j(t)| = \sum_{j=0}^{m-1} |\gamma_j| + |\delta_j|.$$

This means that we can decouple problem $H(t,m)$ into two problems

$$P_1 : \ \min \sum_{j=0}^{m-1} |\gamma_j| \ \text{ s.t. } \sum_{j=0}^{m-1} (-1)^j \gamma_j = c_1, \tag{32}$$

$$P_2 : \ \min \sum_{j=0}^{m-1} |\delta_j| \ \text{ s.t. } \sum_{j=0}^{m-1} (-1)^j \delta_j = c_2. \tag{33}$$

Lemma 1. *The optimal value of P_1 is $|c_1|$. If $c_1 = 0$, the solution of P_1 is uniquely determined. If $c_1 \neq 0$, the solution of P_1 is not uniquely determined. In fact, $(\gamma_0, ..., \gamma_{m-1})$ is a solution of P_1 if and only if*

$$\gamma_j = (-1)^j \lambda_j c_1,\ j \in \{0, ..., m-1\},$$

where for $j \in \{0, ..., m-1\}$ we have $\lambda_j \geq 0$ and $\sum_{i=0}^{m-1} \lambda_i = 1$. The corresponding assertions for P_2 also hold.

Proof. The point with the components γ_j as defined in the Lemma satisfies the equality constraint of P_1 and has the objective value $|c_1|$. Thus the objective value is less than or equal to $|c_1|$. Now take an arbitrary point that satisfies the equality constraint of P_1. Then the triangle inequality implies

$$\sum_{j=0}^{m-1} |\gamma_j| = \sum_{j=0}^{m-1} |(-1)^j \gamma_j| \geq |\sum_{j=0}^{m-1} (-1)^j \gamma_j| = |c_1|.$$

Hence the optimal value of P_1 is greater than or equal to $|c_1|$ and we have proved the assertion. Assume that $c_1 \neq 0$. Let an arbitrary solution of P_1 with the components η_0, η_1,...,η_{m-1} be given. Then we have

$$\sum_{j=0}^{m-1} |\eta_j| = |c_1|.$$

Define $\lambda_j = |\eta_j|/|c_1|$. Then $\lambda_j \geq 0$, $\sum_{i=0}^{m-1} \lambda_i = 1$, and $\eta_j = |c_1|\lambda_j \text{sign}(\eta_j)$. The equation

$$\sum_{j=0}^{m-1} (-1)^j \eta_j = \sum_{j=0}^{m-1} \lambda_j |c_1| (-1)^j \text{sign}(\eta_j) = |c_1| \sum_{j=0}^{m-1} \lambda_j (-1)^j \text{sign}(\eta_j) = c_1$$

holds. Thus

$$\sum_{j=0}^{m-1} \lambda_j (-1)^j \text{sign}(\eta_j) = \text{sign}(c_1).$$

This equation can only hold if for all $j \in \{0, ..., m-1\}$ we have

$$(-1)^j \text{sign}(\eta_j) = \text{sign}(c_1),$$

which implies $\text{sign}(\eta_j) = (-1)^j \text{sign}(c_1)$. Thus we have $\eta_j = (-1)^j \lambda_j c_1$, and the assertion follows. □

7.3 Solution of the Optimal Control Problem

Consider the functions $\alpha_j(t)$, $\beta_j(t)$ defined as the solutions of $H(t, k+1)$ for $t \in (0, \Delta)$ and of $H(t, k)$ for $t \in (\Delta, t_0)$ almost everywhere. Lemma 1 gives values $\alpha_j(t)+\beta_{j+1}(t)$ and $\alpha_{j+1}(t)+\beta_j(t)$ for the solution of problem $H(t,m)$ explicitly. The general solution given in Lemma 1 yields solutions of the form

$$\begin{aligned}
\gamma_j(t) &= (-1)^j\lambda_j(t)[\alpha_0(t)-r] \text{ for } t\in(0,\Delta),\ j\in\{0,...,k\}\\
\gamma_j(t) &= (-1)^j\mu_j(t)[\alpha_0(t)-r] \text{ for } t\in(\Delta,t_0),\ j\in\{0,...,k-1\}\\
\delta_j(t) &= (-1)^j\nu_j(t)[\beta_0(t)+r] \text{ for } t\in(0,\Delta),\ j\in\{0,...,k\}\\
\delta_j(t) &= (-1)^j\omega_j(t)[\beta_0(t)+r] \text{ for } t\in(\Delta,t_0),\ j\in\{0,...,k-1\}.
\end{aligned}$$

Here λ_j and ν_j are functions defined almost everywhere on $(0,\Delta)$ such that

$$\lambda_j(t)\geq 0,\ \nu_j(t)\geq 0,\ \sum_{j=0}^{k}\lambda_j(t)=1=\sum_{j=0}^{k}\nu_j(t),$$

and such that the functions $\lambda_j(\alpha_0-r)$ and $\nu_j(\beta_0+r)$ are in $L^1(0,\Delta)$ for all $j\in\{0,...,k\}$. Moreover μ_j and ω_j are functions defined almost everywhere on (Δ,t_0) such that

$$\mu_j(t)\geq 0,\ \omega_j(t)\geq 0,\ \sum_{j=0}^{k-1}\mu_j(t)=1=\sum_{j=0}^{k-1}\omega_j(t),$$

and such that the functions $\mu_j(\alpha_0-r)$ and $\omega_j(\beta_0+r)$ are in $L^1(\Delta,t_0)$ for all $j\in\{0,...,k-1\}$.

Equations (26), (27) and the definition of γ_j, δ_j imply that the control values corresponding to these functions are given as

$$u_1(t+jt_0)=\gamma_j(t)/2 \text{ if } j \text{ is even,} \tag{34}$$
$$u_1(t+jt_0)=\delta_j(t)/2 \text{ if } j \text{ is odd,} \tag{35}$$

$$u_2(t+jt_0)=\delta_j(t)/2 \text{ if } j \text{ is even,} \tag{36}$$
$$u_2(t+jt_0)=\gamma_j(t)/2 \text{ if } j \text{ is odd.} \tag{37}$$

Now both for u_1 and u_2 we have to consider four different cases, depending on whether t is in the interval $(0,\Delta)$ or the interval (Δ,t_0) and on whether j is even of j is odd. The general solutions given in Lemma 1 correspond to optimal controls of the form

$$u_1(t+jt_0)=\lambda_j(t)[\alpha_0(t)-r]/2 \text{ if } j \text{ is even and } t\in(0,\Delta), \tag{38}$$
$$u_1(t+jt_0)=\mu_j(t)[\alpha_0(t)-r]/2 \text{ if } j \text{ is even and } t\in(\Delta,t_0), \tag{39}$$
$$u_1(t+jt_0)=-\nu_j(t)[\beta_0(t)+r]/2 \text{ if } j \text{ is odd and } t\in(0,\Delta), \tag{40}$$
$$u_1(t+jt_0)=-\omega_j(t)[\beta_0(t)+r]/2 \text{ if } j \text{ is odd and } t\in(\Delta,t_0), \tag{41}$$

$$u_2(t+jt_0)=\nu_j(t)[\beta_0(t)+r]/2 \text{ if } j \text{ is even and } t\in(0,\Delta), \tag{42}$$
$$u_2(t+jt_0)=\omega_j(t)[\beta_0(t)+r]/2 \text{ if } j \text{ is even and } t\in(\Delta,t_0), \tag{43}$$
$$u_2(t+jt_0)=\lambda_j(t)[-\alpha_0(t)+r]/2 \text{ if } j \text{ is odd and } t\in(0,\Delta), \tag{44}$$
$$u_2(t+jt_0)=\mu_j(t)[-\alpha_0(t)+r]/2 \text{ if } j \text{ is odd and } t\in(\Delta,t_0). \tag{45}$$

If $T = k\,t_0$, that is if $\Delta = 0$ the intervals $(0, \Delta)$ vanish.

It only remains to determine the value of the real number r. For this purpose, the control given above is inserted in the objective function $J(u_1, u_2)$ and r is chosen such that $J(u_1, u_2)$ is minimized.

8 Main Result

In this section we state the main result of this paper, which provides an explicit solution to the optimization problem $\mathcal{P}$, that is to say, to the L^1–norm optimal two–point Dirichlet boundary control of the wave equation to the zero position.

Theorem 2. *Assume that T is greater than or equal to $t_0 = L/c$. Consider the Problem $\mathcal{P}$ defined in (1)–(5). Choose a real number r that minimizes*

$$\frac{1}{2}\int_0^{t_0} |\alpha_0(t) - r| + |\beta_0(t) + r|\,dt \tag{46}$$

where α_0 is given by (22) and β_0 is given by (23).

Then a solution of Problem $\mathcal{P}$ is given by controls u_1, u_2 defined in (38)–(45) and, conversely, every solution has this form.

The minimal value of Problem $\mathcal{P}$ is given by the integral (46) with an optimal choice of r. Problem $\mathcal{P}$ admits a unique solution if and only if the minimal value of Problem $\mathcal{P}$ is zero.

Proof. We have presented controls u_1, $u_2 \in L^1(0,T)$ such that the generated state satisfies the end conditions and the corresponding value of the objective function is

$$J(u_1, u_2) = \min_r \frac{1}{2}\int_0^{t_0} |\alpha_0(t) - r| + |\beta_0(t) + r|\,dt.$$

Let v_1, $v_2 \in L^1(0,T)$ be control functions for which the generated state satisfies the end conditions. Then there exists a real number $r = r_0$ such that (12) holds. Suppose that the corresponding functions γ_j, δ_j (as in (34), (37)) do not solve the problem $H(t, k+1)$ almost everywhere on $(0, \Delta)$ (with $\alpha_{k+1} = r_0$, $\beta_{k+1} = -r_0$) or do not solve the problem $H(t,k)$ almost everywhere on (Δ, t_0) (with $\alpha_k = r_0$, $\beta_k = -r_0$). For $t \in (0, \Delta)$, let $h_1(t)$ denote the optimal value of $H(t, k+1)$. Lemma 1 implies that $h_1(t) = [|\alpha_0(t) - r_0| + |\beta_0(t) + r_0|]/2$. For $t \in (\Delta, t_0)$ let $h_2(t)$ denote the optimal value of $H(t,k)$. Lemma 1 implies that $h_2(t) = [|\alpha_0(t) - r_0| + |\beta_0(t) + r_0|]/2$. Then we have

$$\begin{aligned} J(v_1, v_2) &> \int_0^{\Delta} h_1(t)\,dt + \int_{\Delta}^{t_0} h_2(t)\,dt \\ &= \frac{1}{2}\int_0^{t_0} |\alpha_0(t) - r_0| + |\beta_0(t) + r_0|\,dt \\ &\geq J(u_1, u_2). \end{aligned}$$

Hence v_1, v_2 cannot be a solution of $\mathcal{P}$. This yields the assertion that the optimal controls are of the form as stated in the theorem, that is they solve the problem $H(t, k+1)$ almost everywhere on $(0, \Delta)$ (with $\alpha_{k+1} = r$, $\beta_{k+1} = -r$) and solve the problem $H(t, k)$ almost everywhere on (Δ, t_0) (with $\alpha_k = r$, $\beta_k = -r$), where r is chosen as to minimize (46). □

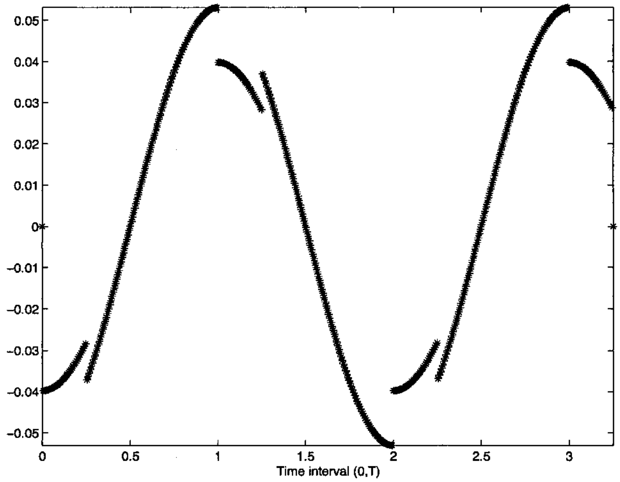

Fig. 1. The optimal control $u_1 = u_2$ in Example 2

9 Examples

In general the value of r for which the integral (46) attains its minimal value is *not* uniquely determined.

Example 1. Assume that $y_0 = c_0$ is constant and $y_1 = 0$, that is the string is initially at rest. Then (22) implies that $\alpha_0(t) = c_0$ and (23) implies that $\beta_0(t) = c_0$, hence we have $\alpha_0(t) = \beta_0(t) = c_0$, and the number r from Theorem 2 minimizes

$$\int_0^{t_0} |c_0 - r| + |c_0 + r| \, dt = t_0(|c_0 - r| + |c_0 + r|).$$

The value $r = 0$, minimizes the integral, since with $r = 0$ the integrand equals $2|c_0|$ and the triangle inequality implies that for all real numbers s we have

$$2|c_0| = |c_0 - s + c_0 + s| + \leq |c_0 - s| + |c_0 + s|.$$

So the optimal value of Problem $\mathcal{P}$ is $|c_0|t_0$. In this case, (38)–(45) imply that for all $j \in \{0, ..., k\}$, $t \in (0, \Delta)$ optimal controls are given by

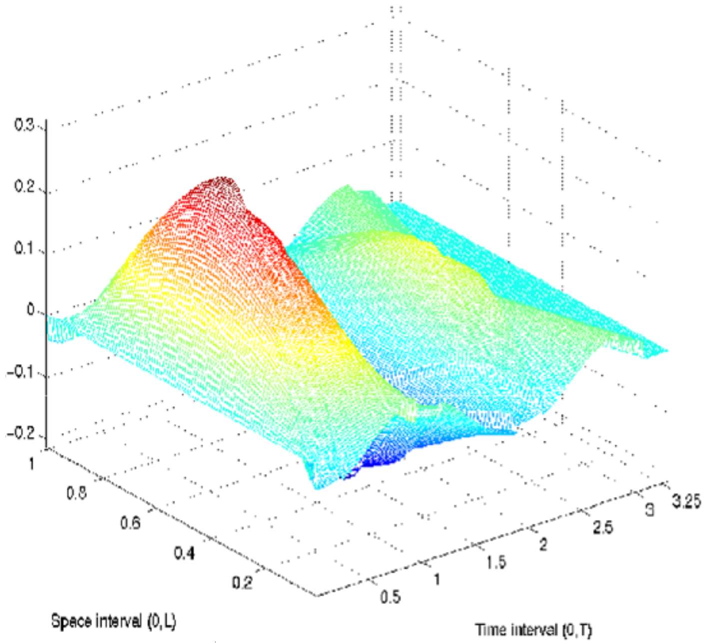

Fig. 2. The optimal state y in Example 2

$$u_1(t + jt_0) = u_2(t + jt_0) = (-1)^j \frac{c_0}{2(k+1)}$$

and for all $j \in \{0, ..., k-1\}$, $t \in (\Delta, t_0)$ optimal controls are given by

$$u_1(t + jt_0) = u_2(t + jt_0) = (-1)^j \frac{c_0}{2k}.$$

If $c_0 > 0$, the optimal value is $t_0 c_0$. With $r \in [-c_0,\ c_0]$, the integral (46) has the value

$$\frac{1}{2}\int_0^{t_0} c_0 - r + c_0 + r \, dt = t_0 c_0,$$

thus also for all $r \in [-c_0, c_0]$ the controls given by (38)–(45) are optimal. Only in the trivial case $c_0 = 0$ where the initial state is already zero, the choice $r = 0$ represents the unique solution.

Example 2. Assume that $y_0(x) = 0$ and $y_1(x) = \sin(x\pi/L)$. Then (22) and (23) imply respectively

$$\alpha_0(t) = \frac{2L}{c\pi}\left[\sin(t\frac{c\pi}{2L})\right]^2, \quad \beta_0(t) = -\frac{2L}{c\pi}\left[\cos(t\frac{c\pi}{2L})\right]^2.$$

Since $\alpha_0(t) = [(2L)/(c\pi)] + \beta_0(t)$, for all real numbers $s \neq 0$ we have

$$|\alpha_0(t) - \frac{L}{c\pi}| + |\beta_0(t) + \frac{L}{c\pi}| = 2|\frac{L}{c\pi} + \beta_0(t)| = |\frac{L}{c\pi} + \frac{L}{c\pi} + \beta_0(t) + \beta_0(t)|$$

$$= |\frac{2L}{c\pi} - s + \beta_0(t) + s + \beta_0(t)| \leq |\frac{2L}{c\pi} - s + \beta_0(t)| + |s + \beta_0(t)| = |\alpha_0(t) - s| + |\beta_0(t) + s|.$$

Hence with the value $r = L/(c\pi)$ the integral from Theorem 2 attains its minimal value, namely

$$\frac{1}{2}\int_0^{t_0} 2|\frac{L}{c\pi} + \beta_0(t)|\, dt$$

and optimal controls are given by (38)–(45) with $r = L/(c\pi)$. Note that since $\alpha_0(t) - r = \beta_0(t) + r$ we have $u_1 = u_2$.

Now let $L = 1$, $c = 1$ and $T = 3.25$, hence $k = 3$. Figure 1 shows the corresponding optimal control $u_1 = u_2$ with $r = 1/\pi$ and Figure 2 shows the state y generated by u_1 and u_2.

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