6

Indization of Categories

In this chapter we develop the theory of ind-objects. The basic reference is [64] where most, if not all, the results which appear here were already obtained (see also [3]). Apart from loc. cit., and despite its importance, it seems difficult to find in the literature a concise exposition of this subject. This chapter is an attempt in this direction.

6.1 Indization of Categories and Functors

Recall that a universe $\mathcal{U}$ is given. When we consider a category, it means a $\mathcal{U}$-category and **Set** is the category of $\mathcal{U}$-sets (see Convention 1.4.1). As far as this has no implications, we will skip this point.

Recall that for a category $\mathcal{C}$, inductive limits in $\mathcal{C}^\wedge := \mathrm{Fct}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$ are denoted by "$\varinjlim$".

Definition 6.1.1. (i) *Let $\mathcal{C}$ be a $\mathcal{U}$-category. An* ind-object *in $\mathcal{C}$ is an object $A \in \mathcal{C}^\wedge$ which is isomorphic to* "$\varinjlim$" α *for some functor $\alpha\colon I \to \mathcal{C}$ with I filtrant and $\mathcal{U}$-small.*

(ii) *We denote by* $\mathrm{Ind}^{\mathcal{U}}(\mathcal{C})$ (*or simply* $\mathrm{Ind}(\mathcal{C})$ *if there is no risk of confusion*) *the full big subcategory of $\mathcal{C}^\wedge$ consisting of ind-objects, and call it the indization of $\mathcal{C}$. We denote by $\iota_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ the natural functor* (*induced by* $\mathrm{h}_{\mathcal{C}}$).

(iii) *Similarly, a* pro-object *in $\mathcal{C}$ is an object $B \in \mathcal{C}^\vee$ which is isomorphic to* "$\varprojlim$" β *for some functor $\beta\colon I^{\mathrm{op}} \to \mathcal{C}$ with I filtrant and small.*

(iv) *We denote by* $\mathrm{Pro}^{\mathcal{U}}(\mathcal{C})$ (*or simply* $\mathrm{Pro}(\mathcal{C})$) *the full big subcategory of $\mathcal{C}^\vee$ consisting of pro-objects.*

Lemma 6.1.2. *The categories* $\mathrm{Ind}(\mathcal{C})$ *and* $\mathrm{Pro}(\mathcal{C})$ *are $\mathcal{U}$-categories.*

Proof. It is enough to treat $\mathrm{Ind}(\mathcal{C})$. Let $A, B \in \mathrm{Ind}(\mathcal{C})$. We may assume that $A \simeq$ "$\varinjlim_{i\in I}$" $\alpha(i)$ and $B \simeq$ "$\varinjlim_{j\in J}$" $\beta(j)$ for small and filtrant categories I and J. In this case $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is isomorphic to a small set by (2.6.4). q.e.d.

We may replace "filtrant and small" by "filtrant and cofinally small" in the above definition.

There is an equivalence

$$\mathrm{Pro}(\mathcal{C}) \simeq (\mathrm{Ind}(\mathcal{C}^{\mathrm{op}}))^{\mathrm{op}} .$$

Hence, we may restrict our study to ind-objects.

Example 6.1.3. Let k be a field and let V denote an infinite-dimensional k-vector space. Consider the contravariant functor on $\mathrm{Mod}(k)$, $W \mapsto V \otimes \mathrm{Hom}_k(W, k)$. It defines an ind-object of $\mathrm{Mod}(k)$ which is not in $\mathrm{Mod}(k)$. Notice that this functor is isomorphic to the functor $V \mapsto$ "$\varinjlim_{V'\subset V}$" V' where V' ranges over the filtrant set of finite-dimensional vector subspaces of V.

Notation 6.1.4. We shall often denote by the capital letters A, B, C, etc. objects of $\mathcal{C}^\wedge$ and as usual by X, Y, Z objects of $\mathcal{C}$.

Recall that for $A \in \mathcal{C}^\wedge$, we introduced the category $\mathcal{C}_A$ and the forgetful functor $\mathrm{j}_A : \mathcal{C}_A \to \mathcal{C}$, and proved the isomorphism $A \simeq$ "$\varinjlim$" j_A (see Proposition 2.6.3).

Proposition 6.1.5. *Let $A \in \mathcal{C}^\wedge$. Then $A \in \mathrm{Ind}(\mathcal{C})$ if and only if $\mathcal{C}_A$ is filtrant and cofinally small.*

Proof. This follows immediately from Proposition 2.6.3 and Proposition 3.2.2. q.e.d.

Applying Definitions 3.3.1 and 3.3.14, we get:

Corollary 6.1.6. *The functor $\iota_{\mathcal{C}} : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ is right exact and right small.*

Proposition 6.1.7. *Assume that a category $\mathcal{C}$ admits finite inductive limits. Then $\mathrm{Ind}(\mathcal{C})$ is the full subcategory of $\mathcal{C}^\wedge$ consisting of functors $A : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ such that A is left exact and $\mathcal{C}_A$ is cofinally small.*

Proof. Apply Propositions 3.3.13 and 6.1.5. q.e.d.

Theorem 6.1.8. *Let $\mathcal{C}$ be a category. The category $\mathrm{Ind}(\mathcal{C})$ admits small filtrant inductive limits and the natural functor $\mathrm{Ind}(\mathcal{C}) \to \mathcal{C}^\wedge$ commutes with such limits.*

Similarly $\mathrm{Pro}(\mathcal{C})$ admits small filtrant projective limits and the natural functor $\mathrm{Pro}(\mathcal{C}) \to \mathcal{C}^\vee$ commutes with such limits.

Proof. Let $\alpha\colon I \to \mathrm{Ind}(\mathcal{C})$ be a functor with I small and filtrant and let $A = \text{“}\varinjlim\text{”}\,\alpha \in \mathcal{C}^\wedge$. It is enough to show that A belongs to $\mathrm{Ind}(\mathcal{C})$. We shall use Proposition 6.1.5.

(i) $\mathcal{C}_A$ is filtrant. By Lemma 3.1.2, it is enough to show that for any finite category J and any functor $\beta\colon J \to \mathcal{C}_A$, there exists $Z \in \mathcal{C}_A$ such that $\varprojlim \mathrm{Hom}_{\mathcal{C}_A}(\beta, Z) \neq \emptyset$. For any $X \in \mathcal{C}_A$, we have

$$\begin{aligned}\mathrm{Hom}_{(\mathcal{C}^\wedge)_A}(X, A) &\simeq \varinjlim_{i\in I} \mathrm{Hom}_{(\mathcal{C}^\wedge)_A}(X, \alpha(i)) \\ &\simeq \varinjlim_{i\in I}\ \varinjlim_{Y\in\mathcal{C}_{\alpha(i)}}\ \mathrm{Hom}_{\mathcal{C}_A}(X, Y)\,.\end{aligned}$$

Since I and $\mathcal{C}_{\alpha(i)}$ are filtrant, $\varinjlim\limits_{i\in I}$ and $\varinjlim\limits_{Y\in\mathcal{C}_{\alpha(i)}}$ commute with finite projective limits by Theorem 3.1.6. Hence, we obtain

$$\begin{aligned}\{\mathrm{pt}\} &\simeq \varprojlim_{j\in J} \mathrm{Hom}_{(\mathcal{C}^\wedge)_A}(\beta(j), A) \\ &\simeq \varinjlim_{i\in I}\ \varinjlim_{Y\in\mathcal{C}_{\alpha(i)}}\ \varprojlim_{j\in J} \mathrm{Hom}_{\mathcal{C}_A}(\beta(j), Y)\,.\end{aligned}$$

Hence, there exist $i \in I$ and $Y \in \mathcal{C}_{\alpha(i)}$ such that $\varprojlim \mathrm{Hom}_{\mathcal{C}_A}(\beta, Y) \neq \emptyset$.

(ii) $\mathcal{C}_A$ is cofinally small. By Proposition 3.2.6, for any $i \in I$, there exists a small subset S_i of $\mathrm{Ob}(\mathcal{C}_{\alpha(i)})$ such that for any $X \in \mathcal{C}_{\alpha(i)}$ there exists a morphism $X \to Y$ with $Y \in S_i$. Let $\varphi_i\colon \mathcal{C}_{\alpha(i)} \to \mathcal{C}_A$ be the canonical functor. Then $S = \bigcup_{i\in I} \varphi_i(S_i)$ is a small subset of $\mathrm{Ob}(\mathcal{C}_A)$ and for any $X \in \mathcal{C}_A$ there exists a morphism $X \to Y$ with $Y \in S$. q.e.d.

Proposition 6.1.9. *Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a functor. There exists a unique functor $IF : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}')$ such that:*

(i) *the restriction of IF to $\mathcal{C}$ is F,*
(ii) *IF commutes with small filtrant inductive limits, that is, if $\alpha\colon I \to \mathrm{Ind}(\mathcal{C})$ is a functor with I small and filtrant, then we have*

$$IF(\text{“}\varinjlim\text{”}\,\alpha) \xrightarrow{\sim} \text{“}\varinjlim\text{”}(IF \circ \alpha)\,.$$

The proof goes as the one of Proposition 2.7.1 and we do not repeat it. The functor IF is given by

$$IF(A) = \text{“}\varinjlim_{(U\to A)\in\mathcal{C}_A}\text{”}\ F(U) \ \text{ for } A \in \mathrm{Ind}(\mathcal{C})\,.$$

Proposition 6.1.9 (i) may be visualized by the commutative diagram below:

$$\begin{array}{ccc}\mathcal{C} & \xrightarrow{\ F\ } & \mathcal{C}' \\ \downarrow & & \downarrow \\ \mathrm{Ind}(\mathcal{C}) & \xrightarrow{\ IF\ } & \mathrm{Ind}(\mathcal{C}').\end{array}$$

Recall that if $A \simeq \text{“}\varinjlim_i\text{”}\, \alpha(i)$, $B \simeq \text{“}\varinjlim_j\text{”}\, \beta(j)$, then (see (2.6.4))

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(A, B) \simeq \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathcal{C}}(\alpha(i), \beta(j)) .$$

The map $IF \colon \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(A, B) \to \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C}')}(IF(A), IF(B))$ is given by

$$\varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathcal{C}}(\alpha(i), \beta(j)) \to \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathcal{C}'}(F(\alpha(i)), F(\beta(j))) . \tag{6.1.1}$$

Remark that if $\mathcal{C}$ is small, the diagram below commutes.

$$\begin{array}{ccc} \mathrm{Ind}(\mathcal{C}) & \xrightarrow{IF} & \mathrm{Ind}(\mathcal{C}') \\ \downarrow & & \downarrow \\ \mathcal{C}^\wedge & \xrightarrow{\widehat{F}} & \mathcal{C}'^\wedge . \end{array}$$

(The functor $\widehat{F}$ is defined in Proposition 2.7.1 and Notation 2.7.2.)

Proposition 6.1.10. *Let $F \colon \mathcal{C} \to \mathcal{C}'$. If F is faithful (resp. fully faithful), so is IF.*

Proof. This follows from (6.1.1). q.e.d.

Proposition 6.1.11. *Let $F \colon \mathcal{C} \to \mathcal{C}'$ and $G \colon \mathcal{C}' \to \mathcal{C}''$ be two functors. Then $I(G \circ F) \simeq IG \circ IF$.*

Proof. The proof is obvious. q.e.d.

Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories. By Proposition 6.1.9, the projection functors $\mathcal{C} \times \mathcal{C}' \to \mathcal{C}$ and $\mathcal{C} \times \mathcal{C}' \to \mathcal{C}'$ define the functor

$$\theta \colon \mathrm{Ind}(\mathcal{C} \times \mathcal{C}') \to \mathrm{Ind}(\mathcal{C}) \times \mathrm{Ind}(\mathcal{C}') \tag{6.1.2}$$

Proposition 6.1.12. *The functor θ in* (6.1.2) *is an equivalence.*

Proof. A quasi-inverse to θ is constructed as follows. To $A \in \mathrm{Ind}(\mathcal{C})$ and $A' \in \mathrm{Ind}(\mathcal{C}')$, associate $\text{“}\varinjlim_{((X \to A),(X' \to A')) \in \mathcal{C}_A \times \mathcal{C}_{A'}}\text{”} (X, X')$. Since $\mathcal{C}_A \times \mathcal{C}_{A'}$ is cofinally small and filtrant, it belongs to $\mathrm{Ind}(\mathcal{C} \times \mathcal{C}')$. q.e.d.

Proposition 6.1.13. *Let $\alpha \colon I \to \mathcal{C}$ and $\beta \colon J \to \mathcal{C}$ be functors with I and J small and filtrant. Let $f \colon \text{“}\varinjlim\text{”}\, \alpha \to \text{“}\varinjlim\text{”}\, \beta$ be a morphism in* $\mathrm{Ind}(\mathcal{C})$*. Then there exist a small and filtrant category K, cofinal functors $p_I \colon K \to I$, $p_J \colon K \to J$ and a morphism of functors $\varphi \colon \alpha \circ p_I \to \beta \circ p_J$ making the diagram below commutative*

$$\begin{array}{ccc} \text{“}\varinjlim\text{”}\,(\alpha\circ p_I) & \xrightarrow{\text{“}\varinjlim\text{”}\,\varphi} & \text{“}\varinjlim\text{”}\,(\beta\circ p_J) \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ \text{“}\varinjlim\text{”}\,\alpha & \xrightarrow{f} & \text{“}\varinjlim\text{”}\,\beta. \end{array} \tag{6.1.3}$$

Proof. Set $A = \text{“}\varinjlim\text{”}\,\alpha$, $B = \text{“}\varinjlim\text{”}\,\beta$, and denote by $\widetilde{\alpha}\colon I \to \mathcal{C}_A$, $\widetilde{\beta}\colon J \to \mathcal{C}_B$ and $\widetilde{f}\colon \mathcal{C}_A \to \mathcal{C}_B$ the functors induced by α, β and f. Consider the category $K := M[I \xrightarrow{\widetilde{f}\circ\widetilde{\alpha}} \mathcal{C}_B \xleftarrow{\widetilde{\beta}} J]$ (see Definition 3.4.1).

The functor $\widetilde{\beta}$ is cofinal by Proposition 2.6.3 (ii), and the categories I and J are small and filtrant by the hypotheses. Proposition 3.4.5 then implies that the category K is filtrant, cofinally small and the projection functors p_I and p_J from K to I and J are cofinal.

We may identify K with the category whose objects are the triplets (i, j, g) of $i \in I$, $j \in J$ and $g\colon \alpha(i) \to \beta(j)$ such that the diagram below commutes

$$\begin{array}{ccc} \alpha(i) & \xrightarrow{g} & \beta(j) \\ \downarrow & & \downarrow \\ \text{“}\varinjlim\text{”}\,\alpha & \xrightarrow{f} & \text{“}\varinjlim\text{”}\,\beta, \end{array}$$

and the morphisms are the natural ones. Then g defines a morphism of functors $\varphi\colon \alpha\circ p_I \to \beta\circ p_J$ such that the diagram (6.1.3) commutes. q.e.d.

Corollary 6.1.14. *Let $f\colon A \to B$ be a morphism in* $\mathrm{Ind}(\mathcal{C})$*. Then there exist a small and filtrant category I and a morphism $\varphi\colon \alpha \to \beta$ of functors from I to $\mathcal{C}$ such that $A \simeq \text{“}\varinjlim\text{”}\,\alpha$, $B \simeq \text{“}\varinjlim\text{”}\,\beta$ and $f = \text{“}\varinjlim\text{”}\,\varphi$.*

We shall extend this result to the case of a pair of parallel arrows. A more general statement for finite diagrams will be given in Sect. 6.4.

Corollary 6.1.15. *Let $f, g\colon A \rightrightarrows B$ be two morphisms in* $\mathrm{Ind}(\mathcal{C})$*. Then there exist a small and filtrant category I and morphisms $\varphi, \psi\colon \alpha \rightrightarrows \beta$ of functors from $I \to \mathcal{C}$ such that $A \simeq \text{“}\varinjlim\text{”}\,\alpha$, $B \simeq \text{“}\varinjlim\text{”}\,\beta$, $f = \text{“}\varinjlim\text{”}\,\varphi$ and $g = \text{“}\varinjlim\text{”}\,\psi$.*

Proof. Let I and J be small filtrant categories and let $\alpha\colon I \to \mathcal{C}$ and $\beta\colon J \to \mathcal{C}$ be two functors such that $A \simeq \text{“}\varinjlim\text{”}\,\alpha$ and $B \simeq \text{“}\varinjlim\text{”}\,\beta$. Denote by $\widetilde{\alpha}\colon I \to \mathcal{C}\times\mathcal{C}$ the functor $i \mapsto \alpha(i)\times\alpha(i)$, and similarly with $\widetilde{\beta}$. Then $(A, A) \simeq \text{“}\varinjlim\text{”}\,\widetilde{\alpha}$ and $(B, B) \simeq \text{“}\varinjlim\text{”}\,\widetilde{\beta}$.

By Proposition 6.1.12, the morphism $(f, g)\colon A\times A \to B\times B$ in $\mathrm{Ind}(\mathcal{C})\times\mathrm{Ind}(\mathcal{C})$ defines a morphism in $\mathrm{Ind}(\mathcal{C}\times\mathcal{C})$. We still denote this morphism by (f, g) and apply Proposition 6.1.13. We find a small and filtrant category K, functors $p_I : K \to I$, $p_J : K \to J$ and a morphism of functors (φ, ψ) from $\widetilde{\alpha} \circ p_I$ to $\widetilde{\beta} \circ p_J$ such that $(f, g) = \text{“}\varinjlim\text{”}\, (\varphi, \psi)$. It follows that $f = \text{“}\varinjlim\text{”}\, \varphi$ and $g = \text{“}\varinjlim\text{”}\, \psi$. q.e.d.

Proposition 6.1.16. (i) *Assume that for any pair of parallel arrows in $\mathcal{C}$, its kernel in $\mathcal{C}^\wedge$ belongs to* $\mathrm{Ind}(\mathcal{C})$*. Then, for any pair of parallel arrows in* $\mathrm{Ind}(\mathcal{C})$*, its kernel in $\mathcal{C}^\wedge$ is its kernel in* $\mathrm{Ind}(\mathcal{C})$*.*

(ii) *Let J be a small set and assume that the product in $\mathcal{C}^\wedge$ of any family indexed by J of objects of $\mathcal{C}$ belongs to* $\mathrm{Ind}(\mathcal{C})$*. Then, for any family indexed by J of objects of* $\mathrm{Ind}(\mathcal{C})$*, its product in $\mathcal{C}^\wedge$ is its product in* $\mathrm{Ind}(\mathcal{C})$*.*

Proof. (i) Let $f, g : A \rightrightarrows B$ be a pair of parallel arrows in $\mathrm{Ind}(\mathcal{C})$. With the notations of Corollary 6.1.14, we may assume that $A = \text{“}\varinjlim\text{”}\, \alpha$, $B = \text{“}\varinjlim\text{”}\, \beta$ and there exist morphisms of functors $\varphi, \psi : \alpha \rightrightarrows \beta$ such that $f = \text{“}\varinjlim\text{”}\, \varphi$ and $g = \text{“}\varinjlim\text{”}\, \psi$. Let γ denote the kernel of (φ, ψ). Then $\text{“}\varinjlim\text{”}\, \gamma$ is a kernel of (f, g) in $\mathcal{C}^\wedge$ and belongs to $\mathrm{Ind}(\mathcal{C})$.
(ii) Let $A_j \in \mathrm{Ind}(\mathcal{C})$, $j \in J$. For each $j \in J$, there exist a small and filtrant category I_j and a functor $\alpha_j : I_j \to \mathcal{C}$ such that $A_j \simeq \text{“}\varinjlim\text{”}\, \alpha_j$. Define the small filtrant category $K = \prod_{j \in J} I_j$ and denote by $\pi_j : K \to I_j$ the natural functor.

Using Corollary 3.1.12 we get the isomorphisms in $\mathcal{C}^\wedge$

$$\prod_{j \in J} A_j \simeq \prod_{j \in J} \text{“}\varinjlim_{i \in I_j}\text{”}\, \alpha_j(i) \simeq \text{“}\varinjlim_{k \in K}\text{”} \prod_{j \in J} \alpha_j(\pi_j(k)) \,.$$

q.e.d.

Corollary 6.1.17. (i) *Assume that the category $\mathcal{C}$ admits finite projective limits. Then the category* $\mathrm{Ind}(\mathcal{C})$ *admits finite projective limits. Moreover, the natural functors* $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ *and* $\mathrm{Ind}(\mathcal{C}) \to \mathcal{C}^\wedge$ *are left exact.*

(ii) *Assume that the category $\mathcal{C}$ admits small projective limits. Then the category* $\mathrm{Ind}(\mathcal{C})$ *admits small projective limits and the natural functors* $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ *and* $\mathrm{Ind}(\mathcal{C}) \to \mathcal{C}^\wedge$ *commute with small projective limits.*

Proposition 6.1.18. (i) *Assume that the category $\mathcal{C}$ admits cokernels, that is, the cokernel of any pair of parallel arrows exists in $\mathcal{C}$. Then* $\mathrm{Ind}(\mathcal{C})$ *admits cokernels.*

(ii) *Assume that $\mathcal{C}$ admits finite coproducts. Then* $\mathrm{Ind}(\mathcal{C})$ *admits small coproducts.*

(iii) *Assume that the category $\mathcal{C}$ admits finite inductive limits. Then* $\mathrm{Ind}(\mathcal{C})$ *admits small inductive limits.*

Proof. (i) Let $f, g : A \rightrightarrows B$ be arrows in $\mathrm{Ind}(\mathcal{C})$. With the notations of Corollary 6.1.14, we may assume that $A = \text{“}\varinjlim\text{”}\, \alpha$, $B = \text{“}\varinjlim\text{”}\, \beta$ and there exist morphisms of functors $\varphi, \psi : \alpha \rightrightarrows \beta$ such that $f = \text{“}\varinjlim\text{”}\, \varphi$ and $g = \text{“}\varinjlim\text{”}\, \psi$. Let λ_i denote the cokernel of $(\alpha(i), \beta(i))$ and let $L \in \mathrm{Ind}(\mathcal{C})$.

Then $\mathrm{Hom}_{\mathcal{C}^\wedge}(\lambda(i), L)$ is the kernel of $\mathrm{Hom}_{\mathcal{C}^\wedge}(\beta(i), L) \rightrightarrows \mathrm{Hom}_{\mathcal{C}^\wedge}(\alpha(i), L)$. Applying the left exact functor $\varprojlim$, we conclude that "$\varinjlim$" λ is a cokernel of ("$\varinjlim$" φ, "$\varinjlim$" ψ).

(ii) The proof that $\mathrm{Ind}(\mathcal{C})$ admits finite coproducts is similar to the proof in (i). The general case follows by Lemma 3.2.9.

(iii) follows from (i), (ii) and the same lemma. q.e.d.

Recall that if $\mathcal{C}$ admits cokernels (resp. finite coproducts, resp. finite inductive limits), then the functor $\iota_{\mathcal{C}} \colon \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ commutes with such limits by Corollary 6.1.6 and Proposition 3.3.2.

Proposition 6.1.19. *Assume that $\mathcal{C}$ admits finite inductive limits and finite projective limits. Then small filtrant inductive limits are exact in* $\mathrm{Ind}(\mathcal{C})$.

Proof. It is enough to check that small filtrant inductive limits commute with finite projective limits in $\mathrm{Ind}(\mathcal{C})$. Since the embedding $\mathrm{Ind}(\mathcal{C}) \to \mathcal{C}^\wedge$ commutes with small filtrant inductive limits and with finite projective limits, this follows from the fact that small filtrant inductive limits are exact in $\mathcal{C}^\wedge$ (see Exercise 3.2). q.e.d.

Remark 6.1.20. (i) The natural functor $\mathrm{Ind}(\mathcal{C}) \to \mathcal{C}^\wedge$ commutes with filtrant inductive limits (Theorem 6.1.8), but it does not commute with inductive limits in general. Indeed, it does not commute with finite coproducts (see Exercise 6.3). Hence, when writing "$\varinjlim$" for an inductive system indexed by a non filtrant category I, the limit should be understood in $\mathcal{C}^\wedge$.
(ii) If $\mathcal{C}$ admits finite inductive limits, then $\mathrm{Ind}(\mathcal{C})$ admits small inductive limits and $\iota_{\mathcal{C}} \colon \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ commutes with finite inductive limits (Corollary 6.1.6 and Proposition 6.1.18) but if $\mathcal{C}$ admits small filtrant inductive limits, $\iota_{\mathcal{C}}$ does not commute with such limits in general. We may summarize these properties by the table below. Here, "○" means that the functors commute, and "×" they do not.

	$\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$	$\mathrm{Ind}(\mathcal{C}) \to \mathcal{C}^\wedge$
finite inductive limits	○	×
finite coproducts	○	×
small filtrant inductive limits	×	○
small coproducts	×	×
small inductive limits	×	×

Since the definition of $\mathrm{Ind}(\mathcal{C})$ makes use of the notion of being small, it depends on the choice of the universe. However, the result below tells us that when replacing a universe $\mathcal{U}$ with a bigger one $\mathcal{V}$, the category of ind-objects of $\mathcal{C}$ in $\mathcal{U}$ is a full subcategory of that of ind-objects of $\mathcal{C}$ in $\mathcal{V}$.

More precisely, consider two universes $\mathcal{U}$ and $\mathcal{V}$ with $\mathcal{U} \subset \mathcal{V}$, and let $\mathcal{C}$ denote a $\mathcal{U}$-category.

Proposition 6.1.21. *The natural functor* $\mathrm{Ind}^{\mathcal{U}}(\mathcal{C}) \to \mathrm{Ind}^{\mathcal{V}}(\mathcal{C})$ *is fully faithful. If $\mathcal{C}$ admits finite inductive limits, then this functor commutes with $\mathcal{U}$-small inductive limits. If $\mathcal{C}$ admits finite (resp. $\mathcal{U}$-small) projective limits, then this functor commutes with such projective limits.*

Proof. The first statement follows from isomorphisms (2.6.4). The functor $\mathrm{Ind}^{\mathcal{U}}(\mathcal{C}) \to \mathrm{Ind}^{\mathcal{V}}(\mathcal{C})$ commutes with finite inductive limits as seen in the proof of Proposition 6.1.18. Since it commutes with $\mathcal{U}$-small filtrant inductive limits, it commutes with $\mathcal{U}$-small inductive limits. Recall that the natural functor $\mathcal{C}^\wedge_{\mathcal{U}} \to \mathcal{C}^\wedge_{\mathcal{V}}$ commutes with $\mathcal{U}$-small projective limits (see Remark 2.6.5). Then the functor $\mathrm{Ind}^{\mathcal{U}}(\mathcal{C}) \to \mathrm{Ind}^{\mathcal{V}}(\mathcal{C})$ commutes with finite (resp. $\mathcal{U}$-small projective) limits by Proposition 6.1.16 if $\mathcal{C}$ admits such limits. q.e.d.

6.2 Representable Ind-limits

Let $\alpha \colon I \to \mathcal{C}$ be a functor with I small and filtrant. We shall study under which conditions the functor "$\varinjlim$" is representable in $\mathcal{C}$.

For each $i \in I$, let us denote by $\rho_i \colon \alpha(i) \to$ "$\varinjlim$" α the natural functor. It satisfies

$$\rho_j \circ \alpha(s) = \rho_i \quad \text{for any } s \colon i \to j\ . \tag{6.2.1}$$

Proposition 6.2.1. *Let $\alpha \colon I \to \mathcal{C}$ be a functor with I small and filtrant and let $Z \in \mathcal{C}$. The conditions below are equivalent:*

(i) "$\varinjlim$" α *is representable by* Z,

(ii) *there exist an $i_0 \in I$ and a morphism $\tau_0 \colon Z \to \alpha(i_0)$ satisfying the property: for any morphism $s \colon i_0 \to i$, there exist a morphism $g \colon \alpha(i) \to Z$ and a morphism $t \colon i \to j$ satisfying*
(a) $g \circ \alpha(s) \circ \tau_0 = \mathrm{id}_Z$,
(b) $\alpha(t) \circ \alpha(s) \circ \tau_0 \circ g = \alpha(t)$.

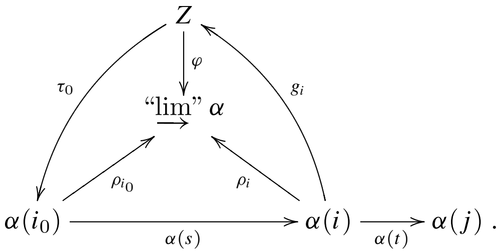
Proof. (i) $\Rightarrow$ (ii) Let $\varphi \colon Z \xrightarrow{\sim}$ "$\varinjlim$" α be an isomorphism. Since we have $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(Z, Z) \simeq \varinjlim_i \mathrm{Hom}_{\mathcal{C}}(Z, \alpha(i))$, there exist $i_0 \in I$ and $\tau_0 \colon Z \to \alpha(i_0)$ such that $\varphi = \rho_{i_0} \circ \tau_0$. For any $i \in I$, the chain of morphisms $\alpha(i) \to$ "$\varinjlim$" $\alpha \xleftarrow{\sim} Z$ defines a morphism $g_i \colon \alpha(i) \to Z$ with $\varphi \circ g_i = \rho_i$. Hence, for any $s \colon i_0 \to i$, we have

$$\varphi \circ g_i \circ \alpha(s) \circ \tau_0 = \rho_i \circ \alpha(s) \circ \tau_0 = \rho_{i_0} \circ \tau_0 = \varphi\ .$$

This shows (ii)-(a). Since I is filtrant and

$$\rho_i \circ \mathrm{id}_{\alpha(i)} = \rho_i = \varphi \circ g_i = \rho_i \circ \alpha(s) \circ \tau_0 \circ g_i \ ,$$

there exists $t : i \to j$ satisfying $\alpha(t) \circ \mathrm{id}_{\alpha(i)} = \alpha(t) \circ (\alpha(s) \circ \tau_0 \circ g_i)$. This is visualized by the diagram



(ii) ⇒ (i) The morphism $\tau_0 : Z \to \alpha(i_0)$ defines the morphism

$$\varphi = \rho_{i_0} \circ \tau_0 : Z \to \text{“}\varinjlim\text{”}\, \alpha \ .$$

To prove that φ is an isomorphism, it is enough to check that φ induces an isomorphism

$$\varphi_X : \mathrm{Hom}_{\mathcal{C}}(X, Z) \xrightarrow{\sim} \varinjlim_i \mathrm{Hom}_{\mathcal{C}}(X, \alpha(i)) \quad \text{for any } X \in \mathcal{C} \ .$$

Injectivity of φ_X. Let $u, v \in \mathrm{Hom}_{\mathcal{C}}(X, Z)$ with $\varphi_X(u) = \varphi_X(v)$. There exists $s : i_0 \to i$ such that $\alpha(s) \circ \tau_0 \circ u = \alpha(s) \circ \tau_0 \circ v$. Then for $g \in \mathrm{Hom}_{\mathcal{C}}(\alpha(i), Z)$ as in (ii),

$$u = g \circ \alpha(s) \circ \tau_0 \circ u = g \circ \alpha(s) \circ \tau_0 \circ v = v \ .$$

Surjectivity of φ_X. Let $w \in \mathrm{Hom}_{\mathcal{C}}(X, \alpha(i))$ and let $s : i_0 \to i$. Take $g : \alpha(i) \to Z$ and $t : i \to j$ as in (ii). Then

$$\alpha(t) \circ w = \alpha(t) \circ \alpha(s) \circ \tau_0 \circ g \circ w \ .$$

The image of w in $\varinjlim_j \mathrm{Hom}_{\mathcal{C}}(X, \alpha(j))$ is $\varphi_X(g \circ w)$. q.e.d.

6.3 Indization of Categories Admitting Inductive Limits

In this section we shall study $\mathrm{Ind}(\mathcal{C})$ in the case where $\mathcal{C}$ admits small filtrant inductive limits. Recall that $\iota_{\mathcal{C}} : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ denotes the natural embedding functor.

Proposition 6.3.1. *Assume that $\mathcal{C}$ admits small filtrant inductive limits.*

(i) *The functor $\iota_{\mathcal{C}} : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ admits a left adjoint $\sigma_{\mathcal{C}} : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$, and if $A \simeq \text{“}\varinjlim\text{”}\, \alpha$, then $\sigma_{\mathcal{C}}(A) \simeq \varinjlim \alpha$.*
(ii) *We have $\sigma_{\mathcal{C}} \circ \iota_{\mathcal{C}} \simeq \mathrm{id}_{\mathcal{C}}$.*

Proof. (i) Let $A \in \mathrm{Ind}(\mathcal{C})$ and let us show that the functor

$$\mathcal{C} \ni X \mapsto \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(A, \iota_{\mathcal{C}}(X))$$

is representable. Let $\alpha\colon I \to \mathcal{C}$ be a functor with I small and filtrant such that $A \simeq \text{“}\varinjlim\text{”}\, \alpha$. Then

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\text{“}\varinjlim\text{”}\, \alpha, \iota_{\mathcal{C}}(X)) &\simeq \varprojlim_{i} \mathrm{Hom}_{\mathcal{C}}(\alpha(i), X) \\ &\simeq \mathrm{Hom}_{\mathcal{C}}(\varinjlim \alpha, X) \,.\end{aligned}$$

(ii) is obvious. q.e.d.

Corollary 6.3.2. *Assume that $\mathcal{C}$ admits small filtrant inductive limits. Then for any functor $F\colon \mathcal{J} \to \mathcal{C}$ there exists a unique (up to unique isomorphism) functor $JF\colon \mathrm{Ind}(\mathcal{J}) \to \mathcal{C}$ such that JF commutes with small filtrant inductive limits and the composition $\mathcal{J} \to \mathrm{Ind}(\mathcal{J}) \to \mathcal{C}$ is isomorphic to F.*

Indeed, JF is given by the composition $\mathrm{Ind}(\mathcal{J}) \xrightarrow{IF} \mathrm{Ind}(\mathcal{C}) \xrightarrow{\sigma_{\mathcal{C}}} \mathcal{C}$.

The next definition will be generalized in Definition 9.2.7.

Definition 6.3.3. *Assume that $\mathcal{C}$ admits small filtrant inductive limits. We say that an object X of $\mathcal{C}$ is of* finite presentation *if for any $\alpha\colon I \to \mathcal{C}$ with I small and filtrant, the natural morphism $\varinjlim \mathrm{Hom}_{\mathcal{C}}(X, \alpha) \to \mathrm{Hom}_{\mathcal{C}}(X, \varinjlim \alpha)$ is an isomorphism, that is, if $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(X, A) \to \mathrm{Hom}_{\mathcal{C}}(X, \sigma_{\mathcal{C}}(A))$ is an isomorphism for any $A \in \mathrm{Ind}(\mathcal{C})$.*

Some authors use the term “compact” instead of “of finite presentation”.

Note that any object of a category $\mathcal{C}$ is of finite presentation in $\mathrm{Ind}(\mathcal{C})$.

Proposition 6.3.4. *Let $F\colon \mathcal{J} \to \mathcal{C}$ be a functor and assume:*

(i) *$\mathcal{C}$ admits small filtrant inductive limits,*
(ii) *F is fully faithful,*
(iii) *for any $Y \in \mathcal{J}$, $F(Y)$ is of finite presentation.*

Then $JF\colon \mathrm{Ind}(\mathcal{J}) \to \mathcal{C}$ is fully faithful.

Proof. Let $\alpha\colon I \to \mathcal{J}$ and $\beta\colon J \to \mathcal{J}$ be two functors with I and J both small and filtrant. Using the hypothesis that $F(\beta(j))$ is of finite presentation for any $j \in J$, we get the chain of isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Ind}(\mathcal{J})}(\text{“}\varinjlim_{j}\text{”}\, \beta(j), \text{“}\varinjlim_{i}\text{”}\, \alpha(i)) &\simeq \varprojlim_{j} \varinjlim_{i} \mathrm{Hom}_{\mathcal{J}}(\beta(j), \alpha(i)) \\ &\simeq \varprojlim_{j} \varinjlim_{i} \mathrm{Hom}_{\mathcal{C}}(F(\beta(j)), F(\alpha(i))) \\ &\simeq \varprojlim_{j} \mathrm{Hom}_{\mathcal{C}}(F(\beta(j)), \varinjlim_{i} F(\alpha(i))) \\ &\simeq \mathrm{Hom}_{\mathcal{C}}(\varinjlim_{j} F(\beta(j)), \varinjlim_{i} F(\alpha(i))) \\ &\simeq \mathrm{Hom}_{\mathcal{C}}(JF(\text{“}\varinjlim_{j}\text{”}\, \beta(j)), JF(\text{“}\varinjlim_{i}\text{”}\, \alpha(i))) \,.\end{aligned}$$

q.e.d.

Let $\mathcal{C}$ be a category which admits small filtrant inductive limits. We denote by $\mathcal{C}^{\mathrm{fp}}$ the full subcategory of $\mathcal{C}$ consisting of objects of finite presentation and by $\rho\colon \mathcal{C}^{\mathrm{fp}} \to \mathcal{C}$ the natural functor. The functor ρ induces a fully faithful functor $I\rho\colon \mathrm{Ind}(\mathcal{C}^{\mathrm{fp}}) \to \mathrm{Ind}(\mathcal{C})$ and we have the diagram of functors

(6.3.1)
$$\begin{array}{ccc} \mathcal{C}^{\mathrm{fp}} & \xrightarrow{\ \rho\ } & \mathcal{C} \\ {\scriptstyle \iota_{\mathcal{C}}}\downarrow & \overset{J\rho}{\nearrow} & {\scriptstyle \iota_{\mathcal{C}}}\downarrow\ \uparrow{\scriptstyle \sigma_{\mathcal{C}}} \\ \mathrm{Ind}(\mathcal{C}^{\mathrm{fp}}) & \xrightarrow[\ I\rho\ ]{} & \mathrm{Ind}(\mathcal{C}). \end{array}$$

Note that the functors $J\rho$ and $I\rho$ are fully faithful. Also note that the diagram (6.3.1) is not commutative in general. More precisely:

(6.3.2)
$$\iota_{\mathcal{C}} \circ J\rho \neq I\rho$$

in general (see Exercise 6.6).

Corollary 6.3.5. *Let $\mathcal{C}$ be a category admitting small filtrant inductive limits and assume that any object of $\mathcal{C}$ is a small filtrant inductive limit of objects of finite presentation. Then the functor $J\rho\colon \mathrm{Ind}(\mathcal{C}^{\mathrm{fp}}) \to \mathcal{C}$ is an equivalence of categories.*

Indeed, the functor $J\rho$ is fully faithful by Proposition 6.3.4 and is essentially surjective by the hypothesis.

A related result to Corollary 6.3.5 will be given in Proposition 9.2.19 below in the framework of π-accessible objects.

Examples 6.3.6. (i) There are equivalences $\mathbf{Set}^f \simeq (\mathbf{Set})^{\mathrm{fp}}$ and $\mathrm{Ind}(\mathbf{Set}^f) \simeq \mathbf{Set}$.
(ii) There are equivalences $\mathrm{Mod}^{\mathrm{fp}}(R) \simeq (\mathrm{Mod}(R))^{\mathrm{fp}}$ and $\mathrm{Ind}(\mathrm{Mod}^{\mathrm{fp}}(R)) \simeq \mathrm{Mod}(R)$ for any ring R (see Exercise 6.8).

Corollary 6.3.7. *In the situation of* Corollary 6.3.5*, the functor $\sigma_{\mathcal{C}}$ admits a left adjoint $\kappa_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$. Moreover:*

(i) *If $\xi\colon I \to \mathcal{C}^{\mathrm{fp}}$ is a functor with I small and filtrant and $X \simeq \varinjlim \xi$ in $\mathcal{C}$, then $\kappa_{\mathcal{C}}(X) \simeq$ "$\varinjlim$" $\rho \circ \xi$,*

(ii) *we have $\sigma_{\mathcal{C}} \circ \kappa_{\mathcal{C}} \simeq \mathrm{id}$,*

(iii) *$\kappa_{\mathcal{C}}$ is fully faithful.*

If there is no risk of confusion, we shall write κ instead of $\kappa_{\mathcal{C}}$.

Proof. (i) Denote by κ' a quasi-inverse of $J\rho$ and set $\kappa = I\rho \circ \kappa'$.

Let $X \in \mathcal{C}$ and let us show that the functor

$$\mathrm{Ind}(\mathcal{C}) \ni A \mapsto \mathrm{Hom}_{\mathcal{C}}(X, \sigma_{\mathcal{C}}(A))$$

is representable by $\kappa(X)$. In the sequel we shall not write $I\rho$ for short.

There exists $\xi\colon J \to \mathcal{C}^{\mathrm{fp}}$ with J small and filtrant such that $X \simeq \varinjlim \xi$. Then $\kappa(X) \simeq$ "$\varinjlim$" ξ. We get the chain of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\kappa(X), A) &\simeq \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\text{“}\varinjlim_j\text{”}\, \xi(j), A) \\
&\simeq \varprojlim_j \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\xi(j), A) \\
&\simeq \varprojlim_j \mathrm{Hom}_{\mathcal{C}}(\xi(j), \sigma_{\mathcal{C}}(A)) \\
&\simeq \mathrm{Hom}_{\mathcal{C}}(\varinjlim_j \xi(j), \sigma_{\mathcal{C}}(A)) \\
&\simeq \mathrm{Hom}_{\mathcal{C}}(X, \sigma_{\mathcal{C}}(A)) \; .
\end{aligned}$$

The other assertions are obvious. q.e.d.

6.4 Finite Diagrams in Ind($\mathcal{C}$)

Let K be a small category. The canonical functor $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ defines the functor

(6.4.1) $$\Phi_0\colon \mathrm{Fct}(K, \mathcal{C}) \to \mathrm{Fct}(K, \mathrm{Ind}(\mathcal{C})) \; .$$

Since $\mathrm{Fct}(K, \mathrm{Ind}(\mathcal{C}))$ admits small filtrant inductive limits, we may apply Corollary 6.3.2, and extend the functor Φ_0 to a functor

(6.4.2) $$\Phi\colon \mathrm{Ind}(\mathrm{Fct}(K, \mathcal{C})) \to \mathrm{Fct}(K, \mathrm{Ind}(\mathcal{C}))$$

which commutes with small filtrant inductive limits.

Proposition 6.4.1. *Assume that K is a finite category. Then the functor Φ in* (6.4.2) *is fully faithful.*

Proof. We shall apply Proposition 6.3.4 to Φ_0. Clearly, the functor Φ_0 is fully faithful and $\mathrm{Fct}(K, \mathrm{Ind}(\mathcal{C}))$ admits small filtrant inductive limits. Hence, it remains to check that given a small and filtrant category I, a functor $\alpha\colon I \to \mathrm{Fct}(K, \mathrm{Ind}(\mathcal{C}))$ and an object $\psi \in \mathrm{Fct}(K, \mathcal{C})$, the map

(6.4.3) $$\varinjlim_i \mathrm{Hom}_{\mathrm{Fct}(K,\mathrm{Ind}(\mathcal{C}))}(\psi, \alpha(i)) \to \mathrm{Hom}_{\mathrm{Fct}(K,\mathrm{Ind}(\mathcal{C}))}(\psi, \varinjlim_i \alpha(i))$$

is bijective. This follows from Lemma 2.1.15 and the chain of isomorphisms

$$\begin{aligned}\varinjlim_i \operatorname{Hom}_{\operatorname{Fct}(K,\operatorname{Ind}(\mathcal{C}))}(\psi, \alpha(i)) &\simeq \varinjlim_i \varprojlim_{(a\to b)\in \operatorname{Mor}_0(K)} \operatorname{Hom}_{\operatorname{Ind}(\mathcal{C})}(\psi(a), \alpha(i)(b))\\ &\simeq \varprojlim_{(a\to b)\in \operatorname{Mor}_0(K)} \varinjlim_i \operatorname{Hom}_{\operatorname{Ind}(\mathcal{C})}(\psi(a), \alpha(i)(b))\\ &\simeq \varprojlim_{(a\to b)\in \operatorname{Mor}_0(K)} \operatorname{Hom}_{\operatorname{Ind}(\mathcal{C})}(\psi(a), \varinjlim_i \alpha(i)(b))\\ &\simeq \operatorname{Hom}_{\operatorname{Fct}(K,\operatorname{Ind}(\mathcal{C}))}(\psi, \varinjlim_i \alpha(i))\ .\end{aligned}$$

Here, we have used the fact that in the category **Set**, small filtrant inductive limits commute with finite projective limits (Theorem 3.1.6). q.e.d.

We shall give a condition in order that the functor Φ in (6.4.2) is an equivalence. We need some preparation.

Consider the category $M[\mathcal{C}_1 \xrightarrow{F} \mathcal{C}_0 \xleftarrow{G} \mathcal{C}_2]$ associated with functors $\mathcal{C}_1 \xrightarrow{F} \mathcal{C}_0 \xleftarrow{G} \mathcal{C}_2$ (see Definition 3.4.1). We set for short:

$$\begin{aligned} M_0 &= M[\mathcal{C}_1 \to \mathcal{C}_0 \leftarrow \mathcal{C}_2]\ ,\\ M_1 &= M[\operatorname{Ind}(\mathcal{C}_1) \to \operatorname{Ind}(\mathcal{C}_0) \leftarrow \operatorname{Ind}(\mathcal{C}_2)]\ .\end{aligned}$$

Then M_1 admits small filtrant inductive limits, and by Proposition 3.4.2 there is a canonical fully faithful functor $M_0 \to M_1$ which thus extend to a functor

(6.4.4) $$\Psi\colon \operatorname{Ind}(M_0) \to M_1$$

commuting with small filtrant inductive limits.

Proposition 6.4.2. *The functor Ψ in* (6.4.4) *is an equivalence of categories.*

Proof. (i) Ψ is fully faithful. Since Ψ commutes with small filtrant inductive limits, it is enough to show that for $X \in M_0$ and a small filtrant inductive system $\{Y_i\}_{I\in I}$ in M_0, we have

(6.4.5) $$\varinjlim_i \operatorname{Hom}_{M_0}(X, Y_i) \xrightarrow{\sim} \operatorname{Hom}_{M_1}(X, \varinjlim \Psi(Y_i))\ .$$

Let us write $X = (X_1, X_2, u)$ with $X_\nu \in \mathcal{C}_\nu$ ($\nu = 1, 2$), $u\colon F(X_1) \to G(X_2)$, and let $Y_i = (Y_1^i, Y_2^i, v_i)$ with $Y_\nu^i \in \mathcal{C}_\nu$, $v_i\colon F(Y_1^i) \to G(Y_2^i)$.

Define the morphisms

$$\begin{aligned} \alpha_i\ &:\ \operatorname{Hom}_{\mathcal{C}_1}(X_1, Y_1^i) \to \operatorname{Hom}_{\mathcal{C}_0}(F(X_1), G(Y_2^i))\\ &\qquad f \mapsto (F(X_1) \xrightarrow{F(f)} F(Y_1^i) \xrightarrow{v_i} G(Y_2^i))\ ,\\ \beta_i\ &:\ \operatorname{Hom}_{\mathcal{C}_2}(X_2, Y_2^i) \to \operatorname{Hom}_{\mathcal{C}_0}(F(X_1), G(Y_2^i))\\ &\qquad g \mapsto (F(X_1) \xrightarrow{u} G(X_2) \xrightarrow{G(g)} G(Y_2^i))\ .\end{aligned}$$

Then

$$\operatorname{Hom}_{M_0}(X, Y_i) = \operatorname{Hom}_{\mathcal{C}_1}(X_1, Y_1^i) \times_{\operatorname{Hom}_{\mathcal{C}_0}(F(X_1), G(Y_2^i))} \operatorname{Hom}_{\mathcal{C}_2}(X_2, Y_2^i) .$$

Since filtrant inductive limits commute with fiber products, we have

$$\begin{aligned}
&\operatorname{Hom}_{M_1}(X, \text{“}\varinjlim_i\text{”}\, Y_i) \\
&\quad\simeq \operatorname{Hom}_{\operatorname{Ind}(\mathcal{C}_1)}(X_1, \text{“}\varinjlim_i\text{”}\, Y_1^i) \\
&\qquad\quad \times_{\operatorname{Hom}_{\operatorname{Ind}(\mathcal{C}_0)}(F(X_1), \text{“}\varinjlim_i\text{”}\, G(Y_2^i))} \operatorname{Hom}_{\operatorname{Ind}(\mathcal{C}_2)}(X_2, \text{“}\varinjlim_i\text{”}\, Y_2^i) \\
&\quad\simeq \varinjlim_i \Big(\operatorname{Hom}_{\mathcal{C}_1}(X_1, Y_1^i) \times_{\operatorname{Hom}_{\mathcal{C}_0}(F(X_1), G(Y_2^i))} \operatorname{Hom}_{\mathcal{C}_2}(X_2, Y_2^i)\Big) \\
&\quad\simeq \varinjlim_i \operatorname{Hom}_{M_0}(X, Y_i) .
\end{aligned}$$

(ii) Ψ is essentially surjective. Let $(X_1, X_2, u) \in M_1$ with $X_1 = \text{“}\varinjlim_{i\in I}\text{”}\, X_1^i$, $X_2 = \text{“}\varinjlim_{j\in J}\text{”}\, X_2^j$, and $u\colon \text{“}\varinjlim_i\text{”}\, F(X_1^i) \to \text{“}\varinjlim_j\text{”}\, G(X_2^j)$. By Proposition 6.1.13 there exist a filtrant category K, cofinal functors $p_I\colon K \to I$ and $p_J\colon K \to J$ and a morphism of functors $v = \{v_k\}_{k\in K}$, $v_k\colon F(X_1^{p_I(k)}) \to G(X_2^{p_J(k)})$ such that $\text{“}\varinjlim_k\text{”}\, v_k = u$. Define $Z^k = (X_1^{p_I(k)}, X_2^{p_J(k)}, v_k)$. Then $Z^k \in M_0$ and $\Psi(\text{“}\varinjlim_k\text{”}\, Z^k) \simeq (X_1, X_2, u)$. q.e.d.

Theorem 6.4.3. *Let K be a finite category such that $\operatorname{Hom}_K(a, a) = \{\mathrm{id}_a\}$ for any $a \in K$. Then the natural functor Φ in* (6.4.2) *is an equivalence.*

Proof. We may assume from the beginning that if two objects in K are isomorphic, then they are identical. Then $\operatorname{Ob}(K)$ has a structure of an ordered set as follows: $a \le b$ if and only if $\operatorname{Hom}_K(a, b) \neq \emptyset$.

Indeed, if $a \le b$ and $b \le a$, then there are morphisms $u\colon a \to b$ and $v\colon b \to a$. Since $v \circ u = \mathrm{id}_a$ and $u \circ v = \mathrm{id}_b$, a and b are isomorphic, hence $a = b$.

We shall prove the result by induction on the cardinal of $\operatorname{Ob}(K)$. If this number is zero, the result is obvious. Otherwise, take a maximal element a of $\operatorname{Ob}(K)$. Then $\operatorname{Hom}_K(a, b) = \emptyset$ for any $b \neq a$. Denote by L the full subcategory of K such that $\operatorname{Ob}(L) = \operatorname{Ob}(K) \setminus \{a\}$ and denote by L_a the category of arrows $b \to a$, with $b \in L$. There is a natural functor $F\colon \operatorname{Fct}(L, \mathcal{C}) \to \operatorname{Fct}(L_a, \mathcal{C})$ associated with $L_a \to L$ and a natural functor $G\colon \mathcal{C} \simeq \operatorname{Fct}(\mathbf{Pt}, \mathcal{C}) \to \operatorname{Fct}(L_a, \mathcal{C})$ associated with the constant functor $L_a \to \mathbf{Pt}$.

There is an equivalence

$$\operatorname{Fct}(K, \mathcal{C}) \simeq M[\operatorname{Fct}(L, \mathcal{C}) \xrightarrow{F} \operatorname{Fct}(L_a, \mathcal{C}) \xleftarrow{G} \mathcal{C}] . \tag{6.4.6}$$

Replacing $\mathcal{C}$ with $\operatorname{Ind}(\mathcal{C})$ and applying Proposition 6.4.2 we get the equivalences

$$(6.4.7)\qquad \mathrm{Fct}(K, \mathrm{Ind}(\mathcal{C})) \simeq M[\mathrm{Fct}(L, \mathrm{Ind}(\mathcal{C})) \xrightarrow{IF} \mathrm{Fct}(L_a, \mathrm{Ind}(\mathcal{C})) \xleftarrow{IG} \mathrm{Ind}(\mathcal{C})]\,,$$

$$(6.4.8)\qquad \mathrm{Ind}(\mathrm{Fct}(K, \mathcal{C})) \simeq M[\mathrm{Ind}(\mathrm{Fct}(L, \mathcal{C})) \xrightarrow{IF} \mathrm{Ind}(\mathrm{Fct}(L_a, \mathcal{C})) \xleftarrow{IG} \mathrm{Ind}(\mathcal{C})]\,.$$

Consider the diagram

$$\begin{array}{ccccc}
\mathrm{Ind}(\mathrm{Fct}(L,\mathcal{C})) & \longrightarrow & \mathrm{Ind}(\mathrm{Fct}(L_a,\mathcal{C})) & \longleftarrow & \mathrm{Ind}(\mathcal{C}) \\
\downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_0} & & \downarrow{\scriptstyle \mathrm{id}_{\mathrm{Ind}(\mathcal{C})}} \\
\mathrm{Fct}(L,\mathrm{Ind}(\mathcal{C})) & \longrightarrow & \mathrm{Fct}(L_a,\mathrm{Ind}(\mathcal{C})) & \longleftarrow & \mathrm{Ind}(\mathcal{C})\,.
\end{array}$$

By the induction hypothesis θ_1 is an equivalence, and by Proposition 6.4.1, θ_0 is fully faithful. It follows that

$$\theta\colon M[\mathrm{Ind}(\mathrm{Fct}(L,\mathcal{C})) \to \mathrm{Ind}(\mathrm{Fct}(L_a,\mathcal{C})) \leftarrow \mathrm{Ind}(\mathcal{C})] \longrightarrow M[\mathrm{Fct}(L,\mathrm{Ind}(\mathcal{C})) \to \mathrm{Fct}(L_a,\mathrm{Ind}(\mathcal{C})) \leftarrow \mathrm{Ind}(\mathcal{C})]$$

is an equivalence of categories by Proposition 3.4.2. The left hand side is equivalent to $\mathrm{Ind}(\mathrm{Fct}(K,\mathcal{C}))$ by (6.4.8), and the right hand side is equivalent to $\mathrm{Fct}(K,\mathrm{Ind}(\mathcal{C}))$ by (6.4.7). q.e.d.

Corollary 6.4.4. *For any category $\mathcal{C}$, the natural functor* $\mathrm{Ind}(\mathrm{Mor}(\mathcal{C})) \to \mathrm{Mor}(\mathrm{Ind}(\mathcal{C}))$ *is an equivalence.*

Proof. Apply Theorem 6.4.3 by taking as K the category $\bullet \to \bullet$. q.e.d.

Exercises

Exercise 6.1. (i) Let $\mathcal{C}$ be a small category and let $A \in \mathrm{Ind}(\mathcal{C})$. Prove that the two conditions below are equivalent.

(a) The functor $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(A, \bullet)$ from $\mathrm{Ind}(\mathcal{C})$ to **Set** commutes with small filtrant inductive limits, i.e., A is of finite presentation in $\mathrm{Ind}(\mathcal{C})$.

(b) There exist $X \in \mathcal{C}$ and morphisms $A \xrightarrow{i} X \xrightarrow{p} A$ such that $p \circ i = \mathrm{id}_A$.

(ii) Prove that any $A \in \mathcal{C}^\wedge$ which satisfies (b) belongs to $\mathrm{Ind}(\mathcal{C})$.
(iii) Prove that $\mathcal{C} \to (\mathrm{Ind}(\mathcal{C}))^{\mathrm{fp}}$ is an equivalence if and only if $\mathcal{C}$ is idempotent complete (see Exercise 2.9).

Exercise 6.2. Prove that if X is an initial (resp. terminal) object in $\mathcal{C}$, then $\iota_{\mathcal{C}}(X)$ is an initial (resp. terminal) object in $\mathrm{Ind}(\mathcal{C})$.

Exercise 6.3. Let $\mathcal{C}$ be a small category and denote by $\emptyset_{\mathcal{C}^\wedge}$ and $\mathrm{pt}_{\mathcal{C}^\wedge}$ the initial and terminal objects of $\mathcal{C}^\wedge$, respectively.
(i) Prove that $\emptyset_{\mathcal{C}^\wedge} \notin \mathrm{Ind}(\mathcal{C})$. (Hint: see Exercise 3.7.)
(ii) Prove that $\mathrm{pt}_{\mathcal{C}^\wedge} \in \mathrm{Ind}(\mathcal{C})$ if and only if $\mathcal{C}$ is filtrant and cofinally small.

Exercise 6.4. Let $\mathcal{C}$ be a category which admits finite inductive limits and denote by $\alpha : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}^\wedge$ the natural functor. Prove that the functor α does not commute with finite inductive limits (see Exercise 6.3).

Exercise 6.5. Prove that $\mathrm{Pro}(\mathbf{Set}^f)$ is equivalent to the category of Hausdorff totally disconnected compact spaces. (Recall that on such spaces, any point has an open and closed neighborhood system.)

Exercise 6.6. Let k be a field, $\mathcal{C} = \mathrm{Mod}(k)$. Let $V = k^{\oplus \mathbb{Z}}$ and $V_n = k^{\oplus I_n}$ where $I_n = \{i \in \mathbb{Z};\, |i| \leq n\}$.
(i) Construct the natural morphism $\text{“}\varinjlim_n\text{”}\, V_n \to V$.
(ii) Show that this morphism is a monomorphism and not an epimorphism.

Exercise 6.7. Let $\mathcal{C}$ be a category which admits small filtrant inductive limits. Let us say that an object X of $\mathcal{C}$ is of finite type if for any functor $\alpha\colon I \to \mathcal{C}$ with I small and filtrant, the natural map $\varinjlim \mathrm{Hom}_{\mathcal{C}}(X, \alpha) \to \mathrm{Hom}_{\mathcal{C}}(X, \varinjlim \alpha)$ is injective. Prove that this definition coincides with the usual one when $\mathcal{C} = \mathrm{Mod}(R)$ for a ring R (see Examples 1.2.4 (iv)).

Exercise 6.8. Let R be a ring.
(i) Prove that $M \in \mathrm{Mod}(R)$ is of finite presentation in the sense of Definition 6.3.3 if and only if it is of finite presentation in the classical sense (see Examples 1.2.4 (iv)), that is, if there exists an exact sequence $R^{\oplus n_1} \to R^{\oplus n_0} \to M \to 0$.
(ii) Prove that any R-module M is a small filtrant inductive limit of modules of finite presentation. (Hint: consider the full subcategory of $(\mathrm{Mod}(A))_M$ consisting of modules of finite presentation and prove it is essentially small and filtrant.)
(iii) Deduce that the functor $J\rho$ defined in Diagram (6.3.1) induces an equivalence $J\rho\colon \mathrm{Ind}(\mathrm{Mod}^{\mathrm{fp}}(R)) \xrightarrow{\sim} \mathrm{Mod}(R)$.

Exercise 6.9. Let $\mathcal{C}$ be a small category, $F\colon \mathcal{C} \to \mathcal{C}'$ a functor and denote by $F_*\colon \mathcal{C}' \to \mathcal{C}^\wedge$ the functor given by $F_*(Y)(U) = \mathrm{Hom}_{\mathcal{C}'}(F(U), Y)$ for $Y \in \mathcal{C}'$, $U \in \mathcal{C}$. Prove that the functor F is right exact if and only if F_* sends $\mathcal{C}'$ to $\mathrm{Ind}(\mathcal{C})$.

Exercise 6.10. Let $\mathcal{C}$ be a category and consider the functor

$$\Phi\colon \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}^\vee \quad \text{given by } A \mapsto \varinjlim_{(X \to A) \in \mathcal{C}_A} \mathrm{k}_{\mathcal{C}}(X)\,.$$

(i) Prove that Φ commutes with small filtrant inductive limits and prove that the composition $\mathcal{C} \xrightarrow{\iota_{\mathcal{C}}} \mathrm{Ind}(\mathcal{C}) \xrightarrow{\Phi} \mathcal{C}^\vee$ is isomorphic to the Yoneda functor $\mathrm{k}_{\mathcal{C}}$.
(ii) Assume that $\mathcal{C}$ admits filtrant inductive limits. Prove that the functor Φ factorizes as $\mathrm{Ind}(\mathcal{C}) \xrightarrow{\sigma_{\mathcal{C}}} \mathcal{C} \xrightarrow{\mathrm{k}_{\mathcal{C}}} \mathcal{C}^\vee$, where $\sigma_{\mathcal{C}}$ is defined in the course of Proposition 6.3.1.

Exercise 6.11. Let $\mathcal{J}$ be a full subcategory of a category $\mathcal{C}$ and let $A \in \mathrm{Ind}(\mathcal{C})$. Prove that A is isomorphic to the image of an object of $\mathrm{Ind}(\mathcal{J})$ if and only if any morphism $X \to A$ in $\mathrm{Ind}(\mathcal{C})$ with $X \in \mathcal{C}$ factors through an object of $\mathcal{J}$.

Exercise 6.12. Let G be a group and let $\mathcal{G}$ be the category with one object denoted by c and morphisms $\mathrm{Hom}_{\mathcal{G}}(c,c) = G$. A G-set is a set S with an action of G. If S and S' are G-sets, a G-equivariant map $f\colon S \to S'$ is a map satisfying $f(gs) = gf(s)$ for all $s \in S$ and all $g \in G$. We denote by G-**Set** the category of G-sets and G-equivariant maps.
(i) Prove that $\mathcal{G}^{\mathrm{op}}$ is equivalent to $\mathcal{G}$.
(ii) Prove that $\mathcal{G}^\wedge$ is equivalent to G-**Set** and that the object c of $\mathcal{G}$ corresponds to the G-set G endowed with the left action of G.
(iii) For a G-set X, prove that $\mathcal{G}_X$ is equivalent to the category $\mathcal{C}$ given by $\mathrm{Ob}(\mathcal{C}) = X$ and $\mathrm{Hom}_{\mathcal{C}}(x,y) = \{g \in G\,;\, y = gx\}$ for $x, y \in X$.
(iv) Prove that $\mathcal{G} \xrightarrow{\sim} \mathrm{Ind}(\mathcal{G})$.