5

Generators and Representability

The aim of this chapter is to give various criteria for a functor with values in **Set** to be representable, and as a by-product, criteria for a functor to have an adjoint.

For that purpose, we need to introduce two important notions. The first one is that of a strict morphism for a category $\mathcal{C}$ which admits finite inductive and finite projective limits. In such a category, there are natural definitions of the coimage and of the image of a morphism, and the morphism is strict if the coimage is isomorphic to the image. A crucial fact for our purpose here is that if $\mathcal{C}$ admits a generator (see below), then the family of strict quotients of any object is a small set.

The second important notion is that of a system of generators (and in particular, a generator) in a category $\mathcal{C}$. If $\mathcal{C}$ admits small inductive limits and G is a generator, then any object $X \in \mathcal{C}$ is a quotient of a small coproduct of copies of G, similarly as any module over a ring A is a quotient of $A^{\oplus I}$ for a small set I.

With these tools in hands, it is then possible to state various theorems of representability. For example, we prove that if $\mathcal{C}$ admits small inductive limits, finite projective limits, a generator and small filtrant inductive limits are stable by base change, then any contravariant functor from $\mathcal{C}$ to **Set** is representable as soon as it sends small inductive limits to projective limits (Theorem 5.3.9).

Many of these results are classical and we refer to [64].

5.1 Strict Morphisms

Definition 5.1.1. *Let $\mathcal{C}$ be a category which admits finite inductive and finite projective limits and let $f\colon X \to Y$ be a morphism in $\mathcal{C}$.*

(i) *The* coimage *of f, denoted by* Coim f, *is given by*

$$\operatorname{Coim} f = \operatorname{Coker}(X \times_Y X \rightrightarrows X)\ .$$

(ii) *The* image *of f, denoted by* $\mathrm{Im}\, f$*, is given by*

$$\mathrm{Im}\, f = \mathrm{Ker}(Y \rightrightarrows Y \sqcup_X Y) \ .$$

Note that the natural morphism $X \to \mathrm{Coim}\, f$ is an epimorphism and the natural morphism $\mathrm{Im}\, f \to Y$ is a monomorphism.

Proposition 5.1.2. *Let $\mathcal{C}$ be a category which admits finite inductive and finite projective limits and let $f\colon X \to Y$ be a morphism in $\mathcal{C}$.*

(i) *There is an isomorphism* $X \underset{X \times_Y X}{\sqcup} X \xrightarrow{\sim} \mathrm{Coim}\, f$*.*

(ii) *There is an isomorphism* $\mathrm{Im}\, f \xrightarrow{\sim} Y \underset{Y \sqcup_X Y}{\times} Y$*.*

(iii) *There is a unique morphism*

$$u\colon \mathrm{Coim}\, f \to \mathrm{Im}\, f \tag{5.1.1}$$

such that the composition $X \to \mathrm{Coim}\, f \xrightarrow{u} \mathrm{Im}\, f \to Y$ *is* f*.*

(iv) *The following three conditions are equivalent:*
(a) f *is an epimorphism,*
(b) $\mathrm{Im}\, f \to Y$ *is an isomorphism,*
(c) $\mathrm{Im}\, f \to Y$ *is an epimorphism.*

Proof. (ii) Set $Z = Y \sqcup_X Y$. We shall prove the isomorphism $\mathrm{Ker}(i_1, i_2\colon Y \rightrightarrows Z) \simeq Y \times_Z Y$. For any $U \in \mathcal{C}$, we have

$$\mathrm{Hom}_{\mathcal{C}}(U, Y \times_Z Y) = \big\{(y_1, y_2)\, ;\, y_1, y_2 \in Y(U), i_1(y_1) = i_2(y_2)\big\} \ .$$

The codiagonal morphism $\sigma\colon Z \to Y$ satisfies $\sigma \circ i_1 = \sigma \circ i_2 = \mathrm{id}_Y$. Hence, $i_1(y_1) = i_2(y_2)$ implies $y_1 = \sigma \circ i_1(y_1) = \sigma \circ i_2(y_2) = y_2$. Therefore we obtain

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(U, Y \times_Z Y) &\simeq \big\{y \in Y(U)\, ;\, i_1(y) = i_2(y)\big\} \\ &\simeq \mathrm{Hom}_{\mathcal{C}}(U, \mathrm{Ker}(i_1, i_2\colon Y \rightrightarrows Z)) \ . \end{aligned}$$

(i) follows from (ii) by reversing the arrows.
(iii) Consider the diagram

$$\begin{tikzcd}
X \times_Y X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[r, shift left, "i_1"] \arrow[r, shift right, "i_2"'] & Y \sqcup_X Y \, . \\
& \mathrm{Coim}\, f \arrow[r, dotted, "u"] \arrow[ur, dotted, "\tilde{f}"] & \mathrm{Im}\, f \arrow[u] &
\end{tikzcd}$$

Since $f \circ p_1 = f \circ p_2$, f factors uniquely as $X \xrightarrow{s} \mathrm{Coim}\, f \xrightarrow{\tilde{f}} Y$. Since $i_1 \circ f = i_1 \circ \tilde{f} \circ s$ and $i_2 \circ f = i_2 \circ \tilde{f} \circ s$ are equal and s is an epimorphism, we obtain $i_1 \circ \tilde{f} = i_2 \circ \tilde{f}$. Hence $\tilde{f}$ factors through $\mathrm{Im}\, f$.

The uniqueness follows from the fact that $X \to \operatorname{Coim} f$ is an epimorphism and $\operatorname{Im} f \to Y$ is a monomorphism.
(iv) Assume that f is an epimorphism. By the construction, the two morphisms $i_1, i_2 \colon Y \to Y \sqcup_X Y$ satisfy $i_1 \circ f = i_2 \circ f$. Since f is an epimorphism, it follows that $i_1 = i_2$. Therefore, $\operatorname{Ker}(i_1, i_2) \simeq Y$.
Conversely, assume that $w \colon \operatorname{Im} f \to Y$ is an epimorphism. Since $i_1 \circ w = i_2 \circ w$, we have $i_1 = i_2$. Consider two morphisms $g_1, g_2 \colon Y \rightrightarrows Z$ such that $g_1 \circ f = g_2 \circ f$. These two morphisms define $g \colon Y \sqcup_X Y \to Z$ and $g_1 = i_1 \circ g = i_2 \circ g = g_2$.
q.e.d.

Examples 5.1.3. (i) Let $\mathcal{C} = \mathbf{Set}$. In this case, the morphism (5.1.1) is an isomorphism, and $\operatorname{Im} f \simeq f(X)$, the set-theoretical image of f.
(ii) Let $\mathcal{C}$ denote the category of topological spaces and let $f \colon X \to Y$ be a continuous map. Then, $\operatorname{Coim} f$ is the space $f(X)$ endowed with the quotient topology of X and $\operatorname{Im} f$ is the space $f(X)$ endowed with topology induced by Y. Hence, (5.1.1) is not an isomorphism in general.

Definition 5.1.4. *Let $\mathcal{C}$ be a category which admits finite inductive limits and finite projective limits. A morphism f is* strict *if $\operatorname{Coim} f \to \operatorname{Im} f$ is an isomorphism.*

Proposition 5.1.5. *Let $\mathcal{C}$ be a category which admits finite inductive limits and finite projective limits and let $f \colon X \to Y$ be a morphism in $\mathcal{C}$.*

(i) *The following five conditions are equivalent*
   (a) *f is a strict epimorphism,*
   (b) $\operatorname{Coim} f \xrightarrow{\sim} Y$,
   (c) *the sequence $X \times_Y X \rightrightarrows X \to Y$ is exact,*
   (d) *there exists a pair of parallel arrows $g, h \colon Z \rightrightarrows X$ such that $f \circ g = f \circ h$ and $\operatorname{Coker}(g, h) \to Y$ is an isomorphism,*
   (e) *for any $Z \in \mathcal{C}$, $\operatorname{Hom}_{\mathcal{C}}(Y, Z)$ is isomorphic to the set of morphisms $u \colon X \to Z$ satisfying $u \circ v_1 = u \circ v_2$ for any pair of parallel morphisms $v_1, v_2 \colon W \rightrightarrows X$ such that $f \circ v_1 = f \circ v_2$.*

(ii) *If f is both a strict epimorphism and a monomorphism, then f is an isomorphism.*
(iii) *The morphism $X \to \operatorname{Coim} f$ is a strict epimorphism.*

Proof. (i) (a) $\Rightarrow$ (b) since $\operatorname{Im} f \xrightarrow{\sim} Y$ by Proposition 5.1.2 (iv).

(i) (b) $\Rightarrow$ (a) is obvious.

(i) (b) $\Leftrightarrow$ (c) is obvious.

(i) (d) $\Rightarrow$ (b). Assume that the sequence $Z \rightrightarrows X \xrightarrow{f} Y$ is exact. Consider the solid diagram

$$\begin{tikzcd} Z \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & X \arrow[r] \arrow[d] & Y \arrow[d, dotted] \\ X \times_Y X \arrow[r, shift left] \arrow[r, shift right] & X \arrow[r] & \operatorname{Coim} f. \end{tikzcd}$$

We get a morphism $Y \to \operatorname{Coim} f$ which is inverse to the natural morphism $\operatorname{Coim} f \to Y$.

(i) (c) $\Rightarrow$ (d) is obvious.

(i) (c) $\Leftrightarrow$ (e). The condition on u in (e) is equivalent to saying that the two compositions $X \times_Y X \rightrightarrows X \xrightarrow{u} Z$ coincide.

(ii) The morphism f decomposes as $X \to \operatorname{Coim} f \to Y$. The first arrow is an isomorphism by Proposition 5.1.2 (iv) (with the arrows reversed) and the second arrow is an isomorphism by (i).

(iii) follows from (i) (d) by the definition of $\operatorname{Coim} f$. q.e.d.

Remark that in Proposition 5.1.5, it is not necessary to assume that $\mathcal{C}$ admits finite inductive and projective limits to formulate condition (i) (e).

Definition 5.1.6. *Let $\mathcal{C}$ be a category. A morphism $f: X \to Y$ is a* strict epimorphism *if condition* (i) (e) *in* Proposition 5.1.5 *is satisfied.*

Note that condition (i) (e) in Proposition 5.1.5 is equivalent to saying that the map

$$\operatorname{Hom}_{\mathcal{C}}(Y, Z) \to \operatorname{Hom}_{\mathcal{C}^\wedge}(\operatorname{Im} \mathrm{h}_{\mathcal{C}}(f), \mathrm{h}_{\mathcal{C}}(Z))$$

is an isomorphism for any $Z \in \mathcal{C}$.

The notion of a strict monomorphism is defined similarly.

Proposition 5.1.7. *Let $\mathcal{C}$ be a category which admits finite inductive limits and finite projective limits. Assume that any epimorphism in $\mathcal{C}$ is strict. Let $f: X \to Y$ be a morphism in $\mathcal{C}$.*

(i) *The morphism* $\operatorname{Coim} f \to Y$ *is a monomorphism.*
(ii) *If f decomposes as $X \xrightarrow{u} I \xrightarrow{v} Y$ with an epimorphism u and a monomorphism v, then I is isomorphic to* $\operatorname{Coim} f$.

Proof. (i) Set $I = \operatorname{Coim} f$ and let $X \xrightarrow{u} I \xrightarrow{v} Y$ be the canonical morphisms. Let w denote the composition $X \to I \to \operatorname{Coim} v$. Since w is a strict epimorphism, $\operatorname{Coim} w$ is isomorphic to $\operatorname{Coim} v$. For a pair of parallel arrows $\varphi, \psi: W \rightrightarrows X$, the condition $u \circ \varphi = u \circ \psi$ is equivalent to the condition $f \circ \varphi = f \circ \psi$. Indeed, if $f \circ \varphi = f \circ \psi$, then (φ, ψ) gives a morphism $W \to X \times_Y X$, and the two compositions $W \to X \times_Y X \rightrightarrows X \to I$ are equal and coincide with $u \circ \varphi$ and $u \circ \psi$.

Hence, these two conditions are also equivalent to $w \circ \varphi = w \circ \psi$. This implies $X \times_{\operatorname{Coim} v} X \simeq X \times_Y X$, and hence

$$I \simeq \operatorname{Coker}(X \times_Y X \rightrightarrows X) \simeq \operatorname{Coker}(X \times_{\operatorname{Coim} v} X \rightrightarrows X) \\ \simeq \operatorname{Coim} w \simeq \operatorname{Coim} v\ .$$

Then Proposition 5.1.2 (iv) (with the arrows reversed) implies that v is a monomorphism.
(ii) Since v is a monomorphism, the canonical morphism $X \times_I X \to X \times_Y X$ is an isomorphism. Hence,

$$\operatorname{Coim} f \simeq \operatorname{Coker}(X \times_Y X \rightrightarrows X) \simeq \operatorname{Coker}(X \times_I X \rightrightarrows X) \\ \simeq \operatorname{Coim}(X \to I) \simeq I\ ,$$

where the last isomorphism follows from the fact that u is a strict epimorphism. q.e.d.

Similarly as in Definition 1.2.18, we set:

Definition 5.1.8. *Let $\mathcal{C}$ be a category and let $X \in \mathcal{C}$.*

(i) *An isomorphism class of a strict epimorphism with source X is called a* strict quotient *of X.*
(ii) *An isomorphism class of a strict monomorphism with target X is called a* strict subobject *of X.*

5.2 Generators and Representability

Recall that, unless otherwise specified, a category means a $\mathcal{U}$-category. In particular, we denote by **Set** the category of $\mathcal{U}$-sets.

Definition 5.2.1. *Let $\mathcal{C}$ be a category.*

(i) *A* system of generators *in $\mathcal{C}$ is a family of objects $\{G_i\}_{i\in I}$ of $\mathcal{C}$ such that I is small and the functor $\mathcal{C} \to$* **Set** *given by $X \mapsto \prod_{i\in I} \operatorname{Hom}_{\mathcal{C}}(G_i, X)$ is conservative, that is, a morphism $f\colon X \to Y$ is an isomorphism as soon as $\operatorname{Hom}_{\mathcal{C}}(G_i, X) \to \operatorname{Hom}_{\mathcal{C}}(G_i, Y)$ is an isomorphism for all $i \in I$.*
If the family $\{G_i\}_{i\in I}$ consists of a single object G, G is called a generator.
(ii) *A* system of cogenerators *(resp.* a cogenerator*) in $\mathcal{C}$ is a system of generators (resp. is a generator) in $\mathcal{C}^{\mathrm{op}}$.*

Note that if $\mathcal{C}$ admits small coproducts and a system of generators $\{G_i\}_{i\in I}$, then it admits a generator, namely $\coprod_i G_i$.

Examples 5.2.2. (i) The object $\{\mathrm{pt}\}$ is a generator in **Set**, and a set consisting of two elements is a cogenerator in **Set**.
(ii) Let A be a ring. Then A is a generator in $\mathrm{Mod}(A)$.
(iii) Let $\mathcal{C}$ be a small category. Then $\mathrm{Ob}(\mathcal{C})$ is a system of generators in $\mathcal{C}^\wedge$, by Corollary 1.4.7.

We shall concentrate our study on categories having a generator. By reversing the arrows, the reader will deduce the corresponding results for categories having a cogenerator.

For $G \in \mathcal{C}$, we shall denote by φ_G the functor

$$\varphi_G := \mathrm{Hom}_{\mathcal{C}}(G, \bullet) \colon \mathcal{C} \to \mathbf{Set} .$$

Note that for $X \in \mathcal{C}$, the identity element of

$$\mathrm{Hom}_{\mathbf{Set}}(\mathrm{Hom}_{\mathcal{C}}(G, X), \mathrm{Hom}_{\mathcal{C}}(G, X)) \simeq \mathrm{Hom}_{\mathcal{C}^\vee}(G^{\coprod \mathrm{Hom}\,(G,X)}, X)$$

defines a canonical morphism in $\mathcal{C}^\vee$

$$G^{\coprod \mathrm{Hom}\,(G,X)} \to X . \tag{5.2.1}$$

Proposition 5.2.3. *Assume that $\mathcal{C}$ admits finite projective limits, small coproducts and a generator G. Then:*

(i) *the functor $\varphi_G = \mathrm{Hom}_{\mathcal{C}}(G, \bullet)$ is faithful,*
(ii) *a morphism $f\colon X \to Y$ in $\mathcal{C}$ is a monomorphism if and only if $\varphi_G(f)\colon \mathrm{Hom}_{\mathcal{C}}(G, X) \to \mathrm{Hom}_{\mathcal{C}}(G, Y)$ is injective,*
(iii) *a morphism $f\colon X \to Y$ in $\mathcal{C}$ is an epimorphism if $\varphi_G(f)\colon \mathrm{Hom}_{\mathcal{C}}(G, X) \to \mathrm{Hom}_{\mathcal{C}}(G, Y)$ is surjective,*
(iv) *for any $X \in \mathcal{C}$ the canonical morphism $G^{\coprod \mathrm{Hom}\,(G,X)} \to X$ defined in* (5.2.1) *is an epimorphism in $\mathcal{C}$,*
(v) *for any $X \in \mathcal{C}$, the family of subobjects* (*see* Definition 1.2.18) *of X is a small set.*

Proof. (i) follows from Proposition 2.2.3 and the fact that $\mathrm{Hom}_{\mathcal{C}}(G, \bullet)$ is left exact.
(ii)–(iii) follow from (i) and Proposition 1.2.12.
(iv) By (iii) it is enough to check that $\mathrm{Hom}_{\mathcal{C}}(G, G^{\coprod \mathrm{Hom}\,(G,X)}) \to \mathrm{Hom}_{\mathcal{C}}(G, X)$ is an epimorphism, which is obvious.
(v) We have a map from the family of subobjects of X to the set of subsets of $\varphi_G(X)$. Since $\varphi_G(X)$ is a small set, it is enough to show that this map is injective. For two subobjects $Y_1 \hookrightarrow X$ and $Y_2 \hookrightarrow X$, $Y_1 \times_X Y_2$ is a subobject of X. Assuming that $\mathrm{Im}(\varphi_G(Y_1) \to \varphi_G(X)) = \mathrm{Im}(\varphi_G(Y_2) \to \varphi_G(X))$, we find

$$\varphi_G(Y_1 \times_X Y_2) \simeq \varphi_G(Y_1) \times_{\varphi_G(X)} \varphi_G(Y_2) \simeq \varphi_G(Y_1) \simeq \varphi_G(Y_2) .$$

Hence, $Y_1 \times_X Y_2 \xrightarrow{\sim} Y_i$ for $i = 1, 2$. Therefore, Y_1 and Y_2 are isomorphic. q.e.d.

Proposition 5.2.4. *Let $\mathcal{C}$ be a category which admits finite projective limits and small coproducts, and assume that any morphism which is both an epimorphism and a monomorphism is an isomorphism. For an object G of $\mathcal{C}$, the following conditions are equivalent.*

(i) *G is a generator,*

(ii) *φ_G is faithful,*
(iii) *for any $X \in \mathcal{C}$, there exist a small set I and an epimorphism $G^{\coprod I} \to X$.*

Proof. We know by Proposition 5.2.3 that (i) $\Rightarrow$ (ii) & (iii).
(ii) $\Rightarrow$ (i). Let $f: X \to Y$ and assume that $\varphi_G(f)$ is an isomorphism. By Proposition 1.2.12, f is a monomorphism and an epimorphism. We conclude that f is an isomorphism by the third hypothesis.
(iii) $\Rightarrow$ (ii). Let $f, g: X \rightrightarrows Y$ and assume that $\varphi_G(f) = \varphi_G(g)$. For any small set I and any morphism $u: G^{\coprod I} \to X$, the two compositions $G^{\coprod I} \to X \rightrightarrows Y$ are equal. If u is an epimorphism, this implies $f = g$. q.e.d.

Theorem 5.2.5. *Let $\mathcal{C}$ be a category which admits small inductive limits and let $F: \mathcal{C}^{\mathrm{op}} \to$ **Set** be a functor. Then F is representable if and only if the two conditions below are satisfied:*

(a) *F commutes with small projective limits (i.e., F sends inductive limits in $\mathcal{C}$ to projective limits in* **Set***),*
(b) *the category $\mathcal{C}_F$ is cofinally small. (The category $\mathcal{C}_F$ is associated with $F \in \mathcal{C}^\wedge$ and $\mathrm{h}_\mathcal{C}: \mathcal{C} \to \mathcal{C}^\wedge$ as in* Definition 1.2.16*. In particular, its objects are the pairs (X, u) of $X \in \mathcal{C}$ and $u \in F(X)$.)*

Proof. (i) Condition (a) is obviously necessary. Moreover, if F is representable, let us say by $Y \in \mathcal{C}$, then the category $\mathcal{C}_F \simeq \mathcal{C}_Y$ admits a terminal object, namely (Y, id_Y).
(ii) Conversely, assume that F satisfies (a) and (b).

By hypothesis (a) and Lemma 2.1.13, $\mathcal{C}_F$ admits small inductive limits.

By hypothesis (b), $\mathcal{C}_F$ is cofinally small. Hence the inductive limit of the identity functor is well-defined in $\mathcal{C}_F$. Denote this object of $\mathcal{C}_F$ by X_0:

$$X_0 = \varinjlim_{X \in \mathcal{C}_F} X\ .$$

Since X_0 is a terminal object of $\mathcal{C}_F$ by Lemma 2.1.11, X_0 is a representative of F by Lemma 1.4.10. q.e.d.

We shall give a condition in order that the condition (b) of Theorem 5.2.5 is satisfied.

Theorem 5.2.6. *Let $\mathcal{C}$ be a category satisfying:*

(i) *$\mathcal{C}$ admits a generator G,*
(ii) *$\mathcal{C}$ admits small inductive limits,*
(iii) *for any $X \in \mathcal{C}$ the family of quotients of X is a small set.*

*Then any functor $F: \mathcal{C}^{\mathrm{op}} \to$ **Set** which commutes with small projective limits is representable.*

Remark 5.2.7. The hypotheses (iii) is not assumed in [64], but the authors could not follow the argument of loc. cit.

Proof. By Theorem 5.2.5, it is enough to check that the category $\mathcal{C}_F$ is cofinally small. Note that F being left exact, this category is filtrant by Proposition 3.3.13.

Set $Z_0 = G^{\coprod F(G)}$. By the assumption on F, we have

$$F(Z_0) \simeq F(G)^{F(G)} \simeq \mathrm{Hom}_{\mathbf{Set}}(F(G), F(G)) .$$

Denote by $u_0 \in F(Z_0)$ the image of $\mathrm{id}_{F(G)}$. Hence, (Z_0, u_0) belongs to $\mathcal{C}_F$. Let $(X, u) \in \mathcal{C}_F$ and set $X_1 = G^{\coprod \mathrm{Hom}\,(G,X)}$. Then the natural morphism $X_1 \to X$ is an epimorphism by Proposition 5.2.3 (iv).

Consider the maps $\mathrm{Hom}_{\mathcal{C}}(G, X) \to \mathrm{Hom}_{\mathbf{Set}}(F(X), F(G)) \to F(G)$ where the second one is associated with $u \in F(X)$. They define the morphism $X_1 = G^{\coprod \mathrm{Hom}\,(G,X)} \to Z_0 = G^{\coprod F(G)}$ and the commutative diagram in $\mathcal{C}^\wedge$

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & Z_0 \arrow[d] \\ X \arrow[r, "u"'] & F . \end{tikzcd}$$

Define X' as $X \coprod_{X_1} Z_0$ and consider the diagram below in which the square is co-Cartesian:

$$\begin{tikzcd} X_1 = G^{\coprod \mathrm{Hom}\,(G,X)} \arrow[r] \arrow[d, two heads] & Z_0 \arrow[d, two heads] \arrow[ddr] & \\ X \arrow[r] \arrow[drr, "u"'] & X' \arrow[dr, dotted, "u'"] & \\ & & F. \end{tikzcd}$$

Since F commutes with projective limits, the dotted arrow may be completed. Since $X_1 \to X$ is an epimorphism, $Z_0 \to X'$ is an epimorphism by Exercise 2.22. Hence, for any $(X, u) \in \mathcal{C}_F$ we have found a morphism $(X, u) \to (X', u')$ in $\mathcal{C}_F$ such that there exists an epimorphism $Z_0 \twoheadrightarrow X'$. By hypothesis (iii) and Proposition 3.2.6, $\mathcal{C}_F$ is cofinally small. q.e.d.

Proposition 5.2.8. *Let $\mathcal{C}$ be a category which admits small inductive limits. Assume that any functor $F : \mathcal{C}^{\mathrm{op}} \to$ **Set** is representable if it commutes with small projective limits. Then:*

(i) *$\mathcal{C}$ admits small projective limits,*
(ii) *a functor $F : \mathcal{C} \to \mathcal{C}'$ admits a right adjoint if and only if it commutes with small inductive limits.*

Proof. (i) Let $\beta : I^{\mathrm{op}} \to \mathcal{C}$ be a projective system indexed by a small category I. Consider the object $F \in \mathcal{C}^\wedge$ given by

$$F(X) = \varprojlim_i \mathrm{Hom}_{\mathcal{C}}(X, \beta(i)) .$$

This functor from $\mathcal{C}^{\mathrm{op}}$ to **Set** commutes with small projective limits in $\mathcal{C}^{\mathrm{op}}$, and hence it is representable.
(ii) For any $Y \in \mathcal{C}'$, the functor $X \mapsto \mathrm{Hom}_{\mathcal{C}'}(F(X), Y)$ commutes with small projective limits, and hence it is representable. q.e.d.

Proposition 5.2.9. *Assume that $\mathcal{C}$ admits finite inductive limits, finite projective limits, and a generator. Then the family of strict quotients of an object $X \in \mathcal{C}$ is a small set.*

Proof. Recall that $f : X \to Y$ is a strict epimorphism if and only if the sequence $X \times_Y X \rightrightarrows X \to Y$ is exact. Hence, we may identify the family of strict quotients of X with a family of subobjects of $X \times X$, and this is a small set by Proposition 5.2.3 (v).

q.e.d.

Corollary 5.2.10. *Assume that the category $\mathcal{C}$ admits small inductive limits, finite projective limits and a generator. Assume moreover that any epimorphism in $\mathcal{C}$ is strict. Then a functor $F : \mathcal{C}^{\mathrm{op}} \to$ **Set** is representable if and only if it commutes with small projective limits.*

Examples 5.2.11. The hypotheses of Corollary 5.2.10 are satisfied by the category **Set** as well as by the category $\mathrm{Mod}(R)$ of modules over a ring R.

5.3 Strictly Generating Subcategories

In Sect. 5.2 we obtained representability results in a category $\mathcal{C}$ when assuming either that the family of quotients of any object is small or that any epimorphism is strict. In this section, we shall get rid of this kind of hypotheses.

Let $\mathcal{C}$ be a category and $\mathcal{F}$ a small full subcategory of $\mathcal{C}$. Then we have the natural functor

$$\varphi : \mathcal{C} \to \mathcal{F}^\wedge, \tag{5.3.1}$$

which associates with $X \in \mathcal{C}$ the functor $\mathcal{F} \ni Y \mapsto \mathrm{Hom}_{\mathcal{C}}(Y, X)$. By the Yoneda Lemma, we have

$$\mathrm{Hom}_{\mathcal{F}^\wedge}(\varphi(X), \varphi(Y)) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

for $X \in \mathcal{F}$ and $Y \in \mathcal{C}$.

By the definition, φ is conservative if and only if $\mathrm{Ob}(\mathcal{F})$ is a system of generators. If moreover $\mathcal{C}$ admits finite projective limits, then φ is faithful by Proposition 2.2.3.

Definition 5.3.1. *Let $\mathcal{C}$ be a category and $\mathcal{F}$ an essentially small full subcategory of $\mathcal{C}$. We say that $\mathcal{F}$ is* strictly generating *in $\mathcal{C}$ if the functor φ in* (5.3.1) *is fully faithful.*

Note that if $\mathcal{F}$ is a strictly generating full subcategory, then $\mathrm{Ob}(\mathcal{F})$ is a system of generators.

Lemma 5.3.2. *Let $\mathcal{C}$ be a category, and let $\mathcal{F}$ and $\mathcal{G}$ be small full subcategories of $\mathcal{C}$. Assume that $\mathcal{F} \subset \mathcal{G}$ and $\mathcal{F}$ is strictly generating. Then $\mathcal{G}$ is also strictly generating.*

Proof. Let $\varphi_{\mathcal{F}} \colon \mathcal{C} \to \mathcal{F}^\wedge$ and $\varphi_{\mathcal{G}} \colon \mathcal{C} \to \mathcal{G}^\wedge$ be the natural functors. Then $\varphi_{\mathcal{F}}$ is fully faithful and it decomposes as

$$\mathcal{C} \xrightarrow{\varphi_{\mathcal{G}}} \mathcal{G}^\wedge \xrightarrow{\iota} \mathcal{F}^\wedge .$$

Hence $\varphi_{\mathcal{G}}$ is faithful. Let us show that the map

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{G}^\wedge}(\varphi_{\mathcal{G}}(X), \varphi_{\mathcal{G}}(Y))$$

is surjective for any $X,\ Y \in \mathcal{C}$. Let $\xi \in \mathrm{Hom}_{\mathcal{G}^\wedge}(\varphi_{\mathcal{G}}(X), \varphi_{\mathcal{G}}(Y))$. Since $\varphi_{\mathcal{F}}$ is fully faithful, there exists $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ such that

$$(5.3.2) \qquad \iota(\xi) = \varphi_{\mathcal{F}}(f) \quad \text{as elements of } \mathrm{Hom}_{\mathcal{F}^\wedge}(\varphi_{\mathcal{F}}(X), \varphi_{\mathcal{F}}(Y)).$$

Let us show that $\xi = \varphi_{\mathcal{G}}(f)$. It is enough to show that, for any $Z \in \mathcal{G}$, the map induced by ξ

$$\xi_Z \colon \mathrm{Hom}_{\mathcal{C}}(Z, X) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Z, Y)$$

coincides with the map $v \mapsto f \circ v$.

Let $v \in \mathrm{Hom}_{\mathcal{C}}(Z, X)$. Then for any $S \in \mathcal{F}$ and $s \colon S \to Z$:

$$\xi_Z(v) \circ s = \xi_S(v \circ s) = \iota(\xi)_S(v \circ s) = f \circ v \circ s \ ,$$

where the last equality follows from (5.3.2). Hence $\varphi_{\mathcal{F}}(\xi_Z(v)) = \varphi_{\mathcal{F}}(f \circ v)$ as elements of $\mathrm{Hom}_{\mathcal{F}^\wedge}(\varphi_{\mathcal{F}}(Z), \varphi_{\mathcal{F}}(Y))$, and the faithfulness of $\varphi_{\mathcal{F}}$ implies $\xi_Z(v) = f \circ v$. q.e.d.

Lemma 5.3.3. *Let $\mathcal{C}$ be a category which admits small inductive limits and let $\mathcal{F}$ be a small full subcategory of $\mathcal{C}$. Then the functor $\varphi \colon \mathcal{C} \to \mathcal{F}^\wedge$ admits a left adjoint $\psi \colon \mathcal{F}^\wedge \to \mathcal{C}$ and for $F \in \mathcal{F}^\wedge$, we have*

$$\psi(F) \simeq \varinjlim_{(Y \to F) \in \mathcal{F}_F} Y \ .$$

Proof. For $X \in \mathcal{C}$ and $F \in \mathcal{F}^\wedge$, we have the chain of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(\varinjlim_{(Y\to F)\in\mathcal{F}_F} Y, X) &\simeq \varprojlim_{(Y\to F)\in\mathcal{F}_F} \mathrm{Hom}_{\mathcal{C}}(Y, X) \\
&\simeq \varprojlim_{(Y\to F)\in\mathcal{F}_F} \mathrm{Hom}_{\mathcal{F}^\wedge}(\varphi(Y), \varphi(X)) \\
&\simeq \mathrm{Hom}_{\mathcal{F}^\wedge}(\varinjlim_{(Y\to F)\in\mathcal{F}_F} \varphi(Y), \varphi(X)) ,
\end{aligned}$$

and $\varinjlim_{(Y\to F)\in\mathcal{F}_F} \varphi(Y) \simeq F$ by Proposition 2.6.3. q.e.d.

Proposition 5.3.4. *Let $\mathcal{C}$ be a category which admits small inductive limits and let $\mathcal{F}$ be a small strictly generating full subcategory of $\mathcal{C}$. Let $\mathcal{E}$ denote the full subcategory of $\mathcal{F}^\wedge$ consisting of objects $F \in \mathcal{F}^\wedge$ such that the functor $\mathcal{C} \ni X \mapsto \mathrm{Hom}_{\mathcal{F}^\wedge}(\varphi(X), F) \in \mathbf{Set}$ commutes with small projective limits. Then $\mathcal{C}$ is equivalent to $\mathcal{E}$ by φ.*

Proof. It is obvious that φ sends $\mathcal{C}$ to $\mathcal{E}$. Hence, it is enough to show that any $F \in \mathcal{E}$ is isomorphic to the image of an object of $\mathcal{C}$ by φ. Let ψ denote the left adjoint to φ constructed in Lemma 5.3.3. By Proposition 4.1.4, it is enough to prove the isomorphism

$$\mathrm{Hom}_{\mathcal{F}^\wedge}(\varphi\psi(G), F) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{F}^\wedge}(G, F)$$

for any $G \in \mathcal{F}^\wedge$ and $F \in \mathcal{E}$. We have the chain of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{F}^\wedge}(\varphi\psi(G), F) &\simeq \mathrm{Hom}_{\mathcal{F}^\wedge}(\varphi(\varinjlim_{(X\to G)\in\mathcal{F}_G} X), F) \\
&\simeq \varprojlim_{(X\to G)\in\mathcal{F}_G} \mathrm{Hom}_{\mathcal{F}^\wedge}(\varphi(X), F) \\
&\simeq \mathrm{Hom}_{\mathcal{F}^\wedge}(\varinjlim_{(X\to G)\in\mathcal{F}_G} \varphi(X), F) \\
&\simeq \mathrm{Hom}_{\mathcal{F}^\wedge}(G, F) ,
\end{aligned}$$

where the second isomorphism follows from the hypothesis $F \in \mathcal{E}$ and the last isomorphism follows from Proposition 2.6.3 (i). q.e.d.

Proposition 5.3.5. *Let $\mathcal{C}$ be a category which admits small inductive limits and assume that there exists a small strictly generating full subcategory of $\mathcal{C}$. Let $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ be a functor. If F commutes with small projective limits, then F is representable.*

Proof. Let $\mathcal{F}$ be a small strictly generating full subcategory of $\mathcal{C}$. Let $\widetilde{F} \in \mathcal{F}^\wedge$ be the restriction of F to $\mathcal{F}$. For $X \in \mathcal{C}$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{F}^\wedge}(\varphi(X), \widetilde{F}) &\simeq \mathrm{Hom}_{\mathcal{F}^\wedge}(\varinjlim_{(Y\to X)\in\mathcal{F}_X} \varphi(Y), \widetilde{F}) \\ &\simeq \varprojlim_{(Y\to X)\in\mathcal{F}_X} \mathrm{Hom}_{\mathcal{F}^\wedge}(\varphi(Y), \widetilde{F}) \\ &\simeq \varprojlim_{(Y\to X)\in\mathcal{F}_X} F(Y) \simeq F(\varinjlim_{(Y\to X)\in\mathcal{F}_X} Y)\,.\end{aligned}$$

Since φ is fully faithful, we have $\varinjlim_{(Y\to X)\in\mathcal{F}_X} Y \simeq \psi\varphi(X) \simeq X$. Hence, we obtain

$$F(X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{F}^\wedge}(\varphi(X), \widetilde{F}) \quad \text{for any } X \in \mathcal{C}\,. \tag{5.3.3}$$

It follows that the functor $\mathcal{C} \ni X \mapsto \mathrm{Hom}_{\mathcal{F}^\wedge}(\varphi(X), \widetilde{F})$ sends small inductive limits to projective limits, and by Proposition 5.3.4 there exists $X_0 \in \mathcal{C}$ such that $\widetilde{F} \simeq \varphi(X_0)$. Then (5.3.3) implies that

$$\begin{aligned}F(X) &\simeq \mathrm{Hom}_{\mathcal{F}^\wedge}(\varphi(X), \widetilde{F}) \\ &\simeq \mathrm{Hom}_{\mathcal{F}^\wedge}(\varphi(X), \varphi(X_0)) \simeq \mathrm{Hom}_{\mathcal{C}}(X, X_0)\end{aligned}$$

for any $X \in \mathcal{C}$. q.e.d.

We shall give several criteria for a small full subcategory $\mathcal{F}$ to be strictly generating.

Theorem 5.3.6. *Let $\mathcal{C}$ be a category satisfying the conditions* (i)–(iii) *below:*

(i) *$\mathcal{C}$ admits small inductive limits and finite projective limits,*
(ii) *small filtrant inductive limits are stable by base change (see* Definition 2.2.6),
(iii) *any epimorphism is strict.*

Let $\mathcal{F}$ be an essentially small full subcategory of $\mathcal{C}$ such that

(a) $\mathrm{Ob}(\mathcal{F})$ *is a system of generators,*
(b) *$\mathcal{F}$ is closed by finite coproducts in $\mathcal{C}$.*

Then $\mathcal{F}$ is strictly generating.

Proof. We may assume from the beginning that $\mathcal{F}$ is small.

(i) As already mentioned, the functor φ in (5.3.1) is conservative and faithful.

(ii) By Proposition 1.2.12, a morphism f in $\mathcal{C}$ is an epimorphism as soon as $\varphi(f)$ is an epimorphism.

(iii) Let us fix $X \in \mathcal{C}$. For a small filtrant inductive system $\{Y_i\}_{i\in I}$ in $\mathcal{C}_X$, we have

$$\varinjlim_i \mathrm{Coim}(Y_i \to X) \xrightarrow{\sim} \mathrm{Coim}(\varinjlim_i Y_i \to X)\,. \tag{5.3.4}$$

Indeed, setting $Y_\infty = \varinjlim_i Y_i$, we have

$$\begin{aligned}\varinjlim_i (Y_i \times_X Y_i) &\simeq \varinjlim_{i_1, i_2} (Y_{i_1} \times_X Y_{i_2}) \simeq \varinjlim_{i_1} \varinjlim_{i_2} (Y_{i_1} \times_X Y_{i_2}) \\ &\simeq \varinjlim_{i_1} (Y_{i_1} \times_X Y_\infty) \simeq Y_\infty \times_X Y_\infty\,.\end{aligned}$$

Here the first isomorphism follows from Corollary 3.2.3 (ii), and the last two isomorphisms follow from hypothesis (ii). Hence we obtain

$$\begin{aligned}\mathrm{Coim}(Y_\infty \to X) &\simeq \mathrm{Coker}(Y_\infty \times_X Y_\infty \rightrightarrows Y_\infty) \\ &\simeq \mathrm{Coker}\big(\varinjlim_i (Y_i \times_X Y_i) \rightrightarrows \varinjlim_i Y_i\big) \\ &\simeq \varinjlim_i \mathrm{Coker}(Y_i \times_X Y_i \rightrightarrows Y_i) \\ &\simeq \varinjlim_i \mathrm{Coim}(Y_i \to X)\,.\end{aligned}$$

(iv) For $Z \in \mathcal{F}_X$, set

$$\eta(Z) = \mathrm{Coim}(Z \to X) := \mathrm{Coker}(Z \times_X Z \rightrightarrows Z)\,.$$

Then η defines a functor $\mathcal{F}_X \to \mathcal{C}_X$. For any $Y \in \mathcal{C}$, we have

$$\mathrm{Hom}_{\mathcal{C}}(\eta(Z), Y) \simeq \mathrm{Ker}\big(\mathrm{Hom}_{\mathcal{C}}(Z, Y) \rightrightarrows \mathrm{Hom}_{\mathcal{C}}(Z \times_X Z, Y)\big)\,.$$

We have $\mathrm{Hom}_{\mathcal{C}}(Z, Y) \simeq \mathrm{Hom}_{\mathcal{F}^\wedge}(\varphi(Z), \varphi(Y))$ by the Yoneda Lemma. On the other hand, the map $\mathrm{Hom}_{\mathcal{C}}(Z \times_X Z, Y) \to \mathrm{Hom}_{\mathcal{F}^\wedge}(\varphi(Z \times_X Z), \varphi(Y)) \simeq \mathrm{Hom}_{\mathcal{F}^\wedge}(\varphi(Z) \times_{\varphi(X)} \varphi(Z), \varphi(Y))$ is injective since φ is faithful. Hence we obtain

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{C}}(\eta(Z), Y) \\ &\quad\simeq \mathrm{Ker}\big(\mathrm{Hom}_{\mathcal{F}^\wedge}(\varphi(Z), \varphi(Y)) \rightrightarrows \mathrm{Hom}_{\mathcal{F}^\wedge}(\varphi(Z) \times_{\varphi(X)} \varphi(Z), \varphi(Y))\big) \\ &\quad\simeq \mathrm{Hom}_{\mathcal{F}^\wedge}\big(\mathrm{Coker}(\varphi(Z) \times_{\varphi(X)} \varphi(Z) \rightrightarrows \varphi(Z)), \varphi(Y)\big) \\ &\quad\simeq \mathrm{Hom}_{\mathcal{F}^\wedge}\big(\mathrm{Im}(\varphi(Z) \to \varphi(X)), \varphi(Y)\big)\,.\end{aligned}$$

(v) Let us denote by I the set of finite subsets of $\mathrm{Ob}(\mathcal{F}_X)$, ordered by inclusion. Regarding I as a category, it is small and filtrant. For $A \in I$, $\xi(A) := \sqcup_{Z \in A} Z$ belongs to $\mathcal{F}_X$ by (b), and ξ defines a functor $I \to \mathcal{F}_X$. Then

$$\varinjlim_{A \in I} \varphi(\xi(A)) \to \varphi(X) \text{ is an epimorphism}\,. \tag{5.3.5}$$

Indeed, for any $S \in \mathcal{F}$ and $u \in \varphi(X)(S) = \mathrm{Hom}_{\mathcal{C}}(S, X)$, u is in the image of $\varphi(\xi(A))(S)$ with $A = \{(S, u)\}$.

(vi) Since $\varinjlim_{A \in I} \varphi(\xi(A)) \to \varphi(X)$ factors through $\varphi(\varinjlim_{A \in I} \xi(A))$, the morphism $\varphi(\varinjlim_{A \in I} \xi(A)) \to \varphi(X)$ is an epimorphism, and (ii) implies that $\varinjlim_{A \in I} \xi(A) \to X$

is an epimorphism, hence a strict epimorphism by the hypothesis. Proposition 5.1.5 (i) implies $\mathrm{Coim}(\varinjlim_{A\in I}\xi(A)\to X)\simeq X$. By (iii), we have

$$\begin{aligned}\varinjlim_{A\in I}\eta(\xi(A)) &= \varinjlim_{A\in I}\mathrm{Coim}(\xi(A)\to X)\\ &\simeq \mathrm{Coim}(\varinjlim_{A\in I}\xi(A)\to X)\simeq X\;.\end{aligned}$$

(vii) For any $Y\in\mathcal{C}$, we obtain the chain of isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(X,Y) &\simeq \mathrm{Hom}_{\mathcal{C}}(\varinjlim_{A\in I}\eta(\xi(A)),Y)\\ &\simeq \varprojlim_{A\in I}\mathrm{Hom}_{\mathcal{C}}(\eta(\xi(A)),Y)\\ &\simeq \varprojlim_{A\in I}\mathrm{Hom}_{\mathcal{F}^\wedge}(\mathrm{Im}(\varphi(\xi(A))\to\varphi(X)),\varphi(Y))\\ &\simeq \mathrm{Hom}_{\mathcal{F}^\wedge}\big(\varinjlim_{A\in I}\big(\mathrm{Im}(\varphi(\xi(A))\to\varphi(X))\big),\varphi(Y)\big)\\ &\simeq \mathrm{Hom}_{\mathcal{F}^\wedge}(\varphi(X),\varphi(Y))\;,\end{aligned}$$

where the last isomorphism follows from (5.3.5). q.e.d.

Remark 5.3.7. See Exercises 5.5–5.8 which show that it is not possible to drop conditions (ii), (iii) or (b) in Theorem 5.3.6.

Theorem 5.3.8. *Let $\mathcal{C}$ be a category and consider the conditions below:*

(i) *$\mathcal{C}$ admits small inductive limits and finite projective limits,*
(ii) *small inductive limits in $\mathcal{C}$ are stable by base change,*
(ii)' *small filtrant inductive limits in $\mathcal{C}$ are stable by base change.*

Let us consider the conditions on an essentially small full subcategory $\mathcal{F}$ of $\mathcal{C}$:

(a) $\mathrm{Ob}(\mathcal{F})$ *is a system of generators,*
(b) *the inclusion functor $\mathcal{F}\hookrightarrow\mathcal{C}$ is right exact.*

Assume either (i), (ii) *and* (a) *or* (i), (ii)', (a) *and* (b). *Then $\mathcal{F}$ is strictly generating.*

Proof. We already know that $\varphi\colon\mathcal{C}\to\mathcal{F}^\wedge$ is conservative and faithful.

Assuming (i), let $\psi\colon\mathcal{F}^\wedge\to\mathcal{C}$ be the functor

$$\mathcal{F}^\wedge\ni F\mapsto \varinjlim_{(X\to F)\in\mathcal{F}_F} X\in\mathcal{C}\;.$$

Then ψ is left adjoint to φ by Lemma 5.3.3. By Proposition 1.5.6 (i), it is enough to show that $\psi\circ\varphi\to\mathrm{id}_{\mathcal{C}}$ is an isomorphism.

(A) First, we assume (i), (ii) and (a).

(A1) We begin by proving that

$$(5.3.6)\quad \begin{cases}\text{for any } X \in \mathcal{C} \text{ and any small inductive system } \{X_i\}_{i\in I} \text{ in } \mathcal{F}_X, \text{ if} \\ \varinjlim_i \varphi(X_i) \to \varphi(X) \text{ is an isomorphism, then } \varinjlim_i X_i \to X \text{ is an iso-} \\ \text{morphism.}\end{cases}$$

Set $X_0 = \varinjlim_i X_i \in \mathcal{C}$ and let $u\colon X_0 \to X$ be the canonical morphism. Since the composition $\varinjlim_i \varphi(X_i) \to \varphi(X_0) \to \varphi(X)$ is an isomorphism, $\varphi(u)\colon \varphi(X_0) \to \varphi(X)$ is an epimorphism. Since φ is conservative by (a), it remains to show that $\varphi(u)$ is a monomorphism.

For $i_1, i_2 \in I$, the two compositions $X_{i_1} \times_X X_{i_2} \to X_{i_\nu} \to X_0$ $(\nu = 1, 2)$ give two morphisms $\xi_1, \xi_2\colon X_{i_1} \times_X X_{i_2} \rightrightarrows X_0$. Then we have a diagram

$$\begin{tikzcd} \varphi(X_{i_1} \times_X X_{i_2}) \arrow[r, shift left] \arrow[r, shift right] & \varinjlim_i \varphi(X_i) \arrow[r] \arrow[rr, bend left, "\sim"] & \varphi(X_0) \arrow[r] & \varphi(X). \end{tikzcd}$$

Hence, the two arrows $\varphi(X_{i_1} \times_X X_{i_2}) \rightrightarrows \varinjlim_i \varphi(X_i)$ coincide, which implies $\varphi(\xi_1) = \varphi(\xi_2)$. Thus we obtain $\xi_1 = \xi_2$. It means that

$$X_{i_1} \times_{X_0} X_{i_2} \to X_{i_1} \times_X X_{i_2}$$

is an isomorphism for any $i_1, i_2 \in I$.

On the other hand, the condition (ii) implies that

$$(5.3.7)\quad \begin{aligned} \varinjlim_{i_1,i_2}(X_{i_1} \times_{X_0} X_{i_2}) &\simeq \varinjlim_{i_1}(X_{i_1} \times_{X_0} \varinjlim_{i_2} X_{i_2}) \\ &\simeq (\varinjlim_{i_1} X_{i_1}) \times_{X_0} (\varinjlim_{i_2} X_{i_2}) \ , \end{aligned}$$

and similarly,

$$(5.3.8)\quad \varinjlim_{i_1,i_2}(X_{i_1} \times_X X_{i_2}) \simeq (\varinjlim_{i_1} X_{i_1}) \times_X (\varinjlim_{i_2} X_{i_2}) \ .$$

Hence, we obtain the isomorphisms

$$\begin{aligned} &\varinjlim_{i_1,i_2}(X_{i_1} \times_{X_0} X_{i_2}) \simeq X_0 \ , \\ &\varinjlim_{i_1,i_2}(X_{i_1} \times_X X_{i_2}) \simeq X_0 \times_X X_0 \ . \end{aligned}$$

Hence, $X_0 \to X_0 \times_X X_0$ is an isomorphism, and this means that $X_0 \to X$ is a monomorphism by Exercise 2.4.

We have proved that $\varphi(X_0) \to \varphi(X)$ is a monomorphism and this completes the proof of (5.3.6).

(A2) Finally we shall show that $\psi \circ \varphi \to \mathrm{id}_{\mathcal{C}}$ is an isomorphism. For any $X \in \mathcal{C}$, we have $\varinjlim_{(Y \to X) \in \mathcal{F}_X} \varphi(Y) \xrightarrow{\sim} \varphi(X)$ by Proposition 2.6.3 (i), and (5.3.6) implies that $\psi\varphi(X) \simeq \varinjlim_{(Y \to X) \in \mathcal{F}_X} Y \simeq X$.

(B) Now, we assume (i), (ii)', (a) and (b). The proof is similar to the former case (A). For $X \in \mathcal{C}$, $\mathcal{F}_X$ is filtrant by (b). Hence, in step (A2), we only need (5.3.6) when I is filtrant. On the other hand, (5.3.6) in the filtrant case follows from (ii)' by the same argument as in (A1). Note that, in case (A), the condition (ii) is used only in proving (5.3.7) and (5.3.8). q.e.d.

Theorem 5.3.9. *Let $\mathcal{C}$ be a category satisfying:*

(i) *$\mathcal{C}$ admits small inductive limits and finite projective limits,*
(ii) *small filtrant inductive limits in $\mathcal{C}$ are stable by base change,*
(iii) *$\mathcal{C}$ admits a generator.*

Then any functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ which commutes with small projective limits is representable.

Proof. Let $\emptyset_{\mathcal{C}}$ be an initial object of $\mathcal{C}$ and let G be a generator of $\mathcal{C}$. We construct by induction an increasing sequence $\{\mathcal{F}_n\}_{n \geq 0}$ of small full subcategories as follows.

$$\begin{aligned}
\mathrm{Ob}(\mathcal{F}_0) &= \{\emptyset_{\mathcal{C}}, G\} \\
\mathrm{Ob}(\mathcal{F}_n) &= \mathrm{Ob}(\mathcal{F}_{n-1}) \bigsqcup \{Y_1 \sqcup_X Y_2\,;\, X \to Y_1 \text{ and } X \to Y_2 \text{ are morphisms} \\
&\qquad\qquad \text{in } \mathcal{F}_{n-1}\} \quad \text{for } n > 0.
\end{aligned}$$

Let $\mathcal{F}$ be the full subcategory of $\mathcal{C}$ with $\mathrm{Ob}(\mathcal{F}) = \bigcup_n \mathrm{Ob}(\mathcal{F}_n)$. Then $\mathcal{F}$ is a small category, $\mathrm{Ob}(\mathcal{F})$ is a system of generators, and $\mathcal{F}$ is closed by finite inductive limits. Hence, Proposition 3.3.3 implies that $\mathcal{F} \to \mathcal{C}$ is right exact, and $\mathcal{F}$ is strictly generating by Theorem 5.3.8. It remains to apply Corollary 5.3.5. q.e.d.

Note that if small filtrant inductive limits in $\mathcal{C}$ are exact, then such limits are stable by base change by Lemma 3.3.9.

Exercises

Exercise 5.1. Let $\mathcal{C}$ be one of the categories $\mathcal{C} = \mathbf{Set}$, $\mathcal{C} = \mathrm{Mod}(R)$ for a ring R, or $\mathcal{C} = \mathcal{D}^\wedge$ for a small category $\mathcal{D}$. Prove that any morphism in $\mathcal{C}$ is strict. Also prove that, when $\mathcal{C} = \mathcal{D}^\wedge$ and f is a morphism in $\mathcal{C}$, $\mathrm{Im}\, f$ is the functor $\mathcal{D} \ni Z \mapsto \mathrm{Im}(f(Z))$.

Exercise 5.2. Assume that a category $\mathcal{C}$ admits finite projective limits and finite inductive limits. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Prove the isomorphism $\mathrm{Hom}_{\mathcal{C}}(\mathrm{Coim}(f), Z) \simeq \mathrm{Hom}_{\mathcal{C}^\wedge}(\mathrm{Im}(\mathrm{h}_{\mathcal{C}}(f)), \mathrm{h}_{\mathcal{C}}(Z))$ for any $Z \in \mathcal{C}$.

Exercise 5.3. Let $\mathcal{C}$ be a category which admits finite inductive limits and finite projective limits. Consider the following conditions on $\mathcal{C}$:

(a) any morphism is strict,
(b) any epimorphism is strict,
(c) for any morphism $f: X \to Y$, $\operatorname{Coim} f \to Y$ is a monomorphism,
(d) any morphism which is both an epimorphism and a monomorphism is an isomorphism,
(e) for any strict epimorphisms $f: X \to Y$ and $g: Y \to Z$, their composition $g \circ f$ is a strict epimorphism.

Prove that (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) + (d) and that (c) $\Leftrightarrow$ (e).
(Hint: (e) $\Rightarrow$ (c). Adapt the proof of Proposition 5.1.7.
(c) $\Rightarrow$ (e). Consider $W = \operatorname{Coim}(g \circ f)$. Using the fact that $W \to Z$ is a monomorphism, deduce that $Y \times_W Y \to Y \times_Z Y$ is an isomorphism.)

Exercise 5.4. Let $\mathcal{C}$ be a category which admits finite inductive limits and finite projective limits. Let $f: X \to Y$ be the composition $X \xrightarrow{g} Z \xrightarrow{h} Y$ where g is a strict epimorphism. Prove that h factors uniquely through $\operatorname{Coim} f \to Y$ such that the composition $X \to Z \to \operatorname{Coim} f$ coincides with the canonical morphism.

Exercise 5.5. Let k be a field and set $\mathcal{F} := \mathrm{Mod}^{\mathrm{f}}(k)$, the full subcategory of $\mathrm{Mod}(k)$ consisting of finite-dimensional vector spaces. For $V \in \mathrm{Mod}(k)$, set $V^* = \mathrm{Hom}_k(V, k)$.
(i) Prove that the functor $V \mapsto V^*$ induces an equivalence of categories $\mathcal{F} \simeq \mathcal{F}^{\mathrm{op}}$.
(ii) Let $V \in \mathrm{Mod}(k)$. Prove the isomorphism $\varprojlim\limits_{(V \to W) \in \mathcal{F}^V} W \simeq V^{**}$.
(iii) Prove that $\mathcal{F}$ is a strictly generating full subcategory of $\mathrm{Mod}(k)$.
(iv) Prove that $\mathrm{Mod}(k)^{\mathrm{op}}$ and $\mathcal{F}^{\mathrm{op}}$ satisfy all hypotheses of Theorem 5.3.6 except condition (ii).
(v) Prove that the functor $\varphi: \mathrm{Mod}(k)^{\mathrm{op}} \to (\mathcal{F}^{\mathrm{op}})^\wedge$ defined in (5.3.1) decomposes as $\mathrm{Mod}(k)^{\mathrm{op}} \xrightarrow{*} \mathrm{Mod}(k) \to \mathcal{F}^\wedge \xrightarrow{\sim} (\mathcal{F}^{\mathrm{op}})^\wedge$.
(vi) Prove that the functor $\varphi: \mathrm{Mod}(k)^{\mathrm{op}} \to (\mathcal{F}^{\mathrm{op}})^\wedge$ is not fully faithful.

Exercise 5.6. Let k be a field and denote by $\mathcal{F}$ the full subcategory of $\mathrm{Mod}(k)$ consisting of the single object $\{k\}$. Prove that $\mathrm{Mod}(k) \to \mathcal{F}^\wedge$ is not fully faithful.

Exercise 5.7. Let A be a ring and denote by $\mathcal{F}$ the full subcategory of $\mathrm{Mod}(A)$ consisting of the two objects $\{A, A^{\oplus 2}\}$. Prove that $\mathrm{Mod}(A) \to \mathcal{F}^\wedge$ is fully faithful.

Exercise 5.8. Let k be a field, let $A = k[x, y]$ and let $\mathcal{C} = \mathrm{Mod}(A)$. Let $\mathfrak{a}$ denote the ideal $\mathfrak{a} = Ax + Ay$. (See also Exercises 8.27–8.29.) Let $\mathcal{C}_0$ be the full subcategory of $\mathcal{C}$ consisting of objects X such that there exists an epimorphism

$\mathfrak{a}^{\oplus I} \twoheadrightarrow X$ for some small set I. Let $\mathcal{F}$ be the full subcategory of $\mathcal{C}_0$ consisting of the objects $\{\mathfrak{a}^{\oplus n}\,;\, n \geq 0\}$. Let $\mathcal{G}$ be the full subcategory of $\mathcal{C}$ consisting of the objects $\{A^{\oplus n}\,;\, n \geq 0\}$.
(i) Prove that $\mathcal{F}$ and $\mathcal{G}$ are equivalent.
(ii) Prove that the functor $\varphi\colon \mathcal{C}_0 \to \mathcal{F}^\wedge$ given by

$$\mathcal{C}_0 \ni X \mapsto (\mathcal{F} \ni Y \mapsto \mathrm{Hom}_{\mathcal{C}}(Y, X))$$

decomposes as $\mathcal{C}_0 \xrightarrow{\xi} \mathrm{Mod}(A) \xrightarrow{\eta} \mathcal{F}^\wedge$ where $\xi(X) = \mathrm{Hom}_A(\mathfrak{a}, X)$ and $\eta(M)(Y) = \mathrm{Hom}_A(Y, \mathfrak{a}) \otimes_A M$ for $Y \in \mathcal{F}$. (In other words, $\eta(M) \in \mathcal{F}^\wedge$ is the functor $\mathcal{F} \ni \mathfrak{a}^{\oplus n} \mapsto M^{\oplus n}$.)
(iii) Prove that η is fully faithful. (Hint: use (i) and Theorem 5.3.6.)
(iv) Prove that φ is not fully faithful.
(v) Prove that $(\mathcal{C}_0, \mathcal{F})$ satisfies all the conditions in Theorem 5.3.6 except condition (iii).
(vi) Prove that any functor $F\colon \mathcal{C}_0^{\mathrm{op}} \to \mathbf{Set}$ commuting with small projective limits is representable. (Hint: use Theorem 5.2.6 or Theorem 5.3.9.)

Exercise 5.9. Let $\mathcal{C}$ be a category with a generator and satisfying the conditions (i) and (ii) in Theorem 5.3.8. Prove that for any $X, Y \in \mathcal{C}$, there exists an object $\mathcal{H}om\,(X, Y)$ in $\mathcal{C}$ which represents the functor $\mathcal{C} \ni Z \mapsto \mathrm{Hom}_{\mathcal{C}}(Z \times X, Y)$.

Exercise 5.10. (i) Let **Arr** be the category given in Notations 1.2.8 (iii), with two objects a and b and one morphism from a to b. Prove that **Arr** satisfies the conditions (i) and (ii) in Theorem 5.3.8, and b is a generator.
(ii) Conversely, let $\mathcal{C}$ be a category which satisfies the conditions (i) and (ii) in Theorem 5.3.8. Moreover assume that there exists a generator G such that $\mathrm{End}_{\mathcal{C}}(G) = \{\mathrm{id}_G\}$. Prove that $\mathcal{C}$ is equivalent to either **Set**, or **Arr** or **Pt**. (Hint: apply Theorem 5.3.8.)

Exercise 5.11. Prove that a functor $F\colon \mathbf{Set} \to \mathbf{Set}$ is representable if F commutes with small projective limits.