4

Tensor Categories

This chapter is devoted to tensor categories which axiomatize the properties of tensor products of vector spaces. Its importance became more evident when quantum groups produced rich examples of non commutative tensor categories and this notion is now used in many areas, mathematical physics, knot theory, computer sciences, etc. Tensor categories and their applications deserve at least a whole book, and we shall be extremely superficial and sketchy here. Among the vast literature on this subject, let us only quote [15, 40].

We begin this chapter by introducing projectors in categories. Then we define and study tensor categories, dual pairs, braidings and the Yang-Baxter equations. We also introduce the notions of a ring in a tensor category and a module over this ring in a category on which the tensor category operates. As a particular case we treat monads, and finally we prove the Bar-Beck theorem.

Most of the notions introduced in this Chapter (with the exception of §4.1) are not necessary for the understanding of the rest of the book, and this chapter may be skipped.

4.1 Projectors

The notion of a projector in linear algebra has its counterpart in Category Theory.

Definition 4.1.1. *Let $\mathcal{C}$ be a category. A* projector (P, ε) *on $\mathcal{C}$ is the data of a functor $P\colon \mathcal{C} \to \mathcal{C}$ and a morphism $\varepsilon\colon \mathrm{id}_{\mathcal{C}} \to P$ such that the two morphisms of functors $\varepsilon \circ P$, $P \circ \varepsilon\colon P \rightrightarrows P^2$ are isomorphisms. Here, $P^2 := P \circ P$.*

Lemma 4.1.2. *If (P, ε) is a projector, then $\varepsilon \circ P = P \circ \varepsilon$.*

Proof. For any $X \in \mathcal{C}$, we have a commutative diagram with solid arrows:

$$\begin{tikzcd} X \arrow[r, "\varepsilon_X"] \arrow[d, "\varepsilon_X"'] & P(X) \arrow[d, "P(\varepsilon_X)"] \arrow[dl, dotted, "u"'] \\ P(X) \arrow[r, "\sim", "\varepsilon_{P(X)}"'] & P^2(X). \end{tikzcd} \tag{4.1.1}$$

Since $\varepsilon_{P(X)}$ is an isomorphism, we can find a morphism $u\colon P(X) \to P(X)$ such that $\varepsilon_{P(X)} \circ u = P(\varepsilon_X)$. Then $u \circ \varepsilon_X = \varepsilon_X$ and the commutative diagram

$$\begin{tikzcd} P(X) \arrow[r, "P(\varepsilon_X)"] \arrow[d, "P(\varepsilon_X)"'] & P^2(X) \arrow[dl, "P(u)"] \\ P^2(X) & \end{tikzcd}$$

implies that $P(u) = \mathrm{id}_{P^2(X)}$. Since $\varepsilon_{P(X)}$ is an isomorphism, we conclude that $u = \mathrm{id}_{P(X)}$ by the commutative diagram

$$\begin{tikzcd} P(X) \arrow[r, "u"] \arrow[d, "\varepsilon_{P(X)}"'] & P(X) \arrow[d, "\varepsilon_{P(X)}"] \\ P^2(X) \arrow[r, "P(u)"] & P^2(X)\,. \end{tikzcd}$$

q.e.d.

Proposition 4.1.3. *Let (P, ε) be a projector on $\mathcal{C}$.*

(i) *For any $X, Y \in \mathcal{C}$, the map*

$$\mathrm{Hom}_{\mathcal{C}}(P(X), P(Y)) \xrightarrow{\circ\varepsilon_X} \mathrm{Hom}_{\mathcal{C}}(X, P(Y))$$

is bijective.

(ii) *The following three conditions on $X \in \mathcal{C}$ are equivalent:*
  (a) *$\varepsilon_X\colon X \to P(X)$ is an isomorphism,*
  (b) $\mathrm{Hom}_{\mathcal{C}}(P(Y), X) \xrightarrow{\circ\varepsilon_Y} \mathrm{Hom}_{\mathcal{C}}(Y, X)$ *is bijective for any $Y \in \mathcal{C}$,*
  (c) *the map in* (b) *is surjective for $Y = X$.*

(iii) *Let $\mathcal{C}_0$ be the full subcategory of $\mathcal{C}$ consisting of objects $X \in \mathcal{C}$ satisfying the equivalent conditions in* (ii). *Then $P(X) \in \mathcal{C}_0$ for any $X \in \mathcal{C}$ and P induces a functor $\mathcal{C} \to \mathcal{C}_0$ which is left adjoint to the inclusion functor $\iota\colon \mathcal{C}_0 \to \mathcal{C}$.*

Proof. (i) The composition

$$\theta\colon \mathrm{Hom}_{\mathcal{C}}(X, P(Y)) \to \mathrm{Hom}_{\mathcal{C}}(P(X), P^2(Y)) \xleftarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(P(X), P(Y))\,,$$

where the second map is given by $\varepsilon_{P(Y)}$, is an inverse of the map $\circ\varepsilon_X$. Indeed, $\theta \circ (\bullet \circ \varepsilon_X)$ and $(\bullet \circ \varepsilon_X) \circ \theta$ are the identities, as seen by the commutative diagrams below.

$$
\begin{tikzcd}
 & P(X) \arrow[r, "u"] \arrow[d, "\varepsilon_{P(X)}"] & P(Y) \arrow[d, "\varepsilon_{P(Y)}", "\sim"'] \\
P(X) \arrow[ur, "\mathrm{id}_{P(X)}"] \arrow[r, "P(\varepsilon_X)"'] & P^2(X) \arrow[r, "P(u)"'] & P^2(Y),
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "v"] \arrow[d, "\varepsilon_X"'] & P(Y) \arrow[d, "\varepsilon_{P(Y)}", "\sim"'] \\
P(X) \arrow[ur, dotted, "\theta(v)"] \arrow[r, "P(v)"'] & P^2(Y).
\end{tikzcd}
$$

(ii) (a) $\Rightarrow$ (b) follows from (i).
(b) $\Rightarrow$ (c) is obvious.
(c) $\Rightarrow$ (a). There exists a morphism $u\colon P(X) \to X$ such that $u \circ \varepsilon_X = \mathrm{id}_X$. Since $(\varepsilon_X \circ u) \circ \varepsilon_X = \varepsilon_X \circ \mathrm{id}_X = \mathrm{id}_{P(X)} \circ \varepsilon_X$, we have $\varepsilon_X \circ u = \mathrm{id}_{P(X)}$ by (i) with $Y = X$. Hence, ε_X is an isomorphism.

(iii) Since $\varepsilon_{P(X)}$ is an isomorphism, $P(X) \in \mathcal{C}_0$ for any $X \in \mathcal{C}$ and P induces a functor $\mathcal{C} \to \mathcal{C}_0$. This functor is a left adjoint to $\iota\colon \mathcal{C}_0 \to \mathcal{C}$ by (i). q.e.d.

Proposition 4.1.4. *Let $R\colon \mathcal{C}' \to \mathcal{C}$ be a fully faithful functor and assume that R admits a left adjoint $L\colon \mathcal{C} \to \mathcal{C}'$. Let $\varepsilon\colon \mathrm{id}_{\mathcal{C}} \to R \circ L$ and $\eta\colon L \circ R \to \mathrm{id}_{\mathcal{C}'}$ be the adjunction morphisms. Set $P = R \circ L\colon \mathcal{C} \to \mathcal{C}$. Then*

(i) *(P, ε) is a projector,*
(ii) *for any $X \in \mathcal{C}$, the following conditions are equivalent:*
 (a) *$\varepsilon_X\colon X \to RL(X)$ is an isomorphism,*
 (b) $\mathrm{Hom}_{\mathcal{C}}(RL(Y), X) \xrightarrow{\circ\varepsilon_Y} \mathrm{Hom}_{\mathcal{C}}(Y, X)$ *is bijective for any $Y \in \mathcal{C}$.*
(iii) *Let $\mathcal{C}_0$ be the full subcategory of $\mathcal{C}$ consisting of objects X satisfying the equivalent conditions in* (ii)*. Then $\mathcal{C}'$ is equivalent to $\mathcal{C}_0$.*

Proof. Since R is fully faithful, η is an isomorphism.
(i) The two compositions

$$
P \underset{P\circ\varepsilon}{\overset{\varepsilon\circ P}{\rightrightarrows}} P^2 \xrightarrow{R\eta L} P
$$

are equal to id_P. Since $R \circ \eta \circ L\colon RLRL \to RL$ is an isomorphism, it follows that $P \circ \varepsilon$ and $\varepsilon \circ P$ are isomorphisms.
(ii) follows from Proposition 4.1.3.
(iii) For $X \in \mathcal{C}'$, the morphism $R(\eta_X)\colon PR(X) = RLR(X) \to R(X)$ is an isomorphism. Since the composition

$$
R(X) \xrightarrow{\varepsilon_{R(X)}} PR(X) \xrightarrow{R(\eta_X)} R(X)
$$

is $\mathrm{id}_{R(X)}$, $\varepsilon_{R(X)}$ is an isomorphism. Hence, R sends $\mathcal{C}'$ to $\mathcal{C}_0$. This functor is fully faithful, and it is essentially surjective since $Y \simeq RL(Y)$ for any $Y \in \mathcal{C}_0$.
q.e.d.

4.2 Tensor Categories

Definition 4.2.1. *A* tensor category *is the data of a category $\mathcal{T}$, a bifunctor $\bullet\otimes\bullet: \mathcal{T}\times\mathcal{T} \to \mathcal{T}$ and an isomorphism of functors $a \in \mathrm{Mor}(\mathrm{Fct}(\mathcal{T}\times\mathcal{T}\times\mathcal{T}, \mathcal{T}))$,*

$$a(X, Y, Z)\colon (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$$

such that the diagram below is commutative for any $X, Y, Z, W \in \mathcal{T}$:

$$(4.2.1)\qquad \begin{tikzcd}
((X \otimes Y) \otimes Z) \otimes W \arrow[r, "a(X\otimes Y,Z,W)"] \arrow[d, "a(X,Y,Z)\otimes W"'] & (X \otimes Y) \otimes (Z \otimes W) \arrow[dd, "a(X,Y,Z\otimes W)"] \\
(X \otimes (Y \otimes Z)) \otimes W \arrow[d, "a(X,Y\otimes Z,W)"'] & \\
X \otimes ((Y \otimes Z) \otimes W) \arrow[r, "X\otimes a(Y,Z,W)"'] & X \otimes (Y \otimes (Z \otimes W)).
\end{tikzcd}$$

Examples 4.2.2. The following $(\mathcal{T}, \otimes, a)$ (with a the obvious one) are tensor categories.
(i) k is a commutative ring, $\mathcal{T} = \mathrm{Mod}(k)$ and $\otimes = \otimes_k$.
(ii) M is a monoid, $\mathcal{T}$ is the discrete category with $\mathrm{Ob}(\mathcal{T}) = M$, $a \otimes b = ab$ for $a, b \in M$.
(iii) A is a k-algebra, $\mathcal{T} = \mathrm{Mod}(A \otimes_k A^{\mathrm{op}})$ and $\otimes = \otimes_A$.
(iv) $\mathcal{C}$ is a category, $\mathcal{T} = \mathrm{Fct}(\mathcal{C}, \mathcal{C})$ and $\otimes = \circ$.
(v) $\mathcal{T}$ is a category which admits finite products and $\otimes = \times$.
(vi) $\mathcal{T}$ is a category which admits finite coproducts and $\otimes = \sqcup$.
(vii) G is a group, k is a field, $\mathcal{T}$ is the category of G-modules over k, that is, the category whose objects are the pairs (V, φ), $V \in \mathrm{Mod}(k)$, $\varphi\colon G \to \mathrm{Aut}_k(V)$ is a morphism of groups, and the morphisms are the natural ones. For $V, W \in \mathcal{T}$, $V \otimes W$ is the tensor product in $\mathrm{Mod}(k)$ endowed with the diagonal action of G given by $g(v \otimes w) = gv \otimes gw$.
(viii) I is a category, $\mathcal{T} = \mathcal{S}(I)$ is the category defined as follows. The objects of $\mathcal{S}(I)$ are the finite sequences of objects of I of length ≥ 1. For $X = (x_1, \dots, x_n)$ and $Y = (y_1, \dots, y_p)$ in $\mathcal{S}(I)$,

$$\mathrm{Hom}_{\mathcal{S}(I)}(X, Y) = \begin{cases} \prod_{i=1}^n \mathrm{Hom}_I(x_i, y_i) & \text{if } n = p\ , \\ \emptyset & \text{otherwise}\ . \end{cases}$$

Hence, $\mathcal{S}(I) \simeq \bigsqcup_{n\geq 1} I^n$.
For two objects $X = (x_1, \dots, x_n)$ and $Y = (y_1, \dots, y_p)$ of $\mathcal{S}(I)$, define $X \otimes Y$ as the sequence $(x_1, \dots, x_n, y_1, \dots, y_p)$.
(ix) k is a commutative ring and, with the notations of Chap. 11, $\mathcal{T} = \mathrm{C}^{\mathrm{b}}(\mathrm{Mod}(k))$ is the category of bounded complexes of k-modules and $X \otimes Y$ is the simple complex associated with the double complex $X \otimes_k Y$.

Let $(\mathcal{T}, \otimes, a)$ be a tensor category. Then $\mathcal{T}^{\mathrm{op}}$ has a structure of a tensor category in an obvious way. Another tensor category structure on $\mathcal{T}$ is obtained as follows. For $X, Y \in \mathcal{T}$, define

$$X \overset{\mathrm{r}}{\otimes} Y := Y \otimes X \ .$$

For $X, Y, Z \in \mathcal{T}$, define

$$a^r(X, Y, Z) : (X \overset{\mathrm{r}}{\otimes} Y) \overset{\mathrm{r}}{\otimes} Z \xrightarrow{\sim} X \overset{\mathrm{r}}{\otimes} (Y \overset{\mathrm{r}}{\otimes} Z)$$

by

$$(X \overset{\mathrm{r}}{\otimes} Y) \overset{\mathrm{r}}{\otimes} Z = Z \otimes (Y \otimes X) \xrightarrow{a(Z,Y,X)^{-1}} (Z \otimes Y) \otimes X = X \overset{\mathrm{r}}{\otimes} (Y \overset{\mathrm{r}}{\otimes} Z) \ .$$

Then $(\mathcal{T}, \overset{\mathrm{r}}{\otimes}, a^r)$ is a tensor category. We call it the *reversed* tensor category of $(\mathcal{T}, \otimes, a)$.

Tensor Functors

Definition 4.2.3. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two tensor categories. A functor of tensor categories* (*or, a* tensor functor) *is a pair (F, ξ_F) where $F : \mathcal{T} \to \mathcal{T}'$ is a functor and ξ_F is an isomorphism of bifunctors*

$$\xi_F : F(\,\cdot\, \otimes \,\cdot\,) \xrightarrow{\sim} F(\,\cdot\,) \otimes F(\,\cdot\,)$$

such that the diagram below commutes for all $X, Y, Z \in \mathcal{T}$:

(4.2.2)
$$\begin{tikzcd} F((X \otimes Y) \otimes Z) \arrow[rr, "F(a(X,Y,Z))"] \arrow[d, "\xi_F(X \otimes Y, Z)"'] & & F(X \otimes (Y \otimes Z)) \arrow[d, "\xi_F(X, Y \otimes Z)"] \\ F(X \otimes Y) \otimes F(Z) \arrow[d, "\xi_F(X,Y) \otimes F(Z)"'] & & F(X) \otimes F(Y \otimes Z) \arrow[d, "F(X) \otimes \xi_F(Y,Z)"] \\ (F(X) \otimes F(Y)) \otimes F(Z) \arrow[rr, "a(F(X),F(Y),F(Z))"'] & & F(X) \otimes (F(Y) \otimes F(Z)). \end{tikzcd}$$

In practice, we omit to write ξ_F.

For two tensor functors $F,\ G : \mathcal{T} \to \mathcal{T}'$, a morphism of tensor functors $\theta : F \to G$ is a morphism of functors such that the diagram below commutes for all $X, Y \in \mathcal{T}$:

$$\begin{tikzcd} F(X \otimes Y) \arrow[r, "\xi_F(X,Y)"] \arrow[d, "\theta_{X \otimes Y}"'] & F(X) \otimes F(Y) \arrow[d, "\theta_X \otimes \theta_Y"] \\ G(X \otimes Y) \arrow[r, "\xi_G(X,Y)"] & G(X) \otimes G(Y) \ . \end{tikzcd}$$

Recall that to a category I we have associated a tensor category $\mathcal{S}(I)$ in Example 4.2.2 (viii). Let us denote by $\iota : I \to \mathcal{S}(I)$ the canonical functor.

Lemma 4.2.4. *let $\mathcal{T}$ be a tensor category, let I be a category and let $\varphi\colon I \to \mathcal{T}$ be a functor. There exists a functor of tensor categories $\Phi\colon \mathcal{S}(I) \to \mathcal{T}$ such that $\Phi \circ \iota \simeq \varphi$. Moreover, Φ is unique up to unique isomorphism.*

Proof. We define by induction on n

$$\Phi\big((i_1, \dots, i_n)\big) = \Phi\big((i_1, \dots, i_{n-1})\big) \otimes \varphi(i_n) .$$

We define the isomorphism

$$\xi_\Phi : \Phi\big((i_1, \dots, i_n) \otimes (j_1, \dots, j_m)\big) \xrightarrow{\sim} \Phi\big((i_1, \dots, i_n)\big) \otimes \Phi\big((j_1, \dots, j_m)\big)$$

by the induction on m as follows:

$$\begin{aligned}
\Phi\big((i_1, \dots, i_n) &\otimes (j_1, \dots, j_m)\big)\\
&\simeq \Phi\big((i_1, \dots, i_n, j_1, \dots, j_m)\big)\\
&\simeq \Phi\big((i_1, \dots, i_n, j_1, \dots, j_{m-1})\big) \otimes \varphi(j_m)\\
&\simeq \Phi\big((i_1, \dots, i_n) \otimes (j_1, \dots, j_{m-1})\big) \otimes \varphi(j_m)\\
&\simeq \Big(\Phi\big((i_1, \dots, i_n)\big) \otimes \Phi\big((j_1, \dots, j_{m-1})\big)\Big) \otimes \varphi(j_m)\\
&\simeq \Phi\big((i_1, \dots, i_n)\big) \otimes \Big(\Phi\big((j_1, \dots, j_{m-1})\big) \otimes \varphi(j_m)\Big)\\
&\simeq \Phi\big((i_1, \dots, i_n)\big) \otimes \Phi\big((j_1, \dots, j_m)\big) .
\end{aligned}$$

It is left to the reader to check that this defines a functor of tensor categories. q.e.d.

Hence, in a tensor category $\mathcal{T}$, it is possible to define the tensor product $X_1 \otimes \cdots \otimes X_n$ for $X_1, \dots, X_n \in \mathcal{T}$ by the formula

$$X_1 \otimes \cdots \otimes X_n = (\cdots((X_1 \otimes X_2) \otimes X_3) \otimes \cdots) \otimes X_n$$

and this does not depend on the order of the parentheses, up to a unique isomorphism.

In the sequel, we shall often omit the parentheses.

Unit Object

Definition 4.2.5. *A* unit object *of a tensor category $\mathcal{T}$ is an object $\mathbf{1}$ of $\mathcal{T}$ endowed with an isomorphism $\varrho\colon \mathbf{1} \otimes \mathbf{1} \xrightarrow{\sim} \mathbf{1}$ such that the functors from $\mathcal{T}$ to $\mathcal{T}$ given by $X \mapsto X \otimes \mathbf{1}$ and $X \mapsto \mathbf{1} \otimes X$ are fully faithful.*

Lemma 4.2.6. *Let $(\mathbf{1}, \varrho)$ be a unit object of $\mathcal{T}$. Then there exist unique functorial isomorphisms $\alpha(X)\colon X \otimes \mathbf{1} \xrightarrow{\sim} X$ and $\beta(X)\colon \mathbf{1} \otimes X \xrightarrow{\sim} X$ satisfying the following properties*

(a) $\alpha(\mathbf{1}) = \beta(\mathbf{1}) = \varrho$,

(b) *the two morphisms* $X \otimes Y \otimes \mathbf{1} \overset{\alpha(X\otimes Y)}{\underset{X\otimes\alpha(Y)}{\rightrightarrows}} X \otimes Y$ *coincide,*

(c) *the two morphisms* $\mathbf{1} \otimes X \otimes Y \overset{\beta(X\otimes Y)}{\underset{\beta(X)\otimes Y}{\rightrightarrows}} X \otimes Y$ *coincide,*

(d) *the two morphisms* $X \otimes \mathbf{1} \otimes Y \overset{\alpha(X)\otimes Y}{\underset{X\otimes\beta(Y)}{\rightrightarrows}} X \otimes Y$ *coincide,*

(e) *the diagram*
$$\begin{tikzcd} \mathbf{1}\otimes X \otimes \mathbf{1} \arrow[r, "\mathbf{1}\otimes\alpha(X)"] \arrow[d, "\beta(X)\otimes\mathbf{1}"'] & \mathbf{1}\otimes X \arrow[d, "\beta(X)"] \\ X\otimes \mathbf{1} \arrow[r, "\alpha(X)"] & X \end{tikzcd}$$
commutes .

Proof. If such α and β exist, then (a) and (d) imply $\alpha(X) \otimes \mathbf{1} = X \otimes \beta(\mathbf{1}) = X \otimes \varrho$, $\alpha(X)$ is uniquely determined because $X \mapsto X \otimes \mathbf{1}$ is fully faithful, and similarly with β.

Proof of the existence of α, β. Since $X \mapsto X \otimes \mathbf{1}$ is fully faithful, there exists a unique morphism $\alpha(X)\colon X \otimes \mathbf{1} \to X$ such that $\alpha(X) \otimes \mathbf{1}\colon X \otimes \mathbf{1} \otimes \mathbf{1} \to X \otimes \mathbf{1}$ coincides with $X \otimes \varrho$. Since $X \otimes \varrho$ is an isomorphism, $\alpha(X)$ is an isomorphism. The morphism β is constructed similarly by $\mathbf{1} \otimes \beta(X) = \varrho \otimes X$.

Proof of (b)–(c). The morphism $X \otimes Y \otimes \varrho\colon X \otimes Y \otimes \mathbf{1} \otimes \mathbf{1} \to X \otimes Y \otimes \mathbf{1}$ coincides with $\alpha(X \otimes Y) \otimes \mathbf{1}$ and also with $X \otimes \alpha(Y) \otimes \mathbf{1}$. Hence, $\alpha(X \otimes Y) = X \otimes \alpha(Y)$. The proof of (c) is similar.

Proof of (e). By the functoriality of α, the diagram in (e) commutes when replacing $\mathbf{1} \otimes \alpha(X)$ in the top row with $\alpha(\mathbf{1} \otimes X)$. Since $\alpha(\mathbf{1} \otimes X) = \mathbf{1} \otimes \alpha(X)$ by (b), we conclude.

Proof of (d). Consider the diagram

$$\begin{tikzcd} & X\otimes\mathbf{1}\otimes\mathbf{1}\otimes Y \arrow[dl, "\alpha(X)\otimes\mathbf{1}\otimes Y"'] \arrow[d, "X\otimes\varrho\otimes Y"] \arrow[dr, "X\otimes\mathbf{1}\otimes\beta(Y)"] & \\ X\otimes\mathbf{1}\otimes Y \arrow[dr, "X\otimes\beta(Y)"'] & X\otimes\mathbf{1}\otimes Y \arrow[l, "\mathrm{id}"'] \arrow[r, "\mathrm{id}"] & X\otimes\mathbf{1}\otimes Y \arrow[dl, "\alpha(X)\otimes Y"] \\ & X\otimes Y. & \end{tikzcd}$$

Since the upper two triangles commute as well as the big square, we obtain $X \otimes \beta(Y) = \alpha(X) \otimes Y$.

Proof of (a). By (d), one has $\alpha(\mathbf{1}) \otimes \mathbf{1} = \mathbf{1} \otimes \beta(\mathbf{1})$. On the other hand, $\alpha(\mathbf{1}) \otimes \mathbf{1} = \mathbf{1} \otimes \varrho$ by the construction of α. Hence, $\mathbf{1} \otimes \beta(\mathbf{1}) = \mathbf{1} \otimes \varrho$. This implies that $\beta(\mathbf{1}) = \varrho$. The proof for α is similar. q.e.d.

Remark 4.2.7. If $(\mathbf{1},\varrho)$ and $(\mathbf{1}',\varrho')$ are unit objects, then there exists a unique isomorphism $\iota\colon \mathbf{1}\to\mathbf{1}'$ compatible with ϱ and ϱ', that is, the diagram

$$\begin{tikzcd}
\mathbf{1}\otimes\mathbf{1} \arrow[r, "\iota\otimes\iota"'] \arrow[d, "\varrho"'] & \mathbf{1}'\otimes\mathbf{1}' \arrow[d, "\varrho'"] \\
\mathbf{1} \arrow[r, "\iota"] & \mathbf{1}'
\end{tikzcd}$$

commutes. Indeed, $\mathbf{1} \xleftarrow{\sim} \mathbf{1}\otimes\mathbf{1}' \xrightarrow{\sim} \mathbf{1}'$ gives ι which satisfies the desired properties.

Remark that all tensor categories in Examples 4.2.2 except (viii) admit a unit object.

Definition 4.2.8. *Let $\mathcal{T}$ be a tensor category with a unit object $(\mathbf{1},\varrho)$. A tensor functor $F\colon \mathcal{T}\to\mathcal{T}'$ is called* unital *if $(F(\mathbf{1}), F(\varrho))$ is a unit object of $\mathcal{T}'$.*

More precisely, $F(\mathbf{1})\otimes F(\mathbf{1}) \xrightarrow{\sim} F(\mathbf{1})$ is given as the composition $F(\mathbf{1})\otimes F(\mathbf{1}) \xleftarrow[\xi_F(\mathbf{1},\mathbf{1})]{\sim} F(\mathbf{1}\otimes\mathbf{1}) \xrightarrow[F(\varrho)]{\sim} F(\mathbf{1})$.

Definition 4.2.9. *Let $\mathcal{T}$ be a tensor category. An action of $\mathcal{T}$ on a category $\mathcal{C}$ is a tensor functor $F\colon \mathcal{T}\to \mathrm{Fct}(\mathcal{C},\mathcal{C})$. If $\mathcal{T}$ has a unit object and $\mathcal{T}\to\mathrm{Fct}(\mathcal{C},\mathcal{C})$ is unital, the action is called unital.*

For $X\in\mathcal{T}$ and $W\in\mathcal{C}$, set $X\otimes W := F(X)(W)$. To give isomorphisms $\xi_F(X,Y)\colon F(X\otimes Y)\xrightarrow{\sim} F(X)\circ F(Y)$ is thus equivalent to give isomorphisms $(X\otimes Y)\otimes W \xrightarrow{\sim} X\otimes(Y\otimes W)$. Hence, to give an action of $\mathcal{T}$ on $\mathcal{C}$ is equivalent to giving a bifunctor $\otimes\colon \mathcal{T}\times\mathcal{C}\to\mathcal{C}$ and isomorphisms $a(X,Y,W)\colon (X\otimes Y)\otimes W \simeq X\otimes(Y\otimes W)$ functorial in $X,Y\in\mathcal{T}$ and $W\in\mathcal{C}$ such that the diagram (4.2.1) commutes for $X,Y,Z\in\mathcal{T}$ and $W\in\mathcal{C}$. In this language, the action is unital if there exists an isomorphism $\eta(X)\colon \mathbf{1}\otimes X\xrightarrow{\sim} X$ functorially in $X\in\mathcal{C}$ such that the diagram

$$\begin{tikzcd}
\mathbf{1}\otimes\mathbf{1}\otimes X \arrow[r, "\varrho\otimes X"] \arrow[d, "\mathbf{1}\otimes\eta(X)"'] & \mathbf{1}\otimes X \arrow[d, "\eta(X)"] \\
\mathbf{1}\otimes X \arrow[r, "\eta(X)"] & X
\end{tikzcd}$$

commutes. (See Exercise 4.8.)

Examples 4.2.10. (i) For a category $\mathcal{C}$, the tensor category $\mathrm{Fct}(\mathcal{C},\mathcal{C})$ acts on $\mathcal{C}$.
(ii) If $\mathcal{T}$ is a tensor category, then $\mathcal{T}$ acts on itself.

Dual Pairs

We shall now introduce the notion of a dual pair and the reader will notice some similarities with that of adjoint functors (see Sect. 4.3).

Definition 4.2.11. *Let $\mathcal{T}$ be a tensor category with a unit object $\mathbf{1}$. Let $X, Y \in \mathcal{T}$ be two objects and $\varepsilon\colon \mathbf{1} \to Y \otimes X$ and $\eta\colon X \otimes Y \to \mathbf{1}$ two morphisms. We say that (X, Y) is a* dual pair *or that X is a* left dual *to Y or Y is a* right dual *to X if the conditions* (a) *and* (b) *below are satisfied:*

(a) *the composition $X \simeq X \otimes \mathbf{1} \xrightarrow{X\otimes\varepsilon} X \otimes Y \otimes X \xrightarrow{\eta\otimes X} \mathbf{1} \otimes X \simeq X$ is the identity of X,*

(b) *the composition $Y \simeq \mathbf{1} \otimes Y \xrightarrow{\varepsilon\otimes Y} Y \otimes X \otimes Y \xrightarrow{Y\otimes\eta} Y \otimes \mathbf{1} \simeq Y$ is the identity of Y.*

Lemma 4.2.12. *If (X, Y) is a dual pair, then for any $Z, W \in \mathcal{T}$, there is an isomorphisms $\mathrm{Hom}_{\mathcal{T}}(Z, W \otimes X) \simeq \mathrm{Hom}_{\mathcal{T}}(Z \otimes Y, W)$ and $\mathrm{Hom}_{\mathcal{T}}(X \otimes Z, W) \simeq \mathrm{Hom}_{\mathcal{T}}(Z, Y \otimes W)$.*

Proof. We shall only prove the first isomorphism.
First, we construct a map $A\colon \mathrm{Hom}_{\mathcal{T}}(Z, W \otimes X) \to \mathrm{Hom}_{\mathcal{T}}(Z \otimes Y, W)$ as follows. Let $u \in \mathrm{Hom}_{\mathcal{T}}(Z, W \otimes X)$. Then $A(u)$ is the composition $Z \otimes Y \xrightarrow{u\otimes Y} W \otimes X \otimes Y \xrightarrow{W\otimes\eta} W \otimes \mathbf{1} \simeq W$.
Next, we construct a map $B\colon \mathrm{Hom}_{\mathcal{T}}(Z\otimes Y, W) \to \mathrm{Hom}_{\mathcal{T}}(Z, W\otimes X)$ as follows. Let $v \in \mathrm{Hom}_{\mathcal{T}}(Z \otimes Y, W)$. Then $B(v)$ is the composition $Z \xrightarrow{\sim} Z \otimes \mathbf{1} \xrightarrow{Z\otimes\varepsilon} Z \otimes Y \otimes X \xrightarrow{v\otimes X} W \otimes X$.
It is easily checked that A and B are inverse to each other. q.e.d.

Remark 4.2.13. (i) Y is a representative of the functor $Z \mapsto \mathrm{Hom}_{\mathcal{T}}(X \otimes Z, \mathbf{1})$ as well as a representative of the functor $W \mapsto \mathrm{Hom}_{\mathcal{T}}(\mathbf{1}, W \otimes X)$.
(ii) $(\bullet \otimes Y, \bullet \otimes X)$ is a pair of adjoint functors, as well as $(X \otimes \bullet, Y \otimes \bullet)$.

Braiding

Definition 4.2.14. *A* braiding*, also called an* R-matrix*, is an isomorphism $X \otimes Y \xrightarrow{\sim} Y \otimes X$ functorially in $X, Y \in \mathcal{T}$, such that the diagrams*

(4.2.3)
$$\begin{tikzcd} X \otimes Y \otimes Z \arrow[r, "R(X,Y)\otimes Z"] \arrow[dr, "R(X,Y\otimes Z)"'] & Y \otimes X \otimes Z \arrow[d, "Y\otimes R(X,Z)"] \\ & Y \otimes Z \otimes X \end{tikzcd}$$

and

(4.2.4)
$$\begin{tikzcd} X\otimes Y\otimes Z \arrow[r, "X\otimes R(Y,Z)"] \arrow[dr, "R(X\otimes Y,Z)"'] & X\otimes Z\otimes Y \arrow[d, "R(X,Z)\otimes Y"] \\ & Z\otimes X\otimes Y \end{tikzcd}$$

commute for all $X, Y, Z \in \mathcal{T}$.

Consider the diagram

(4.2.5)
$$\begin{tikzcd} & X\otimes Y\otimes Z \arrow[dl, "R(X,Y)\otimes Z"'] \arrow[dr, "X\otimes R(Y,Z)"] \arrow[ddl, dotted, "R(X,Y\otimes Z)"] & \\ Y\otimes X\otimes Z \arrow[d, "Y\otimes R(X,Z)"'] & & X\otimes Z\otimes Y \arrow[d, "R(X,Z)\otimes Y"] \arrow[ddl, dotted, "R(X,Z\otimes Y)"'] \\ Y\otimes Z\otimes X \arrow[dr, "R(Y,Z)\otimes X"'] & & Z\otimes X\otimes Y \arrow[dl, "Z\otimes R(X,Y)"] \\ & Z\otimes Y\otimes X. & \end{tikzcd}$$

Lemma 4.2.15. *If R is a braiding, then the solid diagram* (4.2.5) *commutes.*

The commutativity of this diagram may be translated by the so-called "Yang-Baxter equation"

(4.2.6)
$$\begin{aligned}&(R(Y,Z)\otimes X)\circ(Y\otimes R(X,Z))\circ(R(X,Y)\otimes Z)\\ &\quad=(Z\otimes R(X,Y))\circ(R(X,Z)\otimes Y)\circ(X\otimes R(Y,Z))\,.\end{aligned}$$

Proof. Consider the diagram (4.2.5) with the dotted arrows. The triangles $(X\otimes Y\otimes Z,\ Y\otimes X\otimes Z,\ Y\otimes Z\otimes X)$ and $(X\otimes Z\otimes Y,\ Z\otimes X\otimes Y,\ Z\otimes Y\otimes X)$ commute by the definition of a braiding. The square $(X\otimes Y\otimes Z,\ X\otimes Z\otimes Y,\ Y\otimes X\otimes Z,\ Z\otimes Y\otimes X)$ commutes by the functoriality of R. q.e.d.

Note that if R is a braiding, then

$$R(Y,X)^{-1}\colon X\otimes Y \xrightarrow{\sim} Y\otimes X$$

is also a braiding. We denote it by R^{-1}.

Definition 4.2.16. *A tensor category with a braiding R is called a* commutative tensor category *if $R = R^{-1}$, i.e., the composition $X\otimes Y \xrightarrow{R(X,Y)} Y\otimes X \xrightarrow{R(Y,X)} X\otimes Y$ is equal to* $\mathrm{id}_{X\otimes Y}$.

Remark 4.2.17. Commutative tensor categories are called "tensor categories" by some authors and tensor categories are then called *monoidal categories.*

4.3 Rings, Modules and Monads

By mimicking the definition of a monoid in the tensor category **Set** (by Example 4.2.2 (v)), or of a ring in the tensor category $\mathrm{Mod}(\mathbb{Z})$ (see Example 4.2.2 (i)), we introduce the following notion.

Definition 4.3.1. *Let $\mathcal{T}$ be a tensor category with a unit $\mathbf{1}$. A* ring *in $\mathcal{T}$ is a triplet $(A, \mu_A, \varepsilon_A)$ of an object $A \in \mathcal{T}$ and two morphisms $\mu_A\colon A \otimes A \to A$ and $\varepsilon_A\colon \mathbf{1} \to A$ such that the diagrams below commute:*

$$\begin{tikzcd} A \otimes \mathbf{1} \arrow[r, "A\otimes\varepsilon_A"] \arrow[dr, "\sim", "\alpha(A)"'] & A \otimes A \arrow[d, "\mu_A"] \\ & A, \end{tikzcd} \quad \begin{tikzcd} \mathbf{1} \otimes A \arrow[r, "\varepsilon_A\otimes A"] \arrow[dr, "\sim", "\beta(A)"'] & A \otimes A \arrow[d, "\mu_A"] \\ & A, \end{tikzcd} \quad \begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu_A\otimes A"] \arrow[d, "A\otimes\mu_A"'] & A \otimes A \arrow[d, "\mu_A"] \\ A \otimes A \arrow[r, "\mu_A"] & A\,. \end{tikzcd}$$

Note that ε_A is a unit and μ_A is a composition in the case of rings in $\mathrm{Mod}(k)$.

Remark 4.3.2. Some authors call $(A, \mu_A, \varepsilon_A)$ a monoid.

Definition 4.3.3. *Let $\mathcal{T}$ be a tensor category with a unit $\mathbf{1}$ acting unitally on a category $\mathcal{C}$ (see* Definition 4.2.9*). Let $(A, \mu_A, \varepsilon_A)$ be a ring in $\mathcal{T}$.*

(i) *An A-*module *in $\mathcal{C}$ is a pair (M, μ_M) of an object $M \in \mathcal{C}$ and a morphism $\mu_M\colon A \otimes M \to M$ such that the diagrams below in $\mathcal{C}$ commute:*

$$\begin{tikzcd} \mathbf{1} \otimes M \arrow[r, "\varepsilon_A\otimes M"] \arrow[dr] & A \otimes M \arrow[d, "\mu_M"] \\ & M, \end{tikzcd} \quad \begin{tikzcd} A \otimes A \otimes M \arrow[r, "\mu_A\otimes M"] \arrow[d, "A\otimes\mu_M"'] & A \otimes M \arrow[d, "\mu_M"] \\ A \otimes M \arrow[r, "\mu_M"] & M. \end{tikzcd}$$

(ii) *For two A-modules (M, μ_M) and (N, μ_N), a morphism $u\colon (M, \mu_M) \to (N, \mu_N)$ is a morphism $u\colon M \to N$ making the diagram below commutative:*

$$\begin{tikzcd} A \otimes M \arrow[r, "A\otimes u"] \arrow[d, "\mu_M"] & A \otimes N \arrow[d, "\mu_N"] \\ M \arrow[r, "u"] & N\,. \end{tikzcd}$$

Clearly, the family of A-modules in $\mathcal{C}$ forms a category $\mathrm{Mod}(A, \mathcal{C})$ and the forgetful functor $\mathit{for}\colon \mathrm{Mod}(A, \mathcal{C}) \to \mathcal{C}$ is faithful.

Lemma 4.3.4. *Let $\mathcal{T}$ and $\mathcal{C}$ be as in* Definition 4.3.3*, let $(A, \mu_A, \varepsilon_A)$ be a ring in $\mathcal{T}$ and let (M, μ_M) be an A-module in $\mathcal{C}$. Then the diagram below is exact in $\mathcal{C}^\wedge$:*

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, shift left, "\mu_A\otimes M"] \arrow[r, shift right, "A\otimes\mu_M"'] & A \otimes M \arrow[r, "\mu_M"] & M\,. \end{tikzcd}$$

Proof. The morphisms $s\colon M \simeq \mathbf{1} \otimes M \xrightarrow{\varepsilon_A \otimes M} A \otimes M$ and $u\colon A \otimes M \simeq \mathbf{1} \otimes A \otimes M \xrightarrow{\varepsilon_A \otimes A \otimes M} A \otimes A \otimes M$ satisfy

$$\mu_M \circ s = \mathrm{id}_M, \ (A \otimes \mu_M) \circ u = s \circ \mu_M, \ (\mu_A \otimes M) \circ u = \mathrm{id}_{A \otimes M} \ .$$

Hence, it is enough to apply the result of Exercise 2.25. q.e.d.

Recall that, for a category $\mathcal{C}$, the tensor category $\mathrm{Fct}(\mathcal{C}, \mathcal{C})$ acts on $\mathcal{C}$.

Definition 4.3.5. *Let $\mathcal{C}$ be a category. A ring in the tensor category* $\mathrm{Fct}(\mathcal{C}, \mathcal{C})$ *is called a* monad *in $\mathcal{C}$.*

The following lemma gives examples of monads and A-modules.

Lemma 4.3.6. *Let* $\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{C}'$ *be functors such that (L, R) is a pair of adjoint functors. Let $\varepsilon\colon \mathrm{id}_{\mathcal{C}} \to R \circ L$ and $\eta\colon L \circ R \to \mathrm{id}_{\mathcal{C}'}$ be the adjunction morphisms.*

(a) *Set $A := R \circ L$, $\varepsilon_A := \varepsilon$ and $\mu_A := R \circ \eta \circ L$. (Hence, $\mu_A\colon A \circ A = R \circ L \circ R \circ L \to R \circ L = A$.) Then $(A, \mu_A, \varepsilon_A)$ is a monad in $\mathcal{C}$.*
(b) *Let $Y \in \mathcal{C}'$. Set $X = R(Y) \in \mathcal{C}$ and $\mu_X = R(\eta_Y)\colon A(X) = R \circ L \circ R(Y) \xrightarrow{R(\eta(Y))} R(Y) = X$. Then (X, μ_X) is an A-module and the correspondence $Y \mapsto (X, \mu_X)$ defines a functor $\Phi\colon \mathcal{C}' \to \mathrm{Mod}(A, \mathcal{C})$.*

Proof. Leaving the rest of the proof to the reader, we shall only prove the associativity of μ_A, that is, the commutativity of the diagram

$$\begin{tikzcd}
A \circ A \circ A(X) \arrow[r, "\mu_A(A(X))"] \arrow[d, "A(\mu_A(X))"] & A \circ A(X) \arrow[d, "\mu_A(X)"] \\
A \circ A(X) \arrow[r, "\mu_A(X)"] & A(X).
\end{tikzcd}$$

We have $A(\mu_A(X)) = R \circ L \circ R(\eta(L(X)))$, $\mu_A(A(X)) = R(\eta(L \circ R \circ L(X)))$ and $\mu_A(X) = R(\eta(L(X)))$. Setting $B := L \circ R$ and $Y := L(X)$, the above diagram is the image by R of the diagram below

$$\begin{tikzcd}
B \circ B(Y) \arrow[r, "\eta(B(Y))"] \arrow[d, "B(\eta(Y))"] & B(Y) \arrow[d, "\eta(Y)"] \\
B(Y) \arrow[r, "\eta(Y)"] & Y.
\end{tikzcd}$$

The commutativity of this diagram follows from the fact that $\eta\colon B \to \mathrm{id}_{\mathcal{C}'}$ is a morphism of functors. q.e.d.

Lemma 4.3.7. *Let $(A, \mu_A, \varepsilon_A)$ be a monad in $\mathcal{C}$.*

(a) *For any $X \in \mathcal{C}$, $(A(X), \mu_A(X))$ is an A-module.*
(b) *The functor $\mathcal{C} \to \mathrm{Mod}(A, \mathcal{C})$ given by $X \mapsto (A(X), \mu_A(X))$ is a left adjoint of the forgetful functor for*: $\mathrm{Mod}(A, \mathcal{C}) \to \mathcal{C}$.

Proof. (i) is left to the reader.
(ii) We define maps

$$\mathrm{Hom}_{\mathrm{Mod}(A,\mathcal{C})}((A(Y), \mu_A(Y)), (X, \mu_X)) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathrm{Hom}_{\mathcal{C}}(Y, X)$$

as follows. To $v\colon (A(Y), \mu_A(Y)) \to (X, \mu_X)$ we associate $\alpha(v)$, the composition $Y \xrightarrow{\varepsilon_A(Y)} A(Y) \xrightarrow{v} X$.

To $u\colon Y \to X$, we associate $\beta(u)$, the composition $A(Y) \xrightarrow{A(u)} A(X) \xrightarrow{\mu_X} X$. It is easily checked that α and β are well defined and inverse to each other. q.e.d.

The next theorem is due to Barr and Beck.

Theorem 4.3.8. *Let $\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{C}'$ be functors such that (L, R) is a pair of adjoint functors. Let $(A = R \circ L, \varepsilon_A, \mu_A)$ and $\Phi\colon \mathcal{C}' \to \mathrm{Mod}(A, \mathcal{C})$ be as in Lemma 4.3.6. Then the following conditions are equivalent.*

(i) *Φ is an equivalence of categories,*
(ii) *the following two conditions hold:*
(a) *R is conservative,*
(b) *for any pair of parallel arrows $f, g\colon X \rightrightarrows Y$ in $\mathcal{C}'$, if* $\mathrm{Coker}(R(f), R(g))$ *exists in $\mathcal{C}$ and $R(X) \underset{R(g)}{\overset{R(f)}{\rightrightarrows}} R(Y) \longrightarrow \mathrm{Coker}(R(f), R(g))$ is exact in $\mathcal{C}^\wedge$* (*see* Exercise 2.25), *then* $\mathrm{Coker}(f, g)$ *exists and* $\mathrm{Coker}\big(R(f), R(g)\big) \xrightarrow{\sim} R(\mathrm{Coker}(f, g))$.

In particular, if $\mathcal{C}'$ admits finite inductive limits and R is conservative and exact, then $\Phi\colon \mathcal{C}' \to \mathrm{Mod}(A, \mathcal{C})$ is an equivalence of categories.

Proof. (i) $\Rightarrow$ (ii). We may assume that A is a monad in $\mathcal{C}$ and R is the forgetful functor $\mathcal{C}' = \mathrm{Mod}(A, \mathcal{C}) \to \mathcal{C}$. Hence, L is the functor $X \mapsto (A(X), \mu_A(X))$ by Lemma 4.3.7. Then (a) is obvious. Let us show (b). Let $f, g\colon (X, \mu_X) \rightrightarrows (Y, \mu_Y)$ be a pair of parallel arrows and assume that $X \rightrightarrows Y \to Z$ is exact in $\mathcal{C}^\wedge$. Then $A(X) \rightrightarrows A(Y) \to A(Z)$ as well as $A^2(X) \rightrightarrows A^2(Y) \to A^2(Z)$ are exact by Proposition 2.6.4. By the commutativity of the solid diagram with exact rows

$$\begin{tikzcd}
A(X) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\mu_X"] & A(Y) \arrow[r] \arrow[d, "\mu_Y"] & A(Z) \arrow[d, dotted, "w"] \\
X \arrow[r, shift left] \arrow[r, shift right] & Y \arrow[r] & Z,
\end{tikzcd}$$

we find the morphism $w \colon A(Z) \to Z$. It is easily checked that (Z, w) is an A-module and $(Z, w) \simeq \mathrm{Coker}(f, g)$ in $\mathrm{Mod}(A, \mathcal{C})$.

(ii) $\Rightarrow$ (i). Let us construct a quasi-inverse $\Psi \colon \mathrm{Mod}(A, \mathcal{C}) \to \mathcal{C}'$ of Φ. Let $(X, \mu_X) \in \mathrm{Mod}(A, \mathcal{C})$. Applying L to $\mu_X \colon A(X) \to X$, we obtain

$$L \circ R \circ L(X) \underset{\eta(L(X))}{\overset{L(\mu_X)}{\rightrightarrows}} L(X). \tag{4.3.1}$$

Applying R to this diagram we get

$$R \circ L \circ R \circ L(X) \underset{R(\eta(L(X)))}{\overset{R \circ L(\mu_X)}{\rightrightarrows}} R \circ L(X)$$

which is equal to the diagram $A \circ A(X) \underset{\mu_A(X)}{\overset{A(\mu_X)}{\rightrightarrows}} A(X)$.
The sequence

$$A \circ A(X) \underset{\mu_A(X)}{\overset{A(\mu_X)}{\rightrightarrows}} A(X) \xrightarrow{\mu_X} X \tag{4.3.2}$$

is exact in $\mathcal{C}^\wedge$ by Lemma 4.3.4. Therefore, (b) implies that (4.3.1) has a cokernel

$$L \circ R \circ L(X) \underset{\eta(L(X))}{\overset{L(\mu_X)}{\rightrightarrows}} L(X) \xrightarrow{\varphi} Y, \tag{4.3.3}$$

and there exists a commutative diagram

$$\begin{tikzcd}
A(X) = R \circ L(X) \arrow[r, "\mu_X"] \arrow[d, "R(\varphi)"'] & X \arrow[dl, "\psi", "\sim"'] \\
R(Y).
\end{tikzcd}$$

We set $\Phi((X, \mu_X)) = Y$. Since the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, "\varepsilon_A(X) = \varepsilon(X)"'] \arrow[rr, bend left, "\mathrm{id}_X"] & A(X) \arrow[r, "\mu_X"'] \arrow[d, "R(\varphi)"'] & X \arrow[dl, "\psi"] \\
& R(Y),
\end{tikzcd}$$

φ and ψ correspond by the adjunction isomorphism $\mathrm{Hom}_{\mathcal{C}'}(L(X), Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, R(Y))$. This implies that the diagram

$$\begin{tikzcd} A(X) \arrow[r, "RL(\psi)", "\sim"'] \arrow[d, "\mu_X"'] \arrow[dr, "R(\varphi)"] & R \circ L \circ R(Y) \arrow[d, "R(\eta(Y))"] \\ X \arrow[r, "\psi", "\sim"'] & R(Y) \end{tikzcd}$$

commutes. Hence, $\Phi\Psi((X, \mu_X)) \simeq (X, \mu_X)$.

Conversely, for $Y \in \mathcal{C}'$, let us set $(X, \mu_X) = \Phi(Y) = \big(R(Y), R(\eta(Y))\big) \in \mathrm{Mod}(A, \mathcal{C})$. Then the two compositions coincide:

$$L \circ R \circ L \circ R(Y) \underset{\eta(L \circ R(Y))}{\overset{L \circ R(\eta(Y))}{\rightrightarrows}} L \circ R(Y) \xrightarrow{\eta(Y)} Y. \tag{4.3.4}$$

Applying R to this diagram, we find the sequence $A \circ A(X) \rightrightarrows A(X) \to X$ which is exact in $\mathcal{C}^\wedge$ by Lemma 4.3.4. Hence, (b) implies that

$$R(Y) = X \simeq R\big(\mathrm{Coker}(L \circ R \circ L \circ R(Y) \rightrightarrows L \circ R(Y))\big) .$$

Then (a) implies that $Y \simeq \mathrm{Coker}(L \circ R \circ L \circ R(Y) \rightrightarrows L \circ R(Y))$. Hence, $\Psi(\Phi(Y)) \simeq Y$. q.e.d.

Exercises

Exercise 4.1. Let $\mathbf{Pr}$ be the category given in Notations 1.2.8 (v). Let $F\colon \mathbf{Pr} \to \mathbf{Pr}$ be the functor given by $F(u) = \mathrm{id}_c$ for any $u \in \mathrm{Mor}(\mathbf{Pr})$. Let $\varepsilon\colon \mathrm{id}_{\mathbf{Pr}} \to F$ be the morphism of functors given by $\varepsilon_c = p$.
(i) Prove that F and ε are well-defined.
(ii) Prove that $F \circ \varepsilon\colon F \to F^2$ is an isomorphism but $\varepsilon \circ F\colon F \to F^2$ is not an isomorphism.

Exercise 4.2. Let $\mathcal{T}$ be a tensor category with a unit object $\mathbf{1}$. Let $X \in \mathcal{T}$ and $\alpha\colon \mathbf{1} \to X$. Prove that if the compositions $X \simeq \mathbf{1} \otimes X \xrightarrow{\alpha \otimes X} X \otimes X$ and $X \simeq X \otimes \mathbf{1} \xrightarrow{X \otimes \alpha} X \otimes X$ are isomorphisms, then they are equal and the inverse morphism $\mu\colon X \otimes X \to X$ gives a ring structure on X.

Exercise 4.3. Prove that if a tensor category has a unit object, then this object is unique up to unique isomorphism. More precisely, prove the statement in Remark 4.2.7. Also prove that if $(\mathbf{1}, \varrho)$ is a unit object, then $\varrho \otimes \mathbf{1} = \mathbf{1} \otimes \varrho$.

Exercise 4.4. Let $\mathcal{T}$ be a tensor category with a unit $\mathbf{1}$ and a braiding R.
(i) Prove that the diagram below commutes:

$$\begin{tikzcd} \mathbf{1}\otimes X \arrow[rr, "R(\mathbf{1},X)"] \arrow[dr, "\beta(X)"'] & & X\otimes\mathbf{1} \arrow[dl, "\alpha(X)"] \\ & X. & \end{tikzcd}$$

(ii) Prove that $R(\mathbf{1},\mathbf{1}) = \mathrm{id}_{\mathbf{1}\otimes\mathbf{1}}$.

Exercise 4.5. Let k be a field and recall that $k^\times$ denotes the group of its invertible elements. Let L be an additive group and denote by $\mathcal{C}$ the category whose objects are the families

$$\mathrm{Ob}(\mathcal{C}) = \{X = \{X_l\}_{l\in L}; X_l \in \mathrm{Mod}(k),\ \ X_l = 0 \text{ for all but finitely many } l\},$$

the morphisms in $\mathcal{C}$ being the natural ones. For $X = \{X_l\}_{l\in L}$ and $Y = \{Y_l\}_{l\in L}$, define $X\otimes Y$ by $(X\otimes Y)_l = \oplus_{l=l'+l''} X_{l'}\otimes Y_{l''}$.
(i) Let $c\colon L\times L\times L \to k^\times$ be a function. For $X, Y, Z \in \mathcal{C}$, let

$$a_c(X,Y,Z)\colon (X\otimes Y)\otimes Z \to X\otimes(Y\otimes Z)$$

be the isomorphism induced by

$$(X_{l_1}\otimes Y_{l_2})\otimes Z_{l_3} \xrightarrow{c(l_1,l_2,l_3)} X_{l_1}\otimes(Y_{l_2}\otimes Z_{l_3})\ .$$

Prove that $(\mathcal{C},\otimes,a_c)$ is a tensor category if and only if c satisfies the cocycle condition:

(4.3.5) $c(l_1+l_2,l_3,l_4)c(l_1,l_2,l_3+l_4) = c(l_1,l_2,l_3)c(l_1,l_2+l_3,l_4)c(l_2,l_3,l_4)$.

If c satisfies the cocycle condition (4.3.5), we shall denote by $\otimes_c$ the tensor product in the tensor category $(\mathcal{C},\otimes,a_c)$.
(ii) Let b and c be two functions from $L\times L\times L$ to $k^\times$ both satisfying (4.3.5). Let $\varphi\colon L\times L\to k^\times$ be a function and for $X, Y\in\mathcal{C}$, let $\xi(X,Y)\colon X\otimes Y\to X\otimes Y$ be the isomorphism in $\mathcal{C}$ given by

$$X_l\otimes Y_{l'} \xrightarrow{\varphi(l,l')} X_l\otimes Y_{l'}\ .$$

Prove that $(\mathrm{id}_\mathcal{C},\xi)$ is a tensor functor from $(\mathcal{C},\otimes_b,a_b)$ to $(\mathcal{C},\otimes_c,a_c)$ if and only if

$$(4.3.6)\qquad c(l_1,l_2,l_3) = \frac{\varphi(l_2,l_3)\varphi(l_1,l_2+l_3)}{\varphi(l_1,l_2)\varphi(l_1+l_2,l_3)}b(l_1,l_2,l_3)\ .$$

(iii) Assume that c satisfies the cocycle condition (4.3.5) and let $\rho\colon L\times L\to k^\times$ be a function. Let

$$R(X,Y)\colon X\otimes_c Y\to Y\otimes_c X$$

be the isomorphism induced by

$$X_l\otimes Y_{l'} \xrightarrow{\rho(l,l')} Y_{l'}\otimes X_l\ .$$

(a) Prove that R satisfies the Yang-Baxter equation (4.2.6) if

$$c(l_1, l_2, l_3)c(l_2, l_3, l_1)c(l_3, l_1, l_2) = c(l_1, l_3, l_2)c(l_3, l_2, l_1)c(l_2, l_1, l_3) .$$

(b) Prove that R is a braiding if and only if

$$\frac{c(l_1, l_2, l_3)c(l_2, l_3, l_1)}{c(l_2, l_1, l_3)} = \frac{\rho(l_1, l_2)\rho(l_1, l_3)}{\rho(l_1, l_2 + l_3)} = \frac{\rho(l_2 + l_3, l_1)}{\rho(l_2, l_1)\rho(l_3, l_1)} . \tag{4.3.7}$$

(iv) Let $\psi\colon L \to k$ be a function. Define $\theta\colon \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ by setting $\theta_X|_{X_l} = \psi(l)\,\mathrm{id}_{X_l}$. Prove that θ is a morphism of tensor functors if and only if

$$\psi(l_1 + l_2) = \psi(l_1)\psi(l_2) .$$

(v) Let $L = \mathbb{Z}/2\mathbb{Z}$.

(a) Prove that the function c given by

$$c(l_1, l_2, l_3) = \begin{cases} -1 & \text{if } l_1 = l_2 = l_3 = 1 \bmod 2 , \\ 1 & \text{otherwise} \end{cases} \tag{4.3.8}$$

satisfies the cocycle condition (4.3.5).

(b) Assume that there exists an element $i \in k^\times$ such that $i^2 = -1$ and let c be as in (4.3.8). Prove that the solutions of (4.3.7) are given by

$$\rho(l, l') = \begin{cases} \pm i & \text{if } l = l' = 1 \bmod 2 , \\ 1 & \text{otherwise.} \end{cases}$$

(vi) Let $L = \mathbb{Z}/2\mathbb{Z}$. Prove that two tensor categories $(\mathcal{C}, \otimes_c, a_c)$ and $(\mathcal{C}, \otimes_b, a_b)$ with c as in (4.3.8) and $b(l_1, l_2, l_3) = 1$, are not equivalent when k is a field of characteristic different from 2.

(vii) Let $L = \mathbb{Z}/2\mathbb{Z}$, and b as in (vi). Let R be the braiding given by $\rho(l, l') = -1$ or 1 according that $l = l' = 1 \bmod 2$ or not. Prove that $(\mathcal{C}, \otimes_b, a_b)$ is a commutative tensor category. (The objects of $\mathcal{C}$ are called *super* vector spaces.)

Exercise 4.6. Let $\mathcal{T}$ be a tensor category with a unit object $\mathbf{1}$. Prove that if $\theta\colon \mathrm{id}_{\mathcal{T}} \to \mathrm{id}_{\mathcal{T}}$ is an isomorphism of tensor functors, then $\theta_{\mathbf{1}} = \mathrm{id}_{\mathbf{1}}$.

Exercise 4.7. Let $\mathcal{T}$ be a tensor category with a unit object. Prove that if (X, Y) and (X, Y') are dual pairs, then Y and Y' are isomorphic.

Exercise 4.8. Let $\mathcal{T}$ be a tensor category with a unit object $\mathbf{1}$ and acting on a category $\mathcal{C}$. Prove that this action is unital if and only if the functor $\mathcal{C} \ni X \mapsto \mathbf{1} \otimes X \in \mathcal{C}$ is fully faithful.

Exercise 4.9. Let $\mathbf{\Delta}$ be the category of finite totally ordered sets and order-preserving maps (see Definition 11.4.1 and Exercise 1.21).

(i) For $\sigma, \tau \in \mathbf{\Delta}$, define $\sigma \otimes \tau$ as the set $\sigma \sqcup \tau$ endowed with the total order such that $i < j$ for any i in the image of σ and j in the image of τ and $\sigma \to \sigma \sqcup \tau$ and $\tau \to \sigma \sqcup \tau$ are order-preserving. Prove that $\mathbf{\Delta}$ is a tensor category with a unit object.
(ii) Let $R(\sigma, \tau)\colon \sigma \otimes \tau \to \tau \otimes \sigma$ denote the unique isomorphism of these two objects in $\mathbf{\Delta}$. Prove that R defines a commutative tensor category structure on $\mathbf{\Delta}$.
(iii) Let $\mathcal{T}$ be a tensor category with a unit object. Prove that the category of rings in $\mathcal{T}$ is equivalent to the category of unital tensor functors from $\mathbf{\Delta}$ to $\mathcal{T}$.

Exercise 4.10. Let G be a group and let us denote by $\mathcal{G}$ the associated discrete category. A structure of a tensor category on $\mathcal{G}$ is defined by setting $g_1 \otimes g_2 = g_1 g_2$ ($g_1, g_2 \in G$). Let $\mathcal{C}$ be a category. An action of G on $\mathcal{C}$ is a unital action $\psi\colon \mathcal{G} \to \mathrm{Fct}(\mathcal{C}, \mathcal{C})$ of the tensor category $\mathcal{G}$ on $\mathcal{C}$.
(i) Let $T : \mathcal{C} \to \mathcal{C}$ be an auto-equivalence. Show that there exists an action ψ of $\mathbb{Z}$ on $\mathcal{C}$ such that $\psi(1) = T$.
(ii) Let T_1 and T_2 be two auto-equivalences of $\mathcal{S}$ and let $\varphi_{12} : T_1 \circ T_2 \xrightarrow{\sim} T_2 \circ T_1$ be an isomorphism of functors. Show that there exists an action ψ of $\mathbb{Z}^2$ on $\mathcal{C}$ such that $\psi((1,0)) = T_1$ and $\psi((0,1)) = T_2$.
(iii) More generally, let $T_1, \dots, T_n$ be n auto-equivalences of $\mathcal{C}$ for a non-negative integer n, and let $\varphi_{ij} : T_i \circ T_j \xrightarrow{\sim} T_j \circ T_i$ be isomorphisms of functors for $1 \le i < j \le n$. Assume that for any $1 \le i < j < k \le n$, the diagram below commutes

$$\begin{tikzcd}
 & T_i \circ T_j \circ T_k \arrow[dl, "T_i \circ \varphi_{jk}"'] \arrow[dr, "\varphi_{ij} \circ T_k"] & \\
T_i \circ T_k \circ T_j \arrow[d, "\varphi_{ik} \circ T_j"'] & & T_j \circ T_i \circ T_k \arrow[d, "T_j \circ \varphi_{ik}"] \\
T_k \circ T_i \circ T_j \arrow[dr, "T_k \circ \varphi_{ij}"'] & & T_j \circ T_k \circ T_i \arrow[dl, "\varphi_{jk} \circ T_i"] \\
 & T_k \circ T_j \circ T_i\,. &
\end{tikzcd}$$

Denote by $u_1, \dots, u_n$ the canonical basis of $\mathbb{Z}^n$. Prove that there exists an action ψ of $\mathbb{Z}^n$ on $\mathcal{C}$ such that $\psi(u_i) = T_i$ and the composition $T_i \circ T_j \simeq \psi(u_i \otimes u_j) = \psi(u_j \otimes u_i) \xrightarrow{\sim} T_j \circ T_i$ coincides with φ_{ij}.

Exercise 4.11. Let $\mathcal{T}$ be a tensor category with a unit object $(\mathbf{1}, \varrho)$. Let $a \in \mathrm{End}_{\mathcal{T}}(\mathbf{1})$.
(i) Prove that the diagram

$$\begin{tikzcd}
\mathbf{1} \otimes \mathbf{1} \arrow[r, "\mathbf{1} \otimes a"] \arrow[d, "\varrho"'] & \mathbf{1} \otimes \mathbf{1} \arrow[d, "\varrho"] \\
\mathbf{1} \arrow[r, "a"] & \mathbf{1}
\end{tikzcd}$$

commutes and that $\mathbf{1} \otimes a = a \otimes \mathbf{1}$.
(ii) Prove that $\mathrm{End}_{\mathcal{T}}(\mathbf{1})$ is commutative.
(iii) Define

$$\begin{aligned}
R &: \mathrm{End}_{\mathcal{T}}(\mathbf{1}) \to \mathrm{End}_{\,\mathrm{Fct}(\mathcal{T},\mathcal{T})}(\,\bullet \otimes \mathbf{1}) \xleftarrow{\sim} \mathrm{End}_{\,\mathrm{Fct}(\mathcal{T},\mathcal{T})}(\mathrm{id}_{\mathcal{T}})\,, \\
L &: \mathrm{End}_{\mathcal{T}}(\mathbf{1}) \to \mathrm{End}_{\,\mathrm{Fct}(\mathcal{T},\mathcal{T})}(\mathbf{1} \otimes \bullet\,) \xleftarrow{\sim} \mathrm{End}_{\,\mathrm{Fct}(\mathcal{T},\mathcal{T})}(\mathrm{id}_{\mathcal{T}})\,,
\end{aligned}$$

where $R(a)_X \otimes \mathbf{1} = X \otimes a$ and $\mathbf{1} \otimes L(a)_X = a \otimes X$. Prove that if $\mathcal{T}$ has a braiding, then $R = L$.

Exercise 4.12. Let $\mathcal{T}$ be a tensor category with a unit object $(\mathbf{1}, \varrho)$. Let $X, Y \in \mathcal{T}$ and assume that $X \otimes Y \simeq \mathbf{1}$ and $Y \otimes X \simeq \mathbf{1}$. Prove that there exist isomorphisms $\xi : X \otimes Y \xrightarrow{\sim} \mathbf{1}$ and $\eta : Y \otimes X \xrightarrow{\sim} \mathbf{1}$ such that the diagrams below commute.

$$\begin{tikzcd}
X \otimes Y \otimes X \arrow[r, "\xi \otimes X"] \arrow[d, "X \otimes \eta"] & \mathbf{1} \otimes X \arrow[d] \\
X \otimes \mathbf{1} \arrow[r] & X,
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \otimes X \otimes Y \arrow[r, "Y \otimes \xi"] \arrow[d, "\eta \otimes Y"] & Y \otimes \mathbf{1} \arrow[d] \\
\mathbf{1} \otimes Y \arrow[r] & Y.
\end{tikzcd}$$

Exercise 4.13. Let $\mathcal{T}$ be a tensor category with a unit object $(\mathbf{1}, \varrho)$. Assume to be given $X \in \mathcal{T}$, a positive integer n and an isomorphism $\lambda : X^{\otimes n} \xrightarrow{\sim} \mathbf{1}$. Consider the diagram

$$\begin{tikzcd}
X^{\otimes(n+1)} \arrow[r, "X \otimes \lambda"] \arrow[d, "\lambda \otimes X"'] & X \otimes \mathbf{1} \arrow[d] \\
\mathbf{1} \otimes X \arrow[r] & X.
\end{tikzcd} \tag{4.3.9}$$

(i) Assume that (4.3.9) commutes. Prove that there exists a unital functor $\varphi : \mathbb{Z}/n\mathbb{Z} \to \mathcal{T}$ such that $\varphi(1) = X$. Here, the group $\mathbb{Z}/n\mathbb{Z}$ is regarded as a tensor category as in Exercise 4.10.
(ii) Prove that if $\mathcal{T}$ has a braiding, the fact that the diagram (4.3.9) commutes does not depend on the choice of the isomorphism $\lambda : X^{\otimes n} \xrightarrow{\sim} \mathbf{1}$. (Hint: use Exercise 4.11 (iii).)
(iii) Give an example of a braided tensor category $\mathcal{T}$ and (X, λ) such that (4.3.9) does not commute. (Hint: use Exercise 4.5 (v).)