16

Grothendieck Topologies

As already mentioned, sheaves on topological spaces were invented by Leray and this notion was extended to sheaves on categories by Grothendieck who noticed that the notion of sheaves on a topological space X essentially relies on the category Op_X of open subsets of X and on the notion of open coverings, nothing else. Hence to define sheaves on a category $\mathcal{C}$, it is enough to axiomatize the notion of a covering which defines a so-called Grothendieck topology on $\mathcal{C}$.

Notice that, even in the topological case, if $\{U_i\}_{i \in I}$ is a covering of an open subset U, there is no natural object describing it in the category Op_X, but it is possible to consider the coproduct of the U_i's in the category $(\mathrm{Op}_X)^\wedge$. Hence, to define the notion of a covering on $\mathcal{C}$, we work in $\mathcal{C}^\wedge$, the category of presheaves of sets on $\mathcal{C}$.

Here, we first give the axioms of Grothendieck topologies using sieves and then introduce the notions of local epimorphisms and local isomorphisms. We give several examples and study in some details the properties of the family of local isomorphisms, showing in particular that this family is stable by inductive limits.

Important related topics, such as Topos Theory, will not be approached in this book.

References are made to [64].

16.1 Sieves and Local Epimorphisms

Let $\mathcal{C}$ be a category.

Definition 16.1.1. *Let $U \in \mathrm{Ob}(\mathcal{C})$. A* sieve[1] *$S$ over U is a subset of* $\mathrm{Ob}(\mathcal{C}_U)$ *such that the composition $W \to V \to U$ belongs to S as soon as $V \to U$ belongs to S.*

[1] " Un crible" in French

To a sieve S over U, we associate a subobject A_S of U in $\mathcal{C}^\wedge$ by taking

$$A_S(V) = \Big\{ s \in \mathrm{Hom}_{\mathcal{C}}(V,U)\,;\, (V \xrightarrow{s} U) \in S \Big\} \quad \text{for any } V \in \mathcal{C}\,. \tag{16.1.1}$$

If $\mathcal{C}$ is small, we have $A_S = \mathrm{Im}\big(\text{“}\coprod_{(V\to U)\in S}\text{”}\, V \to U\big)$.

Conversely, to an object $A \to U$ of $(\mathcal{C}^\wedge)_U$ we associate a sieve S_A by taking

$$(V \to U) \in S_A \text{ if and only if } V \to U \text{ decomposes as } V \to A \to U\,. \tag{16.1.2}$$

Note that $S_A = S_{\mathrm{Im}(A\to U)}$. Hence, there is a one-to-one correspondence between the family of sieves over U and the family of subobjects of U in $\mathcal{C}^\wedge$.

Definition 16.1.2. *A* Grothendieck topology (*or simply a topology*) *on a category $\mathcal{C}$ is the data of a family $\{\mathcal{S}\mathrm{Cov}_U\}_{U\in\mathrm{Ob}(\mathcal{C})}$, where $\mathcal{S}\mathrm{Cov}_U$ is a family of sieves over U, these data satisfying the axioms* GT1–GT4 *below.*

GT1 $\mathrm{Ob}(\mathcal{C}_U)$ *belongs to* $\mathcal{S}\mathrm{Cov}_U$.

GT2 *If $S_1 \subset S_2 \subset \mathrm{Ob}(\mathcal{C}_U)$ are sieves and if S_1 belongs to $\mathcal{S}\mathrm{Cov}_U$, then S_2 belongs to $\mathcal{S}\mathrm{Cov}_U$.*

GT3 *Let $U \to V$ be a morphism in $\mathcal{C}$. If S belongs to $\mathcal{S}\mathrm{Cov}_V$, then $S \times_V U$ belongs to $\mathcal{S}\mathrm{Cov}_U$. Here,*

$$S \times_V U := \{W \to U\,;\, \textit{the composition } W \to U \to V \textit{ belongs to } \mathcal{S}\mathrm{Cov}_V\}\,.$$

GT4 *Let S and S' be sieves over U. Assume that $S' \in \mathcal{S}\mathrm{Cov}_U$ and that $S \times_U V \in \mathcal{S}\mathrm{Cov}_V$ for any $(V \to U) \in S'$. Then $S \in \mathcal{S}\mathrm{Cov}_U$.*

A sieve S over U is called a covering sieve *if $S \in \mathcal{S}\mathrm{Cov}_U$.*

Definition 16.1.3. *Let $\mathcal{C}$ be a category endowed with a Grothendieck topology.*

(i) *A morphism $A \to U$ in $\mathcal{C}^\wedge$ with $U \in \mathcal{C}$ is called a* local epimorphism *if the sieve S_A given by* (16.1.2) *is a covering sieve over U.*

(ii) *A morphism $A \to B$ in $\mathcal{C}^\wedge$ is called a local epimorphism if for any $V \in \mathcal{C}$ and any morphism $V \to B$, $A \times_B V \to V$ is a local epimorphism.*

Consider a local epimorphism $A \to U$ as in Definition 16.1.3 (i) and let $V \to U$ be a morphism in $\mathcal{C}$. The sieve $S_{A\times_U V} = S_A \times_U V$ is a covering sieve over V by GT3 and it follows that $A \times_U V \to V$ is a local epimorphism. Therefore, if we take $B = U \in \mathcal{C}$ in Definition 16.1.3 (ii), we recover Definition 16.1.3 (i).

The family of local epimorphisms associated with a Grothendieck topology will satisfy the following properties (the verification is left to the reader):

LE1 For any $U \in \mathcal{C}$, $\mathrm{id}_U\colon U \to U$ is a local epimorphism.

LE2 Let $A_1 \xrightarrow{u} A_2 \xrightarrow{v} A_3$ be morphisms in $\mathcal{C}^\wedge$. If u and v are local epimorphisms, then $v \circ u$ is a local epimorphism.

LE3 Let $A_1 \xrightarrow{u} A_2 \xrightarrow{v} A_3$ be morphisms in $\mathcal{C}^\wedge$. If $v \circ u$ is a local epimorphism, then v is a local epimorphism.

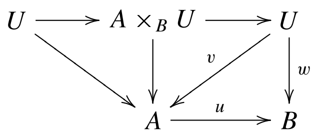
LE4 A morphism $u: A \to B$ in $\mathcal{C}^\wedge$ is a local epimorphism if and only if for any $U \in \mathcal{C}$ and any morphism $U \to B$, the morphism $A \times_B U \to U$ is a local epimorphism.

Conversely, consider a family of morphisms in $\mathcal{C}^\wedge$ satisfying LE1–LE4. Let us say that a sieve S over U is a covering sieve if $A_S \to U$ is a local epimorphism, where A_S is given by (16.1.1). Then it is easily checked that the axioms GT1–GT4 will be satisfied. In other words, a Grothendieck topology can alternatively be defined by starting from a family of morphisms in $\mathcal{C}^\wedge$ satisfying LE1–LE4.

Note that a family of morphisms in $\mathcal{C}^\wedge$ satisfies LE1–LE4 if and only if it satisfies LE2–LE4 and LE1' below:

LE1' If $u: A \to B$ is an epimorphism in $\mathcal{C}^\wedge$, then u is a local epimorphism.

Indeed, LE1' implies LE1. Conversely, assume that $u: A \to B$ is an epimorphism in $\mathcal{C}^\wedge$. If $w: U \to B$ is a morphism with $U \in \mathcal{C}$, there exists $v: U \to A$ such that $w = u \circ v$. Hence, $\mathrm{id}_U: U \to U$ factors as $U \to A \times_B U \to U$. Therefore $A \times_B U \to U$ is a local epimorphism by LE1 and LE3, and this implies that $A \to B$ is a local epimorphism by LE4. This is visualized by:



Definition 16.1.4. *Let $\mathcal{C}$ be a small category and $U \in \mathcal{C}$. Consider two small families of objects of $\mathcal{C}_U$, $\mathcal{S}_1 = \{U_i\}_{i \in I}$ and $\mathcal{S}_2 = \{V_j\}_{j \in J}$. The family $\mathcal{S}_1$ is a refinement of $\mathcal{S}_2$ if for any $i \in I$ there exist $j \in J$ and a morphism $U_i \to V_j$ in $\mathcal{C}_U$. In such a case, we write $\mathcal{S}_1 \preceq \mathcal{S}_2$.*

Note that $\mathcal{S}_1 = \{U_i\}_{i \in I}$ is a refinement of $\mathcal{S}_2 = \{V_j\}_{j \in J}$ if and only if

$$\mathrm{Hom}_{\mathcal{C}_U^\wedge}(\text{“}\coprod_i\text{”}\, U_i, \text{“}\coprod_j\text{”}\, V_j) \neq \emptyset\,. \tag{16.1.3}$$

Definition 16.1.5. *Let $\mathcal{C}$ be a small category which admits fiber products. Assume that $\mathcal{C}$ is endowed with a Grothendieck topology and let $U \in \mathcal{C}$. A small family $\mathcal{S} = \{U_i\}_{i \in I}$ of objects of $\mathcal{C}_U$ is a* covering *of U if the morphism “$\coprod_i$” $U_i \to U$ is a local epimorphism.*

Denote by Cov_U the family of coverings of U. The family of coverings will satisfy the axioms COV1–COV4 below.

COV1 $\{U\}$ belongs to Cov_U.

COV2 If $\mathcal{S}_1 \in \mathrm{Cov}_U$ is a refinement of a family $\mathcal{S}_2 \subset \mathrm{Ob}(\mathcal{C}_U)$, then $\mathcal{S}_2 \in \mathrm{Cov}_U$.

COV3 If $\mathcal{S} = \{U_i\}_{i \in I}$ belongs to Cov_U, then $\mathcal{S} \times_U V := \{U_i \times_U V\}_{i \in I}$ belongs to Cov_V for any morphism $V \to U$ in $\mathcal{C}$.

COV4 If $\mathcal{S}_1 = \{U_i\}_{i \in I}$ belongs to Cov_U, $\mathcal{S}_2 = \{V_j\}_{j \in J}$ is a small family of objects of $\mathcal{C}_U$, and $\mathcal{S}_2 \times_U U_i$ belongs to Cov_{U_i} for any $i \in I$, then $\mathcal{S}_2$ belongs to Cov_U.

Conversely, to a covering $\mathcal{S} = \{U_i\}_{i \in I}$ of U, we associate a sieve S over U by setting

$$S = \{\varphi \in \mathrm{Hom}_{\mathcal{C}}(V, U);\ \varphi \text{ factors through } U_i \to U \text{ for some } i \in I\}\ .$$

If the family of coverings satisfies COV1–COV4, it is easily checked that the associated family of sieves $\mathcal{S}\mathrm{Cov}_U$ will satisfy the axioms GT1–GT4.

In this book, we shall mainly use the notion of local epimorphisms. However, we started by introducing sieves, because this notion does not depend on the choice of a universe.

In the sequel, $\mathcal{C}$ is a category endowed with a Grothendieck topology.

Lemma 16.1.6. *Let $u\colon A \to B$ be a morphism in $\mathcal{C}^\wedge$. The conditions below are equivalent.*

(i) *u is a local epimorphism,*
(ii) *for any $t\colon U \to B$ with $U \in \mathcal{C}$, there exist a local epimorphism $u\colon C \to U$ and a morphism $s\colon C \to A$ such that $u \circ s = t \circ u$,*
(iii) $\mathrm{Im}\, u \to B$ *is a local epimorphism.*

Proof. (i) $\Rightarrow$ (ii) is obvious.
(ii) $\Rightarrow$ (i). Let $C \to U$ be a local epimorphism. It factorizes through $A \times_B U \to U$ by the hypothesis. Therefore $A \times_B U \to U$ is a local epimorphism by LE3 and the result follows from LE4.
(i) $\Rightarrow$ (iii) follows from LE3.
(iii) $\Rightarrow$ (i) follows from LE1' and LE2. q.e.d.

Example 16.1.7. Let X be a topological space, $\mathcal{C}_X := \mathrm{Op}_X$ the category of its open subsets. Note that $\mathcal{C}_X$ admits a terminal object, namely X, and the products of two objects $U, V \in \mathcal{C}_X$ is $U \cap V$. Also note that if U is an open subset of X, then $(\mathcal{C}_X)_U \simeq \mathrm{Op}_U$. We define a Grothendieck topology by deciding that a small family $\mathcal{S} = \{U_i\}_{i \in I}$ of objects of Op_U belongs to Cov_U if $\bigcup_i U_i = U$.

We may also define a Grothendieck topology as follows. A morphism $u\colon A \to B$ in $(\mathcal{C}_X)^\wedge$ is a local epimorphism if for any $U \in \mathrm{Op}_X$ and any $t \in B(U)$, there exist a covering $U = \bigcup_i U_i$ and for each i an $s_i \in A(U_i)$ with $u(s_i) = t|_{U_i}$. (Here, $t|_{U_i}$ is the image of t by $B(U) \to B(U_i)$.) Hence, a morphism $A \to U$ in $(\mathcal{C}_X)^\wedge$ $(U \in \mathcal{C})$ is a local epimorphism if there exists an open covering $U = \bigcup_{i \in I} U_i$ such that $U_i \to U$ factorizes through A for every $i \in I$.

These two definitions give the same topology. We shall call this Grothendieck topology the "associated Grothendieck topology" on X.

Example 16.1.8. For a real analytic manifold X, denote by $\mathcal{C}_{X_{\mathrm{sa}}}$ the full subcategory of $\mathcal{C}_X = \mathrm{Op}_X$ consisting of open subanalytic subsets (see [38] for an exposition). We define a Grothendieck topology on the category $\mathcal{C}_{X_{\mathrm{sa}}}$ by deciding that a small family $\mathcal{S} = \{U_i\}_{i\in I}$ of subobjects of $U \in \mathcal{C}_{\mathrm{sa}}$ belongs to Cov_U if for any compact subset K of X, there is a finite subset $J \subset I$ such that $\bigcup_{j\in J} U_j \cap K = U \cap K$. We call this Grothendieck topology the subanalytic topology on X. This topology naturally arises in Analysis, for example when studying temperate holomorphic functions. References are made to [39].

Examples 16.1.9. Let $\mathcal{C}$ be a category.
(i) We may endow $\mathcal{C}$ with a Grothendieck topology by deciding that the local epimorphisms in $\mathcal{C}^\wedge$ are the epimorphisms. This topology is called the final topology.
(ii) We may endow $\mathcal{C}$ with a Grothendieck topology by deciding that all morphisms are local epimorphisms. This topology is called the initial topology.
(iii) Recall that **Pt** denotes the category with one object c and one morphism. We endow this category with the final topology. Note that this topology is different from the initial one. Indeed, the morphism $\emptyset_{\mathbf{Pt}^\wedge} \to c$ in $\mathbf{Pt}^\wedge$ is a local epimorphism for the initial topology, not for the final one. In other words, the empty covering of pt is a covering for the initial topology, not for the final one.

Examples 16.1.10. The following examples are extracted from [51].
Let G be a finite group and denote by G-**Top** the category of small G-topological spaces. An object is a small topological space X endowed with a continuous action of G, and a morphism $f\colon X \to Y$ is a continuous map which commutes with the action of G. Such an f is said to be G-equivariant.

The category $\mathcal{E}t_G$ is defined as follows. Its objects are those of G-**Top** and its morphisms $f\colon V \to U$ are the G-equivariant maps such that f is a local homeomorphism. Note that $f(V)$ is open in U.

The category $\mathcal{E}t_G$ admits fiber products. If $U \in$ G-**Top**, then the category $\mathcal{E}t_G(U) := (\mathcal{E}t_G)_U$ admits finite projective limits.

(i) The étale topology on $\mathcal{E}t_G$ is defined as follows. A sieve S over $U \in \mathcal{E}t_G$ is a covering sieve if for any x, there exists a morphism $f\colon V \to U$ in S such that $x \in f(V)$.
(ii) The Nisnevich topology on $\mathcal{E}t_G$ is defined as follows. A sieve S over $U \in \mathcal{E}t_G$ is a covering sieve if for any $x \in U$ there exist a morphism $V \to U$ in S and $y \in V$ such that $f(y) = x$ and y has the same isotropy group as x. (The isotropy group G_y of y is the subgroup of G consisting of $g \in G$ satisfying $g \cdot y = y$.)
(iii) The Zariski topology on $\mathcal{E}t_G$ is defined as follows. A sieve S over $U \in \mathcal{E}t_G$ is a covering sieve if for any $x \in U$, there exists an open embedding $f\colon V \to U$ in S such that $x \in f(V)$.

It is easily checked that the axioms of Grothendieck topologies are satisfied in these three cases.

Proposition 16.1.11. (i) *Let $u\colon A \to B$ be a local epimorphism and let $v\colon C \to B$ be a morphism. Then $A \times_B C \to C$ is a local epimorphism.*
(ii) *If $u\colon A \to B$ is a morphism in $\mathcal{C}^\wedge$, $v\colon C \to B$ is a local epimorphism and $w\colon A \times_B C \to C$ is a local epimorphism, then u is a local epimorphism.*

Property (i) is translated by saying that "local epimorphisms are stable by base change" and property (ii) by saying that for $u\colon A \to B$ to be a local epimorphism is a local property on B.

Proof. (i) For any $U \to C$ with $U \in \mathcal{C}$, $(A \times_B C) \times_C U \simeq A \times_B U \to U$ is a local epimorphism.
(ii) It follows from the hypothesis that $v \circ w$ is a local epimorphism. Denote by $s\colon A \times_B C \to A$ the natural morphism. Then $v \circ w = u \circ s$, and u is a local epimorphism by LE3. q.e.d.

Proposition 16.1.12. *Let I be a small category and let $\alpha\colon I \to$ Mor$(\mathcal{C}^\wedge)$ be a functor. Assume that for each $i \in I$, $\alpha(i)\colon A_i \to B_i$ is a local epimorphism. Let $u\colon A \to B$ denote the inductive limit in* Mor$(\mathcal{C}^\wedge)$ *of α. Then u is a local epimorphism.*

Proof. Consider a morphism $v\colon V \to B$ with $V \in \mathcal{C}$. There exists $i \in I$ such that v factorizes as $V \to B_i \to B$. By the hypothesis, $A_i \times_{B_i} V \to V$ is a local epimorphism. Since this morphism factorizes through $A \times_B V \to V$, this last morphism is a local epimorphism by LE3. q.e.d.

16.2 Local Isomorphisms

Consider a morphism $u\colon A \to B$ in $\mathcal{C}^\wedge$. Recall (see Exercise 2.4) that the associated diagonal morphism $A \to A \times_B A$ is a monomorphism. It is an epimorphism if and only if u is a monomorphism. This naturally leads to the following:

Definition 16.2.1. (i) *We say that a morphism $u\colon A \to B$ in $\mathcal{C}^\wedge$ is a* local monomorphism *if $A \to A \times_B A$ is a local epimorphism.*
(ii) *We say that a morphism $u\colon A \to B$ in $\mathcal{C}^\wedge$ is a* local isomorphism *if it is both a local epimorphism and a local monomorphism.*

Example 16.2.2. Let X be a topological space and let $\mathcal{C} = \mathrm{Op}_X$ with the associated Grothendieck topology (see Example 16.1.7). Let $A = \text{“}\coprod_{i \in I}\text{”}\, U_i$ and $B = \text{“}\coprod_{j \in J}\text{”}\, V_j$, where the U_i's and V_j's are open in X. Any morphism $u\colon A \to B$ is induced by a map $\varphi\colon I \to J$ such that $U_i \subset V_{\varphi(i)}$ for all $i \in I$. Notice that

(i) u is a local epimorphism if and only if, for any $j \in J$, $V_j = \bigcup_{i \in \varphi^{-1}(j)} U_i$,

(ii) let U be an open subset, $\{U_i\}_{i\in I}$ an open covering of U, and for each $i, i' \in I$ let $\{W_j\}_{j\in J(i,i')}$ be an open covering of $U_i \cap U_{i'}$. Set

$$C := \mathrm{Coker}\Big(\mathop{\text{“}\coprod\text{”}}_{i,i'\in I,\, j\in J(i,i')} W_j \rightrightarrows \mathop{\text{“}\coprod\text{”}}_{i\in I} U_i \Big) .$$

Then $C \to U$ is a local isomorphism (see Exercise 16.6). Conversely, for any local isomorphism $A \to U$, we can find families $\{U_i\}_{i\in I}$ and $\{W_j\}_{j\in J(i,i')}$ as above such that $C \to U$ factors as $C \to A \to U$. It is a classical result (see [27], Lemma 3.8.1) that if U is normal and paracompact, we can take $W_j = U_i \times_U U_{i'}$, i.e., $C = \mathrm{Im}(\mathop{\text{“}\coprod\text{”}}_{i\in I} U_i \to U)$.

Lemma 16.2.3. (i) *If $u \colon A \to B$ is a monomorphism, then it is a local monomorphism. In particular, a monomorphism which is a local epimorphism is a local isomorphism.*

(ii) *If $u \colon A \to B$ is a local epimorphism, then* $\mathrm{Im}(A \to B) \to B$ *is a local isomorphism.*

(iii) *For a morphism $u \colon A \to B$, the conditions below are equivalent.*

(a) *$u \colon A \to B$ is a local monomorphism,*

(b) *for any diagram $U \rightrightarrows A \to B$ with $U \in \mathcal{C}$ such that the two compositions coincide, there exists a local epimorphism $S \to U$ such that the two compositions $S \to U \rightrightarrows A$ coincide,*

(c) *for any diagram $Z \rightrightarrows A \to B$ with $Z \in \mathcal{C}^\wedge$ such that the two compositions coincide, there exists a local epimorphism $S \to Z$ such that the two compositions $S \to Z \rightrightarrows A$ coincide.*

Proof. (i)–(ii) are obvious.

(iii) Notice first that a morphism $U \to A \times_B A$ is nothing but a diagram $U \rightrightarrows A \to B$ such that the two compositions coincide, and then any diagram $S \to U \rightrightarrows A$ such that the two compositions coincide factorizes as $S \to A \underset{A\times_B A}{\times} U \to U$.

(b) $\Rightarrow$ (a). Let $U \to A \times_B A$ be a morphism. Let $S \to U$ be a local epimorphism such that the two compositions $S \to U \rightrightarrows A$ coincide. Then $S \to U$ factorizes through $A \underset{A\times_B A}{\times} U \to U$ and this morphism will be a local epimorphism. By LE4, this implies (a).

(a) $\Rightarrow$ (c). Given $Z \to A \times_B A$, take $A \underset{A\times_B A}{\times} Z \to Z$ as $S \to Z$. q.e.d.

Lemma 16.2.4. (i) *Let $u \colon A \to B$ be a local monomorphism (resp. local isomorphism) and let $v \colon C \to B$ be a morphism. Then $A \times_B C \to C$ is a local monomorphism (resp. local isomorphism).*

(ii) *Conversely, let $u \colon A \to B$ be a morphism and let $v \colon C \to B$ be a local epimorphism. If $A \times_B C \to C$ is a local monomorphism (resp. local isomorphism), then u is a local monomorphism (resp. local isomorphism).*

(iii) *Let $A_1 \xrightarrow{u} A_2 \xrightarrow{v} A_3$ be morphisms in $\mathcal{C}^\wedge$. If u and v are local monomorphisms, then $v \circ u$ is a local monomorphism.*

(iv) *Let* $A_1 \xrightarrow{u} A_2 \xrightarrow{v} A_3$ *be morphisms in* $\mathcal{C}^\wedge$*. If* $v \circ u$ *is a local epimorphism and* v *is a local monomorphism, then* u *is a local epimorphism.*

(v) *Let* $A_1 \xrightarrow{u} A_2 \xrightarrow{v} A_3$ *be morphisms in* $\mathcal{C}^\wedge$*. If* $v \circ u$ *is a local monomorphism, then* u *is a a local monomorphism.*

(vi) *Let* $A_1 \xrightarrow{u} A_2 \xrightarrow{v} A_3$ *be morphisms in* $\mathcal{C}^\wedge$*. If* $v \circ u$ *is a local monomorphism and* u *is a local epimorphism, then* v *is a local monomorphism.*

(vii) *Let* $A_1 \xrightarrow{u} A_2 \xrightarrow{v} A_3$ *be morphisms in* $\mathcal{C}^\wedge$*. If two of the three morphisms* u*,* v*,* $v \circ u$ *are local isomorphisms, then all are local isomorphisms.*

Proof. (i) (a) Assume that u is a local monomorphism. Let $D = A \times_B C$. Consider the commutative diagram

(16.2.1)
$$\begin{array}{ccccc} D & \xrightarrow{w'} & D \times_C D & \longrightarrow & C \\ \downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle v} \\ A & \xrightarrow{w} & A \times_B A & \longrightarrow & B. \end{array}$$

Since both squares (A, B, C, D) and $(A \times_B A, B, C, D \times_C D)$ are Cartesian, the square $(A, A \times_B A, D \times_C D, D)$ is Cartesian. Since $A \to A \times_B A$ is a local epimorphism, $D \to D \times_C D$ is also a local epimorphism.

(i) (b) Since both local epimorphisms and local monomorphisms are stable by base change, the same result holds for local isomorphisms.

(ii) It is enough to treat the case where $A \times_B C \to C$ is a local monomorphism. In the diagram (16.2.1), h is a local epimorphism. Since w' is a local epimorphism, so is w by Proposition 16.1.11 (ii).

(iii) Consider the diagram

$$\begin{array}{ccccc} A_1 & \xrightarrow{u'} & A_1 \times_{A_2} A_1 & \xrightarrow{w'} & A_1 \times_{A_3} A_1 \\ & & \downarrow & & \downarrow \\ & & A_2 & \xrightarrow{v'} & A_2 \times_{A_3} A_2. \end{array}$$

Since the square is Cartesian and v' is a local epimorphism, w' is also a local epimorphism. Therefore $w' \circ u'$ is again a local epimorphism.

(iv) Consider the Cartesian squares

$$\begin{array}{ccc} A_1 & \xrightarrow[w_1]{} & A_1 \times_{A_3} A_2 \\ \downarrow & & \downarrow \\ A_2 & \xrightarrow{v'} & A_2 \times_{A_3} A_2, \end{array} \qquad \begin{array}{ccc} A_1 \times_{A_3} A_2 & \xrightarrow{w_2} & A_2 \\ \downarrow & & \downarrow{\scriptstyle v} \\ A_1 & \xrightarrow{v \circ u} & A_3. \end{array}$$

Since v' and $v \circ u$ are local epimorphisms, w_1 and w_2 as well as $w_2 \circ w_1 = u$ are local epimorphisms.

(v) Consider the Cartesian square

$$\begin{array}{ccc} A_1 & \xrightarrow{w_2} & A_1 \times_{A_2} A_1 \\ \downarrow & & \downarrow \\ A_1 & \xrightarrow{w_3} & A_1 \times_{A_3} A_1 . \end{array}$$

Since w_3 is a local epimorphism so is w_2.
(vi) The composition of the local epimorphisms $A_1 \times_{A_3} A_1 \to A_2 \times_{A_3} A_1 \to A_2 \times_{A_3} A_2$ is a local epimorphism. Consider the commutative diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{w_2} & A_1 \times_{A_3} A_1 \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle w_3} \\ A_2 & \xrightarrow{v'} & A_2 \times_{A_3} A_2 . \end{array}$$

Hence, $w_3 \circ w_2 = v' \circ u$ is a local epimorphism and this implies that v' is a local epimorphism.
(vii) (a) Assume that u and v are local isomorphisms. Then $v \circ u$ is a local epimorphism by LE2, and a local monomorphism by (iii).
(vii) (b) Assume that v and $v \circ u$ are local isomorphisms. We know by (iv) that u is a local epimorphism. It is a local monomorphism by (v).
(vii) (c) Assume that u and $v \circ u$ are local isomorphisms. We already know that v is a local epimorphism. It is a local monomorphism by (vi). q.e.d.

Notations 16.2.5. (i) We denote by $\mathcal{LI}$ the set of local isomorphisms.
(ii) Following Definition 7.1.9, for $A \in \mathcal{C}^\wedge$, we denote by $\mathcal{LI}_A$ the category given by

$$\mathrm{Ob}(\mathcal{LI}_A) = \{\text{the local isomorphisms } B \to A\} ,$$
$$\mathrm{Hom}_{\mathcal{LI}_A}((B \xrightarrow{u} A), (C \xrightarrow{v} A)) = \{w \colon B \to C; u = v \circ w\} .$$

Note that such a w is necessarily a local isomorphism.
(iii) The category $\mathcal{LI}^A$ is defined similarly.

Lemma 16.2.6. *The family $\mathcal{LI}$ of local isomorphisms in $\mathcal{C}^\wedge$ is a left saturated multiplicative system.*

Proof. Let us check the axioms S1–S5 of Definitions 7.1.5 and 7.1.19. Axiom S1 is obviously satisfied, S2 follows from Lemma 16.2.4 (iii), and S3 (with the arrows reversed, as in Remark 7.1.7) follows from Lemma 16.2.4 (i).
S4 Consider a pair of parallel morphisms $f, g \colon A \rightrightarrows B$ and a local isomorphism $t : B \to C$ such that $t \circ f = t \circ g$. Consider the Cartesian square

$$\begin{array}{ccc} \mathrm{Ker}(f, g) & \longrightarrow & B \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle u} \\ A & \longrightarrow & B \times_C B . \end{array}$$

By the hypothesis, u is a local epimorphism, and it is a monomorphism. Hence u is a local isomorphism. Since local isomorphisms are stable by base change (Lemma 16.2.4 (i)), s is a local isomorphism.
S5 Consider morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

and assume that $g \circ f$ as well as $h \circ g$ are local isomorphisms. It follows that g is both a local epimorphism and a local monomorphism. Then both $g \circ h$ and g are local isomorphisms, and this implies that h is a local isomorphism. q.e.d.

Lemma 16.2.7. *The category $\mathcal{LI}_A$ admits finite projective limits. In particular, $\mathcal{LI}_A$ is cofiltrant.*

Proof. (i) The category $\mathcal{LI}_A$ admits a terminal object, namely $A \xrightarrow{\text{id}} A$.
(ii) The category $\mathcal{LI}_A$ admits fiber products. Indeed, if $C \to B$, $D \to B$ and $B \to A$ are local isomorphisms, then $C \times_B D \to A$ is a local isomorphism by Lemma 16.2.4. q.e.d.

Lemma 16.2.8. *Assume that $\mathcal{C}$ is small. Then, for any $A \in \mathcal{C}^\wedge$, the category $(\mathcal{LI}_A)^{\text{op}}$ is cofinally small.*

Proof. Set $I = \{(U, s); U \in \mathcal{C}, s \in A(U)\}$. For $i = (U, s) \in I$, set $U_i = U$. Note that I is a small set and there exists a canonical epimorphism

$$\text{“}\coprod_{i \in I}\text{”}\, U_i \twoheadrightarrow A.$$

For a subset $J \subset I$, we set

$$C_J = \text{“}\coprod_{j \in J}\text{”}\, U_j \ .$$

Let us consider the set $\mathcal{S}$ of (J, S, v, w) where J is a subset of I, $v\colon C_J \to S$ is an epimorphism and $w\colon S \to A$ is a local isomorphism:

$$C_J \overset{v}{\twoheadrightarrow} S \xrightarrow{w} A \ .$$

By Proposition 5.2.9 and the result of Exercise 5.1, the set of quotients of any object of $\mathcal{C}^\wedge$ is small, and hence $\mathcal{S}$ is a small set. On the other hand we have a map

$$\begin{aligned} \varphi\colon \mathcal{S} &\to \text{Ob}(\mathcal{LI}_A) \ , \\ (C_J \overset{v}{\twoheadrightarrow} S \xrightarrow{w} A) &\mapsto (S \xrightarrow{w} A) \ . \end{aligned}$$

Let us show that $\varphi(\mathcal{S})$ satisfies the condition in Proposition 3.2.6. Let $B \to A$ be a local isomorphism. Set $B_1 = \text{Im}(B \to A)$. Then we have $B_1(U) \subset A(U)$ for any $U \in \mathcal{C}$. Set $J = \big\{(U, s) \in I\,; s \in B_1(U)\big\}$. Then $C_J \to B_1$ is an epimorphism, and it decomposes into $C_J \to B \to B_1$ since $B(U) \to B_1(U)$ is surjective for any $U \in \mathcal{C}$. Thus we obtain the following commutative diagram:

$$\begin{array}{ccccc} & & C_J & \longrightarrow & C_I \\ & \swarrow & \downarrow\!\!\downarrow & & \downarrow \\ B & \twoheadrightarrow & B_1 & \longrightarrow & A. \end{array}$$

Set $S = \mathrm{Im}(C_J \to B)$. Since $B_1 \to A$ and $B \to A$ are local isomorphisms, $B \to B_1$ is a local isomorphism. Since $C_J \to B_1$ is an epimorphism, $C_J \to B$ is a local epimorphism. Therefore $S \to B$ is a local epimorphism, hence a local isomorphism as well as $S \to A$. This shows that $C_J \twoheadrightarrow S \to A$ belongs to $\mathcal{S}$. q.e.d.

16.3 Localization by Local Isomorphisms

In this section, $\mathcal{C}$ is assumed to be a *small* category endowed with a Grothendieck topology. Recall that $\mathcal{LI}$ denotes the set of local isomorphisms. We shall construct a functor

$$(\,\bullet\,)^a \colon \mathcal{C}^\wedge \to \mathcal{C}^\wedge\,.$$

Since $\mathcal{LI}$ is a left multiplicative system and $(\mathcal{LI}_A)^{\mathrm{op}}$ is cofinally small for any $A \in \mathcal{C}^\wedge$, the left localization $(\mathcal{C}^\wedge)_{\mathcal{LI}}$ is a well-defined $\mathcal{U}$-category. We denote as usual by

$$Q \colon \mathcal{C}^\wedge \to (\mathcal{C}^\wedge)_{\mathcal{LI}}$$

the localization functor. For $A \in \mathcal{C}^\wedge$, we define $A^a \in \mathcal{C}^\wedge$ by

$$A^a \colon \mathcal{C} \ni U \mapsto \mathrm{Hom}_{(\mathcal{C}^\wedge)_{\mathcal{LI}}}(Q(U), Q(A))\,.$$

By the definition of $(\mathcal{C}^\wedge)_{\mathcal{LI}}$, we get

$$A^a(U) \simeq \varinjlim_{(B\to U)\in\mathcal{LI}_U} \mathrm{Hom}_{\mathcal{C}^\wedge}(B, A)\,. \tag{16.3.1}$$

For a morphism $U \to U'$ in $\mathcal{C}$, the map $A^a(U') \to A^a(U)$ is given as follows:

$$\begin{aligned} A^a(U') \;\simeq\; & \varinjlim_{(B'\to U')\in\mathcal{LI}_{U'}} \mathrm{Hom}_{\mathcal{C}^\wedge}(B', A) \\ \to\; & \varinjlim_{(B'\to U')\in\mathcal{LI}_{U'}} \mathrm{Hom}_{\mathcal{C}^\wedge}(B' \times_{U'} U, A) \\ \to\; & \varinjlim_{(B\to U)\in\mathcal{LI}_U} \mathrm{Hom}_{\mathcal{C}^\wedge}(B, A) \simeq A^a(U)\,, \end{aligned}$$

where the first morphism is associated with $B' \times_{U'} U \to B'$ and the second one is the natural morphism induced by $\mathcal{LI}_{U'} \ni (B' \to U') \mapsto (B' \times_{U'} U \to U) \in \mathcal{LI}_U$.

The identity morphism $U \xrightarrow{\mathrm{id}} U \in \mathcal{LI}_U$ defines $A(U) \to A^a(U)$. We thus obtain a morphism of functors

(16.3.2) $$\varepsilon\colon \mathrm{id}_{\mathcal{C}^\wedge} \to (\bullet)^a\,.$$

In this section, we shall study the properties of the functor $(\bullet)^a$. Since we shall treat this functor in a more general framework in Chap. 17, we restrict ourselves to the study of the properties that we need later.

Lemma 16.3.1. *Let $u\colon B \to A$ be a morphism in $\mathcal{C}^\wedge$ and let $s\colon B \to U$ be a local isomorphism with $U \in \mathcal{C}$. Denote by $v \in A^a(U)$ the corresponding element* (*using* (16.3.1))*. Then the diagram*

(16.3.3) $$\begin{array}{ccc} B & \xrightarrow{s} & U \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ A & \xrightarrow[\varepsilon(A)]{} & A^a \end{array}$$

commutes.

Proof. It is enough to show that, for any $t\colon V \to B$ with $V \in \mathcal{C}$, we have $\varepsilon(A) \circ u \circ t = v \circ s \circ t$. The element $v \circ s \circ t \in A^a(V)$ is given by the pair $(B \times_U V \to V, B \times_U V \to A)$ of the local isomorphism $B \times_U V \to V$ and the morphism $B \times_U V \to B \xrightarrow{u} A$.

Let $w\colon V \to B \times_U V$ be the morphism such that the composition $V \to B \times_U V \to V$ is id_V and $V \to B \times_U V \to B$ is t. Then w gives a morphism $(V \xrightarrow{\mathrm{id}_V} V) \to (B \times_U V \to V)$ in $\mathcal{LI}_V$. Hence, $v \circ s \circ t$ is given by the pair $(V \xrightarrow{\mathrm{id}_V} V, V \xrightarrow{t} B \xrightarrow{u} A)$, which is equal to $\varepsilon(A) \circ u \circ t$. q.e.d.

Lemma 16.3.2. *For any $A \in \mathcal{C}^\wedge$, the natural morphism $\varepsilon(A)\colon A \to A^a$ is a local isomorphism.*

Proof. (i) Consider a morphism $U \to A^a$. By the definition of A^a, there exist a local isomorphism $B \to U$ and a commutative diagram (16.3.3). Therefore, $A \to A^a$ is a local epimorphism by Lemma 16.1.6.
(ii) Consider a diagram $U \rightrightarrows A \to A^a$ such that the two compositions coincide. The two morphisms $U \rightrightarrows A$ define $s_1, s_2 \in A(U)$ with the same image in $A^a(U)$. Since $\mathcal{LI}_U$ is cofiltrant, there exist a local isomorphism $B \to U$ and a diagram $B \to U \rightrightarrows A$ such that the two compositions coincide. Therefore $A \to A^a$ is a local monomorphism by Lemma 16.2.3 (iii). q.e.d.

Proposition 16.3.3. *Let $w\colon A_1 \to A_2$ be a local isomorphism. Then $w^a\colon A_1^a \to A_2^a$ is an isomorphism.*

Proof. It is enough to show that $A_1^a(U) \to A_2^a(U)$ is bijective for any $U \in \mathcal{C}$.

(i) Injectivity. Let $v_1, v_2 \in A_1^a(U)$ and assume they have the same image in $A_2^a(U)$. Since $\mathcal{LI}_U$ is cofiltrant, there exist a local isomorphism $s\colon B \to U$ and $u_i\colon B \to A_1$ $(i = 1, 2)$ such that (u_i, s) gives $v_i \in A_1^a(U)$. Since $w^a(v_1) = w^a(v_2) \in A_2^a(U)$, there exists a local isomorphism $t\colon B' \to B$ such that the two compositions $B' \longrightarrow B \underset{u_2}{\overset{u_1}{\rightrightarrows}} A_1 \longrightarrow A_2$ coincide. Since $A_1 \to A_2$ is a local monomorphism, there exists a local isomorphism $B'' \to B$ such that the two compositions $B'' \longrightarrow B' \longrightarrow B \underset{u_2}{\overset{u_1}{\rightrightarrows}} A_1$ coincide. Hence, $v_1 = v_2$.

(ii) Surjectivity. Let $v \in A_2^a$. Then v is represented by a local isomorphism $s\colon B \to U$ and a morphism $u\colon B \to A_2$. In the following commutative diagram

$$\begin{array}{ccccc} A_1 \times_{A_2} B & \xrightarrow{w'} & B & \xrightarrow{s} & U \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} & & \\ A_1 & \xrightarrow{w} & A_2 & & \end{array}$$

w' is a local isomorphism and $(u', s \circ w')$ defines an element of $A_1^a(U)$ whose image in $A_2^a(U)$ coincides with v. q.e.d.

Proposition 16.3.4. *Let I be a small category and let $\alpha\colon I \to \mathrm{Mor}(\mathcal{C}^\wedge)$ be a functor. Assume that for each $i \in I$, $\alpha(i)\colon A_i \to B_i$ is a local isomorphism. Let $u\colon A \to B$ denote the inductive limit in $\mathrm{Mor}(\mathcal{C}^\wedge)$ of $\alpha(i)\colon A_i \to B_i$. Then u is a local isomorphism.*

In other words, $\mathcal{LI}$, considered as a full subcategory of $\mathrm{Mor}(\mathcal{C}^\wedge)$, is closed by small inductive limits in $\mathrm{Mor}(\mathcal{C}^\wedge)$.

Proof. Since $A_i^a \to B_i^a$ is an isomorphism by Proposition 16.3.3, we get the following commutative diagram on the left:

$$\begin{array}{ccccc} A_i & \xrightarrow{\varepsilon(A_i)} & A_i^a & \longrightarrow & A^a \\ {\scriptstyle \alpha(i)}\downarrow & \nearrow & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle u^a} \\ B_i & \xrightarrow[\varepsilon(B_i)]{} & B_i^a & \longrightarrow & B^a, \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\varepsilon(A)} & A^a \\ {\scriptstyle u}\downarrow & \overset{v}{\nearrow} & \downarrow{\scriptstyle u^a} \\ B & \xrightarrow[\varepsilon(B)]{} & B^a\,. \end{array}$$

Taking the inductive limit with respect to i, we obtain the commutative diagram on the right. Since $\varepsilon(A) = v \circ u$ is a local isomorphism, v is a local epimorphism. Since $u^a \circ v = \varepsilon(B)$ is a local monomorphism, v is a local monomorphism. Hence v as well as u is a local isomorphism. q.e.d.

Exercises

Exercise 16.1. Prove that the axioms LE1–LE4 are equivalent to the axioms GT1–GT4, and also prove that they are equivalent to the axioms COV1–COV4 when $\mathcal{C}$ is small and admits fiber products.

Exercise 16.2. Prove that the axioms LE1', LE2 and LE4 imply LE3.

Exercise 16.3. Let $\mathcal{C}$ be a category and $\mathcal{C}_0$ a subcategory of $\mathcal{C}$. Let us say that a morphism $u\colon A \to B$ in $\mathcal{C}^\wedge$ is a local epimorphism if for any $U \in \mathcal{C}_0$ and any morphism $U \to B$ in $\mathcal{C}^\wedge$, there exist a morphism $s\colon V \to U$ in $\mathcal{C}_0$ and a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{s} & U \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

in $\mathcal{C}^\wedge$.

Prove that the family of local epimorphisms defined above satisfies the axioms LE1–LE4.

Exercise 16.4. Let $\mathcal{C}$ be a category. Let us say that a morphism $f\colon B \to A$ in $\mathcal{C}^\wedge$ is a local epimorphism if for any morphism $U \to A$ with $U \in \mathcal{C}$, there exist $V \in \mathcal{C}$, an epimorphism $g\colon V \to U$ in $\mathcal{C}$ and a morphism $V \to B$ such that the diagram below commutes:

$$\begin{array}{ccc} V & \xrightarrow{g} & U \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & A. \end{array}$$

(i) Check that the axioms LE1–LE4 are satisfied. We call this topology the epitopology on $\mathcal{C}$.
(ii) Assume that $\mathcal{C}$ admits finite coproducts. Show that it is also possible to define a topology, replacing V above by "$\coprod_{i\in I}$" V_i with I finite, under the condition that $\coprod_{i\in I} V_i \to U$ is an epimorphism in $\mathcal{C}$.

Exercise 16.5. Let $\mathcal{C}$ be a category. Let $\mathcal{LI}$ be a subset of $\mathrm{Ob}(\mathrm{Mor}(\mathcal{C}^\wedge))$ satisfying:

LI 1 every isomorphism belongs to $\mathcal{LI}$,
LI 2 let $A \xrightarrow{u} B \xrightarrow{v} C$ be morphisms in $\mathcal{C}^\wedge$. If two of the morphisms u, v and $v \circ u$ belong to $\mathcal{LI}$, then all belong to $\mathcal{LI}$,
LI 3 a morphism $u\colon A \to B$ in $\mathcal{C}^\wedge$ belongs to $\mathcal{LI}$ if and only if for any $U \in \mathcal{C}$ and any morphism $U \to B$, the morphism $A \times_B U \to U$ belongs to $\mathcal{LI}$.

Let us say that a morphism $u\colon A \to B$ in $\mathcal{C}^\wedge$ is a local epimorphism if the morphism $\mathrm{Im}\, u \to B$ belongs to $\mathcal{LI}$.

Prove that the family of local epimorphisms so defined satisfies LE1–LE4 and $\mathcal{LI}$ coincides with the set of local isomorphisms for this Grothendieck topology.

Hence, we have an alternative definition of Grothendieck topologies, using LI1–LI3.

Exercise 16.6. Let $\mathcal{C}$ be a category endowed with a Grothendieck topology. Let $B \to A$ and $C \to B \times_A B$ be local epimorphisms. Prove that the induced morphism $\mathrm{Coker}(C \rightrightarrows B) \to A$ is a local isomorphism.

Exercise 16.7. Let $\mathcal{C}$ be a small category endowed with a Grothendieck topology and let $A \in \mathcal{C}^\wedge$. Recall the morphism of functors ε of (16.3.2).
(i) Prove that (a, ε) is a projector on $\mathcal{C}^\wedge$ (see Definition 4.1.1).
(ii) Prove that a morphism $A_1 \to A_2$ is a local isomorphism if and only if $A_1^a \to A_2^a$ is an isomorphism.
(iii) Prove that, for any local isomorphism $B_1 \to B_2$, the induced map $\mathrm{Hom}_{\mathcal{C}^\wedge}(B_2, A^a) \to \mathrm{Hom}_{\mathcal{C}^\wedge}(B_1, A^a)$ is bijective.
(iv) Prove that A^a is a terminal object in $\mathcal{LI}^A$.