15

Indization and Derivation of Abelian Categories

In this chapter we study the derived category $D^b(\mathrm{Ind}(\mathcal{C}))$ of the category of ind-objects of the abelian category $\mathcal{C}$. The main difficulty comes from the fact that, as we shall see, the category $\mathrm{Ind}(\mathcal{C})$ does not have enough injectives in general. This difficulty is partly overcome by introducing the weaker notion of "quasi-injective objects", and these objects are sufficient to derive functors on $\mathrm{Ind}(\mathcal{C})$ which are indization of functors on $\mathcal{C}$.

As a byproduct, we shall give a sufficient condition which ensures that the right derived functor of a left exact functor commutes with small filtrant inductive limits.

Finally, we study the relations between $D^b(\mathrm{Ind}(\mathcal{C}))$ and the category $\mathrm{Ind}(D^b(\mathcal{C}))$ of ind-objects of $D^b(\mathcal{C})$.

15.1 Injective Objects in Ind($\mathcal{C}$)

In this chapter, $\mathcal{C}$ is an abelian category and recall that by the hypothesis, $\mathcal{C}$ is a $\mathcal{U}$-category (see Convention 1.4.1). It follows that $\mathrm{Ind}(\mathcal{C})$ is again an abelian $\mathcal{U}$-category.

Recall that we denote by "$\bigoplus$" the coproduct in $\mathrm{Ind}(\mathcal{C})$ (see Notation 8.6.1).

As in Chap. 6, we denote by $\iota_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ the natural functor. This functor is fully faithful and exact. By Proposition 6.3.1, if $\mathcal{C}$ admits small inductive limits, the functor $\iota_{\mathcal{C}}$ admits a left adjoint, denoted by $\sigma_{\mathcal{C}}$. It follows from Proposition 8.6.6 that if the small filtrant inductive limits are exact in $\mathcal{C}$, then the functor $\sigma_{\mathcal{C}}$ is exact.

Proposition 15.1.1. *Assume that $\mathcal{C}$ admits small inductive limits and that small filtrant inductive limits are exact. Let $X \in \mathcal{C}$. Then*

(i) *X is injective in $\mathcal{C}$ if and only if $\iota_{\mathcal{C}}(X)$ is injective in* $\mathrm{Ind}(\mathcal{C})$*,*
(ii) *X is projective in $\mathcal{C}$ if and only if $\iota_{\mathcal{C}}(X)$ is projective in* $\mathrm{Ind}(\mathcal{C})$*.*

Proof. (i) Let X be an injective object of $\mathcal{C}$. For $A \in \mathrm{Ind}(\mathcal{C})$, we have $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(A, \iota_{\mathcal{C}}(X)) \simeq \mathrm{Hom}_{\mathcal{C}}(\sigma_{\mathcal{C}}(A), X)$ and the result follows since $\sigma_{\mathcal{C}}$ is exact. The converse statement is obvious.
(ii) Let X be a projective object of $\mathcal{C}$, let $f\colon A \to B$ be an epimorphism in $\mathrm{Ind}(\mathcal{C})$ and $u\colon X \to B$ a morphism. Let us show that u factors through f. By Proposition 8.6.9, there exist an epimorphism $f'\colon Y \to X$ in $\mathcal{C}$ and a morphism $v\colon Y \to A$ such that $u \circ f' = f \circ v$. Since X is projective, there exists a section $s\colon X \to Y$ of f'. Therefore, $f \circ (v \circ s) = u \circ f' \circ s = u$. This is visualized by the diagram

$$\begin{tikzcd} Y \arrow[r, two heads, shift left, "f'"] \arrow[d, "v"'] & X \arrow[l, dotted, shift left, "s"] \arrow[d, "u"] \\ A \arrow[r, two heads, "f"'] & B . \end{tikzcd}$$

The converse statement is obvious. q.e.d.

In the simple case where $\mathcal{C} = \mathrm{Mod}(k)$ with a field k, we shall show that the category $\mathrm{Ind}(\mathcal{C})$ does not have enough injectives. In the sequel, we shall write $\mathrm{Ind}(k)$ instead of $\mathrm{Ind}(\mathrm{Mod}(k))$, for short.

Proposition 15.1.2. *Assume that k is a field. Let $Z \in \mathrm{Ind}(k)$. Then Z is injective if and only if Z belongs to $\mathrm{Mod}(k)$.*

Proof. Assume that $Z \in \mathrm{Ind}(k)$ is injective. Any object in $\mathrm{Ind}(k)$ is a quotient of "$\bigoplus_i$" M_i with $M_i \in \mathrm{Mod}(k)$, and the natural morphism "$\bigoplus_i$" $M_i \to \bigoplus_i M_i$ is a monomorphism. Since Z is injective, "$\bigoplus_i$" $M_i \to Z$ factorizes through $\bigoplus_i M_i$. Hence we can assume from the beginning that

$$Z = X/Y \text{ with } X \in \mathrm{Mod}(k),\ Y \in \mathrm{Ind}(k).$$

Since $Y \to X$ is a monomorphism, $\sigma_{\mathcal{C}}(Y)$ is a sub-object of X. Hence, there exits a decomposition $X = X' \oplus \sigma_{\mathcal{C}}(Y)$ in $\mathrm{Mod}(k)$. Then $Z = X' \oplus (\sigma_{\mathcal{C}}(Y)/Y)$ and $\sigma_{\mathcal{C}}(Y)/Y$ is injective. Thus we may assume from the beginning that

$$Z = X/Y \text{ with } X \in \mathrm{Mod}(k),\ Y \subset X \text{ and } \sigma_{\mathcal{C}}(Y) = X.$$

Let $\kappa_{\mathcal{C}}\colon \mathrm{Mod}(k) \to \mathrm{Ind}(k)$ be the functor introduced in Sect. 6.3, $V \mapsto$ "$\varinjlim$" W, where W ranges over the family of finite-dimensional vector subspaces of V. Then we have $\kappa_{\mathcal{C}}(V) \subset Y$ for any $V \in \mathrm{Mod}(k)$ with $V \subset X$.

Assuming $Y \neq X$, we shall derive a contradiction. Set

$$\begin{aligned} \mathcal{K} &= \{V\,;\, V \in \mathrm{Mod}(k)\,,\, V \subset Y\}\,, \\ N &= k^{\oplus \mathcal{K}} = \bigoplus_{V \in \mathcal{K}} k e_V\,, \\ \Phi &= \mathrm{Hom}_k(N, X)\,. \end{aligned}$$

For $\varphi \in \Phi$, let N_φ be a copy of N and let $a_\varphi \colon N \xrightarrow{\sim} N_\varphi$ be the isomorphism. We denote by $c_\varphi \colon N \to \bigoplus_{\varphi' \in \Phi} N_{\varphi'}$ the composition $N \underset{a_\varphi}{\xrightarrow{\sim}} N_\varphi \to \bigoplus_{\varphi' \in \Phi} N_{\varphi'}$. Set

$$T = \text{“}\bigoplus_{V \in \mathcal{K}}\text{”}\Big(\bigoplus_{\varphi \in \Phi} k c_\varphi(e_V)\Big) \subset \bigoplus_{\varphi \in \Phi} N_\varphi \; .$$

Then, for any finite subset A of Φ, we have

$$\begin{aligned} T \cap \Big(\bigoplus_{\varphi \in A} N_\varphi\Big) &= \text{“}\bigoplus_{V \in \mathcal{K}}\text{”}\Big(\bigoplus_{\varphi \in A} k c_\varphi(e_V)\Big) \\ &= \bigoplus_{\varphi \in A}\Big(\text{“}\bigoplus_{V \in \mathcal{K}}\text{”}\, k c_\varphi(e_V)\Big) = \bigoplus_{\varphi \in A} \kappa_{\mathcal{C}}(N_\varphi) \; . \end{aligned}$$

Hence, we have a monomorphism

$$\text{“}\bigoplus_{\varphi \in \Phi}\text{”}\big(N_\varphi / \kappa_{\mathcal{C}}(N_\varphi)\big) \hookrightarrow \Big(\bigoplus_{\varphi \in \Phi} N_\varphi\Big)/T \; .$$

Let $f \colon \bigoplus_{\varphi \in \Phi} N_\varphi \to X$ be the morphism defined by $f \circ c_\varphi(u) = \varphi(u)$ for $u \in N$. It induces a morphism

$$\tilde{f} \; : \; \text{“}\bigoplus_{\varphi \in \Phi}\text{”}(N_\varphi / \kappa_{\mathcal{C}}(N_\varphi)) \to Z \; .$$

Since Z is injective, the morphism $\tilde{f}$ factors through $(\bigoplus_{\varphi \in \Phi} N_\varphi)/T$. Note that any object in Mod(k) is a projective object in Ind(k) by Proposition 15.1.1. Hence $\bigoplus_{\varphi \in \Phi} N_\varphi$ is a projective object of Ind(k), and the composition $\bigoplus_{\varphi \in \Phi} N_\varphi \to (\bigoplus_{\varphi \in \Phi} N_\varphi)/T \to Z$ factors through X. Thus we obtain the commutative diagram

$$\begin{tikzcd} & \text{“}\bigoplus_{\varphi \in \Phi}\text{”}\, N_\varphi \arrow[dl] \arrow[dr] & \\ \text{“}\bigoplus_{\varphi \in \Phi}\text{”}\big(N_\varphi / \kappa_{\mathcal{C}}(N_\varphi)\big) \arrow[r, tail] \arrow[dr, "\tilde{f}"'] & \big(\bigoplus_{\varphi \in \Phi} N_\varphi\big)/T \arrow[d] \arrow[dr, phantom, "\mathrm{A}"] & \bigoplus_{\varphi \in \Phi} N_\varphi \arrow[l, two heads] \arrow[d, dotted, "F"] \\ & Z & X. \arrow[l, two heads] \end{tikzcd} \tag{15.1.1}$$

The morphism $F \colon \bigoplus_{\varphi \in \Phi} N_\varphi \to X$ has the following properties:

- $F_\varphi := F \circ c_\varphi \colon N \to X$ satisfies the condition: for any $V \in \mathcal{K}$, there exists $K(V) \in \mathcal{K}$ such that $F_\varphi(e_V) \in K(V)$ for any $\varphi \in \Phi$,
- $G_\varphi := (F_\varphi - \varphi)(N) \subset X$ belongs to $\mathcal{K}$ for any $\varphi \in \Phi$.

Indeed, the first property follows from the fact that the composition

$$\bigoplus_{\varphi\in\Phi} kc_\varphi(e_V) \to \bigoplus_{\varphi\in\Phi} N_\varphi \xrightarrow{F} X \to Z = X/Y$$

vanishes by the commutativity of the square labeled by A in (15.1.1), and the second follows from the fact that the two compositions $N \underset{F_\varphi}{\overset{\varphi}{\rightrightarrows}} X \longrightarrow Z$ coincide.

Hence we have

$$\varphi(e_V) \in K(V) + G_\varphi \text{ for any } V \in \mathcal{K} \text{ and } \varphi \in \Phi\ . \tag{15.1.2}$$

Since $Y \neq X$, we have $K(V) + V \neq X$ for any $V \in \mathcal{K}$. Hence there exists $x(V) \in X$ such that $x(V) \notin K(V)+V$. Define $\varphi_0 \in \Phi$ by $\varphi_0(e_V) = x(V)$. Then for $V = G_{\varphi_0}$, we have

$$\varphi_0(e_V) = x(V) \notin K(V) + V = K(V) + G_{\varphi_0}\ .$$

This contradicts (15.1.2). q.e.d.

Corollary 15.1.3. *The category* Ind(k) *does not have enough injectives.*

Proof. Let us take $V \in \mathrm{Mod}(k)$ with $\dim V = \infty$ and let $U = \kappa_{\mathcal{C}}(V)$. Define $W \in \mathrm{Ind}(k)$ by the exact sequence

$$0 \to U \to V \to W \to 0\ .$$

Then, we have $\sigma_{\mathcal{C}}(W) \simeq 0$, but W does not vanish. Assume that there exists a monomorphism $W \rightarrowtail Z$ with an injective object $Z \in \mathrm{Ind}(k)$. Then Z belongs to $\mathrm{Mod}(k)$ by Proposition 15.1.2. The morphism of functors $\mathrm{id} \to \sigma_{\mathcal{C}}$ (we do not write $\iota_{\mathcal{C}}$) induces the commutative diagram in Ind(k)

$$\begin{tikzcd} W \arrow[r, tail] \arrow[d] & Z \arrow[d] \\ \sigma_{\mathcal{C}}(W) \arrow[r] & \sigma_{\mathcal{C}}(Z). \end{tikzcd}$$

Since $Z \to \sigma_{\mathcal{C}}(Z)$ is an isomorphism, we get $W \simeq 0$, which is a contradiction. q.e.d.

15.2 Quasi-injective Objects

Let $\mathcal{C}$ be an abelian category. We have seen in Sect. 15.1 that the abelian category Ind$(\mathcal{C})$ does not have enough injectives in general. However, quasi-injective objects, which we introduce below, are sufficient for many purposes.

Definition 15.2.1. *Let* $A \in \mathrm{Ind}(\mathcal{C})$*. We say that* A *is* quasi-injective *if the functor*

$$\begin{aligned}\mathcal{C} &\to \mathrm{Mod}(\mathbb{Z}) ,\\ X &\mapsto A(X) = \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(X, A),\end{aligned}$$

is exact.

Clearly, a small filtrant inductive limit of quasi-injective objects is quasi-injective.

Lemma 15.2.2. *Let* $0 \to A' \xrightarrow{f} A \xrightarrow{g} A'' \to 0$ *be an exact sequence in* $\mathrm{Ind}(\mathcal{C})$ *and assume that* A' *is quasi-injective. Then*

(i) *the sequence* $0 \to A'(X) \to A(X) \to A''(X) \to 0$ *is exact for any* $X \in \mathcal{C}$*,*
(ii) A *is quasi-injective if and only if* A'' *is quasi-injective.*

Proof. (i) It is enough to prove the surjectivity of $A(X) \to A''(X)$. Let $u \in A''(X)$. Using Proposition 8.6.9, we get a commutative solid diagram with exact rows and with $Z, Y \in \mathcal{C}$

$$\begin{tikzcd}
0 \arrow[r] & Z \arrow[r, "f'"] \arrow[d, "w"] & Y \arrow[r, "g'"] \arrow[d, "v"] \arrow[dl, dotted, "\varphi"'] & X \arrow[r] \arrow[d, "u"] \arrow[dl, dotted, "\psi"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "f"] & A \arrow[r, "g"] & A'' \arrow[r] & 0 .
\end{tikzcd}$$

Since A' is quasi-injective, there exists a morphism $\varphi\colon Y \to A'$ such that $w = \varphi \circ f'$. Therefore, $(v - f \circ \varphi) \circ f' = v \circ f' - f \circ w = 0$, and the morphism $v - f \circ \varphi$ factors through $\mathrm{Coker}\, f' \simeq X$. Hence, there exists $\psi\colon X \to A$ such that $v - f \circ \varphi = \psi \circ g'$. Then $g \circ \psi \circ g' = g \circ (v - f \circ \varphi) = u \circ g'$, and this implies $u = g \circ \psi$.

(ii) The proof is left as an easy exercise. q.e.d.

Proposition 15.2.3. *Assume that* $\mathcal{C}$ *has enough injectives and let* $A \in \mathrm{Ind}(\mathcal{C})$*. Then the conditions below are equivalent.*

(i) A *is quasi-injective,*
(ii) *there exist a small and filtrant category* J *and a functor* $\alpha\colon J \to \mathcal{C}$ *such that* $A \simeq$ "$\varinjlim$" α *and* $\alpha(j)$ *is injective in* $\mathcal{C}$ *for all* $j \in J$*,*
(iii) *any morphism* $a\colon X \to A$ *with* $X \in \mathcal{C}$ *factorizes through an injective object* Y *of* $\mathcal{C}$*, (i.e.,* $a = b \circ f$ *with* $X \xrightarrow{f} Y \xrightarrow{b} A$*).*

Proof. Let $\mathcal{I}$ denote the full subcategory of $\mathcal{C}$ consisting of injective objects.
(i) $\Rightarrow$ (iii). By the hypothesis, there exists a monomorphism $X \rightarrowtail Y$ with $Y \in \mathcal{I}$. Since A is quasi-injective, $X \to A$ factorizes through Y.
(iii) $\Rightarrow$ (ii) follows from Exercise 6.11.
(ii) $\Rightarrow$ (i). Let $X \in \mathcal{C}$. We have $A(X) \simeq \varinjlim_{j \in J} \mathrm{Hom}_{\mathcal{C}}(X, \alpha(j))$. Since $\alpha(j)$ is injective and the functor $\varinjlim$ is exact, A is exact. q.e.d.

Definition 15.2.4. *We say that* $\mathrm{Ind}(\mathcal{C})$ *has enough quasi-injectives if the full subcategory of quasi-injective objects is cogenerating in* $\mathrm{Ind}(\mathcal{C})$.

Theorem 15.2.5. *Let* $\mathcal{J}$ *be a cogenerating full subcategory of* $\mathcal{C}$. *Then* $\mathrm{Ind}(\mathcal{J})$ *is cogenerating in* $\mathrm{Ind}(\mathcal{C})$.

In order to prove this result, we need a lemma.

Lemma 15.2.6. *For any small subset* $\mathcal{S}$ *of* $\mathrm{Ob}(\mathcal{C})$, *there exists a small fully abelian subcategory* $\mathcal{C}_0$ *of* $\mathcal{C}$ *such that*

(i) $\mathcal{S} \subset \mathrm{Ob}(\mathcal{C}_0)$,
(ii) $\mathcal{C}_0 \cap \mathcal{J}$ *is cogenerating in* $\mathcal{C}_0$.

Proof. We shall define an increasing sequence $\{\mathcal{S}_n\}_{n\geq 0}$ of full subcategories of $\mathcal{C}$ by induction on n. For any $X \in \mathcal{S}$, let us take $I_X \in \mathcal{J}$ and a monomorphism $X \rightarrowtail I_X$. We define $\mathcal{S}_0$ as the full subcategory of $\mathcal{C}$ such that $\mathrm{Ob}(\mathcal{S}_0) = \mathcal{S}$. For $n > 0$, let $\mathcal{S}_n$ be the full subcategory of $\mathcal{C}$ such that

$$\begin{aligned}\mathrm{Ob}(\mathcal{S}_n) = \ & \mathrm{Ob}(\mathcal{S}_{n-1}) \cup \{I_X\,;\, X \in \mathcal{S}_{n-1}\} \cup \{X \oplus Y\,;\, X, Y \in \mathcal{S}_{n-1}\} \\ & \cup \big\{\mathrm{Ker}\, u\,;\, u \in \mathrm{Mor}(\mathcal{S}_{n-1})\big\} \cup \big\{\mathrm{Coker}\, u\,;\, u \in \mathrm{Mor}(\mathcal{S}_{n-1})\big\}\ .\end{aligned}$$

Then $\mathcal{C}_0 = \bigcup_n \mathcal{S}_n$ satisfies the desired conditions. q.e.d.

Proof of Theorem 15.2.5. Let $A \in \mathrm{Ind}(\mathcal{C})$. There exist a small filtrant category I and a functor $\alpha\colon I \to \mathcal{C}$ such that $A \simeq \text{“}\varinjlim\text{”}\,\alpha$. By Lemma 15.2.6, there exists a small fully abelian subcategory $\mathcal{C}_0$ of $\mathcal{C}$ such that $\alpha(i) \in \mathcal{C}_0$ for all $i \in I$ and $\mathcal{J} \cap \mathcal{C}_0$ is cogenerating in $\mathcal{C}_0$. Then $A \in \mathrm{Ind}(\mathcal{C}_0)$, and $\mathrm{Ind}(\mathcal{C}_0)$ admits enough injectives by Corollary 9.6.5. Hence, there exist an injective object B of $\mathrm{Ind}(\mathcal{C}_0)$ and a monomorphism $A \rightarrowtail B$. In order to prove that $B \in \mathrm{Ind}(\mathcal{C}_0 \cap \mathcal{J})$, it is enough to check that any morphism $Z \to B$ with $Z \in \mathcal{C}_0$, factorizes through an object of $\mathcal{C}_0 \cap \mathcal{J}$ (see Exercise 6.11). Take a monomorphism $Z \to Y$ with $Y \in \mathcal{C}_0 \cap \mathcal{J}$. Since B is injective, $Z \to B$ factors through $Z \to Y$. q.e.d.

Corollary 15.2.7. *Let* $\mathcal{C}$ *be an abelian category which admits enough injectives. Then* $\mathrm{Ind}(\mathcal{C})$ *admits enough quasi-injectives.*

15.3 Derivation of Ind-categories

As above, $\mathcal{C}$ denotes an abelian category.

Theorem 15.3.1. (i) *The natural functor* $\mathrm{D}^*(\mathcal{C}) \to \mathrm{D}^*_{\mathcal{C}}(\mathrm{Ind}(\mathcal{C}))$ *is an equivalence for* $* = \mathrm{b}, -$.
(ii) *Assume that* $\mathcal{C}$ *admits small inductive limits and small filtrant inductive limits are exact. Then* $\mathrm{D}^+(\mathcal{C}) \to \mathrm{D}^+_{\mathcal{C}}(\mathrm{Ind}(\mathcal{C}))$ *is an equivalence.*

Proof. (i) By Theorem 13.2.8 (with the arrows reversed), it is enough to show that for any epimorphism $A \twoheadrightarrow Y$ in $\mathrm{Ind}(\mathcal{C})$ with $Y \in \mathcal{C}$, there exist $X \in \mathcal{C}$ and a morphism $X \to A$ such that the composition $X \to Y$ is an epimorphism. This follows from Proposition 8.6.9.
(ii) Let us apply Theorem 13.2.8 and consider a monomorphism $X \rightarrowtail A$ with $X \in \mathcal{C}$ and $A \in \mathrm{Ind}(\mathcal{C})$. Then $X \simeq \sigma_{\mathcal{C}}(X) \to \sigma_{\mathcal{C}}(A)$ is a monomorphism and factors through $X \to A$. (Recall that the functor $\sigma_{\mathcal{C}}$ is defined in Proposition 6.3.1.) q.e.d.

Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor, and let $IF : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}')$ be the associated left exact functor. We shall consider the following hypothesis:

$$\text{there exists an } F\text{-injective subcategory } \mathcal{J} \text{ of } \mathcal{C}\,. \tag{15.3.1}$$

Hypothesis (15.3.1) implies that the right derived functor $R^+F : \mathrm{D}^+(\mathcal{C}) \to \mathrm{D}^+(\mathcal{C}')$ exists and $R^kF : \mathcal{C} \to \mathcal{C}'$ induces a functor $I(R^kF) : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}')$.

Proposition 15.3.2. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor of abelian categories and let $\mathcal{J}$ be an F-injective subcategory of $\mathcal{C}$. Then*

(a) $\mathrm{Ind}(\mathcal{J})$ *is IF-injective,*
(b) *the functor IF admits a right derived functor $R^+(IF) : \mathrm{D}^+(\mathrm{Ind}(\mathcal{C})) \to \mathrm{D}^+(\mathrm{Ind}(\mathcal{C}'))$,*
(c) *the diagram below commutes*

$$\begin{tikzcd} \mathrm{D}^+(\mathcal{C}) \arrow[r, "R^+F"] \arrow[d] & \mathrm{D}^+(\mathcal{C}') \arrow[d] \\ \mathrm{D}^+(\mathrm{Ind}(\mathcal{C})) \arrow[r, "R^+(IF)"] & \mathrm{D}^+(\mathrm{Ind}(\mathcal{C}')), \end{tikzcd}$$

(d) *there is an isomorphism $I(R^kF) \simeq R^k(IF)$ for all $k \in \mathbb{Z}$. In particular, $R^k(IF)$ commutes with small filtrant inductive limits.*

Proof. (a) First, note that $\mathrm{Ind}(\mathcal{J})$ is cogenerating by Theorem 15.2.5. Set

$$\widetilde{\mathcal{J}} := \{A \in \mathrm{Ind}(\mathcal{C}); I(R^kF)(A) \simeq 0 \text{ for all } k > 0\}\,.$$

Since $\widetilde{\mathcal{J}}$ contains $\mathrm{Ind}(\mathcal{J})$, it is cogenerating. Let us check that $\widetilde{\mathcal{J}}$ satisfies the conditions (ii) and (iii) in Corollary 13.3.8. Consider an exact sequence $0 \to A \to B \to C \to 0$ in $\mathrm{Ind}(\mathcal{C})$. By Proposition 8.6.6 (a), there exist a small filtrant category I and an exact sequence of functors from I to $\mathcal{C}$

$$0 \to \alpha \to \beta \to \gamma \to 0 \tag{15.3.2}$$

such that the exact sequence in $\mathrm{Ind}(\mathcal{C})$ is obtained by applying the functor "$\varinjlim$" to (15.3.2). Consider the long exact sequence for $i \in I$

$$0 \to R^0F(\alpha(i)) \to R^0F(\beta(i)) \to R^0F(\gamma(i)) \to R^1F(\alpha(i)) \to \cdots\,.$$

Applying the functor "$\varinjlim$", we obtain the long exact sequence

(15.3.3)
$$0 \to I(R^0F)(A) \to I(R^0F)(B) \to I(R^0F)(C) \to I(R^1F)(A) \to \cdots .$$

Assuming $A, B \in \widetilde{\mathcal{J}}$, we deduce $C \in \widetilde{\mathcal{J}}$. Assuming $A \in \widetilde{\mathcal{J}}$, we deduce the exact sequence $0 \to IF(A) \to IF(B) \to IF(C) \to 0$. Therefore, $\widetilde{\mathcal{J}}$ is IF-injective and it follows from Proposition 13.3.5 (ii) that $\mathrm{Ind}(\mathcal{J})$ is itself IF-injective.

(b) follows from Proposition 13.3.5 (i).

(c) follows from Proposition 13.3.13. Indeed, $R^+(\iota_{\mathcal{C}'} \circ F) \simeq \iota_{\mathcal{C}'} \circ R^+F$ since $\iota_{\mathcal{C}'}$ is exact and $R^+(IF \circ \iota_{\mathcal{C}}) \simeq R^+(IF) \circ \iota_{\mathcal{C}}$ since $\mathrm{Ind}(\mathcal{J})$ contains $\mathcal{J}$ and is IF-injective.

(d) We construct a morphism $I(R^kF) \to R^k(IF)$ as follows. For $A \in \mathrm{Ind}(\mathcal{C})$,
$$I(R^kF)(A) \simeq \mathop{\text{“}\varinjlim\text{”}}_{(X\to A)\in\mathcal{C}_A} R^kF(X) \simeq \mathop{\text{“}\varinjlim\text{”}}_{(X\to A)\in\mathcal{C}_A} R^k(IF)(X) \to R^k(IF)(A) .$$

The isomorphism in (d) obviously holds for $k = 0$. We shall prove that it holds for $k = 1$, then for all k.
Consider an exact sequence $0 \to A \to B \to C \to 0$ with $B \in \widetilde{\mathcal{J}}$. Then $I(R^kF)(B) \simeq 0$ for all $k > 0$ by definition and $R^k(IF)(B) \simeq 0$ for all $k > 0$ since $\widetilde{\mathcal{J}}$ is IF-injective. There exists an exact sequence

(15.3.4)
$$0 \to R^0(IF)(A) \to R^0(IF)(B) \to R^0(IF)(C) \to R^1(IF)(A) \to \cdots .$$

By comparing the exact sequences (15.3.3) and (15.3.4), we get the result for $k = 1$.

We have the isomorphisms $I(R^kF)(A) \simeq I(R^{k-1}F)(C)$ and $R^k(IF)(A) \simeq R^{k-1}(IF)(C)$ for $k \geq 2$. By induction on k, we may assume $I(R^{k-1}F)(C) \simeq R^{k-1}(IF)(C)$. Therefore, $I(R^kF)(A) \simeq R^k(IF)(A)$. q.e.d.

Proposition 15.3.3. *Let $\mathcal{C}$ and $\mathcal{C}'$ be abelian categories admitting small inductive limits and assume that small filtrant inductive limits are exact in $\mathcal{C}$ and $\mathcal{C}'$. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a left exact functor commuting with small filtrant inductive limits and let $\mathcal{J}$ be an F-injective additive subcategory of $\mathcal{C}$ closed by small filtrant inductive limits. Then $R^kF\colon \mathcal{C} \to \mathcal{C}'$ commutes with small filtrant inductive limits for all $k \in \mathbb{Z}$.*

Proof. The functor $\sigma_{\mathcal{C}}\colon \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ is exact and induces a triangulated functor $\mathrm{D}^+(\mathrm{Ind}(\mathcal{C})) \to \mathrm{D}^+(\mathcal{C})$ that we still denote by $\sigma_{\mathcal{C}}$, and similarly with $\mathcal{C}$ replaced with $\mathcal{C}'$. Consider the diagram

(15.3.5)
$$\begin{tikzcd} \mathrm{D}^+(\mathrm{Ind}(\mathcal{C})) \arrow[r, "R^+(IF)"] \arrow[d, "\sigma_{\mathcal{C}}"'] & \mathrm{D}^+(\mathrm{Ind}(\mathcal{C}')) \arrow[d, "\sigma_{\mathcal{C}'}"'] \\ \mathrm{D}^+(\mathcal{C}) \arrow[r, "R^+F"] & \mathrm{D}^+(\mathcal{C}'). \end{tikzcd}$$

We shall show that this diagram commutes. Note that $\sigma_{\mathcal{C}'} \circ IF \simeq F \circ \sigma_{\mathcal{C}}$ by the assumption, and $\sigma_{\mathcal{C}'} \circ R^+(IF) \simeq R^+(\sigma_{\mathcal{C}'} \circ IF)$. Hence, it is enough to show that

$$(R^+F) \circ \sigma_{\mathcal{C}} \simeq R^+(F \circ \sigma_{\mathcal{C}}) \,, \tag{15.3.6}$$

and this follows from Proposition 13.3.13 since $\sigma_{\mathcal{C}}$ sends $\mathrm{Ind}(\mathcal{J})$ to $\mathcal{J}$.

To conclude, consider a small filtrant inductive system $\{X_i\}_{i \in I}$ in $\mathcal{C}$. We have the chain of isomorphisms

$$\begin{aligned} \varinjlim_i R^kF(X_i) &\simeq \sigma_{\mathcal{C}'}(\text{“}\varinjlim_i\text{”}\, R^kF(X_i)) \\ &\simeq \sigma_{\mathcal{C}'} R^k(IF)(\text{“}\varinjlim_i\text{”}\, X_i) \\ &\simeq (R^kF)\sigma_{\mathcal{C}}(\text{“}\varinjlim_i\text{”}\, X_i) \simeq R^kF(\varinjlim_i X_i) \,. \end{aligned}$$

Here, the second isomorphism follows from Proposition 15.3.2 (d) and the third one from the commutativity of (15.3.5). q.e.d.

Notation 15.3.4. We shall denote by $\mathcal{I}_{qinj}$ the full subcategory of $\mathrm{Ind}(\mathcal{C})$ consisting of quasi-injective objects.

Consider the hypothesis

$$\text{the category } \mathcal{I}_{qinj} \text{ is cogenerating in } \mathrm{Ind}(\mathcal{C}) \,. \tag{15.3.7}$$

This condition is a consequence of one of the following hypotheses

$$\mathcal{C} \text{ has enough injectives,} \tag{15.3.8}$$

$$\mathcal{C} \text{ is small} \,. \tag{15.3.9}$$

Indeed, (15.3.8) implies (15.3.7) by Corollary 15.2.7, and (15.3.9) implies (15.3.7) by Theorem 9.6.2.

Proposition 15.3.5. *Assume* (15.3.7) *and let* $F\colon \mathcal{C} \to \mathcal{C}'$ *be a left exact functor. Then the category* $\mathcal{I}_{qinj}$ *of quasi-injective objects is* IF*-injective. In particular,* $R^+(IF)\colon \mathrm{D}^+(\mathrm{Ind}(\mathcal{C})) \to \mathrm{D}^+(\mathrm{Ind}(\mathcal{C}'))$ *exists.*

Proof. (i) We shall verify the hypotheses (i)–(iii) of Corollary 13.3.8. The first one is nothing but (15.3.7).
(ii) follows from Lemma 15.2.2 (ii).
(iii) Consider an exact sequence $0 \to A \to B \to C \to 0$ in $\mathrm{Ind}(\mathcal{C})$ and assume that $A \in \mathcal{I}_{qinj}$. For any $X \in \mathcal{C}$ and any morphism $u\colon X \to C$, Lemma 15.2.2 implies that u factors through $X \xrightarrow{w} B \to C$.

This defines a morphism $F(w)\colon F(X) \to IF(B)$ such that the composition $F(X) \to IF(B) \to IF(C)$ is the canonical morphism. Therefore, we get the exact sequence

$$IF(B) \times_{IF(C)} F(X) \to F(X) \to 0$$

Applying the functor $\underset{(X \to C) \in \mathcal{C}_C}{\text{“}\varinjlim\text{”}}$, we find that $IF(B) \to IF(C)$ is an epimorphism by Lemma 3.3.9. q.e.d.

Corollary 15.3.6. *Assume* (15.3.7). *Then for any $A \in \mathrm{Ind}(\mathcal{C})$ there is a natural isomorphism*

$$\underset{(X \to A) \in \mathcal{C}_A}{\text{“}\varinjlim\text{”}} R^k(IF)(X) \xrightarrow{\sim} R^k(IF)(A) .$$

In particular, $R^k(IF)$ commutes with small filtrant inductive limits.

Proof. Consider the functor $IF \colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}')$. The subcategory $\mathcal{I}_{qinj}$ of $\mathrm{Ind}(\mathcal{C})$ is closed by small filtrant inductive limits and is IF-injective. Hence, the result follows from Proposition 15.3.3. q.e.d.

We consider now a right exact functor $G \colon \mathcal{C} \to \mathcal{C}'$ of abelian categories.

Proposition 15.3.7. *Let $G \colon \mathcal{C} \to \mathcal{C}'$ be a right exact functor of abelian categories and let $\mathcal{K}$ be a G-projective additive subcategory of $\mathcal{C}$. Then*

(a) *the category* $\mathrm{Ind}(\mathcal{K})$ *is IG-projective,*
(b) *the functor IG admits a left derived functor $L^-(IG) \colon \mathrm{D}^-(\mathrm{Ind}(\mathcal{C})) \to \mathrm{D}^-(\mathrm{Ind}(\mathcal{C}'))$,*
(c) *the diagram below commutes*

$$\begin{tikzcd} \mathrm{D}^-(\mathcal{C}) \arrow[r, "L^-G"] \arrow[d] & \mathrm{D}^-(\mathcal{C}') \arrow[d] \\ \mathrm{D}^-(\mathrm{Ind}(\mathcal{C})) \arrow[r, "L^-(IG)"] & \mathrm{D}^-(\mathrm{Ind}(\mathcal{C}')), \end{tikzcd}$$

(d) *there is a natural isomorphism $I(L^kG) \simeq L^k(IG)$ for all $k \in \mathbb{Z}$. In particular, $L^k(IG)$ commutes with small filtrant inductive limits.*

Proof. The proof is very similar to that of Proposition 15.3.2, but we partly repeat it for the reader's convenience.

(a) Set

$$\widetilde{\mathcal{K}} = \{A \in \mathrm{Ind}(\mathcal{C}); I(L^kG)(A) \simeq 0 \text{ for all } k < 0\} .$$

Then $\widetilde{\mathcal{K}}$ contains $\mathrm{Ob}(\mathrm{Ind}(\mathcal{K}))$. Let us show that $\widetilde{\mathcal{K}}$ satisfies the conditions (i)–(iii) (with the arrows reversed) of Corollary 13.3.8.
(i) The category $\widetilde{\mathcal{K}}$ is generating. Indeed, if $A \in \mathrm{Ind}(\mathcal{C})$, there exists an epimorphism $\text{“}\bigoplus_{i \in I}\text{”} X_i \twoheadrightarrow A$ with a small set I and $X_i \in \mathcal{C}$. For each i choose an epimorphism $Y_i \twoheadrightarrow X_i$ with $Y_i \in \mathcal{K}$. Then

$$I(L^kG)(\text{``}\bigoplus_i\text{''}\, Y_i) \simeq \text{``}\bigoplus_i\text{''}\, L^kG(Y_i) \simeq 0$$

for all $k < 0$, hence $\text{``}\bigoplus\limits_i\text{''}\, Y_i \in \widetilde{\mathcal{K}}$.

(ii)–(iii) Consider an exact sequence $0 \to A \to B \to C \to 0$ in $\mathrm{Ind}(\mathcal{C})$. We may assume that this sequence is obtained by applying the functor "$\varinjlim$" to (15.3.2). Consider the long exact sequences for $i \in I$

$$\cdots \to L^{-1}G(\gamma(i)) \to L^0G(\alpha(i)) \to L^0G(\beta(i)) \to L^0G(\gamma(i)) \to 0\,.$$

Applying the functor "$\varinjlim$" , we obtain the long exact sequence

(15.3.10)
$$\cdots \to I(L^{-1}G)(C) \to I(L^0G)(A) \to I(L^0G)(B) \to I(L^0G)(C) \to 0\,.$$

Assuming $B, C \in \widetilde{\mathcal{J}}$, we deduce $A \in \widetilde{\mathcal{J}}$. Assuming $C \in \widetilde{\mathcal{J}}$, we deduce the exact sequence $0 \to IG(A) \to IG(B) \to IG(C) \to 0$.
(b)–(c) go as in Proposition 15.3.2.
(d) The isomorphism in (d) clearly holds for $k = 0$. We shall prove that it holds for $k = 1$, then for all k.
Consider an exact sequence $0 \to A \to B \to C \to 0$ with $B \in \widetilde{\mathcal{K}}$. Then $I(L^kG)(B) \simeq 0$ for all $k < 0$ by definition and $L^k(IG)(B) \simeq 0$ for all $k < 0$ since $\widetilde{\mathcal{K}}$ is IG-projective. There exists an exact sequence

(15.3.11)
$$\cdots \to L^{-1}(IG)(C) \to L^0(IG)(A) \to L^0(IG)(B) \to L^0(IG)(C) \to 0\,.$$

By comparing the exact sequences (15.3.11) and (15.3.10), we get the result for $k = 1$. Then the proof goes as in Proposition 15.3.2. q.e.d.

Theorem 15.3.8. *Assume* (15.3.7).

(i) *The bifunctor* $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}$ *admits a right derived functor*

$$\mathrm{R}^+\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})} \colon \mathrm{D}^+(\mathrm{Ind}(\mathcal{C})) \times \mathrm{D}^-(\mathrm{Ind}(\mathcal{C}))^{\mathrm{op}} \to \mathrm{D}^+(\mathrm{Mod}(\mathbb{Z}))\,.$$

(ii) *Moreover, for* $X \in \mathrm{D}^-(\mathrm{Ind}(\mathcal{C}))$ *and* $Y \in \mathrm{D}^+(\mathrm{Ind}(\mathcal{C}))$,

$$H^0\mathrm{R}^+\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(X, Y) \simeq \mathrm{Hom}_{\mathrm{D}(\mathrm{Ind}(\mathcal{C}))}(X, Y)\,.$$

(iii) $\mathrm{D^b}(\mathcal{C})$ *and* $\mathrm{D^b}(\mathrm{Ind}(\mathcal{C}))$ *are* $\mathcal{U}$*-categories.*

Proof. Let $\mathcal{P}$ denote the full additive subcategory of $\mathrm{Ind}(\mathcal{C})$ defined by:

$$\mathcal{P} = \{A \in \mathrm{Ind}(\mathcal{C}); A \simeq \text{``}\bigoplus_{i\in I}\text{''}\, X_i,\ \ I \text{ small},\ X_i \in \mathcal{C}\}\,.$$

Clearly, the category $\mathcal{P}$ is generating in $\mathrm{Ind}(\mathcal{C})$.

We shall apply Proposition 13.4.4 and Theorem 13.4.1 to the subcategory $\mathcal{I}_{qinj} \times \mathcal{P}^{\mathrm{op}}$ of $\mathrm{Ind}(\mathcal{C}) \times \mathrm{Ind}(\mathcal{C})^{\mathrm{op}}$.
(A) For $B \in \mathcal{P}$, the functor $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(B, \,\bullet\,)$ is exact on $\mathcal{I}_{qinj}$. Indeed, we have

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\text{“}\bigoplus_i\text{”}\, X_i, A) \simeq \prod_i \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(X_i, A)\;,$$

the functor $\prod_i$ is exact on $\mathrm{Mod}(\mathbb{Z})$ and the functor $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(X_i, \,\bullet\,)$ is exact on the category $\mathcal{I}_{qinj}$.
(B) Let A be quasi-injective. In order to see that $\mathcal{P}^{\mathrm{op}}$ is injective with respect to the functor $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\,\bullet\,, A)$, we shall apply Theorem 13.3.7.
Consider an epimorphism $B \twoheadrightarrow P''$ with $P'' \in \mathcal{P}$. We shall show that there exists an exact sequence $0 \to P' \to P \to P'' \to 0$ in $\mathcal{P}$ such that $P \to P''$ factorizes through $B \to P''$. Let $P'' = \text{“}\bigoplus_i\text{”}\, X_i''$. By Proposition 8.6.9, there exist an epimorphism $X_i \twoheadrightarrow X_i''$ and a morphism $X_i \to B$ making the diagram below commutative

$$\begin{tikzcd} X_i \arrow[r, two heads] \arrow[d] & X_i'' \arrow[d] \\ B \arrow[r, two heads] & \text{“}\bigoplus_i\text{”}\, X_i''. \end{tikzcd}$$

Define X_i' as the kernel of $X_i \to X_i''$, and define $P' = \text{“}\bigoplus_i\text{”}\, X_i'$, $P = \text{“}\bigoplus_i\text{”}\, X_i$. Then the sequence $0 \to P' \to P \to P'' \to 0$ is exact.

Let us apply the functor $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\,\bullet\,, A)$ to this sequence. The formula

$$\mathrm{Hom}\,(\text{“}\bigoplus_i\text{”}\, X_i, A) \simeq \prod_i \mathrm{Hom}\,(X_i, A)$$

and the fact that the functor $\prod$ is exact on $\mathrm{Mod}(\mathbb{Z})$ show that the sequence $0 \to \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(P'', A) \to \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(P, A) \to \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(P', A) \to 0$ remains exact.

Hence we have proved (i). The other statements easily follow from (i). q.e.d.

Corollary 15.3.9. *Assume* (15.3.7). *For any $X \in \mathcal{C}$ and $A \in \mathrm{Ind}(\mathcal{C})$, there is an isomorphism*

$$\varinjlim_{(Y \to A) \in \mathcal{C}_A} \mathrm{Ext}^k_{\mathcal{C}}(X, Y) \xrightarrow{\sim} \mathrm{Ext}^k_{\mathrm{Ind}(\mathcal{C})}(X, A)\,.$$

Proof. For $X \in \mathcal{C}$, let $F\colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Mod}(\mathbb{Z})$ be the functor $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(X, \,\bullet\,)$. Then $R^+F\colon \mathrm{D}^+(\mathrm{Ind}(\mathcal{C})) \to \mathrm{D}^+(\mathbb{Z})$ exists and $\mathrm{Ext}^k_{\mathrm{Ind}(\mathcal{C})}(X, \,\bullet\,) \simeq R^kF$. On the other hand, $\mathcal{I}_{qinj}$ being F-injective and closed by small filtrant inductive limits, Proposition 15.3.3 implies the isomorphism $\varinjlim_{(Y \to A) \in \mathcal{C}_A} R^kF(Y) \xrightarrow{\sim} R^kF(A)$. Hence, we obtain

$$\begin{aligned}\mathrm{Ext}^k_{\mathrm{Ind}(\mathcal{C})}(X, A) &\simeq \varinjlim_{(Y \to A) \in \mathcal{C}_A} R^k F(Y) \\ &\simeq \varinjlim_{(Y \to A) \in \mathcal{C}_A} \mathrm{Ext}^k_{\mathrm{Ind}(\mathcal{C})}(X, Y) .\end{aligned}$$

Finally, Theorem 15.3.1 (i) implies $\mathrm{Ext}^k_{\mathrm{Ind}(\mathcal{C})}(X, Y) \simeq \mathrm{Ext}^k_{\mathcal{C}}(X, Y)$. q.e.d.

15.4 Indization and Derivation

In this section we shall study some links between the derived category $\mathrm{D^b}(\mathrm{Ind}(\mathcal{C}))$ and the category $\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))$ associated with an abelian category $\mathcal{C}$. Notice that we do not know whether $\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))$ is a triangulated category.

Throughout this section we assume that $\mathcal{C}$ satisfies condition (15.3.7). Then $\mathrm{D^b}(\mathrm{Ind}(\mathcal{C}))$ and $\mathrm{D^b}(\mathcal{C})$ are $\mathcal{U}$-categories by Theorem 15.3.8.

The shift automorphism $[n]\colon \mathrm{D^b}(\mathcal{C}) \to \mathrm{D^b}(\mathcal{C})$ gives an automorphism of $\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))$ that we denote by the same symbol $[n]$.

Let $\tau^{\le a}$ and $\tau^{\ge b}$ denote the truncation functors from $\mathrm{D^b}(\mathcal{C})$ to itself. They define functors $I\tau^{\le a}$ and $I\tau^{\ge b}$ from $\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))$ to itself. If $A \simeq$ "$\varinjlim_i$" X_i with $X_i \in \mathrm{D^b}(\mathcal{C})$, then $I\tau^{\le a}A \simeq$ "$\varinjlim$" $\tau^{\le a}X_i$ and similarly for $\tau^{\ge b}$.

Let $Y \in \mathrm{D^b}(\mathcal{C})$ and let $A \simeq$ "$\varinjlim_i$" $X_i \in \mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))$. The distinguished triangles in $\mathrm{D^b}(\mathcal{C})$

$$\tau^{<a}X_i \to X_i \to \tau^{\ge a}X_i \to (\tau^{<a}X_i)[1]$$

give rise to morphisms

$$\tau^{<a}A \to A \to \tau^{\ge a}A \to (\tau^{<a}A)[1]$$

and to a long exact sequence

$$\begin{aligned}(15.4.1)\quad \cdots \to \mathrm{Hom}_{\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))}(Y, I\tau^{<a}A) \to \mathrm{Hom}_{\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))}(Y, A) \to \\ \mathrm{Hom}_{\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))}(Y, I\tau^{\ge a}A) \to \mathrm{Hom}_{\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))}(Y, I\tau^{<a}A[1]) \to \cdots .\end{aligned}$$

There are similar long exact sequences corresponding to the other distinguished triangles in Proposition 13.1.15.

Lemma 15.4.1. *Let $\mathcal{A}$ be an additive category and let $n_0, n_1 \in \mathbb{Z}$ with $n_0 \le n_1$. There is a natural equivalence*

$$\mathrm{Ind}(\mathrm{C}^{[n_0,n_1]}(\mathcal{A})) \xrightarrow{\sim} \mathrm{C}^{[n_0,n_1]}(\mathrm{Ind}(\mathcal{A})) .$$

Proof. Let K be the category associated with the ordered set

$$\{n \in \mathbb{Z}\,;\, n_0 \le n \le n_1\} .$$

The natural functors $\mathrm{C}^{[n_0,n_1]}(\mathcal{A}) \to \mathrm{Fct}(K,\mathcal{A})$ and $\mathrm{C}^{[n_0,n_1]}\big(\mathrm{Ind}(\mathcal{A})\big) \to \mathrm{Fct}(K,\mathrm{Ind}(\mathcal{A}))$ are fully faithful, and it follows from Proposition 6.4.1 that $\mathrm{Ind}(\mathrm{C}^{[n_0,n_1]}(\mathcal{A})) \to \mathrm{C}^{[n_0,n_1]}(\mathrm{Ind}(\mathcal{A}))$ is fully faithful.

Let us show that this last functor is essentially surjective. Theorem 6.4.3 implies that $\mathrm{Ind}(\mathrm{Fct}(K,\mathcal{A})) \to \mathrm{Fct}(K,\mathrm{Ind}(\mathcal{A}))$ is an equivalence of categories, and we obtain the quasi-commutative diagram:

$$\begin{tikzcd}
\mathrm{Ind}(\mathrm{C}^{[n_0,n_1]}(\mathcal{A})) \arrow[r, hook, "\text{f.f.}"] \arrow[d, hook, "\text{f.f.}"'] & \mathrm{Ind}(\mathrm{Fct}(K,\mathcal{A})) \arrow[d, "\sim"] \\
\mathrm{C}^{[n_0,n_1]}\big(\mathrm{Ind}(\mathcal{A})\big) \arrow[r, hook, "\text{f.f.}"] & \mathrm{Fct}(K,\mathrm{Ind}(\mathcal{A}))
\end{tikzcd}$$

where the arrows labeled by f.f. are fully faithful functors.

Let $A \in \mathrm{C}^{[n_0,n_1]}(\mathrm{Ind}(\mathcal{A}))$, and regard it as an object of $\mathrm{Ind}(\mathrm{Fct}(K,\mathcal{A}))$. By Exercise 6.11, it is enough to show that for $X \in \mathrm{Fct}(K,\mathcal{A})$, any morphism $u\colon X \to A$ factors through an object of $\mathrm{C}^{[n_0,n_1]}(\mathcal{A})$.

We shall construct by induction on i an object $Y = Y^{n_0} \to \cdots \to Y^i$ in $\mathrm{C}^{[n_0,i]}(\mathcal{A})$ and a diagram $\sigma^{\le i}X \xrightarrow{w} Y \xrightarrow{v} \sigma^{\le i}A$ whose composition is equal to $\sigma^{\le i}(u)$. Assume that we have constructed the diagram of solid arrows

$$\begin{tikzcd}
X^{n_0} \arrow[r] \arrow[d, "w^{n_0}"'] & \cdots \arrow[r] & X^{i-1} \arrow[r] \arrow[d, "w^{i-1}"'] & X^i \arrow[r] \arrow[d, "w^i"'] & X^{i+1} \arrow[d, dotted] \arrow[dd, bend left, "u^{i+1}"] \\
Y^{n_0} \arrow[r] \arrow[d, "v^{n_0}"'] & \cdots \arrow[r] & Y^{i-1} \arrow[r, "d_Y^{i-1}"'] \arrow[d, "v^{i-1}"'] & Y^i \arrow[r, dotted] \arrow[d, "v^i"'] & Z \arrow[d, dotted] \\
X^{n_0} \arrow[r] & \cdots \arrow[r] & A^{i-1} \arrow[r, "d_A^{i-1}"'] & A^i \arrow[r, "d_A^i"'] & A^{i+1}.
\end{tikzcd}$$

Since the category $\mathcal{A}_{A^{i+1}}$ is filtrant, the dotted arrows may be completed making the diagram commutative. Since the composition $d_A^i \circ d_A^{i-1}$ is zero, the composition $Y^{i-1} \to Y^i \to Z \to A^{i+1}$ is zero. This implies that the morphism $Z \to A$ factorizes through a morphism $Z \to Y^{i+1}$ such that the composition $Y^{i-1} \to Y^i \to Y^{i+1}$ is zero. q.e.d.

Recall that $Q\colon \mathrm{C}^{\mathrm{b}}(\mathcal{C}) \to \mathrm{D}^{\mathrm{b}}(\mathcal{C})$ denotes the localization functor. We shall denote by the same letter Q the localization functor $\mathrm{C}^{\mathrm{b}}(\mathrm{Ind}(\mathcal{C})) \to \mathrm{D}^{\mathrm{b}}(\mathrm{Ind}(\mathcal{C}))$.

Proposition 15.4.2. *Assume* (15.3.7)*. Consider integers $n_0, n_1 \in \mathbb{Z}$ with $n_0 \le n_1$ and a small and filtrant inductive system $\{X_i\}_{i\in I}$ in $\mathrm{C}^{[n_0,n_1]}(\mathcal{C})$. Let $Y \in \mathrm{D}^{\mathrm{b}}(\mathcal{C})$. Then:*

$$(15.4.2)\quad \varinjlim_i \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathcal{C})}(Y, Q(X_i)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathrm{Ind}(\mathcal{C}))}(Y, Q(\text{“}\varinjlim_i\text{”}\, X_i)) \,.$$

Proof. By dévissage, we may assume $Y \in \mathcal{C}$. By using the truncation functors we are reduced to prove the isomorphisms below for $Y, X_i \in \mathcal{C}$:

$$\mathrm{Ext}^k_{\mathrm{Ind}(\mathcal{C})}(Y, \text{“}\varinjlim_i\text{”}\, X_i) \simeq \varinjlim_i \mathrm{Ext}^k_{\mathcal{C}}(Y, X_i) \,. \tag{15.4.3}$$

These isomorphisms follow from Corollary 15.3.9. q.e.d.

We define the functor $J\colon \mathrm{D^b}(\mathrm{Ind}(\mathcal{C})) \to (\mathrm{D^b}(\mathcal{C}))^\wedge$ by setting for $A \in \mathrm{D^b}(\mathrm{Ind}(\mathcal{C}))$ and $Y \in \mathrm{D^b}(\mathcal{C})$

$$J(A)(Y) = \mathrm{Hom}_{\mathrm{D^b}(\mathrm{Ind}(\mathcal{C}))}(Y, A) \,. \tag{15.4.4}$$

Hence,

$$J(A) \simeq \text{“}\varinjlim_{(Y\to A)\in \mathrm{D^b}(\mathcal{C})_A}\text{”}\, Y \,.$$

Theorem 15.4.3. *Assume* (15.3.7).

(i) *Consider integers $n_0, n_1 \in \mathbb{Z}$ with $n_0 \le n_1$ and a small and filtrant inductive system $\{X_i\}_{i\in I}$ in $\mathrm{C}^{[n_0,n_1]}(\mathcal{C})$. Setting $A := Q(\text{“}\varinjlim_i\text{”}\, X_i) \in \mathrm{D^b}(\mathrm{Ind}(\mathcal{C}))$, we have $J(A) \simeq \text{“}\varinjlim_i\text{”}\, Q(X_i)$.*

(ii) *The functor J takes its values in $\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))$. In particular, for any $A \in \mathrm{D^b}(\mathrm{Ind}(\mathcal{C}))$, the category $\mathrm{D^b}(\mathcal{C})_A$ is cofinally small and filtrant.*

(iii) *For each $k \in \mathbb{Z}$, the diagram below commutes*

$$\begin{tikzcd}
\mathrm{D^b}(\mathrm{Ind}(\mathcal{C})) \arrow[rr, "J"'] \arrow[dr, "H^k"'] & & \mathrm{Ind}(\mathrm{D^b}(\mathcal{C})) \arrow[dl, "IH^k"] \\
& \mathrm{Ind}(\mathcal{C}). &
\end{tikzcd}$$

Proof. (i) By Proposition 15.4.2, we have for $Y \in \mathrm{D^b}(\mathcal{C})$

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))}(Y, J(A)) &= \mathrm{Hom}_{\mathrm{D^b}(\mathrm{Ind}(\mathcal{C}))}(Y, A) \\
&\simeq \varinjlim_i \mathrm{Hom}_{\mathrm{D^b}(\mathcal{C})}(Y, Q(X_i)) \\
&\simeq \mathrm{Hom}_{\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))}(Y, \text{“}\varinjlim_i\text{”}\, Q(X_i)) \,.
\end{aligned}$$

Therefore, $J(A) \simeq \text{“}\varinjlim_i\text{”}\, Q(X_i)$.

(ii) Let $A \in \mathrm{D^b}(\mathrm{Ind}(\mathcal{C}))$. There exists A' in $\mathrm{C}^{[n_0,n_1]}(\mathrm{Ind}(\mathcal{C}))$ with $A \simeq Q(A')$. Using Lemma 15.4.1 we may write $A' = \text{“}\varinjlim_{i\in I}\text{”}\, X_i$ with a small filtrant inductive system $\{X_i\}_{i\in I}$ in $\mathrm{C}^{[n_0,n_1]}(\mathcal{C})$. Then $J(A) \simeq \text{“}\varinjlim_i\text{”}\, Q(X_i)$ by (i). This object belongs to $\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))$.

(iii) The morphism $IH^k \circ J \to H^k$ is constructed by the sequence of morphisms

$$\begin{aligned} IH^k \circ J(A) &\simeq IH^k(\underset{(Y\to A)\in \mathrm{D^b}(\mathcal{C})_A}{\text{“}\varinjlim\text{”}} Y) \\ &\simeq \underset{(Y\to A)\in \mathrm{D^b}(\mathcal{C})_A}{\text{“}\varinjlim\text{”}} H^k(Y) \to H^k(A)\,. \end{aligned}$$

In order to see that it is an isomorphism, let us take an inductive system $\{X_i\}_{i\in I}$ as above. By (i) we have $J(A) \simeq \text{“}\varinjlim\limits_i\text{”}\, Q(X_i)$. Hence, $IH^k(J(A)) \simeq \text{“}\varinjlim\limits_i\text{”}\, H^k(Q(X_i)) \simeq \text{“}\varinjlim\limits_i\text{”}\, H^k(X_i)$. On the other hand, we have $H^k(A) \simeq H^k(Q(\text{“}\varinjlim\limits_i\text{”}\, X_i)) \simeq H^k(\text{“}\varinjlim\limits_i\text{”}\, X_i) \simeq \text{“}\varinjlim\limits_i\text{”}\, H^k(X_i)$. q.e.d.

Corollary 15.4.4. *Assume* (15.3.7)*. Then the functor* $J\colon \mathrm{D^b}(\mathrm{Ind}(\mathcal{C})) \to \mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))$ *is conservative.*

Remark 15.4.5. The functor $J\colon \mathrm{D^b}(\mathrm{Ind}(\mathcal{C})) \to \mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))$ is not faithful in general (see Exercise 15.2).

Lemma 15.4.6. *Assume* (15.3.7)*. Let* $A,\ B \in \mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))$ *and let* $\varphi\colon A \to B$ *be a morphism in* $\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))$ *such that* $IH^k(\varphi)\colon IH^k(A) \to IH^k(B)$ *is an isomorphism for all* $k \in \mathbb{Z}$*. Assume one of the conditions* (a) *and* (b) *below:*

(a) $A \simeq I\tau^{\geq a}A$ *and* $B \simeq I\tau^{\geq a}B$ *for some* $a \in \mathbb{Z}$*,*
(b) *the homological dimension of* $\mathcal{C}$ *is finite.*

Then φ *is an isomorphism in* $\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))$*.*

Proof. Let $Y \in \mathrm{D^b}(\mathcal{C})$. It is enough to prove that φ induces an isomorphism $\mathrm{Hom}_{\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))}(Y, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))}(Y, B)$.
(i) Assume (a). By the hypothesis, it is enough to prove the isomorphisms

$$\mathrm{Hom}_{\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))}(Y, I\tau^{\geq k}A) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))}(Y, I\tau^{\geq k}B) \tag{15.4.5}$$

for all $k \in \mathbb{Z}$, all $m \in \mathbb{Z}$ and all $Y \in \mathrm{D}^{\leq m}(\mathcal{C})$. Fixing m, let us prove this result by descending induction on k. If $k > m$, then both sides vanish. Assume that $\mathrm{Hom}_{\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))}(Y, I\tau^{\geq k}A) \to \mathrm{Hom}_{\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))}(Y, I\tau^{\geq k}B)$ is an isomorphism for all $k > n$ and all $Y \in \mathrm{D}^{\leq m}(\mathcal{C})$. Applying the long exact sequence (15.4.1) we find a commutative diagram (we shall write Hom instead of $\mathrm{Hom}_{\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))}$ for short)

$$\begin{tikzcd}
\mathrm{Hom}\,(Y[1], I\tau^{>n}A) \arrow[r] \arrow[d] & \mathrm{Hom}\,(Y, IH^n(A)[-n]) \arrow[r] \arrow[d] & \mathrm{Hom}\,(Y, I\tau^{\geq n}A) \arrow[d] \\
\mathrm{Hom}\,(Y[1], I\tau^{>n}B) \arrow[r] & \mathrm{Hom}\,(Y, IH^n(B)[-n]) \arrow[r] & \mathrm{Hom}\,(Y, I\tau^{\geq n}B)
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{Hom}\,(Y, I\tau^{>n}A) \arrow[r] \arrow[d] & \mathrm{Hom}\,(Y, IH^n(A)[1-n]) \arrow[d] \\
{} \arrow[r] & \mathrm{Hom}\,(Y, I\tau^{>n}B) \arrow[r] & \mathrm{Hom}\,(Y, IH^n(B)[1-n])\,.
\end{tikzcd}$$

Since $Y[1]$ and Y belong to $\mathrm{D}^{\leq m}(\mathcal{C})$, the first and the fourth vertical arrows are isomorphisms by the induction hypothesis. The second and the fifth vertical arrows are isomorphisms by the hypothesis. Hence, the third vertical arrow is an isomorphism, and the induction proceeds.
(ii) Assume (b) and let d denote the homological dimension of $\mathcal{C}$. If $Y \in \mathrm{D}^{\geq n_0}(\mathcal{C})$ then $\mathrm{Hom}_{\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))}(Y, \tau^{\leq n}A) \simeq 0$ for $n < n_0 - d$. We get the isomorphism $\mathrm{Hom}_{\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))}(Y, A) \simeq \mathrm{Hom}_{\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))}(Y, \tau^{\geq n}A)$, and similarly with A replaced with B. Then the result follows from the case (i). q.e.d.

Proposition 15.4.7. *Assume that $\mathcal{C}$ and $\mathcal{C}'$ satisfy* (15.3.7). *Consider a triangulated functor $\psi\colon \mathrm{D^b}(\mathrm{Ind}(\mathcal{C})) \to \mathrm{D^b}(\mathrm{Ind}(\mathcal{C}'))$ which satisfies:*

(15.4.6) $H^k\psi\colon \mathrm{Ind}(\mathcal{C})) \to \mathrm{Ind}(\mathcal{C}')$ *commutes with small filtrant inductive limits,*

(15.4.7) ψ *sends* $\mathrm{D}^{\geq 0}(\mathrm{Ind}(\mathcal{C})) \cap \mathrm{D^b}(\mathrm{Ind}(\mathcal{C}))$ *to* $\mathrm{D}^{\geq n}(\mathrm{Ind}(\mathcal{C}))$ *for some n .*

Then there exists a unique functor $\lambda\colon \mathrm{Ind}(\mathrm{D^b}(\mathcal{C})) \to \mathrm{Ind}(\mathrm{D^b}(\mathcal{C}'))$ which commutes with small filtrant “$\varinjlim$” *and such that the diagram below commutes:*

$$\begin{tikzcd}
\mathrm{D^b}(\mathrm{Ind}(\mathcal{C})) \arrow[r, "\psi"'] \arrow[d, "J"'] & \mathrm{D^b}(\mathrm{Ind}(\mathcal{C}')) \arrow[d, "J"'] \\
\mathrm{Ind}(\mathrm{D^b}(\mathcal{C})) \arrow[r, "\lambda"] & \mathrm{Ind}(\mathrm{D^b}(\mathcal{C}')).
\end{tikzcd}$$

Proof. First, notice that (15.4.6) implies that, for $n_0, n_1 \in \mathbb{Z}$ with $n_0 \leq n_1$ and for any small filtrant inductive system $\{X_i\}_{i\in I}$ in $\mathrm{C}^{[n_0,n_1]}(\mathcal{C})$, there is an isomorphism

$$“\varinjlim_i” H^k(\psi \circ Q(X_i)) \simeq H^k(\psi \circ Q(“\varinjlim_i” X_i)) .$$

Denote by $\varphi\colon \mathrm{D^b}(\mathcal{C}) \to \mathrm{Ind}(\mathrm{D^b}(\mathcal{C}'))$ the restriction of $J \circ \psi$ to $\mathrm{D^b}(\mathcal{C})$. The functor φ naturally extends to a functor $\lambda\colon \mathrm{Ind}(\mathrm{D^b}(\mathcal{C})) \to \mathrm{Ind}(\mathrm{D^b}(\mathcal{C}'))$ such that λ commutes with small filtrant inductive limits. We construct a natural morphism of functors

$$u\colon \lambda \circ J \to J \circ \psi$$

as follows. For $A \in \mathrm{D^b}(\mathrm{Ind}(\mathcal{C}))$,

$$\begin{aligned}
\lambda \circ J(A) &\simeq \lambda(\underset{(Y\to A)\in \mathrm{D^b}(\mathcal{C})_A}{“\varinjlim”} Y) \simeq \underset{(Y\to A)\in \mathrm{D^b}(\mathcal{C})_A}{“\varinjlim”} J \circ \psi(Y) \\
&\to J \circ \psi(A) .
\end{aligned}$$

Let us show that u is an isomorphism. Consider a small filtrant inductive system $\{X_i\}_{i\in I}$ in $\mathrm{C}^{[n_0,n_1]}(\mathcal{C})$ such that $A \simeq Q(“\varinjlim_i” X_i) \in \mathrm{D^b}(\mathrm{Ind}(\mathcal{C}))$. We have the chain of isomorphisms

$$\begin{aligned} IH^k(\lambda \circ J(A)) &\simeq \text{“}\varinjlim_i\text{”}\, H^k(\psi(Q(X_i))) \simeq H^k(\psi(A)) \\ &\simeq IH^k(J \circ \psi(A)) \,. \end{aligned}$$

Since $\lambda \circ J(A) \simeq I\tau^{\geq a}(\lambda \circ J(A))$ and $J \circ \psi(A) \simeq I\tau^{\geq a}(J \circ \psi(A))$ for $a \ll 0$, the result follows by Lemma 15.4.6. q.e.d.

Let $\mathcal{T}$ be a full triangulated subcategory of $\mathrm{D^b}(\mathcal{C})$. We identify $\mathrm{Ind}(\mathcal{T})$ with a full subcategory of $\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))$. For $A \in \mathrm{D^b}(\mathrm{Ind}(\mathcal{C}))$, we denote as usual by $\mathcal{T}_A$ the category of arrows $Y \to A$ with $Y \in \mathcal{T}$. We know by Proposition 10.1.18 that $\mathcal{T}_A$ is filtrant.

Notation 15.4.8. Let $\mathcal{T}$ be a full triangulated subcategory of $\mathrm{D^b}(\mathcal{C})$. We denote by $J^{-1}\mathrm{Ind}(\mathcal{T})$ the full subcategory of $\mathrm{D^b}(\mathrm{Ind}(\mathcal{C}))$ consisting of objects $A \in \mathrm{D^b}(\mathrm{Ind}(\mathcal{C}))$ such that $J(A)$ is isomorphic to an object of $\mathrm{Ind}(\mathcal{T})$.

Note that $A \in \mathrm{D^b}(\mathrm{Ind}(\mathcal{C}))$ belongs to $J^{-1}\mathrm{Ind}(\mathcal{T})$ if and only if any morphism $X \to A$ with $X \in \mathrm{D^b}(\mathcal{C})$ factors through an object of $\mathcal{T}$ by Exercise 6.11.

Proposition 15.4.9. *Assume* (15.3.7). *The category* $J^{-1}\mathrm{Ind}(\mathcal{T})$ *is a triangulated subcategory of* $\mathrm{D^b}(\mathrm{Ind}(\mathcal{C}))$.

Proof. Let $A \xrightarrow{f} B \xrightarrow{g} C \to A[1]$ be a d.t. in $\mathrm{D^b}(\mathrm{Ind}(\mathcal{C}))$ with B, C in $J^{-1}\mathrm{Ind}(\mathcal{T})$. Let us show that $A \in J^{-1}\mathrm{Ind}(\mathcal{T})$. Let $u \colon X \to A$ be a morphism with $X \in \mathrm{D^b}(\mathcal{C})$. Since $B \in J^{-1}\mathrm{Ind}(\mathcal{T})$, the composition $X \to A \to B$ factors through $Y \in \mathcal{T}$. We have thus a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & X[1] \arrow[d] \\ A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & A[1] \end{tikzcd}$$

in which the rows are d.t.'s and $X, Z \in \mathrm{D^b}(\mathcal{C})$, $Y \in \mathcal{T}$. Since $C \in J^{-1}\mathrm{Ind}(\mathcal{T})$, the arrow $Z \to C$ factors through $Z' \in \mathcal{T}$. Let us embed the composition $Y \to Z \to Z'$ into a d.t. $X' \to Y \to Z' \to X'[1]$ in $\mathcal{T}$. We thus have a commutative diagram whose rows are d.t.'s

$$\begin{tikzcd} X \arrow[r] \arrow[d, dotted, "v"] & Y \arrow[r] \arrow[d, "\mathrm{id}"] & Z \arrow[r] \arrow[d] & X[1] \arrow[d, dotted, "v[1]"] \\ X' \arrow[r] \arrow[d, dotted, "w"] & Y \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & X'[1] \arrow[d, dotted, "w[1]"] \\ A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & A[1] \,. \end{tikzcd}$$

Since $x := u - w \circ v$ satisfies $x \circ f = 0$, it factors through $C[-1] \to A$. Since $C[-1] \in J^{-1}\mathrm{Ind}(\mathcal{T})$, the morphism $X \to C[-1]$ factors through $X'' \in \mathcal{T}$. Thus $x \colon X \to A$ factors through X''. It follows that $u = x + w \circ v$ factors through $X' \oplus X'' \in \mathcal{T}$. q.e.d.

Exercises

Exercise 15.1. Let $\mathcal{C}$ be an abelian category and assume that $\mathrm{D^b}(\mathcal{C})$ is a $\mathcal{U}$-category. Let $A \in \mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))$ which satisfies the two conditions
(a) there exist $a, b \in \mathbb{Z}$ such that $I\tau^{\leq b}A \xrightarrow{\sim} A \xrightarrow{\sim} I\tau^{\geq a}A$,
(b) $IH^n(A) \in \mathcal{C}$ for any $n \in \mathbb{Z}$.
Prove that $A \in \mathrm{D^b}(\mathcal{C})$. (Hint: argue by induction on $b - a$ and use Exercise 10.14.)

Exercise 15.2. In this exercise, we shall give an example for Remark 15.4.5. Let k be a field and set $\mathcal{C} = \mathrm{Mod}(k)$. Let $J\colon \mathrm{D^b}(\mathrm{Ind}(\mathcal{C})) \to \mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))$ be the canonical functor.
(i) Prove that, for any $X, Y \in \mathrm{Ind}(\mathcal{C})$, $\mathrm{Hom}_{\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))}\big(J(X), J(Y[n])\big) \simeq 0$ for any $n \neq 0$. (Hint: any object of $\mathrm{D^b}(\mathcal{C})$ is a finite direct sum of $Z[m]$'s where $Z \in \mathcal{C}$.)
(ii) Let $Z \in \mathcal{C}$. Prove that the short exact sequence $0 \to \kappa_{\mathcal{C}}(Z) \to Z \to Z/(\kappa_{\mathcal{C}}(Z)) \to 0$ splits in $\mathrm{Ind}(\mathcal{C})$ if and only if Z is a finite-dimensional vector space.
(iii) Deduce that J is not faithful.