14

Unbounded Derived Categories

In this chapter we study the unbounded derived categories of Grothendieck categories, using the results of Chap. 9. We prove the existence of enough homotopically injective objects in order to define unbounded right derived functors, and we prove that these triangulated categories satisfy the hypotheses of the Brown representability theorem. We also study unbounded derived functors in particular for pairs of adjoint functors. We start this study in the framework of abelian categories with translation, then we apply it to the case of the categories of unbounded complexes in abelian categories.

Many of the results in this Chapter are not new and many authors have contributed to the results presented here, in particular, Spaltenstein [65] who first considered unbounded complexes and unbounded derived functors. Other contributions are due to [2, 6, 21, 41, 44], [53]. Note that many of the ideas encountered here come from Topology, and the names of Adams, Bousfield, Kan, Thomason, and certainly many others, should be mentioned.

14.1 Derived Categories of Abelian Categories with Translation

Let $(\mathcal{A}, T)$ be an abelian category with translation. Recall (Definition 13.1.1) that, denoting by $\mathcal{N}$ the triangulated subcategory of the homotopy category $\mathrm{K}_c(\mathcal{A})$ consisting of objects X qis to 0, the derived category $\mathrm{D}_c(\mathcal{A})$ of $(\mathcal{A}, T)$ is the localization $\mathrm{K}_c(\mathcal{A})/\mathcal{N}$. Recall that X is qis to 0 if and only if $T^{-1}X \xrightarrow{T^{-1}d_X} X \xrightarrow{d_X} TX$ is exact.

For $X \in \mathcal{A}_c$, the differential $d_X\colon X \to TX$ is a morphism in $\mathcal{A}_c$. Hence its cohomology $H(X)$ is regarded as an object of $\mathcal{A}_c$ and similarly for $\operatorname{Ker} d_X$ and $\operatorname{Im} d_X$. Note that their differentials vanish.

Proposition 14.1.1. *Assume that $\mathcal{A}$ admits direct sums indexed by a set I and that such direct sums are exact. Then $\mathcal{A}_c$, $\mathrm{K}_c(\mathcal{A})$ and $\mathrm{D}_c(\mathcal{A})$ admit such* *direct sums and the two functors $\mathcal{A}_c \to \mathrm{K}_c(\mathcal{A})$ and $\mathrm{K}_c(\mathcal{A}) \to \mathrm{D}_c(\mathcal{A})$ commute with such direct sums.*

Proof. The result concerning $\mathcal{A}_c$ and $\mathrm{K}_c(\mathcal{A})$ is obvious, and that concerning $\mathrm{D}_c(\mathcal{A})$ follows from Proposition 10.2.8. q.e.d.

For an object X of $\mathcal{A}$, we denote by $M(X)$ the mapping cone of $\mathrm{id}_{T^{-1}X}$, regarding $T^{-1}X$ as an object of $\mathcal{A}_c$ with the zero differential. Hence $M(X)$ is the object $X \oplus T^{-1}X$ of $\mathcal{A}_c$ with the differential

$$d_{M(X)} = \begin{pmatrix} 0 & 0 \\ \mathrm{id}_X & 0 \end{pmatrix} : X \oplus T^{-1}X \to TX \oplus X\ .$$

Therefore $M\colon \mathcal{A} \to \mathcal{A}_c$ is an exact functor. Moreover M is a left adjoint functor to the forgetful functor $\mathcal{A}_c \to \mathcal{A}$ as seen by the following lemma.

Lemma 14.1.2. *For $Z \in \mathcal{A}$ and $X \in \mathcal{A}_c$, we have the isomorphism*

$$\mathrm{Hom}_{\mathcal{A}_c}(M(Z), X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(Z, X). \tag{14.1.1}$$

Proof. The morphism $(u, v)\colon M(Z) \to X$ in $\mathcal{A}_c$ satisfies $d_X \circ (u, v) = T(u, v) \circ d_{M(X)}$ which reads as $d_X \circ u = Tv$ and $d_X \circ v = 0$. Hence it is determined by $u\colon Z \to X$. q.e.d.

Proposition 14.1.3. *Let $\mathcal{A}$ be a Grothendieck category. Then $\mathcal{A}_c$ is again a Grothendieck category.*

Proof. The category $\mathcal{A}_c$ is abelian and admits small inductive limits, and small filtrant inductive limits in $\mathcal{A}_c$ are clearly exact. Moreover, if G is a generator in $\mathcal{A}$, then $M(G)$ is a generator in $\mathcal{A}_c$ by Lemma 14.1.2. q.e.d.

Definition 14.1.4. (i) *An object $I \in \mathrm{K}_c(\mathcal{A})$ is* homotopically injective *if $\mathrm{Hom}_{\mathrm{K}_c(\mathcal{A})}(X, I) \simeq 0$ for all $X \in \mathrm{K}_c(\mathcal{A})$ that is qis to 0.*

(ii) *An object $P \in \mathrm{K}_c(\mathcal{A})$ is* homotopically projective *if P is homotopically injective in $\mathrm{K}_c(\mathcal{A}^{\mathrm{op}})$, that is, if $\mathrm{Hom}_{\mathrm{K}_c(\mathcal{A})}(P, X) \simeq 0$ for all $X \in \mathrm{K}_c(\mathcal{A})$ that is qis to 0.*

We shall denote by $\mathrm{K}_{c,\mathrm{hi}}(\mathcal{A})$ the full subcategory of $\mathrm{K}_c(\mathcal{A})$ consisting of homotopically injective objects and by $\iota\colon \mathrm{K}_{c,\mathrm{hi}}(\mathcal{A}) \to \mathrm{K}_c(\mathcal{A})$ the embedding functor. We denote by $\mathrm{K}_{c,\mathrm{hp}}(\mathcal{A})$ the full subcategory of $\mathrm{K}_c(\mathcal{A})$ consisting of homotopically projective objects.

Note that $\mathrm{K}_{c,\mathrm{hi}}(\mathcal{A})$ is obviously a full triangulated subcategory of $\mathrm{K}_c(\mathcal{A})$.

Lemma 14.1.5. *Let $(\mathcal{A}, T)$ be an abelian category with translation. If $I \in \mathrm{K}_c(\mathcal{A})$ is homotopically injective, then*

$$\mathrm{Hom}_{\mathrm{K}_c(\mathcal{A})}(X, I) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}_c(\mathcal{A})}(X, I)$$

for all $X \in \mathrm{K}_c(\mathcal{A})$.

Proof. Let $X \in \mathrm{K}_c(\mathcal{A})$ and let $X' \to X$ be a qis. Then for $I \in \mathrm{K}_{c,\mathrm{hi}}(\mathcal{A})$, the morphism $\mathrm{Hom}_{\mathrm{K}_c(\mathcal{A})}(X, I) \to \mathrm{Hom}_{\mathrm{K}_c(\mathcal{A})}(X', I)$ is an isomorphism, since there exists a d.t. $X' \to X \to N \to TX$ with N qis to 0 and $\mathrm{Hom}_{\mathrm{K}_c(\mathcal{A})}(N, I) \simeq \mathrm{Hom}_{\mathrm{K}_c(\mathcal{A})}(T^{-1}N, I) \simeq 0$. Therefore, for any $X \in \mathrm{K}_c(\mathcal{A})$ and $I \in \mathrm{K}_{c,\mathrm{hi}}(\mathcal{A})$, we have

$$\mathrm{Hom}_{\mathrm{D}_c(\mathcal{A})}(X, I) \simeq \varinjlim_{(X' \to X) \in \mathrm{Qis}} \mathrm{Hom}_{\mathrm{K}_c(\mathcal{A})}(X', I) \simeq \mathrm{Hom}_{\mathrm{K}_c(\mathcal{A})}(X, I) .$$

q.e.d.

Let us introduce the notation

$$(14.1.2) \quad \mathrm{QM} = \{f \in \mathrm{Mor}(\mathcal{A}_c)\, ;\, f \text{ is both a qis and a monomorphism}\} .$$

Recall (see Definition 9.5.1) that an object $I \in \mathcal{A}_c$ is QM-injective if, for any morphism $f : X \to Y$ in QM, $\mathrm{Hom}_{\mathcal{A}_c}(Y, I) \xrightarrow{\circ f} \mathrm{Hom}_{\mathcal{A}_c}(X, I)$ is surjective.

Proposition 14.1.6. *Let $I \in \mathcal{A}_c$. Then I is* QM*-injective if and only if it satisfies the following two conditions:*

(a) *I is homotopically injective,*
(b) *I is injective as an object of $\mathcal{A}$.*

Proof. (i) Assume that I is QM-injective.
(a) Recall that for a morphism $f : X \to Y$ in $\mathcal{A}_c$, we have constructed a natural monomorphism $\alpha(f) : Y \to \mathrm{Mc}(f)$ in $\mathcal{A}_c$. Let $X \in \mathcal{A}_c$ be qis to 0. Then $u := \alpha(\mathrm{id}_X)$ is a monomorphism and it is also a qis since both X and $\mathrm{Mc}(\mathrm{id}_X)$ are qis to 0. Hence $u \in \mathrm{QM}$, and it follows that any morphism $f : X \to I$ factorizes through $\mathrm{Mc}(\mathrm{id}_X)$. Since $\mathrm{Mc}(\mathrm{id}_X) \simeq 0$ in $\mathrm{K}_c(\mathcal{A})$, the morphism f vanishes in $\mathrm{K}_c(\mathcal{A})$.
(b) Consider a monomorphism $v : U \to V$ in $\mathcal{A}$. The morphism v defines the morphism $M(v) : M(U) \to M(V)$ in $\mathcal{A}_c$ and $M(v)$ belongs to QM. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}_c}(M(V), I) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}_c}(M(U), I) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{Hom}_{\mathcal{A}}(V, I) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(U, I). \end{array}$$

Since $M(v)$ belongs to QM and I is QM-injective, the horizontal arrow on the top is surjective. Hence, the horizontal arrow in the bottom is also surjective, and we conclude that I is injective.

(ii) Assume that I satisfies conditions (a) and (b). Let $f : X \to Y$ be a morphism in $\mathcal{A}_c$ belonging to QM and let $\varphi : X \to I$ be a morphism in $\mathcal{A}_c$. Since I is injective as an object of $\mathcal{A}$, there exists a morphism $\psi : Y \to I$ in $\mathcal{A}$ such that $\varphi = \psi \circ f$. Let $h : T^{-1}Y \to I$ be the morphism in $\mathcal{A}$ given by

$$\begin{aligned} h &= T^{-1}d_I \circ T^{-1}\psi - \psi \circ T^{-1}d_Y \\ &= T^{-1}d_I \circ T^{-1}\psi + \psi \circ d_{T^{-1}Y}. \end{aligned}$$

Then $h: T^{-1}Y \to I$ is a morphism in $\mathcal{A}_c$ and $h \circ T^{-1}f = 0$.

Let us consider an exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathcal{A}_c$. Then, Z is qis to 0. Since $h \circ T^{-1}f = 0$, there exists a morphism $\widetilde{h}: T^{-1}Z \to I$ in $\mathcal{A}_c$ such that $h = \widetilde{h} \circ T^{-1}g$. Since Z is exact and I is homotopically injective, $\widetilde{h}$ is homotopic to zero, i.e., there exists a morphism $\xi: Z \to I$ in $\mathcal{A}$ such that

$$\widetilde{h} = T^{-1}d_I \circ T^{-1}\xi + \xi \circ d_{T^{-1}Z} .$$

Then the morphism $\widetilde{\psi} = \psi - \xi \circ g$ gives a morphism $\widetilde{\psi}: Y \to I$ in $\mathcal{A}_c$ which satisfies $\widetilde{\psi} \circ f = \varphi$. q.e.d.

Now we shall prove the following theorem.

Theorem 14.1.7. *Let $(\mathcal{A}, T)$ be an abelian category with translation and assume that $\mathcal{A}$ is a Grothendieck category. Then, for any $X \in \mathcal{A}_c$, there exists $u: X \to I$ such that $u \in$ QM and I is QM-injective.*

Applying Proposition 14.1.6, we get:

Corollary 14.1.8. *Let $(\mathcal{A}, T)$ be an abelian category with translation and assume that $\mathcal{A}$ is a Grothendieck category. Then for any $X \in \mathcal{A}_c$, there exists a qis $X \to I$ such that I is homotopically injective.*

The proof of Theorem 14.1.7 decomposes into several steps.

Define a subcategory $\mathcal{A}_{c,0}$ of $\mathcal{A}_c$ as follows:

$$\mathrm{Ob}(\mathcal{A}_{c,0}) = \mathrm{Ob}(\mathcal{A}_c),\ \ \mathrm{Mor}(\mathcal{A}_{c,0}) = \mathrm{QM}\ .$$

We shall apply Theorems 9.5.4 and 9.5.5 to the categories $\mathcal{A}_c$ and $\mathcal{A}_{c,0}$ (denoted by $\mathcal{C}$ and $\mathcal{C}_0$ in these theorems).

Let us check that hypothesis (9.5.2) is satisfied. Hypothesis (9.5.2) (i) is satisfied since small filtrant inductive limits are exact and hence $H: \mathcal{A}_c \to \mathcal{A}$ commutes with such limits. Hypothesis (9.5.2) (ii) follows from

$$(14.1.3)\quad \begin{cases} \text{if } u: X \to Y \text{ belongs to QM and } X \to X' \text{ is a morphism in } \mathcal{A}_c, \\ \text{then } u': X' \to X' \oplus_X Y \text{ belongs to QM.} \end{cases}$$

Set $Y' = X' \oplus_X Y$. Then $u': X' \to Y'$ is a monomorphism. Note that u (resp. u') is a qis if and only if $\mathrm{Coker}(u)$ (resp. $\mathrm{Coker}(u')$) is qis to zero. Hence (14.1.3) follows from $\mathrm{Coker}(u) \simeq \mathrm{Coker}(u')$ (Lemma 8.3.11 (b)).

Since $\mathcal{A}_c$ is a Grothendieck category by Proposition 14.1.3, Theorem 9.6.1 implies that there exists an essentially small full subcategory $\mathcal{S}$ of $\mathcal{A}_c$ such that

(14.1.4)
- (i) $\mathcal{S}$ contains a generator of $\mathcal{A}_c$,
- (ii) $\mathcal{S}$ is closed by subobjects and quotients in $\mathcal{A}_c$,
- (iii) for any solid diagram $\begin{array}{ccc} Y' & \overset{g}{\dashrightarrow\!\!\!\!\rightarrow} & Y \\ \downarrow{\scriptstyle\vdots} & & \downarrow \\ X' & \overset{f}{\twoheadrightarrow} & X \end{array}$ in which $f: X' \to X$ is an epimorphism in $\mathcal{A}_c$ and $Y \in \mathcal{S}$, the dotted arrow may be completed to a commutative diagram with $Y' \in \mathcal{S}$ and g an epimorphism,
- (iv) $\mathcal{S}$ is closed by countable direct sums.

In particular, $\mathcal{S}$ is a fully abelian subcategory of $\mathcal{A}_c$ closed by countable inductive limits.

Define the set

$$\mathcal{F}' = \{u: X \to Y; u \in \mathrm{QM},\ \ X, Y \in \mathcal{S}\}\ ,$$

and take $\mathcal{F} \subset \mathcal{F}'$ by collecting a representative of each isomorphism class in $\mathcal{F}'$ (i.e., for the relation of being isomorphic in $\mathrm{Mor}(\mathcal{A}_c)$). Since $\mathcal{S}$ is essentially small, $\mathcal{F}$ is a small subset of $\mathcal{F}'$ such that any $u \in \mathcal{F}'$ is isomorphic to an element of $\mathcal{F}$.

By Theorem 9.6.1, there exists an infinite cardinal π such that if $u: X \to Y$ belongs to $\mathcal{F}$, then $X \in (\mathcal{A}_c)_\pi$. Applying Theorem 9.5.4, we find that for any $X \in \mathcal{A}_c$ there exists a morphism $u: X \to I$ such that $u \in \mathrm{QM}$ and I is $\mathcal{F}$-injective. In order to prove that I is QM-injective, we shall apply Theorem 9.5.5.

For $X \in \mathcal{A}_{c,0}$, an object of $(\mathcal{A}_{c,0})_X$ is given by a monomorphism $Y \rightarrowtail X$. Therefore $(\mathcal{A}_{c,0})_X$ is essentially small by Corollary 8.3.26, and hence hypothesis (9.5.6) is satisfied.

Let us check (9.5.7). We have an exact sequence $0 \to X' \to X \oplus Y' \xrightarrow{w} Y$. Then $\mathrm{Im}\, w \simeq X \oplus_{X'} Y'$ and $h: \mathrm{Im}\, w \to Y$ is a monomorphism. Hence (9.5.7) follows from (14.1.3).

Hypothesis (9.5.8) will be checked in Lemmas 14.1.9–14.1.11 below.

Lemma 14.1.9. *Let $X \in \mathcal{A}_c$ and let $j: V \rightarrowtail X$ be a monomorphism with $V \in \mathcal{S}$. Then there exist $V' \in \mathcal{S}$ and a monomorphism $V' \rightarrowtail X$ such that j decomposes as $V \rightarrowtail V' \rightarrowtail X$ and* $\mathrm{Ker}\big(H(V) \to H(X)\big) \to H(V')$ *vanishes.*

Proof. Since $V \cap \mathrm{Im}(T^{-1}d_X)$ belongs to $\mathcal{S}$, there exists $W \subset T^{-1}X$ such that $W \in \mathcal{S}$ and $(T^{-1}d_X)(W) = V \cap \mathrm{Im}(T^{-1}d_X)$. Set $V' = V + TW$. Then V' is a subobject of X, it belongs to $\mathcal{S}$ and satisfies the desired condition. q.e.d.

Lemma 14.1.10. *Let $X \in \mathcal{A}_c$ and let $j: V \rightarrowtail X$ be a monomorphism with $V \in \mathcal{S}$. Then there exist $V' \in \mathcal{S}$ and a monomorphism $V' \rightarrowtail X$ such that j decomposes as $V \rightarrowtail V' \rightarrowtail X$ and $H(V') \to H(X)$ is a monomorphism.*

Proof. Set $V_0 = V$. Using Lemma 14.1.9, we construct by induction $V_n \in \mathcal{S}$ such that $V_{n-1} \subset V_n \subset X$ and the morphism

$$\mathrm{Ker}\big(H(V_{n-1}) \to H(X)\big) \to \mathrm{Ker}\big(H(V_n) \to H(X)\big)$$

vanishes.

Take $V' = \varinjlim_n V_n \subset X$. Then $V' \in \mathcal{S}$ and

$$\mathrm{Ker}\big(H(V') \to H(X)\big) \simeq \varinjlim_n \mathrm{Ker}\big(H(V_n) \to H(X)\big) \simeq 0 .$$

q.e.d.

Lemma 14.1.11. *Let $f: X \to Y$ be in* QM. *If f satisfies* (9.5.5), *then f is an isomorphism.*

Proof. Let $Z = \mathrm{Coker}\, f$. We get an exact sequence in $\mathcal{A}_c$

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

and Z is qis to 0.

Since $\mathcal{S}$ contains a generator of $\mathcal{A}_c$, it is enough to show that $\mathrm{Hom}_{\mathcal{A}_c}(W, Z) \simeq 0$ for any $W \in \mathcal{S}$. Moreover, replacing W with its image in Z, it is enough to check that any $W \subset Z$ with $W \in \mathcal{S}$ vanishes.

For $W \subset Z$ with $W \in \mathcal{S}$, there exists $W' \in \mathcal{S}$ such that $W \subset W' \subset Z$ and $H(W') \simeq 0$ by Lemma 14.1.10. Let us take $V \subset Y$ with $V \in \mathcal{S}$ and $g(V) = W'$. Set $U = f^{-1}(V)$. Thus we obtain a Cartesian square $\begin{array}{ccc} U & \xrightarrow{s} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$. We have an exact sequence $0 \to U \xrightarrow{s} V \to W' \to 0$. Since W' is qis to zero, $U \xrightarrow{s} V$ belongs to $\mathcal{F}$. Since f satisfies (9.5.5), $V \to Y$ factors through $X \to Y$ and hence $W' = g(V) \simeq 0$. This shows that $W \simeq 0$. q.e.d.

Thus we have proved hypothesis (9.5.8), and the proof of Theorem 14.1.7 is now complete.

Corollary 14.1.12. *Let $(\mathcal{A}, T)$ be an abelian category with translation and assume that $\mathcal{A}$ is a Grothendieck category. Then:*

(i) *the localization functor $Q: \mathrm{K}_c(\mathcal{A}) \to \mathrm{D}_c(\mathcal{A})$ induces an equivalence $\mathrm{K}_{c,\mathrm{hi}}(\mathcal{A}) \xrightarrow{\sim} \mathrm{D}_c(\mathcal{A})$,*
(ii) *the category $\mathrm{D}_c(\mathcal{A})$ is a $\mathcal{U}$-category,*
(iii) *the functor $Q: \mathrm{K}_c(\mathcal{A}) \to \mathrm{D}_c(\mathcal{A})$ admits a right adjoint $R_q: \mathrm{D}_c(\mathcal{A}) \to \mathrm{K}_c(\mathcal{A})$, $Q \circ R_q \simeq \mathrm{id}$, and R_q is the composition of $\iota: \mathrm{K}_{c,\mathrm{hi}}(\mathcal{A}) \to \mathrm{K}_c(\mathcal{A})$ and a quasi-inverse of $Q \circ \iota$,*
(iv) *for any triangulated category $\mathcal{D}$, any triangulated functor $F: \mathrm{K}_c(\mathcal{A}) \to \mathcal{D}$ admits a right localization $RF: \mathrm{D}_c(\mathcal{A}) \to \mathcal{D}$, and $RF \simeq F \circ R_q$.*

Proof. (i) The functor $Q\colon \mathrm{K}_{c,\mathrm{hi}}(\mathcal{A}) \to \mathrm{D}_c(\mathcal{A})$ is fully faithful by Lemma 14.1.5 and essentially surjective by Corollary 14.1.8.
(ii)–(iii) follow immediately.
(iv) follows from Proposition 7.3.2. q.e.d.

14.2 The Brown Representability Theorem

We shall show that the hypotheses of the Brown representability theorem (Theorem 10.5.2) are satisfied for $\mathrm{D}_c(\mathcal{A})$ when $\mathcal{A}$ is a Grothendieck abelian category with translation. Note that $\mathrm{D}_c(\mathcal{A})$ admits small direct sums and the localization functor $Q\colon \mathrm{K}_c(\mathcal{A}) \to \mathrm{D}_c(\mathcal{A})$ commutes with such direct sums by Proposition 14.1.1.

Theorem 14.2.1. *Let $(\mathcal{A}, T)$ be an abelian category with translation and assume that $\mathcal{A}$ is a Grothendieck category. Then the triangulated category $\mathrm{D}_c(\mathcal{A})$ admits small direct sums and a system of t-generators.*

Applying Theorem 10.5.2, we obtain

Corollary 14.2.2. *Let $(\mathcal{A}, T)$ be an abelian category with translation and assume that $\mathcal{A}$ is a Grothendieck category. Let $G\colon (\mathrm{D}_c(\mathcal{A}))^{\mathrm{op}} \to \mathrm{Mod}(\mathbb{Z})$ be a cohomological functor which commutes with small products (i.e., $G(\bigoplus_i X_i) \simeq \prod_i G(X_i)$ for any small family $\{X_i\}_i$ in $\mathrm{D}_c(\mathcal{A})$). Then G is representable.*

Applying Corollary 10.5.3, we obtain:

Corollary 14.2.3. *Let $(\mathcal{A}, T)$ be an abelian category with translation and assume that $\mathcal{A}$ is a Grothendieck category. Let $\mathcal{D}$ be a triangulated category and let $F\colon \mathrm{D}_c(\mathcal{A}) \to \mathcal{D}$ be a triangulated functor. Assume that F commutes with small direct sums. Then F admits a right adjoint.*

We shall prove a slightly more general statement than Theorem 14.2.1. Let $\mathcal{I}$ be a full subcategory of $\mathcal{A}$ closed by subobjects, quotients and extensions in $\mathcal{A}$, and also by small direct sums. Similarly to Definition 13.2.7, let us denote by $\mathrm{D}_{c,\mathcal{I}}(\mathcal{A})$ the full subcategory of $\mathrm{D}_c(\mathcal{A})$ consisting of objects $X \in \mathrm{D}_c(\mathcal{A})$ such that $H(X) \in \mathcal{I}$. Then $\mathrm{D}_{c,\mathcal{I}}(\mathcal{A})$ is a full triangulated subcategory of $\mathrm{D}_c(\mathcal{A})$ closed by small direct sums.

Proposition 14.2.4. *The triangulated category $\mathrm{D}_{c,\mathcal{I}}(\mathcal{A})$ admits a system of t-generators.*

In proving Proposition 14.2.4, we need preliminary lemmas. Recall that there exists an essentially small fully abelian subcategory $\mathcal{S}$ of $\mathcal{A}_c$ satisfying (14.1.4).

Lemma 14.2.5. *Assume that $X \in \mathcal{A}_c$ satisfies $H(X) \in \mathcal{S}$. Then there exists a morphism $j\colon Y \to X$ with $Y \in \mathcal{S}$ and $j \in$* QM.

Proof. There exists $S \in \mathcal{S}$ such that $S \subset \operatorname{Ker} d_X$ and that the composition $S \to \operatorname{Ker} d_X \to H(X)$ is an epimorphism. Since the differential of S vanishes, $H(S)$ is isomorphic to S and $H(S) \to H(X)$ is an epimorphism. By Lemma 14.1.10, there exists $Y \in \mathcal{S}$ such that $S \subset Y \subset X$ and $H(Y) \to H(X)$ is a monomorphism. Hence, $H(Y) \to H(X)$ is an isomorphism. q.e.d.

Lemma 14.2.6. *Let $X \in \mathcal{A}_c$ with $H(X) \in \mathcal{I}$. If $\operatorname{Hom}_{\mathrm{D}_{c,\mathcal{I}}(\mathcal{A})}(Y, X) \simeq 0$ for all $Y \in \mathcal{S}$ such that $H(Y) \in \mathcal{I}$, then X is qis to zero.*

Proof. It is enough to show that $\operatorname{Hom}_{\mathcal{A}_c}(S, H(X)) \simeq 0$ for all $S \in \mathcal{S}$. Let us show that any $u \colon S \to H(X)$ vanishes. Replacing S with the image of u, we may assume that u is a monomorphism. Since $\operatorname{Ker} d_X \to H(X)$ is an epimorphism, there exists $S' \in \mathcal{S}$ such that $S' \subset \operatorname{Ker} d_X$ and that the image of the composition $S' \to \operatorname{Ker} d_X \to H(X)$ is equal to S. By Lemma 14.1.10, there exists $V \in \mathcal{S}$ such that $S' \subset V \subset X$ and $H(V) \to H(X)$ is a monomorphism. Hence $H(V)$ belongs to $\mathcal{I}$. Since $\operatorname{Hom}_{\mathrm{D}_{c,\mathcal{I}}(\mathcal{A})}(V, X) \simeq 0$ by the assumption, the morphism $V \to X$ vanishes in $\mathrm{D}_c(\mathcal{A})$. Taking the cohomology, we find that $H(V) \to H(X)$ vanishes. Since the differentials of S' and S vanish, we have $H(S') \simeq S'$ and $H(S) \simeq S$. Since the composition $H(S') \to H(V) \to H(X)$ vanishes, the composition $S' \twoheadrightarrow S \xrightarrow{u} H(X)$ vanishes. Hence $u = 0$. q.e.d.

Proof of Proposition 14.2.4. Denote by $\mathcal{T}$ the subset of $\mathrm{D}_{c,\mathcal{I}}(\mathcal{A})$ consisting of the image of objects $Y \in \mathcal{S}$ such that $H(Y) \in \mathcal{I}$. We shall show that $\mathcal{T}$ is a system of t-generators in $\mathrm{D}_{c,\mathcal{I}}(\mathcal{A})$.
(i) $\mathcal{T}$ is a system of generators. Indeed, $\operatorname{Hom}_{\mathrm{D}_{c,\mathcal{I}}(\mathcal{A})}(Y, X) \simeq 0$ for all $Y \in \mathcal{T}$ implies that $X \simeq 0$ by Lemma 14.2.6.
(ii) We shall check condition (iii)′ in Remark 10.5.4. Consider a small set I and a morphism $C \to \bigoplus_{i \in I} X_i$ in $\mathrm{D}_{c,\mathcal{I}}(\mathcal{A})$, with $C \in \mathcal{T}$. This morphism is represented by morphisms in $\mathcal{A}_c$:

$$C \xleftarrow{u} Y \to \bigoplus_{i \in I} X_i$$

where $Y \in \mathcal{A}_c$ and u is a qis. By Lemma 14.2.5, there exists a qis $C' \to Y$ with $C' \in \mathcal{S}$. Replacing C with C', we may assume from the beginning that we have a morphism $C \to \bigoplus_{i \in I} X_i$ in $\mathcal{A}_c$. Set $Y_i = \operatorname{Im}(C \to X_i)$. Then Y_i belongs to $\mathcal{S}$. By Lemma 14.1.10, there exists $C_i \in \mathcal{S}$ such that $Y_i \subset C_i \subset X_i$ and that $H(C_i) \to H(X_i)$ is a monomorphism. Then $H(C_i)$ belongs to $\mathcal{I}$. By the result of Exercise 8.35, the morphism $C \to \bigoplus_i X_i$ factorizes through $\bigoplus_i Y_i \to \bigoplus_i X_i$, and hence through $\bigoplus_i C_i \to \bigoplus_i X_i$. q.e.d.

14.3 Unbounded Derived Category

From now on and until the end of this chapter, $\mathcal{C}$, $\mathcal{C}'$, etc. are abelian categories.

We shall apply the results in the preceding Sects. 14.1 and 14.2 to the abelian category with translation $\mathcal{A} := \mathrm{Gr}(\mathcal{C})$. Then we have $\mathcal{A}_c \simeq \mathrm{C}(\mathcal{C})$, $\mathrm{K}_c(\mathcal{A}) \simeq \mathrm{K}(\mathcal{C})$ and $\mathrm{D}_c(\mathcal{A}) \simeq \mathrm{D}(\mathcal{C})$. Assume that $\mathcal{C}$ admits direct sums indexed by a set I and that such direct sums are exact. Then, clearly, $\mathrm{Gr}(\mathcal{C})$ has the same properties. It then follows from Proposition 14.1.1 that $\mathrm{C}(\mathcal{C})$, $\mathrm{K}(\mathcal{C})$ and $\mathrm{D}(\mathcal{C})$ also admit such direct sums and the two functors $\mathrm{C}(\mathcal{C}) \to \mathrm{K}(\mathcal{C})$ and $\mathrm{K}(\mathcal{C}) \to \mathrm{D}(\mathcal{C})$ commute with such direct sums.

We shall write $\mathrm{K}_{\mathrm{hi}}(\mathcal{C})$ for $\mathrm{K}_{c,\mathrm{hi}}(\mathcal{A})$. Hence $\mathrm{K}_{\mathrm{hi}}(\mathcal{C})$ is the full subcategory of $\mathrm{K}(\mathcal{C})$ consisting of homotopically injective objects. Let us denote by $\iota\colon \mathrm{K}_{\mathrm{hi}}(\mathcal{C}) \to \mathrm{K}(\mathcal{C})$ the embedding functor. Similarly we denote by $\mathrm{K}_{\mathrm{hp}}(\mathcal{C})$ the full subcategory of $\mathrm{K}(\mathcal{C})$ consisting of homotopically projective objects. Recall that $I \in \mathrm{K}(\mathcal{C})$ is homotopically injective if and only if $\mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(X, I) \simeq 0$ for all $X \in \mathrm{K}(\mathcal{C})$ that is qis to 0.

Note that an object $I \in \mathrm{K}^+(\mathcal{C})$ whose components are all injective is homotopically injective in view of Lemma 13.2.4.

Let $\mathcal{C}$ be a Grothendieck abelian category. Then $\mathcal{A} := \mathrm{Gr}(\mathcal{C})$ is also a Grothendieck category. Applying Corollary 14.1.8 and Theorem 14.2.1, we get the following theorem.

Theorem 14.3.1. *Let $\mathcal{C}$ be a Grothendieck category.*

(i) *if $I \in \mathrm{K}(\mathcal{C})$ is homotopically injective, then we have an isomorphism*

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(X, I) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(X, I) \quad \textit{for any } X \in \mathrm{K}(\mathcal{C}) ,$$

(ii) *for any $X \in \mathrm{C}(\mathcal{C})$, there exists a qis $X \to I$ such that I is homotopically injective,*

(iii) *the localization functor $Q\colon \mathrm{K}(\mathcal{C}) \to \mathrm{D}(\mathcal{C})$ induces an equivalence*

$$\mathrm{K}_{\mathrm{hi}}(\mathcal{C}) \xrightarrow{\sim} \mathrm{D}(\mathcal{C}) ,$$

(iv) *the category $\mathrm{D}(\mathcal{C})$ is a $\mathcal{U}$-category,*

(v) *the functor $Q\colon \mathrm{K}(\mathcal{C}) \to \mathrm{D}(\mathcal{C})$ admits a right adjoint $R_q\colon \mathrm{D}(\mathcal{C}) \to \mathrm{K}(\mathcal{C})$, $Q \circ R_q \simeq \mathrm{id}$, and R_q is the composition of $\iota\colon \mathrm{K}_{\mathrm{hi}}(\mathcal{C}) \to \mathrm{K}(\mathcal{C})$ and a quasi-inverse of $Q \circ \iota$,*

(vi) *for any triangulated category $\mathcal{D}$, any triangulated functor $F\colon \mathrm{K}(\mathcal{C}) \to \mathcal{D}$ admits a right localization $RF\colon \mathrm{D}(\mathcal{C}) \to \mathcal{D}$ and $RF \simeq F \circ R_q$,*

(vii) *the triangulated category $\mathrm{D}(\mathcal{C})$ admits small direct sums and a system of t-generators,*

(viii) *any cohomological functor $G\colon (\mathrm{D}(\mathcal{C}))^{\mathrm{op}} \to \mathrm{Mod}(\mathbb{Z})$ is representable as soon as G commutes with small products (i.e., $G(\bigoplus_i X_i) \simeq \prod_i G(X_i)$ for any small family $\{X_i\}_i$ in $\mathrm{D}(\mathcal{C})$),*

(ix) *for any triangulated category $\mathcal{D}$, any triangulated functor $F\colon \mathrm{D}(\mathcal{C}) \to \mathcal{D}$ admits a right adjoint as soon as F commutes with small direct sums.*

Corollary 14.3.2. *Let k be a commutative ring and let $\mathcal{C}$ be a Grothendieck k-abelian category. Then $\big(\mathrm{K_{hi}}(\mathcal{C}), \mathrm{K}(\mathcal{C})^{\mathrm{op}}\big)$ is $\mathrm{Hom}_{\mathcal{C}}$-injective, and the functor $\mathrm{Hom}_{\mathcal{C}}$ admits a right derived functor $\mathrm{RHom}_{\mathcal{C}}\colon \mathrm{D}(\mathcal{C}) \times \mathrm{D}(\mathcal{C})^{\mathrm{op}} \to \mathrm{D}(k)$.*

Moreover, $H^0(\mathrm{RHom}_{\mathcal{C}}(X, Y)) \simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(X, Y)$ for $X, Y \in \mathrm{D}(\mathcal{C})$.

Proof. (i) The functor $\mathrm{Hom}_{\mathcal{C}}$ defines a functor $\mathrm{Hom}^{\bullet}_{\mathcal{C}}\colon \mathrm{K}(\mathcal{C}) \times \mathrm{K}(\mathcal{C})^{\mathrm{op}} \to \mathrm{K}(k)$ and $H^0(\mathrm{Hom}^{\bullet}_{\mathcal{C}}) \simeq \mathrm{Hom}_{\mathrm{K}(\mathcal{C})}$ by Proposition 11.7.3. Let $I \in \mathrm{K_{hi}}(\mathcal{C})$. If $X \in \mathrm{K}(\mathcal{C})$ is qis to 0, we find $\mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(X, I) \simeq 0$. Moreover, if $I \in \mathrm{K_{hi}}(\mathcal{C})$ is qis to 0, then I is isomorphic to 0. Therefore $\big(\mathrm{K_{hi}}(\mathcal{C}), \mathrm{K}(\mathcal{C})^{\mathrm{op}}\big)$ is $\mathrm{Hom}_{\mathcal{C}}$-injective, and we can apply Corollary 10.3.11 to the functor $\mathrm{Hom}^{\bullet}_{\mathcal{C}}\colon \mathrm{K}(\mathcal{C}) \times \mathrm{K}(\mathcal{C})^{\mathrm{op}} \to \mathrm{K}(k)$ and conclude.

(ii) The last assertion follows from Theorem 13.4.1. q.e.d.

Remark 14.3.3. Let $\mathcal{I}$ be a full subcategory of a Grothendieck category $\mathcal{C}$ and assume that $\mathcal{I}$ is closed by subobjects, quotients and extensions in $\mathcal{C}$, and also by small direct sums. Then by Proposition 14.2.4, the triangulated category $\mathrm{D}_{\mathcal{I}}(\mathcal{C})$ admits small direct sums and a system of t-generators. Hence $\mathrm{D}_{\mathcal{I}}(\mathcal{C}) \to \mathrm{D}(\mathcal{C})$ has a right adjoint.

We shall now give another criterion for the existence of derived functors in the unbounded case, when the functor has finite cohomological dimension.

Proposition 14.3.4. *Let $\mathcal{C}$ and $\mathcal{C}'$ be abelian categories and $F\colon \mathcal{C} \to \mathcal{C}'$ a left exact functor. Let $\mathcal{J}$ be an F-injective full additive subcategory of $\mathcal{C}$ satisfying the finiteness condition* (13.2.1)*. Then*

(i) $\mathrm{K}(\mathcal{J})$ *is* $\mathrm{K}(F)$*-injective. In particular, the functor F admits a right derived functor $RF\colon \mathrm{D}(\mathcal{C}) \to \mathrm{D}(\mathcal{C}')$ and*

$$RF(X) \simeq \mathrm{K}(F)(Y) \quad \text{for } (X \to Y) \in \mathrm{Qis} \text{ with } Y \in \mathrm{K}(\mathcal{J}) .$$

(ii) *Assume that $\mathcal{C}$ and $\mathcal{C}'$ admit direct sums indexed by a set I and such direct sums are exact.* (*Hence,* $\mathrm{D}(\mathcal{C})$ *and* $\mathrm{D}(\mathcal{C}')$ *admit such direct sums by* Proposition 10.2.8.) *If F commutes with direct sums indexed by I and $\mathcal{J}$ is closed by such direct sums, then $RF\colon \mathrm{D}(\mathcal{C}) \to \mathrm{D}(\mathcal{C}')$ commutes with such direct sums.*

Note that by Proposition 13.3.5, the conditions on the full additive subcategory $\mathcal{J}$ are equivalent to the conditions (a)–(c) below:

(14.3.1)
- (a) $\mathcal{J}$ is cogenerating in $\mathcal{C}$,
- (b) there exists a non-negative integer d such that if $Y^0 \to Y^1 \to \cdots \to Y^d \to 0$ is an exact sequence and $Y^j \in \mathcal{J}$ for $j < d$, then $Y^d \in \mathcal{J}$,
- (c) for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$ with $X', X \in \mathcal{J}$, the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact.

Proof. (i) By Proposition 13.2.6, it remains to prove that if $X \in \mathrm{C}(\mathcal{J})$ is exact, then $F(X)$ is exact. Consider the truncated complex

$$X^{i-d} \to \cdots \to X^{i-1} \to \operatorname{Coker} d_X^{i-1} \to 0 .$$

By the assumption, $\operatorname{Coker} d_X^{i-1}$ belongs to $\mathcal{J}$. Hence,

$$\tau^{\geq i} X := 0 \to \operatorname{Coker} d_X^{i-1} \to X^{i+1} \to \cdots$$

belongs to $\mathrm{K}^+(\mathcal{J})$ and is an exact complex. Therefore,

$$0 \to F(\operatorname{Coker} d_X^{i-1}) \to F(X^{i+1}) \to \cdots$$

is exact.
(ii) Let $\{X_i\}_{i \in I}$ be a family of objects in $\mathrm{C}(\mathcal{C})$. For each $i \in I$, choose a qis $X_i \to Y_i$ with $Y_i \in \mathrm{C}(\mathcal{J})$. Since direct sums indexed by I are exact in $\mathrm{C}(\mathcal{C})$, $\bigoplus_i X_i \to \bigoplus_i Y_i$ is a qis, and by the hypothesis, $\bigoplus_i Y_i$ belongs to $\mathrm{C}(\mathcal{J})$. Then $Q(\bigoplus_i X_i) \simeq \bigoplus_i Q(X_i)$ by Proposition 14.1.1 and

$$RF(\bigoplus_i X_i) \simeq F(\bigoplus_i Y_i) \simeq \bigoplus_i F(Y_i) \simeq \bigoplus_i RF(X_i)$$

in $\mathrm{D}(\mathcal{C}')$. q.e.d.

Corollary 14.3.5. *Let $\mathcal{C}$ and $\mathcal{C}'$ be abelian categories and let $F\colon \mathcal{C} \to \mathcal{C}'$ and $F'\colon \mathcal{C}' \to \mathcal{C}''$ be left exact functors of abelian categories. Let $\mathcal{J}$ and $\mathcal{J}'$ be full additive subcategories of $\mathcal{C}$ and $\mathcal{C}'$ respectively, and assume that $\mathcal{J}$ satisfies the conditions* (a)–(c) *of* (14.3.1) *and similarly for $\mathcal{J}'$ with respect to $\mathcal{C}', \mathcal{C}'', F'$. Assume moreover that $F(\mathcal{J}) \subset \mathcal{J}'$. Then $R(F' \circ F) \simeq RF' \circ RF$.*

Remark 14.3.6. Applying Proposition 14.3.4 and Corollary 14.3.5 with $\mathcal{C}$, $\mathcal{C}'$ and $\mathcal{C}''$ replaced with the opposite categories, we obtain similar results for left derived functors of right exact functors.

By Proposition 14.3.4 together with Theorem 14.3.1, we obtain the following corollary.

Corollary 14.3.7. *Let $\mathcal{C}$ be a Grothendieck category and let $F\colon \mathcal{C} \to \mathcal{C}'$ be a left exact functor of abelian categories which commutes with small direct sums. Let $\mathcal{J}$ be a full additive subcategory of $\mathcal{C}$ satisfying the conditions* (a)–(c) *of* (14.3.1). *Assume moreover that $\mathcal{J}$ is closed by small direct sums. Then $RF\colon \mathrm{D}(\mathcal{C}) \to \mathrm{D}(\mathcal{C}')$ admits a right adjoint.*

14.4 Left Derived Functors

In this section, we shall give a criterion for the existence of the left derived functor $LG\colon \mathrm{D}(\mathcal{C}) \to \mathrm{D}(\mathcal{C}')$ of an additive functor $G\colon \mathcal{C} \to \mathcal{C}'$ of abelian categories, assuming that G admits a right adjoint.

Let $\mathcal{C}$ be an abelian category. We shall assume

(14.4.1) $\mathcal{C}$ admits small direct sums and small direct sums are exact in $\mathcal{C}$.

Hence, by Proposition 14.1.1, $\mathrm{C}(\mathcal{C})$, $\mathrm{K}(\mathcal{C})$ and $\mathrm{D}(\mathcal{C})$ admit small direct sums.

Note that Grothendieck categories satisfy (14.4.1).

Lemma 14.4.1. *Assume* (14.4.1) *and let $\mathcal{P}$ be a full additive generating subcategory of $\mathcal{C}$. For any $X \in \mathrm{C}(\mathcal{C})$, there exists a quasi-isomorphism $X' \to X$ such that X' is the mapping cone of a morphism $Q \to P$, where P and Q are countable direct sums of objects of* $\mathrm{C}^-(\mathcal{P})$.

Proof. By Lemma 13.2.1 (with the arrows reversed), for each $n \in \mathbb{Z}$, there exists a quasi-isomorphism $p_n \colon P_n \to \tau^{\le n}X$ with $P_n \in \mathrm{C}^-(\mathcal{P})$. Then there exists a quasi-isomorphism

$$Q_n \to \mathrm{Mc}\big(P_n \bigoplus P_{n+1} \xrightarrow{(p_n, -p_{n+1})} \tau^{\le n+1}X\big)[-1]$$

with $Q_n \in \mathrm{C}^-(\mathcal{P})$. Hence, we have a commutative diagram in $\mathrm{K}^-(\mathcal{C})$:

$$\begin{array}{ccccc} Q_n & \longrightarrow & P_n & \longrightarrow & \tau^{\le n}X \\ & \searrow & & & \downarrow \\ & & P_{n+1} & \longrightarrow & \tau^{\le n+1}X\,. \end{array}$$

By the octahedral axiom of triangulated categories, there exists a d.t. in $\mathrm{K}(\mathcal{C})$

$$\mathrm{Mc}(P_{n+1} \to P_n \bigoplus P_{n+1}) \to \mathrm{Mc}(P_{n+1} \to \tau^{\le n+1}X) \\ \to \mathrm{Mc}(P_n \bigoplus P_{n+1} \to \tau^{\le n+1}X) \xrightarrow{+1}\ .$$

Since $P_{n+1} \to \tau^{\le n+1}X$ is a qis, the morphism

$$\mathrm{Mc}(P_n \bigoplus P_{n+1} \to \tau^{\le n+1}X\big)[-1] \to \mathrm{Mc}(P_{n+1} \to P_n \bigoplus P_{n+1})$$

is an isomorphism in $\mathrm{D}(\mathcal{C})$. Hence, $Q_n \to P_n$ is a qis.

Set $Q = \bigoplus_{n\in\mathbb{Z}} Q_n$ and $P = \bigoplus_{n\in\mathbb{Z}} P_n$. Then $Q_n \to P_n$ and $Q_n \to P_{n+1}$ define morphisms $u_0, u_1 \colon Q \to P$. Set

$$R := \mathrm{Mc}(Q \xrightarrow{u_0 - u_1} P)\ .$$

There is a d.t. $Q \to P \to R \to Q[1]$. Since the composition $Q \xrightarrow{u_0-u_1} P \to X$ is zero in $\mathrm{K}(\mathcal{C})$, $P \to X$ factors as $P \to R \to X$ in $\mathrm{K}(\mathcal{C})$. Let us show that $R \to X$ is a qis. For $i \in \mathbb{Z}$, set $\varphi_i := H^i(u_0 - u_1)$. We have an exact sequence

$$H^i(Q) \xrightarrow{\varphi_i} H^i(P) \to H^i(R) \to H^{i+1}(Q) \xrightarrow{\varphi_{i+1}} H^{i+1}(P)\ .$$

The hypothesis (14.4.1) implies

$$H^i(Q) \simeq \bigoplus_{n\in\mathbb{Z}} H^i(Q_n) \simeq \bigoplus_{i\le n} H^i(X)\,,$$
$$H^i(P) \simeq \bigoplus_{n\in\mathbb{Z}} H^i(P_n) \simeq \bigoplus_{i\le n} H^i(X)\,.$$

Hence, φ_{i+1} is a monomorphism by Exercise 8.37. Note that $\mathrm{id}-\sigma$ in Exercise 8.37 corresponds to φ_i and $X_0 \to X_1 \to \cdots$ corresponds to $H^i(X) \xrightarrow{\mathrm{id}} H^i(X) \to \cdots$. Therefore, $\operatorname{Coker}\varphi_i \simeq \varinjlim_n H^i(P_n) \to H^i(X)$ is an isomorphism. Hence, $H^i(R) \to H^i(X)$ is an isomorphism. q.e.d.

Lemma 14.4.2. *Assume* (14.4.1). *Let $\mathcal{P}$ be the full subcategory of $\mathcal{C}$ consisting of projective objects and let $\widetilde{\mathcal{P}}$ be the smallest full triangulated subcategory of* $\mathrm{K}(\mathcal{C})$ *closed by small direct sums and containing* $\mathrm{K}^-(\mathcal{P})$. *Then any object of $\widetilde{\mathcal{P}}$ is homotopically projective.*

Proof. The full subcategory $\mathrm{K}_{\mathrm{hp}}(\mathcal{C})$ of $\mathrm{K}(\mathcal{C})$ consisting of homotopically projective objects is closed by small direct sums and contains $\mathrm{K}^-(\mathcal{P})$. Hence, it contains $\widetilde{\mathcal{P}}$. q.e.d.

Theorem 14.4.3. *Let $\mathcal{C}$ be an abelian category satisfying* (14.4.1) *and admitting enough projectives. Then,*

(i) *for any $X \in \mathrm{K}(\mathcal{C})$, there exist $P \in \mathrm{K}_{\mathrm{hp}}(\mathcal{C})$ and a qis $P \to X$,*
(ii) *for any additive functor $G\colon \mathcal{C} \to \mathcal{C}'$, the left derived functor $LG\colon \mathrm{D}(\mathcal{C}) \to \mathrm{D}(\mathcal{C}')$ exists, and $LG(X) \simeq G(X)$ if X is homotopically projective.*

Proof. Apply Lemmas 14.4.1 and 14.4.2. q.e.d.

By reversing the arrows in Theorem 14.4.3, we obtain

Theorem 14.4.4. *Let $\mathcal{C}$ be an abelian category. Assume that $\mathcal{C}$ admits enough injectives, small products exist in $\mathcal{C}$ and such products are exact in $\mathcal{C}$. Then*

(i) *for any $X \in \mathrm{K}(\mathcal{C})$, there exist $I \in \mathrm{K}_{\mathrm{hi}}(\mathcal{C})$ and a qis $X \to I$,*
(ii) *for any additive functor $F\colon \mathcal{C} \to \mathcal{C}'$, the right derived functor $RF\colon \mathrm{D}(\mathcal{C}) \to \mathrm{D}(\mathcal{C}')$ exists, and $RF(X) \simeq F(X)$ if X is homotopically injective.*

Note that Grothendieck categories always admit small products, but small products may not be exact.

Theorem 14.4.5. *Let k be a commutative ring and let $G\colon \mathcal{C} \to \mathcal{C}'$ and $F\colon \mathcal{C}' \to \mathcal{C}$ be k-additive functors of k-abelian categories such that (G, F) is a pair of adjoint functors. Assume that $\mathcal{C}'$ is a Grothendieck category and $\mathcal{C}$ satisfies* (14.4.1). *Let $\mathcal{P}$ be a G-projective full subcategory of $\mathcal{C}$.*

(a) *Let $\widetilde{\mathcal{P}}$ be the smallest full triangulated subcategory of $\mathrm{K}(\mathcal{C})$ closed by small direct sums and containing $\mathrm{K}^-(\mathcal{P})$. Then $\widetilde{\mathcal{P}}$ is $\mathrm{K}(G)$-projective.*
(b) *The left derived functor $LG\colon \mathrm{D}(\mathcal{C}) \to \mathrm{D}(\mathcal{C}')$ exists and (LG, RF) is a pair of adjoint functors.*

(c) *We have an isomorphism in* $\mathrm{D}(k)$*, functorial with respect to* $X \in \mathrm{D}(\mathcal{C})$ *and* $Y \in \mathrm{D}(\mathcal{C}')$:

$$\mathrm{RHom}_{\mathcal{C}}(X, RF(Y)) \simeq \mathrm{RHom}_{\mathcal{C}'}(LG(X), Y) .$$

Proof. (i) Let us denote by $\widetilde{\mathcal{P}}'$ the full subcategory of $\mathrm{K}(\mathcal{C})$ consisting of objects X such that

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(X, F(I)) \to \mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(X, F(I)) \tag{14.4.2}$$

is bijective for any homotopically injective object $I \in \mathrm{C}(\mathcal{C}')$. Then $\widetilde{\mathcal{P}}'$ is a triangulated subcategory of $\mathrm{K}(\mathcal{C})$ closed by small direct sums.
(ii) Let us show that $\widetilde{\mathcal{P}}'$ contains $\mathrm{K}^-(\mathcal{P})$. If $X \in \mathrm{K}^-(\mathcal{P})$, then $\mathrm{Qis}_X \cap \mathrm{K}^-(\mathcal{P})_X$ is co-cofinal to Qis_X, and hence we have

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(X, F(I)) \simeq \varinjlim_{(X' \to X) \in \mathrm{Qis}, X' \in \mathrm{K}^-(\mathcal{P})} \mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(X', F(I)) .$$

Let $X' \to X$ be a qis with $X' \in K^-(\mathcal{P})$. Let X'' be the mapping cone of $X' \to X$. Then X'' is an exact complex in $\mathrm{K}^-(\mathcal{P})$. Hence

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(X'', F(I)) \simeq \mathrm{Hom}_{\mathrm{K}(\mathcal{C}')}(G(X''), I) \simeq 0 ,$$

where the second isomorphism follows from the fact that $\mathcal{P}$ being G-projective, $G(X'')$ is an exact complex. Hence, for $X, X' \in \mathrm{K}^-(\mathcal{P})$ and for a qis $X' \to X$, the map $\mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(X, F(I)) \to \mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(X', F(I))$ is bijective. It follows that the map in (14.4.2) is bijective.
(iii) By (ii), $\widetilde{\mathcal{P}}'$ contains $\widetilde{\mathcal{P}}$.
(iv) We shall prove that if $X \in \widetilde{\mathcal{P}}'$ is exact, then $G(X) \simeq 0$ in $\mathrm{D}(\mathcal{C}')$. Indeed, for any homotopically injective object I in $\mathrm{C}(\mathcal{C}')$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D}(\mathcal{C}')}(G(X), I) &\simeq \mathrm{Hom}_{\mathrm{K}(\mathcal{C}')}(G(X), I) \simeq \mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(X, F(I)) \\ &\simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(X, F(I)) \simeq 0 .\end{aligned}$$

(v) By Lemma 14.4.1, for every $X \in \mathrm{C}(\mathcal{C})$, there exists a quasi-isomorphism $P \to X$ with $P \in \widetilde{\mathcal{P}}$. Hence $\widetilde{\mathcal{P}}$ is $\mathrm{K}(G)$-projective and LG exists. Moreover, we have $LG(X) \simeq G(X)$ for any $X \in \widetilde{\mathcal{P}}$. For a homotopically injective object $I \in \mathrm{C}(\mathcal{C}')$ and $X \in \widetilde{\mathcal{P}}$, we have

$$\begin{aligned}\mathrm{RHom}_{\mathcal{C}}(X, RF(I)) &\simeq \mathrm{Hom}^{\bullet}_{\mathcal{C}}(X, F(I)) \\ &\simeq \mathrm{Hom}^{\bullet}_{\mathcal{C}'}(G(X), I) \simeq \mathrm{RHom}_{\mathcal{C}'}(LG(X), I) .\end{aligned}$$

Hence we obtain (c). By taking the cohomologies, we obtain (b). q.e.d.

Corollary 14.4.6. *Let* $\mathcal{C}$ *and* $\mathcal{C}'$ *be Grothendieck categories and let* $G\colon \mathcal{C} \to \mathcal{C}'$ *be an additive functor commuting with small inductive limits. Assume that there exists a* G*-projective subcategory* $\mathcal{P}$ *of* $\mathcal{C}$*. Then*

(i) $LG\colon \mathrm{D}(\mathcal{C}) \to \mathrm{D}(\mathcal{C}')$ *exists and commutes with small direct sums,*
(ii) *for any small filtrant inductive system* $\alpha\colon I \to \mathcal{C}$, $\varinjlim H^n(LG(\alpha)) \to H^n(LG(\varinjlim \alpha))$ *is an isomorphism for all* $n \in \mathbb{Z}$.

Proof. (i) By Theorem 8.3.27, G admits a right adjoint functor and we may apply Theorem 14.4.5.
(ii) Let $\mathcal{P}'$ be the full subcategory of $\mathcal{C}$ consisting of left G-acyclic objects (see Remark 13.3.6). Then $\mathcal{P}'$ is also G-projective by Lemma 13.3.12 and closed by small direct sums by (i). For each $i \in I$, let us take an epimorphism $P_i \twoheadrightarrow \alpha(i)$ with $P_i \in \mathcal{P}'$. Set $p_0(i) = \bigoplus_{i' \to i} P_{i'}$. Then $p_0\colon I \to \mathcal{C}$ is a functor and $p_0 \to \alpha$ is an epimorphism in $\mathrm{Fct}(I, \mathcal{C})$. It is easily checked that $\varinjlim p_0 \simeq \bigoplus_{i \in I} P_i$ (see Exercise 2.21). By this procedure, we construct an exact sequence in $\mathrm{Fct}(I, \mathcal{C})$

$$\cdots \to p_{n+1} \to p_n \to \cdots \to p_0 \to \alpha \to 0 \qquad (14.4.3)$$

such that any $p_k(i)$ as well as $\varinjlim_i p_k(i)$ belongs to $\mathcal{P}'$.

Define the complex in $\mathrm{Fct}(I, \mathcal{C})$

$$p_\bullet := \cdots \to p_{n+1} \to p_n \to \cdots \to p_0 \to 0\,.$$

Hence we have

$$H^n(LG(\varinjlim \alpha)) \simeq H^n(G(\varinjlim p_\bullet)) \simeq \varinjlim H^n(G(p_\bullet)) \simeq \varinjlim H^n(LG(\alpha))\,.$$

q.e.d.

Proposition 14.4.7. *Let* $\mathcal{C}, \mathcal{C}', \mathcal{C}''$ *be Grothendieck categories and let* $F\colon \mathcal{C} \to \mathcal{C}'$, $F'\colon \mathcal{C}' \to \mathcal{C}''$, $G\colon \mathcal{C}' \to \mathcal{C}$, $G'\colon \mathcal{C}'' \to \mathcal{C}'$ *be additive functors such that* (G, F) *and* (G', F') *are pairs of adjoint functors. Assume that there exist a* G*-projective subcategory* $\mathcal{P}'$ *of* $\mathcal{C}'$ *and a* G'*-projective subcategory* $\mathcal{P}''$ *of* $\mathcal{C}''$ *such that* $G'(\mathcal{P}'') \subset \mathcal{P}'$. *Then* $R(F' \circ F) \to RF' \circ RF$ *and* $LG \circ LG' \to L(G \circ G')$ *are isomorphisms of functors.*

Proof. Since $R(F' \circ F)$, RF', RF are left adjoint functors to $L(G \circ G')$, LG', LG, it is enough to prove the isomorphism $LG \circ LG' \xrightarrow{\sim} L(G \circ G')$. Let $\widetilde{\mathcal{P}}''$ (resp. $\widetilde{\mathcal{P}}'$) denote the smallest full triangulated subcategory of $\mathrm{K}(\mathcal{C}'')$ (resp. $\mathrm{K}(\mathcal{C}')$) closed by small direct sums and containing $\mathrm{K}^-(\mathcal{P}'')$ (resp. $\mathrm{K}^-(\mathcal{P}')$). Then $\widetilde{\mathcal{P}}''$ (resp. $\widetilde{\mathcal{P}}'$) is projective with respect to the functor $\mathrm{K}(G')$ (resp. $\mathrm{K}(G)$). Moreover, $\mathrm{K}(G')(\widetilde{\mathcal{P}}'') \subset \widetilde{\mathcal{P}}'$. Hence $LG \circ LG' \to L(G \circ G')$ is an isomorphism by Proposition 10.3.5. q.e.d.

Theorem 14.4.8. *Let* k *be a commutative ring and let* $\mathcal{C}_1$, $\mathcal{C}_2$ *and* $\mathcal{C}_3$ *be* k*-abelian categories. We assume that* $\mathcal{C}_3$ *is a Grothendieck category and that* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *satisfy* (14.4.1). *Let* $G\colon \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_3$, $F_1\colon \mathcal{C}_2^{\mathrm{op}} \times \mathcal{C}_3 \to \mathcal{C}_1$ *and*

$F_2\colon \mathcal{C}_1^{\mathrm{op}}\times\mathcal{C}_3 \to \mathcal{C}_2$ be k-additive functors. Assume that there are isomorphisms, functorial with respect to $X_i \in \mathcal{C}_i$ $(i = 1, 2, 3)$:

$$\begin{aligned}(14.4.4)\qquad \mathrm{Hom}_{\mathcal{C}_3}(G(X_1, X_2), X_3) &\simeq \mathrm{Hom}_{\mathcal{C}_1}(X_1, F_1(X_2, X_3))\\ &\simeq \mathrm{Hom}_{\mathcal{C}_2}(X_2, F_2(X_1, X_3))\,.\end{aligned}$$

Let $\mathrm{K}(G)\colon \mathrm{K}(\mathcal{C}_1)\times\mathrm{K}(\mathcal{C}_2) \to \mathrm{K}(\mathcal{C}_3)$ be the triangulated functor associated with $\mathrm{tot}_\oplus G(X_1, X_2)$, and let $\mathrm{K}(F_1)\colon \mathrm{K}(\mathcal{C}_2)^{\mathrm{op}} \times \mathrm{K}(\mathcal{C}_3) \to \mathrm{K}(\mathcal{C}_1)$ be the triangulated functor associated with $\mathrm{tot}_\pi F_1(X_2, X_3)$ and similarly for $\mathrm{K}(F_2)$.
Let $\mathcal{P}_i \subset \mathcal{C}_i$ $(i = 1, 2)$ be a full subcategory such that $(\mathcal{P}_1, \mathcal{P}_2)$ is $\mathrm{K}(G)$-projective. Denote by $\widetilde{\mathcal{P}}_i$ the smallest full triangulated subcategory of $\mathrm{K}(\mathcal{C}_i)$ that contains $\mathrm{K}^-(\mathcal{P}_i)$ and is closed by small direct sums $(i = 1, 2)$.

Then:

- (i) *$(\widetilde{\mathcal{P}}_1, \widetilde{\mathcal{P}}_2)$ is $\mathrm{K}(G)$-projective. In particular $LG\colon \mathrm{D}(\mathcal{C}_1) \times \mathrm{D}(\mathcal{C}_2) \to \mathrm{D}(\mathcal{C}_3)$ exists and $LG(X_1, X_2) \simeq \mathrm{K}(G)(X_1, X_2)$ for $X_1 \in \widetilde{\mathcal{P}}_1$ and $X_2 \in \widetilde{\mathcal{P}}_2$.*
- (ii) *$(\widetilde{\mathcal{P}}_2^{\mathrm{op}}, \mathrm{K}_{\mathrm{hi}}(\mathcal{C}_3))$ is $\mathrm{K}(F_1)$-injective. In particular, $RF_1\colon \mathrm{D}(\mathcal{C}_2)^{\mathrm{op}}\times\mathrm{D}(\mathcal{C}_3) \to \mathrm{D}(\mathcal{C}_1)$ exists and $RF_1(X_2, X_3) \simeq \mathrm{K}(F_1)(X_2, X_3)$ for $X_2 \in \widetilde{\mathcal{P}}_2$ and $X_3 \in \mathrm{K}_{\mathrm{hi}}(\mathcal{C}_3)$. Similar statements hold for F_2.*
- (iii) *There are isomorphisms, functorial with respect to $X_i \in \mathrm{D}(\mathcal{C}_i)$ $(i = 1, 2, 3)$*

$$\begin{aligned}(14.4.5)\quad \mathrm{Hom}_{\mathrm{D}(\mathcal{C}_3)}(LG(X_1, X_2), X_3) &\simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{C}_1)}(X_1, RF_1(X_2, X_3))\\ &\simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{C}_2)}(X_2, RF_2(X_1, X_3)),\end{aligned}$$

  and

$$\begin{aligned}(14.4.6)\quad \mathrm{RHom}_{\mathcal{C}_3}(LG(X_1, X_2), X_3) &\simeq \mathrm{RHom}_{\mathcal{C}_1}(X_1, RF_1(X_2, X_3))\\ &\simeq \mathrm{RHom}_{\mathcal{C}_2}(X_2, RF_2(X_1, X_3))\,.\end{aligned}$$

- (iv) *Moreover, if $\mathcal{P}_i = \mathcal{C}_i$ for $i = 1$ or $i = 2$, we can take $\widetilde{\mathcal{P}}_i = \mathrm{K}(\mathcal{C}_i)$ in* (i) *and* (ii).

Proof. In the sequel, we shall write for short G and F_i instead of $\mathrm{K}(G)$ and $\mathrm{K}(F_i)$, respectively. The isomorphism (14.4.4) gives rise to an isomorphism

$$(14.4.7)\qquad \mathrm{Hom}_{\mathrm{K}(\mathcal{C}_3)}(G(X_1, X_2), X_3) \simeq \mathrm{Hom}_{\mathrm{K}(\mathcal{C}_1)}(X_1, F_1(X_2, X_3))$$

functorial with respect to $X_i \in \mathrm{K}(\mathcal{C}_i)$ $(i = 1, 2, 3)$.

Note also that for any $X_2 \in \mathcal{C}_2$, the functor $X_1 \mapsto G(X_1, X_2)$ commutes with small direct sums. Indeed this functor has a right adjoint $X_3 \mapsto F_1(X_2, X_3)$.

(a) Let us first prove the following statement:

$$(14.4.8)\qquad \begin{array}{l}\text{if } X_1 \in \mathrm{K}^-(\mathcal{P}_1) \text{ is an exact complex and } X_2 \in \widetilde{\mathcal{P}}_2\,,\\ \text{then } G(X_1, X_2) \text{ is exact.}\end{array}$$

Indeed, for such an X_1, the category

$$\widetilde{\mathcal{P}}_2' = \left\{Y \in \mathrm{K}(\mathcal{C}_2)\,;\, G(X_1, Y) \text{ is exact}\right\}$$

is a triangulated subcategory of $\mathrm{K}(\mathcal{C}_2)$ which contains $\mathrm{K}^-(\mathcal{P}_2)$ and is closed by small direct sums. Hence, $\widetilde{\mathcal{P}}_2'$ contains $\widetilde{\mathcal{P}}_2$.

(b) Set

$$\begin{aligned}\widetilde{\mathcal{P}}_1' = \{X_1 \in \mathrm{K}(\mathcal{C}_1);\ &\mathrm{Hom}_{\mathrm{K}(\mathcal{C}_1)}(X_1, F_1(X_2, X_3)) \to \mathrm{Hom}_{\mathrm{D}(\mathcal{C}_1)}(X_1, F_1(X_2, X_3)) \\ &\text{is an isomorphism for all } X_2 \in \widetilde{\mathcal{P}}_2, X_3 \in \mathrm{K}_{\mathrm{hi}}(\mathcal{C}_3)\}\,.\end{aligned}$$

Let us show that $\widetilde{\mathcal{P}}_1 \subset \widetilde{\mathcal{P}}_1'$.

Since the category $\widetilde{\mathcal{P}}_1'$ is a full triangulated subcategory of $\mathrm{K}(\mathcal{C}_1)$ closed by small direct sums, it is enough to show that $\mathrm{K}^-(\mathcal{P}_1) \subset \widetilde{\mathcal{P}}_1'$. If $Y_1 \in \mathrm{K}^-(\mathcal{P}_1)$ is exact, then

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{C}_1)}(Y_1, F_1(X_2, X_3)) \simeq \mathrm{Hom}_{\mathrm{K}(\mathcal{C}_3)}(G(Y_1, X_2), X_3) \simeq 0\,, \tag{14.4.9}$$

where the last isomorphism follows from (14.4.8) and $X_3 \in \mathrm{K}_{\mathrm{hi}}(\mathcal{C}_3)$. Hence, if $X_1' \to X_1$ is a qis in $\mathrm{K}^-(\mathcal{P}_1)$, then

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{C}_1)}(X_1, F_1(X_2, X_3)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{K}(\mathcal{C}_1)}(X_1', F_1(X_2, X_3))\,.$$

Hence we obtain for any $X_1 \in \mathrm{K}^-(\mathcal{P}_1)$

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{D}(\mathcal{C}_1)}(X_1, F_1(X_2, X_3)) \\ &\qquad \simeq \varinjlim_{(X_1' \to X_1) \in \mathrm{Qis} \cap \mathrm{K}^-(\mathcal{P}_1)} \mathrm{Hom}_{\mathrm{K}(\mathcal{C}_1)}(X_1', F_1(X_2, X_3)) \\ &\qquad \simeq \mathrm{Hom}_{\mathrm{K}(\mathcal{C}_1)}(X_1, F_1(X_2, X_3))\,.\end{aligned} \tag{14.4.10}$$

Thus $\mathrm{K}^-(\mathcal{P}_1) \subset \widetilde{\mathcal{P}}_1'$ and hence $\widetilde{\mathcal{P}}_1 \subset \widetilde{\mathcal{P}}_1'$.

(c) Next let us show

$$\begin{aligned}&\text{for } X_i \in \widetilde{\mathcal{P}}_i \ (i = 1, 2) \text{ and } X_3 \in \mathrm{K}_{\mathrm{hi}}(\mathcal{C}_3), \text{ we have} \\ &\mathrm{Hom}_{\mathrm{D}(\mathcal{C}_3)}(G(X_1, X_2), X_3) \simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{C}_1)}(X_1, F_1(X_2, X_3))\,.\end{aligned} \tag{14.4.11}$$

There are isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D}(\mathcal{C}_3)}(G(X_1, X_2), X_3) &\simeq \mathrm{Hom}_{\mathrm{K}(\mathcal{C}_3)}(G(X_1, X_2), X_3) \\ &\simeq \mathrm{Hom}_{\mathrm{K}(\mathcal{C}_1)}(X_1, F_1(X_2, X_3)) \\ &\simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{C}_1)}(X_1, F_1(X_2, X_3))\,.\end{aligned}$$

Here the first isomorphism follows from $X_3 \in \mathrm{K}_{\mathrm{hi}}(\mathcal{C}_3)$ and the last isomorphism follows from $\widetilde{\mathcal{P}}_1 \subset \widetilde{\mathcal{P}}_1'$.

(d) Let us prove (i). It is enough to show that for $X_i \in \widetilde{\mathcal{P}}_i$ $(i = 1, 2)$, $G(X_1, X_2)$ is exact as soon as X_1 or X_2 is exact. Assume that X_1 is exact. Then, for any $X_3 \in \mathrm{K_{hi}}(\mathcal{C}_3)$, we have by (14.4.11)

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{C}_3)}(G(X_1, X_2), X_3) \simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{C}_1)}(X_1, F_1(X_2, X_3)) \simeq 0 .$$

This implies that $G(X_1, X_2)$ is exact. The proof in the case where X_2 is exact is similar.

(e) Let us prove (ii). It is enough to show that for $X_2 \in \widetilde{\mathcal{P}}_2$ and $X_3 \in \mathrm{K_{hi}}(\mathcal{C}_3)$, $F_1(X_2, X_3)$ is exact as soon as X_2 or X_3 is exact.
(e1) Assume that X_2 is exact. For any $X_1 \in \widetilde{\mathcal{P}}_1$, $G(X_1, X_2)$ is exact by (i), and hence $\mathrm{Hom}_{\mathrm{D}(\mathcal{C}_1)}(X_1, F_1(X_2, X_3)) \simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{C}_3)}(G(X_1, X_2), X_3)$ vanishes. This implies that $F_1(X_2, X_3)$ is exact.
(e2) Assume that $X_3 \in \mathrm{K_{hi}}(\mathcal{C}_3)$ is exact. Then $X_3 \simeq 0$ in $\mathrm{K}(\mathcal{C}_3)$ and $F_1(X_2, X_3)$ is exact.

(f) Let us show (iii). The isomorphisms (14.4.5) immediately follow from (14.4.11). The adjunction morphism $X_1 \to RF_1(X_2, LG(X_1, X_2))$ induces the morphisms

$$\begin{aligned}\mathrm{RHom}_{\mathcal{C}_3}(LG(X_1, X_2), X_3) &\to \mathrm{RHom}_{\mathcal{C}_1}\big(RF_1(X_2, LG(X_1, X_2)), RF_1(X_2, X_3)\big)\\ &\to \mathrm{RHom}_{\mathcal{C}_1}(X_1, RF_1(X_2, X_3)) .\end{aligned}$$

By taking the cohomologies, it induces isomorphisms by (14.4.5) and Theorem 13.4.1.

(g) Let us prove (iv). Assume $\mathcal{P}_1 = \mathrm{K}^-(\mathcal{C}_1)$.
(g1) Let us show that $(\mathrm{K}(\mathcal{C}_1), \widetilde{\mathcal{P}}_2)$ is $\mathrm{K}(G)$-projective. For that purpose it is enough to show that $G(X_1, X_2)$ is exact for $X_1 \in \mathrm{K}(\mathcal{C}_1)$ and $X_2 \in \widetilde{\mathcal{P}}_2$ as soon as X_1 or X_2 is exact. Since $\tau^{\leq n}X_1$ or X_2 is exact and $(\widetilde{\mathcal{P}}_1, \widetilde{\mathcal{P}}_2)$ is $\mathrm{K}(G)$-projective, $G(\tau^{\leq n}X_1, X_2)$ is exact. Hence $G(X_1, X_2) \simeq \varinjlim_n G(\tau^{\leq n}X_1, X_2)$ is exact.

(g2) Let us show that $(\mathrm{K}(\mathcal{C}_1)^{\mathrm{op}}, \mathrm{K_{hi}}(\mathcal{C}_3))$ is $\mathrm{K}(F_2)$-injective. Let $X_1 \in \mathrm{K}(\mathcal{C}_1)$ and $X_3 \in \mathrm{K_{hi}}(\mathcal{C}_3)$. If X_3 is exact, then $X_3 \simeq 0$, and hence $F_2(X_1, X_3)$ is exact. If $X_1 \in \mathrm{K}(\mathcal{C}_1)$ is exact, then for any $X_2 \in \widetilde{\mathcal{P}}_2$ we have

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{C}_2)}(X_2, F_2(X_1, X_3)) \simeq \mathrm{Hom}_{\mathrm{K}(\mathcal{C}_3)}(G(X_1, X_2), X_3) \simeq 0 ,$$

where the last isomorphism follows from the fact that $G(X_1, X_2)$ is exact by (g1). Hence $F_2(X_1, X_3)$ is exact. q.e.d.

Corollary 14.4.9. *Let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ be Grothendieck categories. Let $G\colon \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_3$ be an additive functor which commutes with small inductive limits with respect to each variable. Let $\mathcal{P}_i \subset \mathcal{C}_i$ $(i = 1, 2)$ be a full subcategory such that $(\mathcal{P}_1, \mathcal{P}_2)$ is G-projective. Denote by $\widetilde{\mathcal{P}}_i$ the smallest full triangulated subcategory of $\mathrm{K}(\mathcal{C}_i)$ that contains $\mathrm{K}^-(\mathcal{P}_i)$ and is closed by small direct sums $(i = 1, 2)$. Let $\mathrm{K}(G)\colon \mathrm{K}(\mathcal{C}_1) \times \mathrm{K}(\mathcal{C}_2) \to \mathrm{K}(\mathcal{C}_3)$ be the functor associated with $\mathrm{tot}_\oplus G(X_1, X_2)$. Then*

(i) $(\widetilde{\mathcal{P}}_1, \widetilde{\mathcal{P}}_2)$ *is* $\mathrm{K}(G)$*-projective. In particular* $LG\colon \mathrm{D}(\mathcal{C}_1) \times \mathrm{D}(\mathcal{C}_2) \to \mathrm{D}(\mathcal{C}_3)$ *exists and* $LG(X_1, X_2) \simeq G(X_1, X_2)$ *for* $X_1 \in \widetilde{\mathcal{P}}_1$ *and* $X_2 \in \widetilde{\mathcal{P}}_2$.

(ii) LG *commutes with small direct sums.*

(iii) *Moreover, if* $\mathcal{P}_i = \mathcal{C}_i$ *for* $i = 1$ *or* $i = 2$*, we can take* $\widetilde{\mathcal{P}}_i = \mathrm{K}(\mathcal{C}_i)$.

Proof. By Theorem 8.3.27, the two functors $X_1 \mapsto G(X_1, X_2)$ and $X_2 \mapsto G(X_1, X_2)$ have right adjoints. q.e.d.

Example 14.4.10. Let R denote a k-algebra. The functor $\bullet \otimes_R \bullet \colon \mathrm{Mod}(R^{\mathrm{op}}) \times \mathrm{Mod}(R) \to \mathrm{Mod}(k)$ defines a functor

$$\bullet \otimes_R \bullet \colon \mathrm{K}(\mathrm{Mod}(R^{\mathrm{op}})) \times \mathrm{K}(\mathrm{Mod}(R)) \to \mathrm{K}(\mathrm{Mod}(k)) , \quad (X^\bullet, Y^\bullet) \mapsto \mathrm{tot}_\oplus(X^\bullet \otimes_R Y^\bullet) . \tag{14.4.12}$$

Then

$$\begin{aligned}\mathrm{Hom}_k(N \otimes_R M, L) &\simeq \mathrm{Hom}_{R^{\mathrm{op}}}(N, \mathrm{Hom}_k(M, L)) \\ &\simeq \mathrm{Hom}_R(M, \mathrm{Hom}_k(N, L))\end{aligned}$$

for any $N \in \mathrm{Mod}(R^{\mathrm{op}})$, $M \in \mathrm{Mod}(R)$ and $L \in \mathrm{Mod}(k)$.

Let $\mathcal{P}_{proj}$ denote the full additive subcategory of $\mathrm{Mod}(R)$ consisting of projective modules and $\widetilde{\mathcal{P}}_{proj}$ the smallest full triangulated subcategory of $\mathrm{K}(\mathrm{Mod}(R))$ closed by small direct sums and containing $\mathrm{K}^-(\mathcal{P}_{proj})$. We may apply Theorem 14.4.8 with $\mathcal{C}_1 = \mathrm{Mod}(R^{\mathrm{op}})$, $\mathcal{C}_2 = \mathrm{Mod}(R)$ and $\mathcal{C}_3 = \mathrm{Mod}(k)$. Then $(\mathrm{K}(\mathrm{Mod}(R^{\mathrm{op}})), \widetilde{\mathcal{P}}_{proj})$ is $(\bullet \otimes_R \bullet)$-projective and the functor in (14.4.12) admits a left derived functor

$$\bullet \overset{\mathrm{L}}{\otimes}_R \bullet \colon \mathrm{D}(R^{\mathrm{op}}) \times \mathrm{D}(R) \to \mathrm{D}(k) ,$$

and

$$N \overset{\mathrm{L}}{\otimes}_R M \simeq \mathrm{tot}_\oplus(N \otimes_R P) \quad \text{for} \quad P \in \widetilde{\mathcal{P}}_{proj}, \quad (P \to M) \in \mathrm{Qis} .$$

Moreover, the functor

$$\mathrm{Hom}_k(\bullet, \bullet) \colon \mathrm{K}(\mathrm{Mod}(R))^{\mathrm{op}} \times \mathrm{K}(\mathrm{Mod}(k)) \to \mathrm{K}(\mathrm{Mod}(R^{\mathrm{op}}))$$

admits a right adjoint functor and we have

$$\begin{aligned}\mathrm{RHom}_k(N \overset{\mathrm{L}}{\otimes}_R M, L) &\simeq \mathrm{RHom}_{R^{\mathrm{op}}}(N, \mathrm{RHom}_k(M, L)) \\ &\simeq \mathrm{RHom}_R(M, \mathrm{RHom}_k(N, L))\end{aligned}$$

for any $N \in \mathrm{D}(R^{\mathrm{op}})$, $M \in \mathrm{D}(R)$ and $L \in \mathrm{D}(k)$.

Exercises

Exercise 14.1. Let $\mathcal{C}$ be an abelian category and let $a \leq b$ be integers.
(i) Prove that for $X \in \mathrm{D}^{\geq b}(\mathcal{C})$ and $Y \in \mathrm{D}^{\leq a}(\mathcal{C})$, any morphism $f\colon X \to Y$ in $\mathrm{D}(\mathcal{C})$ decomposes as $X \to U[-b] \to V[-a] \to Y$ for some $U, V \in \mathcal{C}$. (Hint: to prove the existence of V, represent X by an object of $\mathrm{C}^{\geq b}(\mathcal{C})$ and use $\sigma^{\geq a}$.)
(ii) Assume that $\mathrm{hd}(\mathcal{C}) < b-a$. Prove that $\mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(X, Y) \simeq 0$ for $X \in \mathrm{D}^{\geq b}(\mathcal{C})$ and $Y \in \mathrm{D}^{\leq a}(\mathcal{C})$.

Exercise 14.2. Let $\mathcal{C}$ be an abelian category with enough projectives and which satisfies (14.4.1). Let $\mathcal{P}$ denote the full subcategory of $\mathcal{C}$ consisting of projective objects. Denote by $\widetilde{\mathcal{P}}$ the smallest full triangulated category of $\mathrm{K}(\mathcal{C})$ that contains $\mathrm{K}^-(\mathcal{P})$ and is closed by small direct sums. Prove that the derived functor $\mathrm{RHom}_{\mathcal{C}}\colon \mathrm{D}(\mathcal{C}) \times \mathrm{D}(\mathcal{C})^{\mathrm{op}} \to \mathrm{D}(\mathbb{Z})$ exists and prove that if $P \to X$ is a qis in $\mathrm{K}(\mathcal{C})$ with $P \in \widetilde{\mathcal{P}}$, then $\mathrm{RHom}_{\mathcal{C}}(X, Y) \simeq \mathrm{tot}_\pi(\mathrm{Hom}^{\bullet,\bullet}_{\mathcal{C}}(P, Y))$.

Exercise 14.3. Let $\mathcal{C}$ be an abelian category which admits countable direct sums and assume that such direct sums are exact. Let $X \in \mathrm{D}(\mathcal{C})$.
(i) Prove that there is a d.t. in $\mathrm{D}(\mathcal{C})$:

$$\bigoplus_{n\geq 0} \tau^{\leq n}X \xrightarrow{\mathrm{id}-\sigma} \bigoplus_{n\geq 0} \tau^{\leq n}X \xrightarrow{w} X \xrightarrow{v} \bigoplus_{n\geq 0} \tau^{\leq n}X[1]\,, \tag{14.4.13}$$

where σ is defined in Notation 10.5.10 and w is induced by the canonical morphisms $\tau^{\leq n}X \to X$.
(ii) Assume further that the cohomological dimension of $\mathcal{C}$ is less than or equal to 1. Prove that any $X \in \mathrm{D}(\mathcal{C})$ is isomorphic to $\bigoplus_{n\in\mathbb{Z}} H^n(X)[-n]$. (Hint: applying Exercise 14.1 to $\tau^{<n}X \to X \to \tau^{\geq n}X \xrightarrow{+1}$, construct $H^n(X)[-n] \to X$.)

Exercise 14.4. Let k be a field, $A = k[x, y]$, $\mathcal{C} = \mathrm{Mod}(A)$ and denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{C})$ the full triangulated subcategory of $\mathrm{D}(\mathcal{C})$ consisting of objects X such that $H^j(X)$ is finitely generated over A for any $j \in \mathbb{Z}$. Let $L_0 = A$, $L = A \bigoplus A$, and consider the exact sequence $0 \to L_0 \xrightarrow{\varphi} L \xrightarrow{\psi} L_0 \to k \to 0$ in Exercise 13.21. Let $p := \varphi \circ \psi\colon L \to L$ and denote by X the object of $\mathrm{K}(\mathcal{C})$:

$$X := 0 \to L_0 \xrightarrow{\varphi} L \xrightarrow{p} L \xrightarrow{p} L \to \cdots$$

where L_0 stands in degree -2.
(i) For $Z \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{C})$ and $Y_n \in \mathrm{D}(\mathcal{C})$ $(n \in \mathbb{Z})$, prove the isomorphism

$$\bigoplus_n \mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(Z, Y_n) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(Z, \bigoplus_n Y_n)\,.$$

(ii) Prove that

(a) $H^i(X) \simeq k$ for $i \geq 0$ and $H^i(X) \simeq 0$ for $i < 0$,

(b) for $i \geq 0$ and the d.t. in $\mathrm{D}(\mathcal{C})$

$$H^i(X)[-i] \to \tau^{\leq i+1}\tau^{\geq i}(X) \to H^{i+1}(X)[-i-1] \xrightarrow{u_i} H^i(X)[-i+1] \,,$$

the morphism u_i does not vanish in $\mathrm{D}(\mathcal{C})$.

(iii) Prove that the object $\tau^{\leq n}X$ of $\mathrm{K}(\mathcal{C})$ is isomorphic to the complex

$$0 \to L_0 \xrightarrow{\varphi} L \xrightarrow{p} L \xrightarrow{p} L \to \cdots \xrightarrow{p} L \to L_0 \to 0$$

where L_0 on the right stands in degree n and L_0 on the left in degree -2.
(iv) Prove the isomorphism $\tau^{\geq n}X \simeq X[-n]$ in $\mathrm{D}(\mathcal{C})$ for $n \geq 0$.
(v) Prove that for any $n > 0$ and any morphism $f : X \to X[n]$ in $\mathrm{D}(\mathcal{C})$, $H^i(f)$ vanishes for all $i \in \mathbb{Z}$. (Hint: use the commutative diagram

$$\begin{array}{ccc} H^k(X) & \longrightarrow & H^{k-1}(X)[2] \\ \downarrow{\scriptstyle H^k(f)} & & \downarrow{\scriptstyle H^{k-1}(f)} \\ H^k(X[n]) & \longrightarrow & H^{k-1}(X[n])[2] \end{array}$$

deduced from (ii) (b).)
(vi) Prove that the morphism v in (14.4.13) does not vanish in $\mathrm{D}(\mathcal{C})$ using the following steps.

(a) If $v = 0$, then there exists $s : X \to \bigoplus_n \tau^{\leq n}X$ such that $w \circ s = \mathrm{id}_X$.

(b) For any $a > 0$, there exists b such that the composition $\tau^{<a}X \to X \xrightarrow{s} \bigoplus_n \tau^{\leq n}X$ factors through $\bigoplus_{n<b} \tau^{\leq n}X \to \bigoplus_n \tau^{\leq n}X$. (Hint: use (i).)

(c) For any $a > 0$, there exist $b > 0$ and morphisms $\tau^{\geq a}X \to X$ and $X \to \tau^{\leq b}X$ such that the composition $X \to \tau^{\geq a}X \bigoplus \tau^{\leq b}X \to X$ is id_X. (Hint: s is the sum of two morphisms $X \to \bigoplus_{n<b} \tau^{\leq n}X$ and $X \to \bigoplus_{n\geq b} \tau^{\leq n}X$.)

(d) For any $a > 0$, there exists a morphism $\tau^{\geq a}X \to X$ such that the composition $\tau^{\geq a}X \to X \to \tau^{\geq b}X$ is the canonical morphism for some $b > a$.

(e) Using (v) and (iv), conclude.

(vii) Prove that $\tau^{\leq n}v = 0$ in $\mathrm{D}(\mathcal{C})$ for all $n \in \mathbb{Z}$.
(viii) Prove that the natural functor $\mathrm{D}^+(\mathcal{C}) \to \mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))$, given by $X \mapsto$ "$\varinjlim_n$" $\tau^{\leq n}X$, is not faithful.

Exercise 14.5. Let $\mathcal{C}$ be an abelian category and let $\mathrm{Gr}(\mathcal{C})$ be the associated graded category (see Definition 11.3.1). Consider the functor

$$\begin{aligned} \Theta : \mathrm{Gr}(\mathcal{C}) &\longrightarrow \mathrm{D}(\mathcal{C}) \\ \{X^n\}_{n\in\mathbb{Z}} &\mapsto \bigoplus_n X^n[-n] \,. \end{aligned}$$

(i) Prove that Θ is an equivalence if and only if $\mathcal{C}$ is semisimple.
(ii) Prove that Θ is essentially surjective if and only if $\mathcal{C}$ is hereditary.

Exercise 14.6. Let $\mathcal{C}$ be an abelian category which has enough injectives and denote by $\mathcal{I}_{\mathcal{C}}$ the full additive subcategory of injective objects of $\mathcal{C}$. Assume moreover that $\mathcal{C}$ has finite homological dimension (see Exercise 13.8). Prove that any $X \in \mathrm{K}(\mathcal{I}_{\mathcal{C}})$ is homotopically injective.

Exercise 14.7. Let k be a commutative ring and $\mathcal{C} = \mathrm{Mod}(k)$. Let $x \in k$ be a non-zero-divisor. Consider the additive functor $F: \mathcal{C} \to \mathcal{C}$ given by $M \mapsto x \cdot M$ (see Example 8.3.19). Prove that $RF \simeq \mathrm{id}_{\mathrm{D}(\mathcal{C})}$, $LF \simeq \mathrm{id}_{\mathrm{D}(\mathcal{C})}$ and the canonical morphism $LF \to RF$ (see (7.3.3)) is given by the multiplication by x.

Exercise 14.8. Let $\mathcal{C}$ be a Grothendieck category. Prove that an object I of $\mathrm{C}(\mathcal{C})$ is an injective object if and only if I is homotopic to zero and all I^n are injective objects of $\mathcal{C}$. (Hint: consider $I \to \mathrm{Mc}(\mathrm{id}_I)$.)