10

Triangulated Categories

Triangulated categories play an increasing role in mathematics and this subject deserves a whole book.

In this chapter we define and give the main properties of triangulated categories and cohomological functors and prove in particular that the localization of a triangulated category is still triangulated. We also show that under natural hypotheses, the Kan extension of a cohomological functor remains cohomological.

Then we study triangulated categories admitting small direct sums. Such categories are studied by many authors, in particular [6] and [53]. Here, we prove the so-called "Brown representability theorem" [11] in the form due to Neeman [53], more precisely, a variant due to [44], which asserts that any cohomological contravariant functor defined on a triangulated category admitting small direct sums and a suitable system of generators is representable as soon as it sends small direct sums to products. (The fact that Brown's theorem could be adapted to triangulated categories was also noticed by Keller [42].)

There also exist variants of the Brown representability theorem for triangulated categories which do not admit small direct sums. For results in this direction, we refer to [8].

We ask the reader to wait until Chap. 11 to encounter examples of triangulated categories. In fact, it would have been possible to formulate the important Theorem 11.3.8 below before defining triangulated categories, by listing the properties which become the axioms of these categories. We have chosen to give the axioms first in order to avoid repetitions, and also because the scope of triangulated categories goes much beyond the case of complexes in additive categories.

We do not treat here t-structures on triangulated categories and refer to the original paper [4] (see also [38] for an exposition). Another important closely related subject which is not treated here is the theory of A_∞-algebras (see [41, 43]).

10.1 Triangulated Categories

Definition 10.1.1. (i) *A category with* translation $(\mathcal{D}, T)$ *is a category* $\mathcal{D}$ *endowed with an equivalence of categories* $T\colon \mathcal{D} \xrightarrow{\sim} \mathcal{D}$. *The functor* T *is called the translation functor.*

(ii) *A functor of categories with translation* $F\colon (\mathcal{D}, T) \to (\mathcal{D}', T')$ *is a functor* $F\colon \mathcal{D} \to \mathcal{D}'$ *together with an isomorphism* $F \circ T \simeq T' \circ F$. *If* $\mathcal{D}$ *and* $\mathcal{D}'$ *are additive categories and* F *is additive, we say that* F *is a functor of additive categories with translation.*

(iii) *Let* $F, F'\colon (\mathcal{D}, T) \to (\mathcal{D}', T')$ *be two functors of categories with translation. A morphism* $\theta\colon F \to F'$ *of functors of categories with translation is a morphism of functors such that the diagram below commutes:*

$$\begin{array}{ccc} F \circ T & \xrightarrow{\theta \circ T} & F' \circ T \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ T' \circ F & \xrightarrow{T' \circ \theta} & T' \circ F' . \end{array}$$

(iv) *A subcategory with translation* $(\mathcal{D}', T')$ *of* $(\mathcal{D}, T)$ *is a category with translation such that* $\mathcal{D}'$ *is a subcategory of* $\mathcal{D}$ *and the translation functor* T' *is the restriction of* T.

(v) *Let* $(\mathcal{D}, T)$, $(\mathcal{D}', T')$ *and* $(\mathcal{D}'', T'')$ *be additive categories with translation. A bifunctor of additive categories with translation* $F\colon \mathcal{D} \times \mathcal{D}' \to \mathcal{D}''$ *is an additive bifunctor endowed with functorial isomorphisms*

$$\theta_{X,Y}\colon F(TX, Y) \xrightarrow{\sim} T''F(X, Y) \text{ and } \theta'_{X,Y}\colon F(X, T'Y) \xrightarrow{\sim} T''F(X, Y)$$

for $(X, Y) \in \mathcal{D} \times \mathcal{D}'$ *such that the diagram below anti-commutes* (*see* Definition 8.2.20):

$$\begin{array}{ccc} F(TX, T'Y) & \xrightarrow{\theta_{X,T'Y}} & T''F(X, T'Y) \\ \downarrow{\scriptstyle\theta'_{TX,Y}} & \text{ac} & \downarrow{\scriptstyle T''\theta'_{X,Y}} \\ T''F(TX, Y) & \xrightarrow[T''\theta_{X,Y}]{} & T''^2F(X, Y) . \end{array}$$

Remark 10.1.2. The anti-commutativity of the diagram above will be justified in Chapter 11 (see Proposition 11.2.11 and Lemma 11.6.3).

Notations 10.1.3. (i) We shall denote by T^{-1} a quasi-inverse of T. Then T^n is well defined for $n \in \mathbb{Z}$. These functors are unique up to unique isomorphism.
(ii) If there is no risk of confusion, we shall write $\mathcal{D}$ instead of $(\mathcal{D}, T)$ and TX instead of $T(X)$.

Definition 10.1.4. *Let $(\mathcal{D}, T)$ be an additive category with translation. A* triangle *in $\mathcal{D}$ is a sequence of morphisms*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX \, . \tag{10.1.1}$$

A morphism of triangles is a commutative diagram:

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & TX \\
{\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle T(\alpha)} \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & TX' \, .
\end{array}$$

Remark 10.1.5. For $\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1$, the triangle $X \xrightarrow{\varepsilon_1 f} Y \xrightarrow{\varepsilon_2 g} Z \xrightarrow{\varepsilon_3 h} TX$ is isomorphic to the triangle (10.1.1) if $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$, but if $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$, it is not isomorphic to the triangle (10.1.1) in general.

Definition 10.1.6. *A* triangulated category *is an additive category $(\mathcal{D}, T)$ with translation endowed with a family of triangles, called* distinguished triangles (d.t. *for short*), *this family satisfying the axioms* TR0 – TR5 *below.*

TR0 *A triangle isomorphic to a* d.t. *is a* d.t.

TR1 *The triangle $X \xrightarrow{\mathrm{id}_X} X \to 0 \to TX$ is a* d.t.

TR2 *For all $f \colon X \to Y$, there exists a* d.t. *$X \xrightarrow{f} Y \to Z \to TX$.*

TR3 *A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ is a* d.t. *if and only if $Y \xrightarrow{-g} Z \xrightarrow{-h} TX \xrightarrow{-T(f)} TY$ is a* d.t.

TR4 *Given two* d.t.*'s $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} TX'$ and morphisms $\alpha \colon X \to X'$ and $\beta \colon Y \to Y'$ with $f' \circ \alpha = \beta \circ f$, there exists a morphism $\gamma \colon Z \to Z'$ giving rise to a morphism of* d.t.*'s:*

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & TX \\
{\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\vdots & & \downarrow{\scriptstyle T(\alpha)} \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & TX'.
\end{array}$$

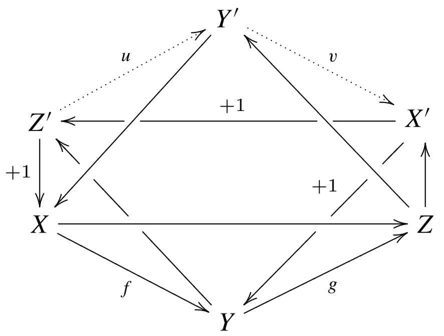
TR5 *Given three* d.t.*'s*

$$X \xrightarrow{f} Y \xrightarrow{h} Z' \to TX \, ,$$
$$Y \xrightarrow{g} Z \xrightarrow{k} X' \to TY \, ,$$
$$X \xrightarrow{g \circ f} Z \xrightarrow{l} Y' \to TX,$$

there exists a d.t. *$Z' \xrightarrow{u} Y' \xrightarrow{v} X' \xrightarrow{w} TZ'$ making the diagram below commutative:*

(10.1.2)
$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{h} & Z' & \longrightarrow & TX \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle g} & & \vdots{\scriptstyle u} & & \downarrow{\scriptstyle \mathrm{id}} \\
X & \xrightarrow{g\circ f} & Z & \xrightarrow{l} & Y' & \longrightarrow & TX \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} & & \vdots{\scriptstyle v} & & \downarrow{\scriptstyle T(f)} \\
Y & \xrightarrow{g} & Z & \xrightarrow{k} & X' & \longrightarrow & TY \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle T(h)} \\
Z' & \xrightarrow{u} & Y' & \xrightarrow{v} & X' & \xrightarrow{w} & TZ' .
\end{array}$$

Diagram (10.1.2) is often called the *octahedron diagram*. Indeed, it can be written using the vertices of an octahedron.



Here, for example, $X' \xrightarrow{+1} Y$ means a morphism $X' \to TY$.

Notation 10.1.7. The translation functor T is called the suspension functor by the topologists.

Remark 10.1.8. The morphism γ in TR4 is not unique and this is the origin of many troubles. See the paper [7] for an attempt to overcome this difficulty.

Definition 10.1.9. (i) *A* triangulated functor *of triangulated categories F: $(\mathcal{D}, T) \to (\mathcal{D}', T')$ is a functor of additive categories with translation sending distinguished triangles to distinguished triangles. If moreover F is an equivalence of categories, F is called an equivalence of triangulated categories.*

(ii) *Let $F, F' \colon (\mathcal{D}, T) \to (\mathcal{D}', T')$ be triangulated functors. A morphism $\theta \colon F \to F'$ of triangulated functors is a morphism of functors of additive categories with translation.*

(iii) *A triangulated subcategory $(\mathcal{D}', T')$ of $(\mathcal{D}, T)$ is an additive subcategory with translation of $\mathcal{D}$ (i.e., the functor T' is the restriction of T) such that it is triangulated and that the inclusion functor is triangulated.*

Remark 10.1.10. (i) A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ is *anti-distinguished* if the triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{-h} TX$ is distinguished. Then $(\mathcal{D}, T)$ endowed with the family of anti-distinguished triangles is triangulated. If we denote by $(\mathcal{D}^{\mathrm{ant}}, T)$ this triangulated category, then $(\mathcal{D}^{\mathrm{ant}}, T)$ and $(\mathcal{D}, T)$ are equivalent as triangulated categories (see Exercise 10.10).
(ii) Consider the contravariant functor $\mathrm{op}\colon \mathcal{D} \to \mathcal{D}^{\mathrm{op}}$, and define $T^{\mathrm{op}} = \mathrm{op} \circ T^{-1} \circ \mathrm{op}^{-1}$. Let us say that a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T^{\mathrm{op}}(X)$ in $\mathcal{D}^{\mathrm{op}}$ is distinguished if its image $Z^{\mathrm{op}} \xrightarrow{g^{\mathrm{op}}} Y^{\mathrm{op}} \xrightarrow{f^{\mathrm{op}}} X^{\mathrm{op}} \xrightarrow{T(h^{\mathrm{op}})} TZ^{\mathrm{op}}$ by op is distinguished. (Here, we write op instead of op^{-1} for short.) Then $(\mathcal{D}^{\mathrm{op}}, T^{\mathrm{op}})$ is a triangulated category.

Proposition 10.1.11. *If* $X \xrightarrow{f} Y \xrightarrow{g} Z \to TX$ *is a* d.t. *then* $g \circ f = 0$.

Proof. Applying TR1 and TR4 we get a commutative diagram:

$$\begin{array}{ccccccc}
X & \xrightarrow{\mathrm{id}} & X & \longrightarrow & 0 & \longrightarrow & TX \\
{\scriptstyle\mathrm{id}}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow & & {\scriptstyle\mathrm{id}}\downarrow \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & TX\,.
\end{array}$$

Then $g \circ f$ factorizes through 0. q.e.d.

Definition 10.1.12. *Let* $(\mathcal{D}, T)$ *be a triangulated category and* $\mathcal{C}$ *an abelian category. An additive functor* $F\colon \mathcal{D} \to \mathcal{C}$ *is* cohomological *if for any* d.t. $X \to Y \to Z \to TX$ *in* $\mathcal{D}$*, the sequence* $F(X) \to F(Y) \to F(Z)$ *is exact in* $\mathcal{C}$*.*

Proposition 10.1.13. *For any* $W \in \mathcal{D}$*, the two functors* $\mathrm{Hom}_{\mathcal{D}}(W, \bullet)$ *and* $\mathrm{Hom}_{\mathcal{D}}(\bullet, W)$ *are cohomological.*

Proof. Let $X \to Y \to Z \to TX$ be a d.t. and let $W \in \mathcal{D}$. We want to show that

$$\mathrm{Hom}\,(W, X) \xrightarrow{f\circ} \mathrm{Hom}\,(W, Y) \xrightarrow{g\circ} \mathrm{Hom}\,(W, Z)$$

is exact, i.e. : for all $\varphi\colon W \to Y$ such that $g \circ \varphi = 0$, there exists $\psi\colon W \to X$ such that $\varphi = f \circ \psi$. This means that the dotted arrows below may be completed, and this follows from the axioms TR4 and TR3.

$$\begin{array}{ccccccc}
W & \xrightarrow{\mathrm{id}} & W & \longrightarrow & 0 & \longrightarrow & TW \\
\vdots & & {\scriptstyle\varphi}\downarrow & & \downarrow & & \vdots \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & TX\,.
\end{array}$$

By replacing $\mathcal{D}$ with $\mathcal{D}^{\mathrm{op}}$, we obtain the assertion for $\mathrm{Hom}\,(\bullet, W)$. q.e.d.

Remark 10.1.14. By TR3, a cohomological functor gives rise to a long exact sequence:

$$(10.1.3) \qquad \cdots \to F(T^{-1}Z) \to F(X) \to F(Y) \to F(Z) \to F(TX) \to \cdots .$$

Proposition 10.1.15. *Consider a morphism of* d.t.*'s:*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & TX \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle T(\alpha)} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & TX' . \end{array}$$

If α and β are isomorphisms, then so is γ.

Proof. Apply $\mathrm{Hom}\,(W, \bullet)$ to this diagram and write $\widetilde{X}$ instead of $\mathrm{Hom}\,(W, X)$, $\widetilde{\alpha}$ instead of $\mathrm{Hom}\,(W, \alpha)$, etc. We get the commutative diagram:

$$\begin{array}{ccccccccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} & \xrightarrow{\widetilde{g}} & \widetilde{Z} & \xrightarrow{\widetilde{h}} & \widetilde{TX} & \xrightarrow{\widetilde{T(f)}} & \widetilde{TY} \\ \downarrow{\scriptstyle\widetilde{\alpha}} & & \downarrow{\scriptstyle\widetilde{\beta}} & & \downarrow{\scriptstyle\widetilde{\gamma}} & & \downarrow{\scriptstyle\widetilde{T(\alpha)}} & & \downarrow{\scriptstyle\widetilde{T(\beta)}} \\ \widetilde{X'} & \xrightarrow{\widetilde{f'}} & \widetilde{Y'} & \xrightarrow{\widetilde{g'}} & \widetilde{Z'} & \xrightarrow{\widetilde{h'}} & \widetilde{TX'} & \xrightarrow{\widetilde{T(f)}} & \widetilde{TY'} . \end{array}$$

The rows are exact in view of the Proposition 10.1.13, and $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widetilde{T(\alpha)}$ and $\widetilde{T(\beta)}$ are isomorphisms. Therefore $\widetilde{\gamma} = \mathrm{Hom}\,(W, \gamma)\colon \mathrm{Hom}\,(W, Z) \to \mathrm{Hom}\,(W, Z')$ is an isomorphism by Lemma 8.3.13. This implies that γ is an isomorphism by Corollary 1.4.7. q.e.d.

Corollary 10.1.16. *Let $\mathcal{D}'$ be a full triangulated subcategory of $\mathcal{D}$.*

(i) *Consider a triangle $X \xrightarrow{f} Y \to Z \to TX$ in $\mathcal{D}'$ and assume that this triangle is distinguished in $\mathcal{D}$. Then it is distinguished in $\mathcal{D}'$.*
(ii) *Consider a* d.t. *$X \to Y \to Z \to TX$ in $\mathcal{D}$ with X and Y in $\mathcal{D}'$. Then Z is isomorphic to an object of $\mathcal{D}'$.*

Proof. There exists a d.t. $X \xrightarrow{f} Y \to Z' \to TX$ in $\mathcal{D}'$. Then $X \xrightarrow{f} Y \to Z \to TX$ is isomorphic to $X \xrightarrow{f} Y \to Z' \to TX$ in $\mathcal{D}$ by TR4 and Proposition 10.1.15. q.e.d.

By Proposition 10.1.15, we obtain that the object Z given in TR2 is unique up to isomorphism. As already mentioned, the fact that this isomorphism is not unique is the source of many difficulties (e.g., gluing problems in sheaf theory). Let us give a criterion which ensures, in some very special cases, the uniqueness of the third term of a d.t.

Proposition 10.1.17. *In the situation of* TR4 *assume that* $\mathrm{Hom}_{\mathcal{D}}(Y, X') = 0$ *and* $\mathrm{Hom}_{\mathcal{D}}(TX, Y') = 0$*. Then γ is unique.*

Proof. We may replace α and β by the zero morphisms and prove that in this case, γ is zero.

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & TX \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle 0} \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & TX' .
\end{array}$$

We shall apply Proposition 10.1.13. Since $h' \circ \gamma = 0$, γ factorizes through g', i.e., there exists $u\colon Z \to Y'$ with $\gamma = g' \circ u$. Similarly, since $\gamma \circ g = 0$, γ factorizes through h, i.e., there exists $v\colon TX \to Z'$ with $\gamma = v \circ h$.

By TR4, there exists a morphism w defining a morphism of d.t.'s:

$$\begin{array}{ccccccc}
Y & \xrightarrow{g} & Z & \xrightarrow{h} & TX & \xrightarrow{-T(f)} & TY \\
 & \swarrow{\scriptstyle u} & & \swarrow{\scriptstyle v} & & \swarrow{\scriptstyle w} & \\
Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & TX' & \longrightarrow & TY' .
\end{array}$$

By the hypothesis, $w = 0$. Hence v factorizes through Y', and by the hypothesis this implies that $v = 0$. Therefore, $\gamma = 0$. q.e.d.

Proposition 10.1.18. *Let $\mathcal{T}$ and $\mathcal{D}$ be triangulated categories and let $F\colon \mathcal{T} \to \mathcal{D}$ be a triangulated functor. Then F is exact* (*see* Definition 3.3.1).

Proof. (i) Let us show that F is right exact, that is, for any $X \in \mathcal{D}$, the category $\mathcal{T}_X$ is filtrant.
(a) The category $\mathcal{T}_X$ is non empty since it contains the object $0 \to X$.
(b) Let (Y_0, s_0) and (Y_1, s_1) be two objects in $\mathcal{T}_X$ with $Y_i \in \mathcal{T}$ and $s_i : F(Y_i) \to X$, $i = 0, 1$. The morphisms s_0 and s_1 define $s\colon F(Y_0 \oplus Y_1) \to X$. Hence, we obtain morphisms $(Y_i, s_i) \to (Y_0 \oplus Y_1, s)$ for $i = 0, 1$.
(c) Consider a pair of parallel arrows $f, g\colon (Y_0, s_0) \rightrightarrows (Y_1, s_1)$ in $\mathcal{T}_X$. Let us embed $f - g\colon Y_0 \to Y_1$ in a d.t. $Y_0 \xrightarrow{f-g} Y_1 \xrightarrow{h} Y \to TY_0$. Since $s_1 \circ F(f) = s_1 \circ F(g)$, Proposition 10.1.13 implies that the morphism $s_1\colon F(Y_1) \to X$ factorizes as $F(Y_1) \to F(Y) \xrightarrow{t} X$. Hence, the two compositions $(Y_0, s_0) \rightrightarrows (Y_1, s_1) \to (Y, t)$ coincide.
(ii) Replacing $F\colon \mathcal{T} \to \mathcal{D}$ with $F^{\mathrm{op}} : \mathcal{T}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$, we find that F is left exact. q.e.d.

Proposition 10.1.19. *Let $\mathcal{D}$ be a triangulated category which admits direct sums indexed by a set I. Then direct sums indexed by I commute with the translation functor T, and a direct sum of distinguished triangles indexed by I is a distinguished triangle.*

Proof. The first assertion is obvious since T is an equivalence of categories.

Let $D_i : X_i \to Y_i \to Z_i \to TX_i$ be a family of d.t.'s indexed by $i \in I$. Let D be the triangle

$$\oplus_{i\in I} D_i : \ \oplus_i X_i \to \oplus_i Y_i \to \oplus_i Z_i \to \oplus_i TX_i .$$

By TR2 there exists a d.t. $D' : \oplus_i X_i \to \oplus_i Y_i \to Z \to T(\oplus_i X_i)$. By TR3 there exist morphisms of triangles $D_i \to D'$ and they induce a morphism $D \to D'$. Let $W \in \mathcal{D}$ and let us show that the morphism $\mathrm{Hom}_{\mathcal{D}}(D', W) \to \mathrm{Hom}_{\mathcal{D}}(D, W)$ is an isomorphism. This will imply the isomorphism $D \xrightarrow{\sim} D'$ by Corollary 1.4.7. Consider the commutative diagram of complexes

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\mathcal{D}}(T(\oplus_i Y_i), W) & \to & \mathrm{Hom}_{\mathcal{D}}(T(\oplus_i X_i), W) & \to & \mathrm{Hom}_{\mathcal{D}}(Z, W) & \to \\
\downarrow & & \downarrow & & \downarrow & \\
\mathrm{Hom}_{\mathcal{D}}(\oplus_i T Y_i, W) & \to & \mathrm{Hom}_{\mathcal{D}}(\oplus_i T X_i, W) & \to & \mathrm{Hom}_{\mathcal{D}}(\oplus_i Z_i, W) & \to
\end{array}$$

$$\begin{array}{ccccc}
\to & \mathrm{Hom}_{\mathcal{D}}(\oplus_i Y_i, W) & \to & \mathrm{Hom}_{\mathcal{D}}(\oplus_i X_i, W) \\
& \downarrow & & \downarrow \\
\to & \mathrm{Hom}_{\mathcal{D}}(\oplus_i Y_i, W) & \to & \mathrm{Hom}_{\mathcal{D}}(\oplus_i X_i, W).
\end{array}$$

The first row is exact since the functor $\mathrm{Hom}_{\mathcal{D}}$ is cohomological. The second row is isomorphic to

$$\prod_i \mathrm{Hom}_{\mathcal{D}}(TY_i, W) \to \prod_i \mathrm{Hom}_{\mathcal{D}}(TX_i, W) \to \prod_i \mathrm{Hom}_{\mathcal{D}}(Z_i, W)$$
$$\to \prod_i \mathrm{Hom}_{\mathcal{D}}(Y_i, W) \to \prod_i \mathrm{Hom}_{\mathcal{D}}(X_i, W) .$$

Since the functor $\prod_i$ is exact on $\mathrm{Mod}(\mathbb{Z})$, this complex is exact. Since the vertical arrows except the middle one are isomorphisms, the middle one is an isomorphism by Lemma 8.3.13. q.e.d.

As particular cases of Proposition 10.1.19, we get:

Corollary 10.1.20. *Let $\mathcal{D}$ be a triangulated category.*

(i) *Let $X_1 \to Y_1 \to Z_1 \to TX_1$ and $X_2 \to Y_2 \to Z_2 \to TX_2$ be two* d.t.*'s. Then $X_1 \oplus X_2 \to Y_1 \oplus Y_2 \to Z_1 \oplus Z_2 \to TX_1 \oplus TX_2$ is a* d.t.

(ii) *Let $X, Y \in \mathcal{D}$. Then $X \to X \oplus Y \to Y \xrightarrow{0} TX$ is a* d.t.

10.2 Localization of Triangulated Categories

Let $\mathcal{D}$ be a triangulated category, $\mathcal{N}$ a full saturated subcategory. (Recall that $\mathcal{N}$ is saturated if $X \in \mathcal{D}$ belongs to $\mathcal{N}$ whenever X is isomorphic to an object of $\mathcal{N}$.)

Lemma 10.2.1. (a) *Let $\mathcal{N}$ be a full saturated triangulated subcategory of $\mathcal{D}$. Then* $\mathrm{Ob}(\mathcal{N})$ *satisfies conditions* N1–N3 *below.*

N1 $0 \in \mathcal{N}$,

N2 $X \in \mathcal{N}$ *if and only if* $TX \in \mathcal{N}$,

N3 *if* $X \to Y \to Z \to TX$ *is a* d.t. *in* $\mathcal{D}$ *and* $X, Z \in \mathcal{N}$ *then* $Y \in \mathcal{N}$.

(b) *Conversely, let* $\mathcal{N}$ *be a full saturated subcategory of* $\mathcal{D}$ *and assume that* $\mathrm{Ob}(\mathcal{N})$ *satisfies conditions* N1–N3 *above. Then the restriction of* T *and the collection of* d.t.*'s* $X \to Y \to Z \to TX$ *in* $\mathcal{D}$ *with* X, Y, Z *in* $\mathcal{N}$ *make* $\mathcal{N}$ *a full saturated triangulated subcategory of* $\mathcal{D}$. *Moreover it satisfies*

N'3 *if* $X \to Y \to Z \to TX$ *is a* d.t. *in* $\mathcal{D}$ *and two objects among* X, Y, Z *belong to* $\mathcal{N}$, *then so does the third one.*

Proof. (a) Assume that $\mathcal{N}$ is a full saturated triangulated subcategory of $\mathcal{D}$. Then N1 and N2 are clearly satisfied. Moreover N3 follows from Corollary 10.1.16 and the hypothesis that $\mathcal{N}$ is saturated.
(b) Let $\mathcal{N}$ be a full subcategory of $\mathcal{D}$ satisfying N1–N3. Then N'3 follows from N2 and N3.
(i) Let us prove that $\mathcal{N}$ is saturated. Let $f\colon X \xrightarrow{\sim} Y$ be an isomorphism with $X \in \mathcal{N}$. The triangle $X \xrightarrow{f} Y \to 0 \to TX$ being isomorphic to the d.t. $X \xrightarrow{\mathrm{id}_X} X \to 0 \to TX$, it is itself a d.t. Hence, $Y \in \mathcal{N}$.
(ii) Let $X, Y \in \mathcal{N}$. Since $X \to X \oplus Y \to Y \xrightarrow{0} TX$ is a d.t., we find that $X \oplus Y$ belongs to $\mathcal{N}$, and it follows that $\mathcal{N}$ is a full additive subcategory of $\mathcal{D}$.
(iii) The axioms of triangulated categories are then easily checked. q.e.d.

Definition 10.2.2. *A* null system *in* $\mathcal{D}$ *is a full saturated subcategory* $\mathcal{N}$ *such that* $\mathrm{Ob}(\mathcal{N})$ *satisfies the conditions* N1–N3 *in* Lemma 10.2.1 (a).

We associate a family of morphisms to a null system as follows. Define:

(10.2.1)

$$\mathcal{N}Q := \{f\colon X \to Y;\ \text{there exists a d.t. } X \to Y \to Z \to TX \text{ with } Z \in \mathcal{N}\}.$$

Theorem 10.2.3. (i) $\mathcal{N}Q$ *is a right and left multiplicative system.*

(ii) *Denote by* $\mathcal{D}_{\mathcal{N}Q}$ *the localization of* $\mathcal{D}$ *by* $\mathcal{N}Q$ *and by* $Q\colon \mathcal{D} \to \mathcal{D}_{\mathcal{N}Q}$ *the localization functor. Then* $\mathcal{D}_{\mathcal{N}Q}$ *is an additive category endowed with an automorphism* (*the image of* T, *still denoted by* T).

(iii) *Define a* d.t. *in* $\mathcal{D}_{\mathcal{N}Q}$ *as being isomorphic to the image of a* d.t. *in* $\mathcal{D}$ *by* Q. *Then* $\mathcal{D}_{\mathcal{N}Q}$ *is a triangulated category and* Q *is a triangulated functor.*

(iv) *If* $X \in \mathcal{N}$, *then* $Q(X) \simeq 0$.

(v) *Let* $F\colon \mathcal{D} \to \mathcal{D}'$ *be a triangulated functor of triangulated categories such that* $F(X) \simeq 0$ *for any* $X \in \mathcal{N}$. *Then* F *factors uniquely through* Q.

One shall be aware that $\mathcal{D}_{\mathcal{N}Q}$ is a big category in general.

Notation 10.2.4. We will write $\mathcal{D}/\mathcal{N}$ instead of $\mathcal{D}_{\mathcal{N}Q}$.

Proof. (i) Since the opposite category of $\mathcal{D}$ is again triangulated and $\mathcal{N}^{\mathrm{op}}$ is a null system in $\mathcal{D}^{\mathrm{op}}$, it is enough to check that $\mathcal{N}Q$ is a right multiplicative system. Let us check the conditions S1–S4 in Definition 7.1.5.

S1: if $f\colon X \to Y$ is an isomorphism, the triangle $X \xrightarrow{f} Y \to 0 \to TX$ is a d.t. and we deduce $f \in \mathcal{N}Q$.

S2: Let $f\colon X \to Y$ and $g\colon Y \to Z$ be in $\mathcal{N}Q$. By TR3, there are d.t.'s $X \xrightarrow{f} Y \to Z' \to TX$, $Y \xrightarrow{g} Z \to X' \to TY$, and $X \xrightarrow{g\circ f} Z \to Y' \to TX$. By TR5, there exists a d.t. $Z' \to Y' \to X' \to TZ'$. Since Z' and X' belong to $\mathcal{N}$, so does Y'.

S3: Let $f\colon X \to Y$ and $s\colon X \to X'$ be two morphisms with $s \in \mathcal{N}Q$. By the hypothesis, there exists a d.t. $W \xrightarrow{h} X \xrightarrow{s} X' \to TW$ with $W \in \mathcal{N}$. By TR2, there exists a d.t. $W \xrightarrow{f\circ h} Y \xrightarrow{t} Z \to TW$, and by TR4, there exists a commutative diagram

$$\begin{array}{ccccccc} W & \xrightarrow{h} & X & \xrightarrow{s} & X' & \longrightarrow & TW \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow \\ W & \xrightarrow[f\circ h]{} & Y & \xrightarrow[t]{} & Z & \longrightarrow & TW\,. \end{array}$$

Since $W \in \mathcal{N}$, we get $t \in \mathcal{N}Q$.

S4: Replacing f by $f - g$, it is enough to check that if there exists $s \in \mathcal{N}Q$ with $f \circ s = 0$, then there exists $t \in \mathcal{N}Q$ with $t \circ f = 0$. Consider the diagram

$$\begin{array}{ccccccc} X' & \xrightarrow{s} & X & \xrightarrow{k} & Z & \longrightarrow & TX' \\ & & & \searrow^{f} & \vdots{\scriptstyle h} & & \\ & & & & Y & & \\ & & & & \vdots{\scriptstyle t} & & \\ & & & & Y'\,. & & \end{array}$$

Here, the row is a d.t. with $Z \in \mathcal{N}$. Since $s \circ f = 0$, the arrow h, making the diagram commutative, exists by Proposition 10.1.13. There exists a d.t. $Z \to Y \xrightarrow{t} Y' \to TZ$ by TR2. We thus obtain $t \in \mathcal{N}Q$ since $Z \in \mathcal{N}$. Finally, $t \circ h = 0$ implies that $t \circ f = t \circ h \circ k = 0$.

(ii) follows from the result of Exercise 8.4.

(iii) Axioms TR0–TR3 are obviously satisfied. Let us prove TR4. With the notations of TR4, and using the result of Exercise 7.4, we may assume that there exists a commutative diagram in $\mathcal{D}$ of solid arrows, with s and t in $\mathcal{N}Q$

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & TX \\ \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \beta'} & & \vdots{\scriptstyle \gamma'} & & \downarrow{\scriptstyle T(\alpha')} \\ X_1 & \xrightarrow{f_1} & Y_1 & \overset{g_1}{\cdots\to} & Z_1 & \cdots\to & TX_1 \\ \uparrow{\scriptstyle s} & & \uparrow{\scriptstyle t} & A & \vdots{\scriptstyle u} & B & \uparrow{\scriptstyle T(s)} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \longrightarrow & TX'\,. \end{array}$$

After having embedded $f_1 \colon X_1 \to Y_1$ in a d.t., we construct the commutative squares labeled by A and B with $u \in \mathcal{N}Q$ by using the result of Exercise 10.6. (In diagram (10.5.5) of this exercise, if Z^0 and Z^1 are in $\mathcal{N}$, then so is Z^2.) Then we construct the morphism γ' using TR4.

Let us prove TR5. Consider two morphisms in $\mathcal{D}/\mathcal{N}$: $f \colon X \to Y$ and $g \colon Y \to Z$. We may represent them by morphisms in $\mathcal{D}$: $\tilde{f} \colon \tilde{X} \to \tilde{Y}$ and $\tilde{g} \colon \tilde{Y} \to \tilde{Z}$. Then apply TR5 (in $\mathcal{D}$) and take the image in $\mathcal{D}/\mathcal{N}$ of the octahedron diagram (10.1.2).

(iv) Consider a d.t. $0 \to X \to X \to T(0)$. The morphism $0 \to X$ belongs to $\mathcal{N}Q$. Hence, $Q(0) \to Q(X)$ is an isomorphism.

(v) is obvious. q.e.d.

Let $\mathcal{N}$ be a null system and let $X \in \mathcal{D}$. The categories $\mathcal{N}Q_X$ and $\mathcal{N}Q^X$ attached to the multiplicative system $\mathcal{N}Q$ (see Sect. 7.1) are given by:

(10.2.2) $\quad \mathrm{Ob}(\mathcal{N}Q^X) = \{s \colon X \to X'; s \in \mathcal{N}Q\}$,

(10.2.3) $\quad \mathrm{Hom}_{\mathcal{N}Q^X}((s \colon X \to X'), (s' \colon X \to X'')) = \{h \colon X' \to X''; h \circ s = s'\}$

and similarly for $\mathcal{N}Q_X$.

Remark 10.2.5. It follows easily from TR5 that the morphism h in (10.2.3) belongs to $\mathcal{N}Q$. Therefore, by considering $\mathcal{N}Q$ as a subcategory of $\mathcal{D}$, the category $\mathcal{N}Q^X$ is the category given by Definition 1.2.16 (with respect to the identity functor $\mathrm{id} \colon \mathcal{N}Q \to \mathcal{N}Q$). The same result holds for $\mathcal{N}Q_X$.

By Lemma 7.1.10 the categories $(\mathcal{N}Q_X)^{\mathrm{op}}$ and $\mathcal{N}Q^X$ are filtrant, and by the definition of the localization functor we get

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}/\mathcal{N}}(X,Y) &\simeq \varinjlim_{(Y\to Y')\in\mathcal{N}Q} \mathrm{Hom}_{\mathcal{D}}(X,Y') \\ &\simeq \varinjlim_{(X'\to X)\in\mathcal{N}Q} \mathrm{Hom}_{\mathcal{D}}(X',Y) \\ &\simeq \varinjlim_{(Y\to Y')\in\mathcal{N}Q,(X'\to X)\in\mathcal{N}Q} \mathrm{Hom}_{\mathcal{D}}(X',Y') \, .\end{aligned}$$

Now consider a full triangulated subcategory $\mathcal{I}$ of $\mathcal{D}$. We shall write $\mathcal{N} \cap \mathcal{I}$ for the full subcategory whose objects are $\mathrm{Ob}(\mathcal{N}) \cap \mathrm{Ob}(\mathcal{I})$. This is clearly a null system in $\mathcal{I}$.

Proposition 10.2.6. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{N}$ a null system, $\mathcal{I}$ a full triangulated subcategory of $\mathcal{D}$. Assume condition* (i) *or* (ii) *below:*

(i) *any morphism $Y \to Z$ with $Y \in \mathcal{I}$ and $Z \in \mathcal{N}$ factorizes as $Y \to Z' \to Z$ with $Z' \in \mathcal{N} \cap \mathcal{I}$,*

(ii) *any morphism $Z \to Y$ with $Y \in \mathcal{I}$ and $Z \in \mathcal{N}$ factorizes as $Z \to Z' \to Y$ with $Z' \in \mathcal{N} \cap \mathcal{I}$.*

Then $\mathcal{I}/(\mathcal{N} \cap \mathcal{I}) \to \mathcal{D}/\mathcal{N}$ is fully faithful.

Proof. We may assume (ii), the case (i) being deduced by considering $\mathcal{D}^{\mathrm{op}}$. We shall apply Proposition 7.2.1. Let $f\colon X \to Y$ is a morphism in $\mathcal{N}Q$ with $X \in \mathcal{I}$. We shall show that there exists $g\colon Y \to W$ with $W \in \mathcal{I}$ and $g \circ f \in \mathcal{N}Q$. The morphism f is embedded in a d.t. $X \to Y \to Z \to TX$ with $Z \in \mathcal{N}$. By the hypothesis, the morphism $Z \to TX$ factorizes through an object $Z' \in \mathcal{N} \cap \mathcal{I}$. We may embed $Z' \to TX$ in a d.t. in $\mathcal{I}$ and obtain a commutative diagram of d.t.'s by TR4:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & TX \\ \downarrow{\scriptstyle \mathrm{id}} & & \vdots{\scriptstyle g} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ X & \xrightarrow{g\circ f} & W & \longrightarrow & Z' & \longrightarrow & TX\,. \end{array}$$

Since Z' belongs to $\mathcal{N}$, we get that $g \circ f \in \mathcal{N}Q \cap \mathrm{Mor}(\mathcal{I})$. q.e.d.

Proposition 10.2.7. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{N}$ a null system, $\mathcal{I}$ a full triangulated subcategory of $\mathcal{D}$, and assume conditions* (i) *or* (ii) *below:*

(i) *for any $X \in \mathcal{D}$, there exists a morphism $X \to Y$ in $\mathcal{N}Q$ with $Y \in \mathcal{I}$,*
(ii) *for any $X \in \mathcal{D}$, there exists a morphism $Y \to X$ in $\mathcal{N}Q$ with $Y \in \mathcal{I}$.*

Then $\mathcal{I}/(\mathcal{N} \cap \mathcal{I}) \to \mathcal{D}/\mathcal{N}$ is an equivalence of categories.

Proof. Apply Corollary 7.2.2. q.e.d.

Proposition 10.2.8. *Let $\mathcal{D}$ be a triangulated category admitting direct sums indexed by a set I and let $\mathcal{N}$ be a null system closed by such direct sums. Let $Q\colon \mathcal{D} \to \mathcal{D}/\mathcal{N}$ denote the localization functor. Then $\mathcal{D}/\mathcal{N}$ admits direct sums indexed by I and the localization functor $Q\colon \mathcal{D} \to \mathcal{D}/\mathcal{N}$ commutes with such direct sums.*

Proof. Let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{D}$. It is enough to show that $Q(\oplus_i X_i)$ is the direct sum of the family $Q(X_i)$, i.e., the map

$$\mathrm{Hom}_{\mathcal{D}/\mathcal{N}}(Q(\bigoplus_{i\in I} X_i), Y) \to \prod_{i\in I} \mathrm{Hom}_{\mathcal{D}/\mathcal{N}}(Q(X_i), Y)$$

is bijective for any $Y \in \mathcal{D}$.
(i) Surjectivity. Let $u_i \in \mathrm{Hom}_{\mathcal{D}/\mathcal{N}}(Q(X_i), Y)$. The morphism u_i is represented by a morphism $u_i'\colon X_i' \to Y$ in $\mathcal{D}$ together with a d.t. $X_i' \xrightarrow{v_i} X_i \xrightarrow{w_i} Z_i \to TX_i'$ in $\mathcal{D}$ with $Z_i \in \mathcal{N}$. We get a morphism $\oplus_i X_i' \to Y$ and a d.t. $\oplus_i X_i' \to \oplus_i X_i \to \oplus_i Z_i \to T(\oplus_i X_i')$ in $\mathcal{D}$ with $\oplus_i Z_i \in \mathcal{N}$.

(ii) Injectivity. Assume that the composition $Q(X_i) \to Q(\oplus_{i'} X_{i'}) \xrightarrow{u} Q(Y)$ is zero for every $i \in I$. By the definition, the morphism u is represented by morphisms $u'\colon \oplus_i X_i \xrightarrow{u'} Y' \xleftarrow{s} Y$ with $s \in \mathcal{N}Q$. Using the result of Exercise 10.11, we can find $Z_i \in \mathcal{N}$ such that $v_i'\colon X_i \to Y'$ factorizes as $X_i \to Z_i \to Y'$. Then $\oplus_i X_i \to Y'$ factorizes as $\oplus_i X_i \to \oplus_i Z_i \to Y'$. Since $\oplus_i Z_i \in \mathcal{N}$, $Q(u) = 0$. q.e.d.

10.3 Localization of Triangulated Functors

Let $F\colon \mathcal{D} \to \mathcal{D}'$ be a functor of triangulated categories, $\mathcal{N}$ and $\mathcal{N}'$ null systems in $\mathcal{D}$ and $\mathcal{D}'$, respectively. The right or left localization of F (when it exists) is defined by mimicking Definition 7.3.1, replacing "functor" by "triangulated functor".

In the sequel, $\mathcal{D}$ (resp. $\mathcal{D}'$, $\mathcal{D}''$) is a triangulated category and $\mathcal{N}$ (resp. $\mathcal{N}'$, $\mathcal{N}''$) is a null system in this category. We denote by $Q\colon \mathcal{D} \to \mathcal{D}/\mathcal{N}$ (resp. $Q'\colon \mathcal{D}' \to \mathcal{D}'/\mathcal{N}'$, $Q''\colon \mathcal{D}'' \to \mathcal{D}''/\mathcal{N}''$) the localization functor and by $\mathcal{N}'Q$ (resp. $\mathcal{N}''Q$) the family of morphisms in $\mathcal{D}'$ (resp. $\mathcal{D}''$) defined in (10.2.1).

Definition 10.3.1. *We say that a triangulated functor $F\colon \mathcal{D} \to \mathcal{D}'$ is right* (*resp. left*) *localizable with respect to* $(\mathcal{N}, \mathcal{N}')$ *if $Q' \circ F\colon \mathcal{D} \to \mathcal{D}'/\mathcal{N}'$ is universally right* (*resp. left*) *localizable with respect to the multiplicative system $\mathcal{N}Q$* (*see* Definition 7.3.1). *Recall that it means that, for any $X \in \mathcal{D}$, $\text{“}\varinjlim_{(X\to Y)\in\mathcal{N}Q^X}\text{”}\, Q'F(Y)$* (*resp.* $\text{“}\varprojlim_{(Y\to X)\in\mathcal{N}Q_X}\text{”}\, Q'F(Y)$) *is representable in $\mathcal{D}'/\mathcal{N}'$. If there is no risk of confusion, we simply say that F is right* (*resp. left*) *localizable or that RF exists.*

Definition 10.3.2. *Let $F\colon \mathcal{D} \to \mathcal{D}'$ be a triangulated functor of triangulated categories, $\mathcal{N}$ and $\mathcal{N}'$ null systems in $\mathcal{D}$ and $\mathcal{D}'$, and $\mathcal{I}$ a full triangulated subcategory of $\mathcal{D}$. Consider the conditions* (i), (ii), (iii) *below.*

(i) *For any $X \in \mathcal{D}$, there exists a morphism $X \to Y$ in $\mathcal{N}Q$ with $Y \in \mathcal{I}$.*
(ii) *For any $X \in \mathcal{D}$, there exists a morphism $Y \to X$ in $\mathcal{N}Q$ with $Y \in \mathcal{I}$.*
(iii) *For any $Y \in \mathcal{N} \cap \mathcal{I}$, $F(Y) \in \mathcal{N}'$.*

Then

(a) *if conditions* (i) *and* (iii) *are satisfied, we say that the subcategory $\mathcal{I}$ is* F-injective *with respect to $\mathcal{N}$ and $\mathcal{N}'$,*
(b) *if conditions* (ii) *and* (iii) *are satisfied, we say that the subcategory $\mathcal{I}$ is* F-projective *with respect to $\mathcal{N}$ and $\mathcal{N}'$.*

If there is no risk of confusion, we omit "with respect to $\mathcal{N}$ and $\mathcal{N}'$".

Note that if $F(\mathcal{N}) \subset \mathcal{N}'$, then $\mathcal{D}$ is both F-injective and F-projective.

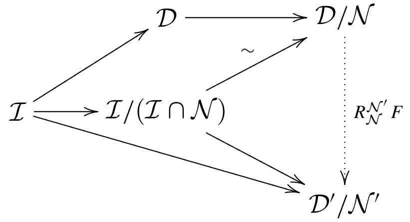
Proposition 10.3.3. *Let $F\colon \mathcal{D} \to \mathcal{D}'$ be a triangulated functor of triangulated categories, $\mathcal{N}$ and $\mathcal{N}'$ null systems in $\mathcal{D}$ and $\mathcal{D}'$, and $\mathcal{I}$ a full triangulated category of $\mathcal{D}$.*

(a) *If $\mathcal{I}$ is F-injective with respect to $\mathcal{N}$ and $\mathcal{N}'$, then F is right localizable and its right localization is a triangulated functor.*
(b) *If $\mathcal{I}$ is F-projective with respect to $\mathcal{N}$ and $\mathcal{N}'$, then F left localizable and its left localization is a triangulated functor.*

Proof. Apply Proposition 7.3.2. q.e.d.

Notation 10.3.4. (i) We denote by $R_{\mathcal{N}}^{\mathcal{N}'}F\colon \mathcal{D}/\mathcal{N} \to \mathcal{D}'/\mathcal{N}'$ the right localization of F with respect to $(\mathcal{N}, \mathcal{N}')$. If there is no risk of confusion, we simply write RF instead of $R_{\mathcal{N}}^{\mathcal{N}'}F$.
(ii) We denote by $L_{\mathcal{N}}^{\mathcal{N}'}F\colon \mathcal{D}/\mathcal{N} \to \mathcal{D}'/\mathcal{N}'$ the left localization of F with respect to $(\mathcal{N}, \mathcal{N}')$. If there is no risk of confusion, we simply write LF instead of $L_{\mathcal{N}}^{\mathcal{N}'}F$.

If $\mathcal{I}$ is F-injective, $R_{\mathcal{N}}^{\mathcal{N}'}F$ may be defined by the diagram:



and

$$R_{\mathcal{N}}^{\mathcal{N}'}F(X) \simeq F(Y) \quad \text{for } (X \to Y) \in \mathcal{N}Q \text{ with } Y \in \mathcal{I}\,. \tag{10.3.1}$$

Similarly, if $\mathcal{I}$ is F-projective, the diagram above defines $L_{\mathcal{N}}^{\mathcal{N}'}F$ and

$$L_{\mathcal{N}}^{\mathcal{N}'}F(X) \simeq F(Y) \quad \text{for } (Y \to X) \in \mathcal{N}Q \text{ with } Y \in \mathcal{I}\,. \tag{10.3.2}$$

Proposition 10.3.5. *Let $F\colon \mathcal{D} \to \mathcal{D}'$ and $F'\colon \mathcal{D}' \to \mathcal{D}''$ be triangulated functors of triangulated categories and let $\mathcal{N}$, $\mathcal{N}'$ and $\mathcal{N}''$ be null systems in $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}''$, respectively.*

(i) *Assume that $R_{\mathcal{N}}^{\mathcal{N}'}F$, $R_{\mathcal{N}'}^{\mathcal{N}''}F'$ and $R_{\mathcal{N}}^{\mathcal{N}''}(F'\circ F)$ exist. Then there is a canonical morphism in* $\mathrm{Fct}(\mathcal{D}/\mathcal{N}, \mathcal{D}''/\mathcal{N}'')$*:*

$$R_{\mathcal{N}}^{\mathcal{N}''}(F'\circ F) \to R_{\mathcal{N}'}^{\mathcal{N}''}F' \circ R_{\mathcal{N}}^{\mathcal{N}'}F\,. \tag{10.3.3}$$

(ii) *Let $\mathcal{I}$ and $\mathcal{I}'$ be full triangulated subcategories of $\mathcal{D}$ and $\mathcal{D}'$, respectively. Assume that $\mathcal{I}$ is F-injective with respect to $\mathcal{N}$ and $\mathcal{N}'$, $\mathcal{I}'$ is F'-injective with respect to $\mathcal{N}'$ and $\mathcal{N}''$, and $F(\mathcal{I}) \subset \mathcal{I}'$. Then $\mathcal{I}$ is $(F'\circ F)$-injective with respect to $\mathcal{N}$ and $\mathcal{N}''$, and* (10.3.3) *is an isomorphism.*

Proof. (i) By Definition 7.3.1, there are a bijection

$$\begin{aligned}\mathrm{Hom}\,(R_{\mathcal{N}}^{\mathcal{N}''}(F'\circ F), R_{\mathcal{N}'}^{\mathcal{N}''}F' \circ R_{\mathcal{N}}^{\mathcal{N}'}F)&\\ \simeq \mathrm{Hom}\,(Q''\circ F'\circ F, R_{\mathcal{N}'}^{\mathcal{N}''}F' \circ R_{\mathcal{N}}^{\mathcal{N}'}F\circ Q)&\,,\end{aligned}$$

and natural morphisms of functors

$$Q''\circ F' \to R_{\mathcal{N}'}^{\mathcal{N}''}F' \circ Q', \quad Q'\circ F \to R_{\mathcal{N}}^{\mathcal{N}'}F \circ Q\,.$$

We deduce the canonical morphisms

$$Q'' \circ F' \circ F \to R_{\mathcal{N}'}^{\mathcal{N}''} F' \circ Q' \circ F \to R_{\mathcal{N}'}^{\mathcal{N}''} F' \circ R_{\mathcal{N}}^{\mathcal{N}'} F \circ Q$$

and the result follows.
(ii) The fact that $\mathcal{I}$ is $(F' \circ F)$-injective follows immediately from the definition. Let $X \in \mathcal{D}$ and consider a morphism $X \to Y$ in $\mathcal{N}Q$ with $Y \in \mathcal{I}$. Then $R_{\mathcal{N}}^{\mathcal{N}'} F(X) \simeq F(Y)$ by (10.3.1) and $F(Y) \in \mathcal{I}'$ by the hypothesis. Hence $(R_{\mathcal{N}'}^{\mathcal{N}''} F')(F(Y)) \simeq F'F(Y)$ by (10.3.1) and we find

$$(R_{\mathcal{N}'}^{\mathcal{N}''} F')(R_{\mathcal{N}}^{\mathcal{N}'} F(X)) \simeq F'F(Y) .$$

On the other hand, $R_{\mathcal{N}}^{\mathcal{N}''}(F' \circ F)(X) \simeq F'F(Y)$ by (10.3.1) since $\mathcal{I}$ is $(F' \circ F)$-injective. q.e.d.

Triangulated Bifunctors

Definition 10.3.6. *Let $(\mathcal{D}, T)$, $(\mathcal{D}', T')$ and $(\mathcal{D}'', T'')$ be triangulated categories. A* triangulated bifunctor $F : \mathcal{D} \times \mathcal{D}' \to \mathcal{D}''$ *is a bifunctor of additive categories with translation (see* Definition 10.1.1 (v)) *which sends* d.t.*'s in each argument to* d.t.*'s.*

Definition 10.3.7. *Let $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}''$ be triangulated categories, $\mathcal{N}$, $\mathcal{N}'$ and $\mathcal{N}''$ null systems in $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}''$, respectively. We say that a triangulated bifunctor $F : \mathcal{D} \times \mathcal{D}' \to \mathcal{D}''$ is right (resp. left) localizable with respect to $(\mathcal{N} \times \mathcal{N}', \mathcal{N}'')$ if $Q'' \circ F : \mathcal{D} \times \mathcal{D}' \to \mathcal{D}''/\mathcal{N}''$ is universally right (resp. left) localizable with respect to the multiplicative system $\mathcal{N}Q \times \mathcal{N}'Q$ (see* Remark 7.4.5). *If there is no risk of confusion, we simply say that F is right (resp. left) localizable.*

Notation 10.3.8. We denote by $R_{\mathcal{N} \times \mathcal{N}'}^{\mathcal{N}''} F : \mathcal{D}/\mathcal{N} \times \mathcal{D}'/\mathcal{N}' \to \mathcal{D}''/\mathcal{N}''$ the right localization of F with respect to $(\mathcal{N} \times \mathcal{N}', \mathcal{N}'')$, if it exists. If there is no risk of confusion, we simply write RF. We use similar notations for the left localization.

Definition 10.3.9. *Let $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}''$ be triangulated categories, $\mathcal{N}$, $\mathcal{N}'$ and $\mathcal{N}''$ null systems in $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}''$, respectively, and $\mathcal{I}, \mathcal{I}'$ full triangulated subcategories of $\mathcal{D}$ and $\mathcal{D}'$, respectively. Let $F : \mathcal{D} \times \mathcal{D}' \to \mathcal{D}''$ be a triangulated bifunctor. The pair $(\mathcal{I}, \mathcal{I}')$ is F-*injective *with respect to $(\mathcal{N}, \mathcal{N}', \mathcal{N}'')$ if*

(i) *$\mathcal{I}'$ is $F(Y, \cdot\,)$-injective with respect to $\mathcal{N}'$ and $\mathcal{N}''$ for any $Y \in \mathcal{I}$,*
(ii) *$\mathcal{I}$ is $F(\,\cdot\,, Y')$-injective with respect to $\mathcal{N}$ and $\mathcal{N}''$ for any $Y' \in \mathcal{I}'$.*

These two conditions are equivalent to saying that

(a) for any $X \in \mathcal{D}$, there exists a morphism $X \to Y$ in $\mathcal{N}Q$ with $Y \in \mathcal{I}$,
(b) for any $X' \in \mathcal{D}'$, there exists a morphism $X' \to Y'$ in $\mathcal{N}'Q$ with $Y' \in \mathcal{I}'$,

(c) $F(X, X')$ belongs to $\mathcal{N}''$ for $X \in \mathcal{I}$, $X' \in \mathcal{I}'$ as soon as X belongs to $\mathcal{N}$ or X' belongs to $\mathcal{N}'$.

The property for $(\mathcal{I}, \mathcal{I}')$ of being *F-projective* is defined similarly.

Proposition 10.3.10. *Let $\mathcal{D}, \mathcal{N}, \mathcal{I}$, $\mathcal{D}', \mathcal{N}', \mathcal{I}'$, $\mathcal{D}'', \mathcal{N}''$ and F be as in* Definition 10.3.9. *Assume that $(\mathcal{I}, \mathcal{I}')$ is F-injective with respect to $(\mathcal{N}, \mathcal{N}')$. Then F is right localizable, its right localization $R^{\mathcal{N}''}_{\mathcal{N}\mathcal{N}'}F$ is a triangulated bifunctor*

$$R^{\mathcal{N}''}_{\mathcal{N}\mathcal{N}'}F \colon \mathcal{D}/\mathcal{N} \times \mathcal{D}'/\mathcal{N}' \to \mathcal{D}''/\mathcal{N}'' ,$$

and moreover

$$(10.3.4) \quad R^{\mathcal{N}''}_{\mathcal{N}\mathcal{N}'}F(X, X') \simeq F(Y, Y') \text{ for } (X \to Y) \in \mathcal{N}Q \text{ and } (X' \to Y') \in \mathcal{N}'Q \text{ with } Y \in \mathcal{I},\ Y' \in \mathcal{I}'.$$

Of course, there exists a similar result by replacing "injective" with "projective" and reversing the arrows in (10.3.4).

Corollary 10.3.11. *Let $\mathcal{D}$, $\mathcal{N}$, $\mathcal{I}$, $\mathcal{D}'$, $\mathcal{N}'$, and $\mathcal{D}''$, $\mathcal{N}''$ be as in* Proposition 10.3.10. *Let $F \colon \mathcal{D} \times \mathcal{D}' \to \mathcal{D}''$ be a triangulated bifunctor. Assume that*

(i) *$F(\mathcal{I}, \mathcal{N}') \subset \mathcal{N}''$,*
(ii) *for any $X' \in \mathcal{D}'$, $\mathcal{I}$ is $F(\,\cdot\,, X')$-injective with respect to $\mathcal{N}$.*

Then F is right localizable. Moreover,

$$R^{\mathcal{N}''}_{\mathcal{N}\mathcal{N}'}F(X, X') \simeq R^{\mathcal{N}''}_{\mathcal{N}}F(\,\bullet\,, X')(X) .$$

Here again, there is a similar statement by replacing "injective" with "projective".

10.4 Extension of Cohomological Functors

In this section, we consider two triangulated categories $\mathcal{T}$ and $\mathcal{D}$, a triangulated functor $\varphi \colon \mathcal{T} \to \mathcal{D}$, an abelian category $\mathcal{A}$, and a cohomological functor $F \colon \mathcal{T} \to \mathcal{A}$. For $X \in \mathcal{D}$, we denote as usual by $\mathcal{T}_X$ the category whose objects are the pairs (Y, u) of objects $Y \in \mathcal{T}$ and morphisms $u \colon \varphi(Y) \to X$.

We make the hypotheses:

$$(10.4.1) \quad \begin{cases} \mathcal{A} \text{ admits small filtrant inductive limits and such limits are exact ,} \\ \mathcal{T}_X \text{ is cofinally small for any } X \in \mathcal{D} . \end{cases}$$

Note that the functor $\varphi \colon \mathcal{T} \to \mathcal{D}$ is exact by Proposition 10.1.18. Hence, Theorem 3.3.18 asserts that the functor $\varphi_* \colon \mathrm{Fct}(\mathcal{D}, \mathcal{A}) \to \mathrm{Fct}(\mathcal{T}, \mathcal{A})$ admits a left adjoint $\varphi^\dagger$ such that for $F \colon \mathcal{D} \to \mathcal{A}$ we have

$$(10.4.2) \quad \varphi^\dagger F(X) = \varinjlim_{(\varphi(Y) \to X) \in \mathcal{T}_X} F(Y) ,$$

and there is a natural morphism of functors

$$F \to (\varphi^\dagger F) \circ \varphi \,. \tag{10.4.3}$$

Theorem 10.4.1. *Let $\varphi\colon \mathcal{T} \to \mathcal{D}$ be a triangulated functor of triangulated categories, let $\mathcal{A}$ be an abelian category, and assume* (10.4.1). *Let $F\colon \mathcal{T} \to \mathcal{A}$ be a cohomological functor. Then the functor $\varphi^\dagger F$ is additive and cohomological.*

Proof. (i) Let us first show that $\varphi^\dagger F$ is additive. By Proposition 8.2.15, it is enough to show that $\varphi^\dagger F(X_1 \oplus X_2) \to \varphi^\dagger F(X_1) \oplus \varphi^\dagger F(X_2)$ is an isomorphism for any $X_1, X_2 \in \mathcal{D}$. Let $\xi\colon \mathcal{T}_{X_1} \times \mathcal{T}_{X_2} \to \mathcal{T}_{X_1 \oplus X_2}$ be the functor given by $\big((\varphi(Y_1) \to X_1), (\varphi(Y_2) \to X_2)\big) \mapsto (\varphi(Y_1 \oplus Y_2) \to X_1 \oplus X_2)$. Then ξ has a left adjoint $\eta\colon \mathcal{T}_{X_1 \oplus X_2} \to \mathcal{T}_{X_1} \times \mathcal{T}_{X_2}$ given by $(\varphi(Y) \to X_1 \oplus X_2) \mapsto \big((\varphi(Y) \to X_1 \oplus X_2 \to X_1), (\varphi(Y) \to X_1 \oplus X_2 \to X_2)\big)$. Hence ξ is a cofinal functor by Lemma 3.3.10. Moreover, the canonical functor $\mathcal{T}_{X_1} \times \mathcal{T}_{X_2} \to \mathcal{T}_{X_i}$ $(i = 1, 2)$ is cofinal. Hence we obtain

$$\begin{aligned}
\varphi^\dagger F(X_1 \oplus X_2) &\simeq \varinjlim_{Y \in \mathcal{T}_{X_1 \oplus X_2}} F(Y) \\
&\simeq \varinjlim_{(Y_1, Y_2) \in \mathcal{T}_{X_1} \oplus \mathcal{T}_{X_2}} F(Y_1 \oplus Y_2) \\
&\simeq \varinjlim_{(Y_1, Y_2) \in \mathcal{T}_{X_1} \oplus \mathcal{T}_{X_2}} F(Y_1) \oplus F(Y_2) \\
&\simeq \big(\varinjlim_{Y_1 \in \mathcal{T}_{X_1}} F(Y_1)\big) \oplus \big(\varinjlim_{Y_2 \in \mathcal{T}_{X_2}} F(Y_2)\big) \\
&\simeq \varphi^\dagger F(X_1) \oplus \varphi^\dagger F(X_2) \,.
\end{aligned}$$

(ii) Let us show that $\varphi^\dagger F$ is cohomological. We shall denote by X, Y, Z objects of $\mathcal{D}$ and by X_0, Y_0, Z_0 objects of $\mathcal{T}$.

By Proposition 10.1.18, the functor φ is exact. This result together with Corollary 3.4.6 implies that:

(a) for $X \in \mathcal{D}$ the category $\mathcal{T}_X$ is filtrant and cofinally small,
(b) for a morphism $g\colon Y \to Z$ in $\mathcal{D}$, the category $\mathrm{Mor}(\mathcal{T})_g$ is filtrant, cofinally small, and the two natural functors from $\mathrm{Mor}(\mathcal{T})_g$ to $\mathcal{T}_Y$ and $\mathcal{T}_Z$ are cofinal.

By (b), for a morphism $g\colon Y \to Z$ in $\mathcal{D}$, we get

$$\varphi^\dagger F(Y) \simeq \varinjlim_{(Y_0 \to Z_0) \in \mathrm{Mor}(\mathcal{T})_g} F(Y_0), \quad \varphi^\dagger F(Z) \simeq \varinjlim_{(Y_0 \to Z_0) \in \mathrm{Mor}(\mathcal{T})_g} F(Z_0) \,.$$

Moreover, since small filtrant inductive limits are exact in $\mathcal{A}$,

$$\mathrm{Ker}\, \varphi^\dagger F(g) \simeq \mathrm{Ker}\big(\varinjlim_{g_0 \in \mathrm{Mor}(\mathcal{T})_g} F(g_0)\big) \simeq \varinjlim_{g_0 \in \mathrm{Mor}(\mathcal{T})_g} (\mathrm{Ker}\, F(g_0)). \tag{10.4.4}$$

Now consider a d.t. $X \xrightarrow{f} Y \xrightarrow{g} Z \to TX$ in $\mathcal{D}$. Let $(Y_0 \xrightarrow{g_0} Z_0) \in \mathrm{Mor}(\mathcal{T})_g$. Embed g_0 in a d.t. $X_0 \xrightarrow{f_0} Y_0 \xrightarrow{g_0} Z_0 \to TX_0$. In the diagram below, we may complete the dotted arrows in order to get a morphism of d.t.'s:

$$\begin{array}{ccccccc} \varphi(X_0) & \xrightarrow{\varphi(f_0)} & \varphi(Y_0) & \xrightarrow{\varphi(g_0)} & \varphi(Z_0) & \longrightarrow & T(\varphi(X_0)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & TX\,. \end{array}$$

Applying the functor $\varphi^\dagger F$, and using the morphism of functors $F \to \varphi^\dagger F \circ \varphi$ (see (10.4.3)), we get a commutative diagram in $\mathcal{A}$ in which the row in the top is exact

$$\begin{array}{ccccc} F(X_0) & \xrightarrow{F(f_0)} & F(Y_0) & \xrightarrow{F(g_0)} & F(Z_0) \\ \downarrow & & \downarrow & & \downarrow \\ \varphi^\dagger F(X) & \xrightarrow{\varphi^\dagger F(f)} & \varphi^\dagger F(Y) & \xrightarrow{\varphi^\dagger F(g)} & \varphi^\dagger F(Z)\,. \end{array}$$

We have a morphism $\mathrm{Coker}(F(f_0)) \to \mathrm{Coker}(\varphi^\dagger F(f))$. Since $F(X_0) \to F(Y_0) \to F(Z_0)$ is exact, the morphism $\mathrm{Ker}(F(g_0)) \to \mathrm{Coker}(F(f_0))$ vanishes and hence $\mathrm{Ker}(F(g_0)) \to \mathrm{Coker}(\varphi^\dagger F(f))$ vanishes. By (10.4.4), the morphism $\mathrm{Ker}(\varphi^\dagger F(g)) \to \mathrm{Coker}(\varphi^\dagger F(f))$ vanishes, which means that the sequence $\varphi^\dagger F(X) \xrightarrow{\varphi^\dagger F(f)} \varphi^\dagger F(Y) \xrightarrow{\varphi^\dagger F(g)} \varphi^\dagger F(Z)$ is exact. q.e.d.

10.5 The Brown Representability Theorem

In this section we shall give a sufficient condition for the representability of contravariant cohomological functors on triangulated categories admitting small direct sums. Recall (Proposition 10.1.19) that in such categories, a small direct sum of d.t.'s is a d.t.

Definition 10.5.1. *Let $\mathcal{D}$ be a triangulated category admitting small direct sums. A* system of t-generators *$\mathcal{F}$ in $\mathcal{D}$ is a small family of objects of $\mathcal{D}$ satisfying conditions* (i) *and* (ii) *below.*

(i) *$\mathcal{F}$ is a system of generators* (*see* Definition 5.2.1), *or equivalently, $\mathcal{F}$ is a small family of objects of $\mathcal{D}$ such that for any $X \in \mathcal{D}$ with* $\mathrm{Hom}_{\mathcal{D}}(C, X) \simeq 0$ *for all $C \in \mathcal{F}$, we have $X \simeq 0$.*

(ii) *For any* countable *set I and any family $\{u_i \colon X_i \to Y_i\}_{i \in I}$ of morphisms in $\mathcal{D}$, the map* $\mathrm{Hom}_{\mathcal{D}}(C, \oplus_i X_i) \xrightarrow{\oplus_i u_i} \mathrm{Hom}_{\mathcal{D}}(C, \oplus_i Y_i)$ *vanishes for every $C \in \mathcal{F}$ as soon as* $\mathrm{Hom}_{\mathcal{D}}(C, X_i) \xrightarrow{u_i} \mathrm{Hom}_{\mathcal{D}}(C, Y_i)$ *vanishes for every $i \in I$ and every $C \in \mathcal{F}$.*

Note that the equivalence in (i) follows from the fact that, for a d.t. $X \xrightarrow{f} Y \to Z \to TX$, f is an isomorphism if and only if $Z \simeq 0$ (see Exercise 10.1).

Theorem 10.5.2. [The Brown representability Theorem] *Let $\mathcal{D}$ be a triangulated category admitting small direct sums and a system of t-generators $\mathcal{F}$.*

(i) *Let* $H : \mathcal{D}^{\mathrm{op}} \to \mathrm{Mod}(\mathbb{Z})$ *be a cohomological functor which commutes with small products (i.e., for any small family* $\{X_i\}_{i\in I}$ *in* $\mathrm{Ob}(\mathcal{D})$*, we have* $H(\oplus_i X_i) \xrightarrow{\sim} \prod_i H(X_i)$*). Then* H *is representable.*
(ii) *Let* $\mathcal{K}$ *be a full triangulated subcategory of* $\mathcal{D}$ *such that* $\mathcal{F} \subset \mathrm{Ob}(\mathcal{K})$ *and* $\mathcal{K}$ *is closed by small direct sums. Then the natural functor* $\mathcal{K} \to \mathcal{D}$ *is an equivalence.*

Similarly to the other representability theorems (see e.g. §5.2), this theorem implies the following corollary.

Corollary 10.5.3. *Let* $\mathcal{D}$ *be a triangulated category admitting small direct sums and a system of t-generators.*

(i) $\mathcal{D}$ *admits small products.*
(ii) *Let* $F : \mathcal{D} \to \mathcal{D}'$ *be a triangulated functor of triangulated categories. Assume that* F *commutes with small direct sums. Then* F *admits a right adjoint* G*, and* G *is triangulated.*

Proof. (i) For a small family $\{X_i\}_{i\in I}$ of objects in $\mathcal{D}$, the functor

$$Z \mapsto \prod_i \mathrm{Hom}_{\mathcal{D}}(Z, X_i)$$

is cohomological and commutes with small products. Hence it is representable.
(ii) For each $Y \in \mathcal{D}'$, the functor $X \mapsto \mathrm{Hom}_{\mathcal{D}'}(F(X), Y)$ is representable by Theorem 10.5.2. Hence F admits a right adjoint. Finally G is triangulated by the result of Exercise 10.3. q.e.d.

Remark 10.5.4. Condition (ii) in Definition 10.5.1 can be reformulated in many ways. Each of the following conditions is equivalent to (ii):

(ii)′ for any countable set I and any family $\{u_i : X_i \to Y_i\}_{i\in I}$ of morphisms in $\mathcal{D}$, the map $\mathrm{Hom}_{\mathcal{D}}(C, \oplus_i X_i) \xrightarrow{\oplus_i u_i} \mathrm{Hom}_{\mathcal{D}}(C, \oplus_i Y_i)$ is surjective for every $C \in \mathcal{F}$ as soon as $\mathrm{Hom}_{\mathcal{D}}(C, X_i) \xrightarrow{u_i} \mathrm{Hom}_{\mathcal{D}}(C, Y_i)$ is surjective for every $i \in I$ and every $C \in \mathcal{F}$.
(ii)″ for any countable set I and any family $\{u_i : X_i \to Y_i\}_{i\in I}$ of morphisms in $\mathcal{D}$, the map $\mathrm{Hom}_{\mathcal{D}}(C, \oplus_i X_i) \xrightarrow{\oplus_i u_i} \mathrm{Hom}_{\mathcal{D}}(C, \oplus_i Y_i)$ is injective for every $C \in \mathcal{F}$ as soon as $\mathrm{Hom}_{\mathcal{D}}(C, X_i) \xrightarrow{u_i} \mathrm{Hom}_{\mathcal{D}}(C, Y_i)$ is injective for every $i \in I$ and every $C \in \mathcal{F}$.

Indeed if we take a d.t. $X \to Y \to Z \to TX$, then we have an equivalence

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{D}}(C, X) \to \mathrm{Hom}_{\mathcal{D}}(C, Y) \text{ vanishes} \\ \iff &\mathrm{Hom}_{\mathcal{D}}(C, Y) \to \mathrm{Hom}_{\mathcal{D}}(C, Z) \text{ is injective} \\ \iff &\mathrm{Hom}_{\mathcal{D}}(C, T^{-1}Z) \to \mathrm{Hom}_{\mathcal{D}}(C, X) \text{ is surjective .} \end{aligned}$$

Condition (ii) is also equivalent to the following condition:

(iii) for any countable set I, any family $\{X_i\}_{i\in I}$ in $\mathcal{D}$, any $C \in \mathcal{F}$ and any morphism $f: C \to \oplus_{i\in I} X_i$, there exists a family of morphisms $u_i: C_i \to X_i$ such that f decomposes into $C \to \oplus_i C_i \xrightarrow{\oplus u_i} \oplus_i X_i$ and each C_i is a small direct sum of objects in $\mathcal{F}$.

Indeed, let $\mathcal{S}$ be the full subcategory of $\mathcal{D}$ consisting of small direct sums of objects in $\mathcal{F}$. If a morphism $X \to Y$ in $\mathcal{D}$ satisfies the condition that $\mathrm{Hom}_{\mathcal{D}}(C, X) \to \mathrm{Hom}_{\mathcal{D}}(C, Y)$ vanishes for every $C \in \mathcal{F}$, then the same condition holds for every $C \in \mathcal{S}$. Hence it is easy to see that (iii) implies (ii). Conversely assume that (ii)′ is true. For a countable family of objects X_i in $\mathcal{D}$ set $C_i = \bigoplus_{C\in\mathcal{F}} C^{\oplus X_i(C)}$. Then $C_i \in \mathcal{S}$, and the canonical morphism $C_i \to X_i$ satisfies the condition that any morphism $C \to X_i$ with $C \in \mathcal{F}$ factors through $C_i \to X_i$. Hence (ii)′ implies that $\mathrm{Hom}_{\mathcal{D}}(C, \oplus_i C_i) \to \mathrm{Hom}_{\mathcal{D}}(C, \oplus_i X_i)$ is surjective. Hence any morphism $C \to \oplus_i X_i$ factors through $\oplus_i C_i \to \oplus_i X_i$.

Note that condition (iii) is a consequence of the following condition (iii)′, which is sufficient in most applications.

(iii)′ for any countable set I, any family $\{X_i\}_{i\in I}$ in $\mathcal{D}$, any $C \in \mathcal{F}$ and any morphism $f: C \to \oplus_{i\in I} X_i$, there exists a family of morphisms $u_i: C_i \to X_i$ with $C_i \in \mathcal{F}$ such that f decomposes into $C \to \oplus_i C_i \xrightarrow{\oplus u_i} \oplus_i X_i$.

Summing up, for a small family $\mathcal{F}$ of objects of $\mathcal{D}$, we have

$$\text{(ii)} \Leftrightarrow \text{(ii)}' \Leftrightarrow \text{(ii)}'' \Leftrightarrow \text{(iii)} \Leftarrow \text{(iii)}' .$$

The Brown representability theorem was proved by Neeman [53] under condition (iii)′, and later by Krause [44] under the condition (ii).

The rest of the section is devoted to the proof of the theorem.

Functors Commuting with Small Products

Let $\mathcal{S}$ be an additive $\mathcal{U}$-category which admits small direct sums. Let $\mathcal{S}^{\wedge,\mathrm{add}}$ be the category of additive functors from $\mathcal{S}^{\mathrm{op}}$ to $\mathrm{Mod}(\mathbb{Z})$. The category $\mathcal{S}^{\wedge,\mathrm{add}}$ is a big abelian category. By Proposition 8.2.12, $\mathcal{S}^{\wedge,\mathrm{add}}$ is regarded as a full subcategory of $\mathcal{S}^{\wedge}$.

A complex $F' \to F \to F''$ in $\mathcal{S}^{\wedge,\mathrm{add}}$ is exact if and only if $F'(X) \to F(X) \to F''(X)$ is exact for every $X \in \mathcal{S}$. Let $\mathcal{S}^{\wedge,\mathrm{prod}}$ be the full subcategory of $\mathcal{S}^{\wedge,\mathrm{add}}$ consisting of additive functors F commuting with small products, namely the canonical map $F(\oplus_i X_i) \to \prod_i F(X_i)$ is bijective for any small family $\{X_i\}_i$ of objects in $\mathcal{S}$.

Lemma 10.5.5. *The full category $\mathcal{S}^{\wedge,\mathrm{prod}}$ is a fully abelian subcategory of $\mathcal{S}^{\wedge,\mathrm{add}}$ closed by extension.*

Proof. It is enough to show that, for an exact complex $F_1 \to F_2 \to F_3 \to F_4 \to F_5$ in $\mathcal{S}^{\wedge,\mathrm{add}}$, if F_j belongs to $\mathcal{S}^{\wedge,\mathrm{prod}}$ for $j \neq 3$, then F_3 also belongs to

$\mathcal{S}^{\wedge,\mathrm{prod}}$ (see Remark 8.3.22). For a small family $\{X_i\}$ of objects in $\mathcal{S}$, we have an exact diagram in $\mathrm{Mod}(\mathbb{Z})$

$$\begin{array}{ccccccccc} F_1(\oplus_i X_i) & \to & F_2(\oplus_i X_i) & \to & F_3(\oplus_i X_i) & \to & F_4(\oplus_i X_i) & \to & F_5(\oplus_i X_i) \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ \prod_i F_1(X_i) & \to & \prod_i F_2(X_i) & \to & \prod_i F_3(X_i) & \to & \prod_i F_4(X_i) & \to & \prod_i F_5(X_i) . \end{array}$$

Since the vertical arrows are isomorphisms except the middle one, the five lemma (Lemma 8.3.13) implies that the middle arrow is an isomorphism. q.e.d.

Now assume that

(10.5.1) there exists a small full subcategory $\mathcal{S}_0$ of $\mathcal{S}$ such that any object of $\mathcal{S}$ is a small direct sum of objects of $\mathcal{S}_0$.

Hence a complex $F' \to F \to F''$ in $\mathcal{S}^{\wedge,\mathrm{prod}}$ is exact if and only if $F'(X) \to F(X) \to F''(X)$ is exact for every $X \in \mathcal{S}_0$. In particular the restriction functor $\mathcal{S}^{\wedge,\mathrm{prod}} \to \mathcal{S}_0^{\wedge,\mathrm{add}}$ is exact, faithful and conservative. Hence, the category $\mathcal{S}^{\wedge,\mathrm{prod}}$ is a $\mathcal{U}$-category.

Let $\varphi\colon \mathcal{S} \to \mathcal{S}^{\wedge,\mathrm{prod}}$ be the functor which associates to $X \in \mathcal{S}$ the functor $\mathcal{S} \ni C \mapsto \mathrm{Hom}_{\mathcal{S}}(C, X)$. This functor commutes with small products. Since $\mathcal{S}^{\wedge,\mathrm{prod}} \to \mathcal{S}^{\wedge}$ is fully faithful, φ is a fully faithful additive functor by the Yoneda lemma.

Lemma 10.5.6. *Assume* (10.5.1). *Then, for any $F \in \mathcal{S}^{\wedge,\mathrm{prod}}$ we can find an object $X \in \mathcal{S}$ and an epimorphism $\varphi(X) \twoheadrightarrow F$.*

Proof. For any $C \in \mathcal{S}_0$, set $X_C = C^{\oplus F(C)}$. Then we have

$$F(X_C) \simeq F(C)^{F(C)} = \mathrm{Hom}_{\mathbf{Set}}(F(C), F(C)) .$$

Hence $\mathrm{id}_{F(C)}$ gives an element $s_C \in F(X_C) \simeq \mathrm{Hom}_{\mathcal{S}^{\wedge,\mathrm{prod}}}(\varphi(X_C), F)$. Since the composition

$$F(C) \to \mathrm{Hom}_{\mathcal{S}}(C, C) \times F(C) \to \mathrm{Hom}_{\mathcal{S}}(C, X_C) \simeq \varphi(X_C)(C) \to F(C)$$

is the identity, the map $\varphi(X_C)(C) \to F(C)$ is surjective. Set $X = \oplus_{C \in \mathcal{S}_0} X_C$. Then $(s_C)_C \in \prod_C F(X_C) \simeq F(X)$ gives a morphism $\varphi(X) \to F$ and $\varphi(X)(C) \to F(C)$ is surjective for any $C \in \mathcal{S}_0$. Hence $\varphi(X) \to F$ is an epimorphism. q.e.d.

Lemma 10.5.7. *Assume* (10.5.1).

(i) *The functor $\varphi\colon \mathcal{S} \to \mathcal{S}^{\wedge,\mathrm{prod}}$ commutes with small direct sums.*
(ii) *The abelian category $\mathcal{S}^{\wedge,\mathrm{prod}}$ admits small direct sums, and hence it admits small inductive limits.*

Proof. (i) For a small family $\{X_i\}_i$ of objects in $\mathcal{S}$ and $F \in \mathcal{S}^{\wedge,\mathrm{prod}}$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{S}^{\wedge,\mathrm{prod}}}(\varphi(\oplus_i X_i), F) &\simeq F(\oplus_i X_i)\\ &\simeq \prod_i F(X_i) \simeq \prod_i \mathrm{Hom}_{\mathcal{S}^{\wedge,\mathrm{prod}}}(\varphi(X_i), F)\ .\end{aligned}$$

(ii) For a small family $\{F_i\}_i$ of objects in $\mathcal{S}^{\wedge,\mathrm{prod}}$, there exists an exact sequence $\varphi(X_i) \to \varphi(Y_i) \to F_i \to 0$ with $X_i, Y_i \in \mathcal{S}$ by Lemma 10.5.6. Since φ is fully faithful, there is a morphism $u_i : X_i \to Y_i$ which induces the morphism $\varphi(X_i) \to \varphi(Y_i)$. Then we have

$$\begin{aligned}\mathrm{Coker}(\varphi(\oplus_i X_i) \xrightarrow{\oplus_i u_i} \varphi(\oplus_i Y_i)) &\simeq \mathrm{Coker}(\oplus_i \varphi(X_i) \to \oplus_i \varphi(Y_i))\\ &\simeq \oplus_i \mathrm{Coker}(\varphi(X_i) \to \varphi(Y_i)) \simeq \oplus_i F_i\ .\end{aligned}$$

q.e.d.

Note that, for a small family $\{F_i\}_i$ of objects in $\mathcal{S}^{\wedge,\mathrm{prod}}$ and $X \in \mathcal{S}$, the map $\oplus_i(F_i(X)) \to (\oplus_i F_i)(X)$ may be not bijective.

Proof of Theorem 10.5.2

Now let us come back to the original situation. Let $\mathcal{D}$ be a triangulated category admitting small direct sums and a system of t-generators $\mathcal{F}$. By replacing $\mathcal{F}$ with $\bigcup_{n\in\mathbb{Z}} T^n\mathcal{F}$, we may assume from the beginning that $T\mathcal{F} = \mathcal{F}$. Let $\mathcal{S}$ be the full subcategory of $\mathcal{D}$ consisting of small direct sums of objects in $\mathcal{F}$. Then $\mathcal{S}$ is an additive category which admits small direct sums. Moreover, $T\mathcal{S} = \mathcal{S}$, and T induces an automorphism $T : \mathcal{S}^{\wedge,\mathrm{prod}} \to \mathcal{S}^{\wedge,\mathrm{prod}}$ by $(TF)(C) = F(T^{-1}C)$ for $F \in \mathcal{S}^{\wedge,\mathrm{prod}}$ and $C \in \mathcal{S}$. By its construction, $\mathcal{S}$ satisfies condition (10.5.1), and hence $\mathcal{S}^{\wedge,\mathrm{prod}}$ is an abelian $\mathcal{U}$-category and Lemmas 10.5.5–10.5.7 hold. Note that a complex $F' \to F \to F''$ in $\mathcal{S}^{\wedge,\mathrm{prod}}$ is exact if and only if $F'(C) \to F(C) \to F''(C)$ is exact for any $C \in \mathcal{F}$.

We shall extend the functor $\varphi : \mathcal{S} \to \mathcal{S}^{\wedge,\mathrm{prod}}$ to the functor $\widetilde{\varphi} : \mathcal{D} \to \mathcal{S}^{\wedge,\mathrm{prod}}$ defined by $\widetilde{\varphi}(X)(C) = \mathrm{Hom}_{\mathcal{D}}(C, X)$ for $X \in \mathcal{D}$ and $C \in \mathcal{S}$. Then $\widetilde{\varphi}$ commutes with T. Note that although $\varphi : \mathcal{S} \to \mathcal{S}^{\wedge,\mathrm{prod}}$ is fully faithful, the functor $\widetilde{\varphi} : \mathcal{D} \to \mathcal{S}^{\wedge,\mathrm{prod}}$ is not faithful in general.

In the proof of the lemma below, we use the fact that $\mathcal{F}$ satisfies the condition (ii) in Definition 10.5.1.

Lemma 10.5.8. (i) *The functor $\widetilde{\varphi} : \mathcal{D} \to \mathcal{S}^{\wedge,\mathrm{prod}}$ is a cohomological functor.*

(ii) *The functor $\widetilde{\varphi} : \mathcal{D} \to \mathcal{S}^{\wedge,\mathrm{prod}}$ commutes with countable direct sums.*

(iii) *Let $\{X_i \to Y_i\}$ be a countable family of morphisms in $\mathcal{D}$. If $\widetilde{\varphi}(X_i) \to \widetilde{\varphi}(Y_i)$ is an epimorphism for all i, then $\widetilde{\varphi}(\oplus_i X_i) \to \widetilde{\varphi}(\oplus_i Y_i)$ is an epimorphism.*

Proof. (i) is obvious.
Let us first prove (iii). For all $C \in \mathcal{F}$, the map $\mathrm{Hom}_{\mathcal{D}}(C, X_i) \to \mathrm{Hom}_{\mathcal{D}}(C, Y_i)$ is surjective. Hence Remark 10.5.4 (ii)$'$ implies that $\mathrm{Hom}_{\mathcal{D}}(C, \oplus_i X_i) \to \mathrm{Hom}_{\mathcal{D}}(C, \oplus_i Y_i)$ is surjective.

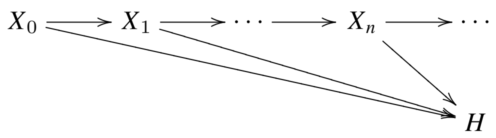
Finally let us prove (ii). Let $\{X_i\}_i$ be a countable family of objects of $\mathcal{D}$. Then we can find an epimorphism $\varphi(Y_i) \twoheadrightarrow \widetilde{\varphi}(X_i)$ in $\mathcal{S}^{\wedge,\mathrm{prod}}$ with $Y_i \in \mathcal{S}$ by Lemma 10.5.6. Let $W_i \to Y_i \to X_i \to TW_i$ be a d.t. Then take an epimorphism $\varphi(Z_i) \twoheadrightarrow \widetilde{\varphi}(W_i)$ with $Z_i \in \mathcal{S}$. Hence $\varphi(\oplus_i Z_i) \to \widetilde{\varphi}(\oplus_i W_i)$ and $\varphi(\oplus_i Y_i) \to \widetilde{\varphi}(\oplus_i X_i)$ are epimorphisms by (iii). On the other hand, $\oplus_i W_i \to \oplus_i Y_i \to \oplus_i X_i \to T(\oplus_i W_i)$ is a d.t., and hence $\widetilde{\varphi}(\oplus_i W_i) \to \varphi(\oplus_i Y_i) \to \widetilde{\varphi}(\oplus_i X_i)$ is exact by (i). Hence, $\varphi(\oplus_i Z_i) \to \varphi(\oplus_i Y_i) \to \widetilde{\varphi}(\oplus_i X_i) \to 0$ is exact. By Lemma 10.5.7, we have $\varphi(\oplus_i Z_i) \simeq \oplus_i \varphi(Z_i)$ and similarly for Y_i. Since $\varphi(Z_i) \to \varphi(Y_i) \to \widetilde{\varphi}(X_i) \to 0$ is exact for all i, $\oplus_i \varphi(Z_i) \to \oplus_i \varphi(Y_i) \to \oplus_i \widetilde{\varphi}(X_i) \to 0$ is also exact, from which we conclude that $\widetilde{\varphi}(\oplus_i X_i) \simeq \oplus_i \widetilde{\varphi}(X_i)$. q.e.d.

Let $H : \mathcal{D}^{\mathrm{op}} \to \mathrm{Mod}(\mathbb{Z})$ be a cohomological functor commuting with small products. The restriction of H to $\mathcal{S}^{\mathrm{op}}$ defines $H_0 \in \mathcal{S}^{\wedge,\mathrm{prod}}$.

In the lemma below, we regard $\mathcal{D}$ as a full subcategory of $\mathcal{D}^{\wedge}$.

Lemma 10.5.9. *Let H and $\mathcal{K}$ be as in* Theorem 10.5.2. *Then there exists a commutative diagram in $\mathcal{D}^{\wedge}$*

(10.5.2)



such that $X_n \in \mathcal{K}$ and $\mathrm{Im}\big(\widetilde{\varphi}(X_n) \to \widetilde{\varphi}(X_{n+1})\big) \xrightarrow{\sim} H_0$ *in $\mathcal{S}^{\wedge,\mathrm{prod}}$.*

Proof. We can take $X_0 \in \mathcal{S}$ and an epimorphism $\varphi(X_0) \twoheadrightarrow H_0$ in $\mathcal{S}^{\wedge,\mathrm{prod}}$ by Lemma 10.5.6. We shall construct $X_n \in \mathcal{K}$ inductively as follows. Assume that $X_0 \to X_1 \to \cdots \to X_n \to H$ has been constructed and $\mathrm{Im}\big(\widetilde{\varphi}(X_i) \to \widetilde{\varphi}(X_{i+1})\big) \xrightarrow{\sim} H_0$ for $0 \le i < n$. Let us take an exact sequence $\varphi(Z_n) \to \widetilde{\varphi}(X_n) \to H_0 \to 0$ with $Z_n \in \mathcal{S}$. Then take a d.t. $Z_n \to X_n \to X_{n+1} \to TZ_n$. Since Z_n and X_n belong to $\mathcal{K}$, X_{n+1} also belongs to $\mathcal{K}$. Since $Z_n \to X_n \to H$ vanishes and H is cohomological, $X_n \to H$ factors through $X_n \to X_{n+1}$. Since $\widetilde{\varphi}(Z_n) \to \widetilde{\varphi}(X_n) \to \widetilde{\varphi}(X_{n+1})$ is exact, we obtain that $\mathrm{Im}\big(\widetilde{\varphi}(X_n) \to \widetilde{\varphi}(X_{n+1})\big) \simeq \mathrm{Coker}\big(\widetilde{\varphi}(Z_n) \to \widetilde{\varphi}(X_n)\big) \simeq H_0$. q.e.d.

Notation 10.5.10. Consider a functor $X : \mathbb{N} \to \mathcal{D}$, that is, a sequence of morphisms $X_0 \xrightarrow{f_0} X_1 \to \cdots \to X_n \xrightarrow{f_n} X_{n+1} \to \cdots$ in $\mathcal{D}$. Denote by

(10.5.3) $$\mathrm{sh}_X : \oplus_{n \ge 0} X_n \to \oplus_{n \ge 0} X_n$$

the morphism obtained as the composition

$$\oplus_{n \ge 0} X_n \xrightarrow{\oplus f_n} \oplus_{n \ge 0} X_{n+1} \simeq \oplus_{n \ge 1} X_n \hookrightarrow \oplus_{n \ge 0} X_n \ .$$

Consider a d.t.

$$\oplus_{n\geq 0}X_n \xrightarrow{\mathrm{id}-\mathrm{sh}_X} \oplus_{n\geq 0}X_n \to Z \to T(\oplus_{n\geq 0}X_n)\ . \tag{10.5.4}$$

In the literature, Z is called the *homotopy colimit* of the inductive system $\{X_n, f_n\}_n$ and denoted by $\mathrm{hocolim}(X)$. Note that this object is unique up to isomorphism, but not up to unique isomorphism. Hence, $\{X_n, f_n\}_n \mapsto Z$ is not a functor.

Consider the functor $X\colon \mathbb{N} \to \mathcal{D}$ given by Lemma 10.5.9 and let sh_X be as in (10.5.3). Since $H(\oplus_{n\geq 0}X_n) \simeq \prod_{n\geq 0} H(X_n)$, the morphisms $X_n \to H$ define the morphism $\oplus_{n\geq 0}X_n \to H$. The commutativity of (10.5.2) implies that the composition $\oplus_{n\geq 0}X_n \xrightarrow{\mathrm{id}-\mathrm{sh}_X} \oplus_{n\geq 0}X_n \to H$ vanishes.

Lemma 10.5.11. *The sequence*

$$0 \to \widetilde{\varphi}(\oplus_{n\geq 0}X_n) \xrightarrow{\mathrm{id}-\mathrm{sh}_X} \widetilde{\varphi}(\oplus_{n\geq 0}X_n) \to H_0 \to 0\ .$$

is exact in $\mathcal{S}^{\wedge,\mathrm{prod}}$.

Proof. Note that we have $\widetilde{\varphi}(\oplus_{n\geq 0}X_n) \simeq \oplus_{n\geq 0}\widetilde{\varphi}(X_n)$ by Lemma 10.5.8. Since $\mathrm{Im}\big(\widetilde{\varphi}(X_n) \to \widetilde{\varphi}(X_{n+1})\big) \simeq H_0$, we have "$\varinjlim_n$" $\widetilde{\varphi}(X_n) \simeq H_0$. Then $\varinjlim_n \widetilde{\varphi}(X_n) \simeq H_0$ and the the above sequence is exact by Exercise 8.37. q.e.d.

Lemma 10.5.12. *There exist* $Z \in \mathcal{K}$ *and a morphism* $Z \to H$ *which induces an isomorphism* $Z(C) \xrightarrow{\sim} H(C)$ *for every* $C \in \mathcal{F}$.

Proof. Let Z be as in (10.5.4). Since H is cohomological, $\oplus_{n\geq 0}X_n \to H$ factors through Z. Set $X = \oplus_{n\geq 0}X_n$. Since $\widetilde{\varphi}$ is cohomological, we have an exact sequence in $\mathcal{S}^{\wedge,\mathrm{prod}}$:

$$\begin{array}{ccccccccc}
\widetilde{\varphi}(X) & \xrightarrow{\mathrm{id}-\mathrm{sh}_X} & \widetilde{\varphi}(X) & \longrightarrow & \widetilde{\varphi}(Z) & \longrightarrow & \widetilde{\varphi}(TX) & \xrightarrow{\widetilde{\varphi}(T(\mathrm{id}-\mathrm{sh}_X))} & \widetilde{\varphi}(TX) \\
 & & & & & & \downarrow\wr & & \downarrow\wr \\
 & & & & & & T(\widetilde{\varphi}(X)) & \xrightarrow{T(\widetilde{\varphi}(\mathrm{id}-\mathrm{sh}_X))} & T(\widetilde{\varphi}(X))\ .
\end{array}$$

Applying Lemma 10.5.11, we find that the last right arrows are monomorphisms. Hence we have

$$\widetilde{\varphi}(Z) \simeq \mathrm{Coker}(\widetilde{\varphi}(X) \xrightarrow{\mathrm{id}-\mathrm{sh}_X} \widetilde{\varphi}(X)) \simeq H_0\ ,$$

where the last isomorphism follows from Lemma 10.5.11. q.e.d.

Lemma 10.5.13. *The natural functor* $\mathcal{K} \to \mathcal{D}$ *is an equivalence.*

Proof. This functor being fully faithful, it remains to show that it is essentially surjective. Let $X \in \mathcal{D}$. Applying Lemma 10.5.12 to the functor $H = \mathrm{Hom}_{\mathcal{D}}(\,\cdot\,, X)$, we get $Z \in \mathcal{K}$ and a morphism $Z \to X$ which induces an isomorphism $Z(C) \xrightarrow{\sim} X(C)$ for all $C \in \mathcal{F}$. Since $\mathcal{F}$ is a system of generators, $Z \xrightarrow{\sim} X$. q.e.d.

Lemma 10.5.14. *Let Z be as in* Lemma 10.5.12*. Then $Z \to H$ is an isomorphism.*

Proof. Let $\mathcal{K}$ denote the full subcategory of $\mathcal{D}$ consisting of objects Y such that $Z(T^nY) \to H(T^nY)$ is an isomorphism for any $n \in \mathbb{Z}$. Then $\mathcal{K}$ contains $\mathcal{F}$, is closed by small direct sums and is a triangulated subcategory of $\mathcal{D}$. Therefore $\mathcal{K} = \mathcal{D}$ by Lemma 10.5.13. q.e.d.

The proof of Theorem 10.5.2 is complete.

Exercises

Exercise 10.1. Let $X \xrightarrow{f} Y \to Z \to TX$ be a d.t. in a triangulated category. Prove that f is an isomorphism if and only if Z is isomorphic to 0.

Exercise 10.2. Let $\mathcal{D}$ be a triangulated category and consider a commutative diagram in $\mathcal{D}$:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & TX \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle T(\alpha)} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & TX' . \end{array}$$

Assume that α and β are isomorphisms, $T(f') \circ h' = 0$, and the first row is a d.t. Prove that the second row is also a d.t. under one of the hypotheses:
(i) for any $P \in \mathcal{D}$, the sequence below is exact:

$$\mathrm{Hom}\,(P, X') \to \mathrm{Hom}\,(P, Y') \to \mathrm{Hom}\,(P, Z') \to \mathrm{Hom}\,(P, TX') ,$$

(ii) for any $P \in \mathcal{D}$, the sequence below is exact:

$$\mathrm{Hom}\,(TX', P) \to \mathrm{Hom}\,(Z', P) \to \mathrm{Hom}\,(Y', P) \to \mathrm{Hom}\,(X', P) .$$

Exercise 10.3. Let $F : \mathcal{D} \to \mathcal{D}'$ be a triangulated functor and assume that F admits an adjoint G. Prove that G is triangulated. (Hint: use Exercise 10.2.)

Exercise 10.4. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ be a d.t. in a triangulated category.
(i) Prove that if $h = 0$, this d.t. is isomorphic to $X \to X \oplus Z \to Z \xrightarrow{0} TX$.
(ii) Prove the same result by assuming now that there exists $k : Y \to X$ with $k \circ f = \mathrm{id}_X$.

Exercise 10.5. Let $f : X \to Y$ be a monomorphism in a triangulated category $\mathcal{D}$. Prove that there exist $Z \in \mathcal{D}$ and an isomorphism $h : Y \xrightarrow{\sim} X \oplus Z$ such that the composition $X \to Y \to X \oplus Z$ is the canonical morphism.

Exercise 10.6. In a triangulated category $\mathcal{D}$ consider the diagram of solid arrows

$$
\begin{array}{ccccccc}
X^0 & \xrightarrow{u} & X^1 & \xrightarrow{v} & X^2 & \xrightarrow{w} & TX^0 \\
\downarrow{\scriptstyle f} & & \downarrow & & \vdots & & \downarrow{\scriptstyle T(f)} \\
Y^0 & \longrightarrow & Y^1 & \longrightarrow & Y^2 & \longrightarrow & TY^0 \\
\downarrow{\scriptstyle g} & & \downarrow & & \vdots & & \downarrow{\scriptstyle T(g)} \\
Z^0 & \dashrightarrow & Z^1 & \dashrightarrow & Z^2 & \dashrightarrow & TZ^0 \\
\downarrow{\scriptstyle h} & & \downarrow & & \vdots & ac & \downarrow{\scriptstyle -T(h)} \\
TX^0 & \xrightarrow[T(u)]{} & TX^1 & \xrightarrow[T(v)]{} & TX^2 & \xrightarrow[-T(w)]{} & T^2X^0 .
\end{array} \tag{10.5.5}
$$

Assume that the two first rows and columns are d.t.'s. Show that the dotted arrows may be completed in order that all squares are commutative except the one labeled "ac" which is anti-commutative (see Definition 8.2.20), all rows and all columns are d.t.'s. (Hint: see [4], Proposition 1.1.11.)

Exercise 10.7. Let $\mathcal{D}$ be a triangulated category, $\mathcal{C}$ an abelian category, $F, G \colon \mathcal{D} \to \mathcal{C}$ two cohomological functors and $\theta \colon F \to G$ a morphism of functors. Define the full subcategory $\mathcal{T}$ of $\mathcal{D}$ consisting of objects $X \in \mathcal{D}$ such that $\theta(T^k(X)) \colon F(T^k(X)) \to G(T^k(X))$ is an isomorphism for all $k \in \mathbb{Z}$. Prove that $\mathcal{T}$ is triangulated. (Hint: use Lemma 8.3.13.)

Exercise 10.8. Let $\mathcal{D}$ be a triangulated category, $\mathcal{A}$ an abelian category and $F \colon \mathcal{D} \to \mathcal{A}$ a cohomological functor. Prove that F is exact.

Exercise 10.9. Let $\mathcal{D}$ be a triangulated category. Denote by $F \colon \mathcal{D} \to \mathcal{D}$ the translation functor T. By choosing a suitable isomorphism of functors $F \circ T \simeq T \circ F$, prove that F induces an equivalence of triangulated categories.

Exercise 10.10. Let $\mathcal{D}$ be a triangulated category and define the triangulated category $\mathcal{D}^{\mathrm{ant}}$ as follows: a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ is distinguished in $\mathcal{D}^{\mathrm{ant}}$ if and only if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{-h} TX$ is distinguished in $\mathcal{D}$. Prove that $\mathcal{D}$ and $\mathcal{D}^{\mathrm{ant}}$ are equivalent as triangulated categories.

Exercise 10.11. Let $\mathcal{D}$ be a triangulated category, $\mathcal{N}$ a null system, and let $Q \colon \mathcal{D} \to \mathcal{D}/\mathcal{N}$ be the canonical functor.
(i) Let $f \colon X \to Y$ be a morphism in $\mathcal{D}$ and assume that $Q(f) = 0$ in $\mathcal{D}/\mathcal{N}$. Prove that there exists $Z \in \mathcal{N}$ such that f factorizes as $X \to Z \to Y$.
(ii) For $X \in \mathcal{D}$, prove that $Q(X) \simeq 0$ if and only if there exists Y such that $X \oplus Y \in \mathcal{N}$ and this last condition is equivalent to $X \oplus TX \in \mathcal{N}$.

Exercise 10.12. Let $F \colon \mathcal{D} \to \mathcal{D}'$ be a triangulated functor of triangulated categories. Let $\mathcal{N}$ be the full subcategory of $\mathcal{D}$ consisting of objects $X \in \mathcal{D}$ such that $F(X) \simeq 0$.

(i) Prove that $\mathcal{N}$ is a null system and F factorizes uniquely as $\mathcal{D} \to \mathcal{D}/\mathcal{N} \to \mathcal{D}'$.
(ii) Prove that if $X \oplus Y \in \mathcal{N}$, then $X \in \mathcal{N}$ and $Y \in \mathcal{N}$.

Exercise 10.13. Let $\mathcal{D}$ be a triangulated category admitting countable direct sums, let $X \in \mathcal{D}$ and let $p\colon X \to X$ be a projector (i.e., $p^2 = p$). Define the functor $\alpha\colon \mathbb{N} \to \mathcal{D}$ by setting $\alpha(n) = X$ and $\alpha(n \to n+1) = p$.
(i) Prove that $\varinjlim \alpha$ exists in $\mathcal{D}$ and is isomorphic to $\mathrm{hocolim}(\alpha)$. (See Notation 10.5.10.)
(ii) Deduce that $\mathcal{D}$ is idempotent complete. (See [53].)

Exercise 10.14. Let $\mathcal{D}$ be a triangulated category and let I be a filtrant category. Let $\alpha \xrightarrow{f} \beta \xrightarrow{g} \gamma \xrightarrow{h} T \circ \alpha$ be morphisms of functors from I to $\mathcal{D}$ such that $\alpha(i) \xrightarrow{f(i)} \beta(i) \xrightarrow{g(i)} \gamma(i) \xrightarrow{h(i)} T(\alpha(i))$ is a d.t. for all $i \in I$. Prove that if "$\varinjlim$" α and "$\varinjlim$" β are representable by objects of $\mathcal{D}$, then so is "$\varinjlim$" γ and the induced triangle "$\varinjlim$" $\alpha \to$ "$\varinjlim$" $\beta \to$ "$\varinjlim$" $\gamma \to T($"$\varinjlim$" $\alpha)$ is a d.t. (Hint: construct a morphism of d.t.'s

$$\begin{array}{ccccccc}
\text{“}\varinjlim\text{”}\,\alpha & \longrightarrow & \text{“}\varinjlim\text{”}\,\beta & \longrightarrow & Z & \longrightarrow & T(\text{“}\varinjlim\text{”}\,\alpha) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\alpha(i) & \longrightarrow & \beta(i) & \longrightarrow & \gamma(i) & \longrightarrow & T(\alpha(i))
\end{array}$$

for some $i \in I$.)

Exercise 10.15. Let $\mathcal{D}$ be a triangulated category, $\mathcal{N}$ a null system, and let $\mathcal{N}^{\perp r}$ (resp. $\mathcal{N}^{\perp l}$) be the full subcategory of $\mathcal{D}$ consisting of objects Y such that $\mathrm{Hom}_{\mathcal{D}}(Z, Y) \simeq 0$ (resp. $\mathrm{Hom}_{\mathcal{D}}(Y, Z) \simeq 0$) for all $Z \in \mathcal{N}$.
(i) Prove that $\mathcal{N}^{\perp r}$ and $\mathcal{N}^{\perp l}$ are null systems in $\mathcal{D}$.
(ii) Prove that $\mathrm{Hom}_{\mathcal{D}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}/\mathcal{N}}(X, Y)$ for any $X \in \mathcal{D}$ and any $Y \in \mathcal{N}^{\perp r}$.

In the sequel, we assume that $X \oplus Y \in \mathcal{N}$ implies $X \in \mathcal{N}$ and $Y \in \mathcal{N}$.
(iii) Prove that the following conditions are equivalent:

(a) $\mathcal{N}^{\perp r} \to \mathcal{D}/\mathcal{N}$ is an equivalence,
(b) $\mathcal{D} \to \mathcal{D}/\mathcal{N}$ has a right adjoint,
(c) $\iota\colon \mathcal{N} \to \mathcal{D}$ has a right adjoint R,
(d) for any $X \in \mathcal{D}$, there exist $X' \in \mathcal{N}$, $X'' \in \mathcal{N}^{\perp r}$ and a d.t. $X' \to X \to X'' \to TX'$,
(e) $\mathcal{N} \to \mathcal{D}/\mathcal{N}^{\perp r}$ is an equivalence,
(f) $\mathcal{D} \to \mathcal{D}/\mathcal{N}^{\perp r}$ has a left adjoint and $\mathcal{N} \simeq (\mathcal{N}^{\perp r})^{\perp l}$,
(g) $\iota'\colon \mathcal{N}^{\perp r} \to \mathcal{D}$ has a left adjoint L and $\mathcal{N} \simeq (\mathcal{N}^{\perp r})^{\perp l}$.

(iv) Assume that the equivalent conditions (a)–(g) in (iii) are satisfied. Let $L\colon \mathcal{D} \to \mathcal{N}^{\perp r}$, $R\colon \mathcal{D} \to \mathcal{N}$, $\iota\colon \mathcal{N} \to \mathcal{D}$ and $\iota'\colon \mathcal{N}^{\perp r} \to \mathcal{D}$ be as above.

(a) Prove that there exists a morphism of functors $\iota' \circ L \to T \circ \iota \circ R$ such that $\iota R(X) \to X \to \iota' L(X) \to T(\iota R(X))$ is a d.t. for all $X \in \mathcal{D}$.
(b) Let $\widetilde{\mathcal{D}}$ be the category whose objects are the triplets (X', X'', u) with $X' \in \mathcal{N}$, $X'' \in \mathcal{N}^{\perp r}$ and u is a morphism $X'' \to TX'$ in $\mathcal{D}$. A morphism $(X', X'', u) \to (Y', Y'', v)$ in $\widetilde{\mathcal{D}}$ is a pair $(w'\colon X' \to Y', w''\colon X'' \to Y'')$ making the diagram below commutative

$$\begin{array}{ccc} X'' & \xrightarrow{u} & TX' \\ {\scriptstyle w''}\downarrow & & \downarrow{\scriptstyle T(w')} \\ Y'' & \xrightarrow{v} & TY' \,. \end{array}$$

Define an equivalence of categories $\mathcal{D} \xrightarrow{\sim} \widetilde{\mathcal{D}}$.

Exercise 10.16. (i) Let $\mathcal{D}$ be a triangulated category. Assume that $\mathcal{D}$ is abelian.

(a) Prove that $\mathcal{D}$ is a semisimple abelian category (see Definition 8.3.16). (Hint: use Exercise 10.5.)
(b) Prove that any triangle in $\mathcal{D}$ is a direct sum of three triangles

$$\begin{array}{ccccccc} X & \xrightarrow{\mathrm{id}_X} & X & \to & 0 & \to & TX\,, \\ 0 & \to & Y & \xrightarrow{\mathrm{id}_Y} & Y & \to & T(0), \qquad \text{and} \\ T^{-1}Z & \to & 0 & \to & Z & \xrightarrow{\mathrm{id}_Z} & T(T^{-1}Z). \end{array}$$

(ii) Conversely let $(\mathcal{C}, T)$ be a category with translation and assume that $\mathcal{C}$ is a semisimple abelian category. We say that a triangle in $\mathcal{C}$ is distinguished if it is a direct sum of three triangles as in (i) (b). Prove that $\mathcal{C}$ is a triangulated category.