Introduction

The aim of this book is to describe the topics outlined in the preface, categories, homological algebra and sheaves. We also present the main features and key results in related topics which await a similar full-scale treatment such as, for example, tensor categories, triangulated categories, stacks.

The general theory of categories and functors, with emphasis on inductive and projective limits, tensor categories, representable functors, ind-objects and localization is dealt with in Chaps. 1–7.

Homological algebra, including additive, abelian, triangulated and derived categories, is treated in Chaps. 8–15. Chapter 9 provides the tools (using transfinite induction) which will be used later for presenting unbounded derived categories.

Sheaf theory is treated in Chaps. 16–19 in the general framework of Grothendieck topologies. In particular, the results of Chap. 14 are applied to the study of the derived category of the category of sheaves on a ringed site. We also sketch an approach to the more sophisticated subject of stacks (roughly speaking, sheaves with values in the 2-category of categories) and introduce the important notion of twisted sheaves.

Of necessity we have excluded many exciting developments and applications such as n-categories, operads, A_∞-categories, model categories, among others. Without doubt these new areas will soon be intensively treated in the literature, and it is our hope that the present work will provide a basis for their understanding.

We now proceed to a more detailed outline of the contents of the book.

Chapter 1. We begin by defining the basic notions of categories and functors, illustrated with many classical examples. There are some set–theoretical dangers and to avoid contradictions, we work in a given universe. Universes are presented axiomatically, referring to [64] for a more detailed treatment. Among other concepts introduced in this chapter are morphisms of functors, equivalences of categories, representable functors, adjoint functors and so on. We introduce in particular the category $\mathrm{Fct}(I, \mathcal{C})$ of functors from a small category I to a category $\mathcal{C}$ in a universe $\mathcal{U}$, and look briefly at the 2-category $\mathcal{U}$-**Cat** of all $\mathcal{U}$-categories.

Here, the key result is the Yoneda lemma showing that a category $\mathcal{C}$ may be embedded in the category $\mathcal{C}^\wedge$ of all contravariant functors from $\mathcal{C}$ to **Set**, the category of sets. This allows us in a sense to reduce category theory to set theory and leads naturally to the notion of a representable functor. The category $\mathcal{C}^\wedge$ enjoys most of the properties of the category **Set**, and it is often extremely convenient, if not necessary, to replace $\mathcal{C}$ by $\mathcal{C}^\wedge$, just as in analysis, we are lead to replace functions by generalized functions.

Chapters 2 and 3. Inductive and projective limits are the most important concepts dealt with in this book. They can be seen as the essential tool of category theory, corresponding approximately to the notions of union and intersection in set theory. Since students often find them difficult to master, we provide many detailed examples. The category **Set** is not equivalent to its opposite category, and projective and inductive limits in **Set** behave very differently. Note that inductive and projective limits in a category are both defined as representable functors of *projective* limits in the category **Set**.

Having reached this point we need to construct the *Kan extension* of functors. Consider three categories $J, I, \mathcal{C}$ and a functor $\varphi\colon J \to I$. The functor φ defines by composition a functor φ_* from $\mathrm{Fct}(I,\mathcal{C})$ to $\mathrm{Fct}(J,\mathcal{C})$, and we can construct a right or left adjoint for this functor by using projective or inductive limits. These constructions will systematically be used in our presentation of sheaf theory and correspond to the operations of direct or inverse images of sheaves.

Next, we cover two essential tools for the study of limits in detail: cofinal functors (roughly analogous to the notion of extracted sequences in analysis) and filtrant[1] categories (which generalizes the notion of a directed set). As we shall see in this book, filtrant categories are of fundamental importance.

We define right exact functors (and similarly by reversing the arrows, left exact functors). Given that finite inductive limits exist, a functor is right exact if and only if it commutes with such limits.

Special attention is given to the category **Set** and to the study of filtrant inductive limits in **Set**. We prove in particular that inductive limits in **Set** indexed by a small category I commute with finite projective limits if and only if I is filtrant.

Chapter 4. Tensor categories axiomatize the properties of tensor products of vector spaces. Nowadays, tensor categories appear in many areas, mathematical physics, knot theory, computer science among others. They acquired popular attention when it was found that quantum groups produce rich examples of non-commutative tensor categories. Tensor categories and their applications in themselves merit an extended treatment, but we content ourselves

[1] Some authors use the terms “filtered” or “filtering”. We have chosen to keep the French word.

here with a rapid treatment referring the reader to [15, 40] and [59] from the vast literature on this subject.

Chapter 5. We give various criteria for a functor with values in **Set** to be representable and, as a by-product, obtain criteria under which a functor will have an adjoint. This necessitates the introduction of two important notions: strict morphisms and systems of generators (and in particular, a generator) in a category $\mathcal{C}$. References are made to [64].

Chapter 6. The Yoneda functor, which sends a category $\mathcal{C}$ to $\mathcal{C}^\wedge$, enjoys many pleasing properties, such as that of being fully faithful and commuting with projective limits, but it is not right exact.

The category $\mathrm{Ind}(\mathcal{C})$ of ind-objects of $\mathcal{C}$ is the subcategory of $\mathcal{C}^\wedge$ consisting of small and filtrant inductive limits of objects in $\mathcal{C}$. This category has many remarkable properties: it contains $\mathcal{C}$ as a full subcategory, admits small filtrant inductive limits, and the functor from $\mathcal{C}$ to $\mathrm{Ind}(\mathcal{C})$ induced by the Yoneda functor is now right exact. On the other hand, we shall show in Chap. 15 that in the abelian case, $\mathrm{Ind}(\mathcal{C})$ does not in general have enough injective objects when we remain in a given universe.

This theory, introduced in [64] (see also [3] for complementary material) was not commonly used until recently, even by algebraic geometers, but matters are rapidly changing and ind-objects are increasingly playing an important role.

Chapter 7. The process of localization appears everywhere and in many forms in mathematics. Although natural, the construction is not easy in a categorical setting. As usual, it is easier to embed than to form quotient.

If a category $\mathcal{C}$ is localized with respect to a family of morphisms $\mathcal{S}$, the morphisms of $\mathcal{S}$ become isomorphisms in the localized category $\mathcal{C}_\mathcal{S}$ and if $F\colon \mathcal{C} \to \mathcal{A}$ is a functor which sends the morphisms in $\mathcal{S}$ to isomorphisms in $\mathcal{A}$, then F will factor uniquely through the natural functor $Q\colon \mathcal{C} \to \mathcal{C}_\mathcal{S}$. This is the aim of localization. We construct the localization of $\mathcal{C}$ when $\mathcal{S}$ satisfies suitable conditions, namely, when $\mathcal{S}$ is a (right or left) multiplicative system.

Interesting features appear when we try to localize a functor F that is defined on $\mathcal{C}$ with values in some category $\mathcal{A}$, and does not map the arrows in $\mathcal{S}$ to isomorphisms in $\mathcal{A}$. Even in this case, we can define the right or left localization of the functor F under suitable conditions. We interpret the right localization functor as a left adjoint to the composition with the functor Q, and this adjoint exists if $\mathcal{A}$ admits inductive limits. It is then a natural idea to replace the category $\mathcal{A}$ with that of ind-objects of $\mathcal{A}$, and check whether the localization of F at $X \in \mathcal{C}$ is representable in $\mathcal{A}$. This is the approach taken by Deligne [17] which we follow here.

Localization is an essential step in constructing derived categories. A classical reference for localization is [24].

Chapter 8. The standard example of abelian categories is the category $\mathrm{Mod}(R)$ of modules over a ring R. Additive categories present a much weaker structure which appears for example when considering special classes of modules (e.g. the category of projective modules over the ring R is additive but not abelian).

The concept of abelian categories emerged in the early 1950s (see [13]). They inherit all the main properties of the category $\mathrm{Mod}(R)$ and form a natural framework for the development of homological algebra, as is shown in the subsequent chapters. Of particular importance are the Grothendieck categories, that is, abelian categories which admit (exact) small filtrant inductive limits and a generator. We prove in particular the Gabriel-Popescu theorem (see [54]) which asserts that a Grothendieck category may be embedded into the category of modules over the ring of endomorphisms of a generator.

We also study the abelian category $\mathrm{Ind}(\mathcal{C})$ of ind-objects of an abelian category $\mathcal{C}$ and show in particular that the category $\mathrm{Ind}(\mathcal{C})$ is abelian and that the natural functor $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ is exact. Finally we prove that under suitable hypotheses, the Kan extension of a right (or left) exact functor defined on an additive subcategory of an abelian category is also exact. Classical references are the book [14] by Cartan-Eilenberg, and Grothendieck's paper [28] which stresses the role of abelian categories, derived functors and injective objects.

An important source of historical information on this period is given in [16] by two of the main contributors.

Chapter 9. In this chapter we extend many results on filtrant inductive limits to the case of π-filtrant inductive limits, for an infinite cardinal π. An object X is π-accessible if $\mathrm{Hom}_{\mathcal{C}}(X, \bullet)$ commutes with π-filtrant inductive limits. We specify conditions which ensure that the category $\mathcal{C}_\pi$ of π-accessible objects is small and that the category of its ind-objects is equivalent $\mathcal{C}$. These techniques are used to prove that, under suitable hypotheses, given a family $\mathcal{F}$ of morphisms in a category $\mathcal{C}$, there are *enough $\mathcal{F}$-injective* objects.

Some arguments developed here were initiated in Grothendieck's paper [28] and play an essential role in the theory of model categories (see [56] and [32]). They are used in Chap. 14 in proving that the derived category of a Grothendieck category admits enough homotopically injective objects.

Here, we give two important applications. The first one is the fact that a Grothendieck category possesses enough injective objects. The second one is the Freyd-Mitchell theorem which asserts that any small abelian category may be embedded in the category of modules over a suitable ring. References are made to [64]. Accessible objects are also discussed in [1, 23] and [49].

Chapter 10. Triangulated categories first appeared implicitly in papers on stable homotopy theory after the work of Puppe [55], until Verdier axiomatized the properties of these categories (we refer to the preface by L. Illusie of [69] for more historical comments). Triangulated categories are now very popular and are part of the basic language in various branches of mathematics, especially algebraic geometry (see e.g. [57, 70]), algebraic topology and representation theory (see e.g. [35]). They appeared in analysis in the early 1970s under the influence of Mikio Sato (see [58]) and more recently in symplectic geometry after Kontsevich expressed mirror symmetry (see [43]) using this language.

A category endowed with an automorphism T is called here a category with translation. In such a category, a triangle is a sequence of morphisms $X \to Y \to Z \to T(X)$. A triangulated category is an additive category with translation endowed with a family of so-called *distinguished triangles* satisfying certain axioms. Although the first example of a triangulated category only appears in the next chapter, it seems worthwhile to develop this very elegant and easy formalism here for its own sake.

In this chapter, we study the localization of triangulated categories and the construction of cohomological functors in some detail. We also give a short proof of the Brown representability theorem [11], in the form due to Neeman [53], which asserts that, under suitable hypotheses, a contravariant cohomological functor defined on a triangulated category which sends small direct sums to products is representable.

We do not treat t-structures here, referring to the original paper [4] (see [38] for an expository treatment).

Chapter 11. It is perhaps the main idea of homological algebra to replace an object in a category $\mathcal{C}$ by a complex of objects of $\mathcal{C}$, the components of which have "good properties". For example, when considering the tensor product and its derived functors, we replace a module by a complex of projective (or flat) modules and, when considering the global-section functor and its derived functors, we replace a sheaf by a complex of flabby sheaves.

It is therefore natural to study the category $\mathrm{C}(\mathcal{C})$ of complexes of objects of an additive category $\mathcal{C}$. This category inherits an automorphism, the *shift functor*, called the "suspension" by algebraic topologists. Other basic constructions borrowed from algebraic topology are that of the *mapping cone* of a morphism and that of *homotopy* of complexes. In fact, in order to be able to work, i.e., to form commutative diagrams, we have to make morphisms in $\mathrm{C}(\mathcal{C})$ which are homotopic to zero, actually isomorphic to zero. This defines the homotopy category $\mathrm{K}(\mathcal{C})$ and the main result (stated in the slightly more general framework of additive categories with translation) is that $\mathrm{K}(\mathcal{C})$ is triangulated.

Many complexes, such as Čech complexes in sheaf theory (see Chap. 18 below), are obtained naturally by simplicial construction. Here, we construct complexes associated with simplicial objects and give a criterion for these complexes to be homotopic to zero.

When considering bifunctors on additive categories, we are rapidly lead to consider the category $\mathrm{C}(\mathrm{C}(\mathcal{C}))$ of complexes of complexes (i.e., double complexes), and so on. We explain here how a diagonal procedure allows us, under suitable hypotheses, to reduce a double complex to a simple one. Delicate questions of signs arise and necessitate careful treatment.

Chapter 12. When $\mathcal{C}$ is abelian, we can define the j-th cohomology object $H^j(X)$ of a complex X. The main result is that the functor H^j is cohomological, that is, sends distinguished triangles in $\mathrm{K}(\mathcal{C})$ to long exact sequences in $\mathcal{C}$.

When a functor F with values in $\mathcal{C}$ is defined on the category of finite sets, it is possible to attach to F a complex in $\mathcal{C}$, generalizing the classical notion of Koszul complexes. We provide the tools needed to calculate the cohomology of such complexes and treat some examples such as distributive families of subobjects.

We also study the cohomology of a double complex, replacing the Leray's traditional spectral sequences by an intensive use of the truncation functors. We find this approach much easier and perfectly adequate in practice.

Chapter 13. Constructing the derived category of an abelian category is easy with the tools now at hand. It is nothing more than the localization of the homotopy category $\mathrm{K}(\mathcal{C})$ with respect to exact complexes.

Here we give the main constructions and results concerning derived categories and functors, including some new results.

Despite their popularity, derived categories are sometimes supposed difficult. A possible reason for this reputation is that to date there has been no systematic, pedagogical treatment of the theory. The classical texts on derived categories are the famous Hartshorne Notes [31], or Verdier's résumé of his thesis [68] (of which the complete manuscript has been published recently [69]). Apart from these, there are a few others which may be found in particular in the books [25, 38] and [71]. Recall that the original idea of derived categories goes back to Grothendieck.

Chapter 14. Using the results of Chap. 9, we study the (unbounded) derived category $\mathrm{D}(\mathcal{C})$ of a Grothendieck category $\mathcal{C}$. First, we show that any complex in a Grothendieck category is quasi-isomorphic to a *homotopically injective complex* and we deduce the existence of right derived functors in $\mathrm{D}(\mathcal{C})$. We then prove that the Brown representability theorem holds in $\mathrm{D}(\mathcal{C})$ and discuss the existence of left derived functors, as well as the composition of (right or left) derived functors and derived adjunction formulas.

Spaltenstein [65] was the first to consider unbounded complexes and the corresponding derived functors. The (difficult) result which asserts that the Brown representability theorem holds in the derived category of a Grothendieck category seems to be due to independently to [2] and [21] (see also [6, 42, 53] and [44]). Note that most of the ideas presented here come from topology, in which context the names of Adams, Bousfield, Kan, Thomason among others should be mentioned.

Chapter 15. We study here the derived category of the category $\mathrm{Ind}(\mathcal{C})$ of ind-objects of an abelian category $\mathcal{C}$. Things are not easy since in the simple case where $\mathcal{C}$ is the category of vector spaces over a field k, the category $\mathrm{Ind}(\mathcal{C})$ does not have enough injective objects. In order to overcome this difficulty, we introduce the notion of quasi-injective objects. We show that under suitable hypotheses, there are enough such objects and that they allow us to derive functors. We also study some links between the derived category of $\mathrm{Ind}(\mathcal{C})$ and that of ind-objects of the derived category of $\mathcal{C}$. Note that the category of ind-objects of a triangulated category does not seem to be triangulated.

Most of the results in this chapter are new and we hope that they may be useful. They are so when applied to the construction of ind-sheaves, for which we refer to [39].

Chapter 16. The notion of sheaves relies on that of coverings and a Grothendieck topology on a category is defined by axiomatizing the notion of coverings.

In this chapter we give the axioms for Grothendieck topologies using sieves and then introduce the notions of local epimorphisms and local isomorphisms. We give several examples and study the properties of the family of local isomorphisms in detail, showing in particular that this family is stable under inductive limits. The classical reference is [64].

Chapter 17. A site X is a category $\mathcal{C}_X$ endowed with a Grothendieck topology. A presheaf F on X with values in a category $\mathcal{A}$ is a contravariant functor on $\mathcal{C}_X$ with values in $\mathcal{A}$, and a presheaf F is a sheaf if, for any local isomorphism $A \to U$, $F(U) \to F(A)$ is an isomorphism. When $\mathcal{C}_X$ is the category of open subsets of a topological space X, we recover a familiar notion.

Here, we construct the sheaf F^a associated with a presheaf F with values in a category $\mathcal{A}$ satisfying suitable properties. We also study restriction and extension of sheaves, direct and inverse images, and internal $\mathcal{H}om$. However, we do not enter the theory of Topos, referring to [64] (see also [48] for further exciting developments).

Chapter 18. When $\mathcal{O}_X$ is a sheaf of rings on a site X, we define the category $\mathrm{Mod}(\mathcal{O}_X)$ of sheaves of $\mathcal{O}_X$-modules. This is a Grothendieck category to which we may apply the tools obtained in Chap. 14.

In this Chapter, we construct the unbounded derived functors $R\mathcal{H}om_{\mathcal{O}_X}$ of internal hom, $\overset{\mathrm{L}}{\otimes}_{\mathcal{O}_X}$ of tensor product, Rf_* of direct image and Lf^* of inverse image (these two last functors being associated with a morphism f of ringed sites) and we study their relations. Such constructions are well-known in the case of bounded derived categories, but the unbounded case, initiated by Spaltenstein [65], is more delicate.

We do not treat proper direct images and duality for sheaves. Indeed, there is no such theory for sheaves on abstract sites, where the construction in the algebraic case for which we refer to [17], differs from that in the topological case for which we refer to [38].

Chapter 19. The notion of constant functions is not local and it is more natural (and useful) to consider locally constant functions. The presheaf of such functions is in fact a sheaf, called a constant sheaf. There are however sheaves which are locally, but not globally, isomorphic to this constant sheaf, and this leads us to the fundamental notion of locally constant sheaves, or local systems. The orientation sheaf on a real manifold is a good example of such a sheaf. We consider similarly categories which are locally equivalent to the category of sheaves, which leads us to the notions of stacks and twisted sheaves.

A stack on a site X is, roughly speaking, a sheaf of categories, or, more precisely, a sheaf with values in the 2-category of all $\mathcal{U}$-categories of a given universe $\mathcal{U}$. Indeed, it would be possible to consider higher objects (n-stacks), but we do not pursue this matter here. This new field of mathematics was first explored in the sixties by Grothendieck and Giraud (see [26]) and after having been long considered highly esoteric, it is now the object of intense activity from algebraic geometry to theoretical physics. Note that 2-categories were first introduced by Bénabou (see [5]), a student of an independent-minded category theorist, Charles Ehresmann.

This last chapter should be understood as a short presentation of possible directions in which the theory may develop.