7

A Multiobjective Simplex Method

An MOLP with two objectives can be conveniently solved using the parametric Simplex method presented in Algorithm 6.2. With three or more objectives, however, this is no longer possible because we deal with at least two parameters in the objective function $c(\lambda)$.

7.1 Algebra of Multiobjective Linear Programming

In this section we consider the general MOLP

$$\begin{aligned} \min \quad & Cx \\ \text{subject to} \quad & Ax = b \\ & x \geqq 0. \end{aligned} \tag{7.1}$$

For $\lambda \in \mathbb{R}^p_>$ we denote by LP(λ) the weighted sum linear program

$$\min \left\{ \lambda^T Cx : Ax = b, x \geqq 0 \right\}. \tag{7.2}$$

We use the notation $\bar{C} = C - C_{\mathcal{B}} A_{\mathcal{B}}^{-1} A$ for the reduced cost matrix with respect to basis $\mathcal{B}$ and $R := \bar{C}_{\mathcal{N}}$ for the nonbasic part of the reduced cost matrix. Note that $\bar{C}_{\mathcal{B}} = 0$ according to (6.15) and is therefore uninteresting. Proofs in this section will make use of Theorem 6.11. These results are multicriteria analogies of well known linear programming results, or necessary extensions to cope with the increased complexity of multiobjective compared to single objective linear programming.

Lemma 7.1. *If $\mathcal{X}_E \neq \emptyset$ then $\mathcal{X}$ has an efficient basic feasible solution.*

Proof. By Theorem 6.11 there is some $\lambda \in \mathbb{R}^p_>$ such that $\min_{x \in \mathcal{X}} \lambda^T Cx$ has an optimal solution. But by Theorem 6.13 the LP(λ) $\min_{x \in \mathcal{X}} \lambda^T Cx$ has a optimal basic feasible solution solution, which is an efficient solution of the MOLP by Theorem 6.6. $\square$

Lemma 7.1 justifies the definition of an efficient basis.

Definition 7.2. *A feasible basis $\mathcal{B}$ is called* efficient basis *if $\mathcal{B}$ is an optimal basis of LP(λ) for some $\lambda \in \mathbb{R}^p_>$.*

We now look at pivoting among efficient bases. We say that a pivot is a *feasible pivot* if the solution obtained after the pivot step is feasible, even if the pivot element $\tilde{A}_{rs} < 0$.

Definition 7.3. *Two bases $\mathcal{B}$ and $\hat{\mathcal{B}}$ are called* adjacent *if one can be obtained from the other by a single pivot step.*

Definition 7.4.
1. *Let $\mathcal{B}$ be an efficient basis. Variable $x_j, j \in \mathcal{N}$ is called* efficient nonbasic variable *at $\mathcal{B}$ if there exists a $\lambda \in \mathbb{R}^p_>$ such that $\lambda^T R \geqq 0$ and $\lambda^T r^j = 0$, where r^j is the column of R corresponding to variable x_j.*
2. *Let $\mathcal{B}$ be an efficient basis and let x_j be an efficient nonbasic variable. Then a feasible pivot from $\mathcal{B}$ with x_j entering the basis is called an* efficient pivot *with respect to $\mathcal{B}$ and x_j.*

The system $\lambda^T R \geqq 0, \lambda^T r^j = 0$ is the general form of the equations we used to compute the critical λ values in parametric linear programming that were used to derive (6.23): We chose s such that $\bar{c}(\lambda) \geq 0, \bar{c}(\lambda)_s = 0$.

Proposition 7.5. *Let $\mathcal{B}$ be an efficient basis. There exists an efficient non-basic variable at $\mathcal{B}$.*

Proof. Because $\mathcal{B}$ is an efficient basis there exists $\lambda > 0$ such that $\lambda^T R \geqq 0$. Thus the set $\mathcal{L} := \{\lambda > 0 : \lambda^T R \geqq 0\}$ is not empty. We have to show that there is $\lambda \in \mathcal{L}$ and $j \in \mathcal{N}$ such that $\lambda^T r^j = 0$.

First we observe that there is no column r of R such that $r \leq 0$. There also must be at least one column with positive and negative elements, because of the general assumption (6.2). Now let $\lambda^* \in \mathcal{L}$. In particular $\lambda^{*T} \geqq 0$. Let $\lambda' \in \mathbb{R}^p_>$ be such that $\mathcal{I} := \{i \in \mathcal{N} : \lambda'^T r^j < 0\} \neq \emptyset$. Such a λ must exist, because R contains at least one negative entry.

We define $\phi : \mathbb{R} \to \mathbb{R}^{|\mathcal{N}|}$ by

$$\phi_i(t) := (t\lambda^{*T} + (1-t)\lambda'^T)r^i, i \in \mathcal{N}.$$

Thus, $\phi(0) = \lambda'^T R$ and $\phi(1) = \lambda^{*T} R \geqq 0$. For each $i \in \mathcal{N} \setminus \mathcal{I}$ we have that $\phi_i(t) \geq 0$ for all $t \in [0,1]$. For all $i \in \mathcal{I}$ there exists some $t_i \in [0,1]$ such that

$$\phi_i(t) \begin{cases} < 0, t \in [0, t_i) \\ = 0, t = t_i \\ \geq 0, t \in [t_i, 1]. \end{cases}$$

With $t^* := \max\{t_i : i \in \mathcal{I}\}$ we have that $\phi_i(t^*) \geqq 0$ and $\phi_i(t^*) = 0$ for some $i \in \mathcal{I}$. Thus $\hat{\lambda} := t\lambda^* + (1-t)\lambda' \in \mathcal{L}$ and the proof is complete. □

Example 7.6. It might appear that any nonbasic variable such that r^j contains positive and negative entries is an efficient nonbasic variable. This is not the case, as the following example shows. Let

$$R = \begin{pmatrix} -4 & 1 \\ -2 & 1 \end{pmatrix}.$$

Then there is no $\lambda \in \mathbb{R}^2_>$ such that $\lambda^T R \geqq 0$ and $\lambda^T r^2 = 0$. The latter equation means $\lambda_2 = 2\lambda_1$. Then $\lambda^T r^1 \geq 0$ would require $-2\lambda_2 \geq 0$, an impossibility.

□

Lemma 7.7. *Let $\mathcal{B}$ be an efficient basis and x_j be an efficient nonbasic variable. Then any efficient pivot from $\mathcal{B}$ leads to an adjacent efficient basis $\hat{\mathcal{B}}$.*

Proof. Let x_j be the entering variable at basis $\mathcal{B}$. Because x_j is an efficient nonbasic variable, we have $\lambda \in \mathbb{R}^p_>$ with $\lambda^T R \geqq 0$ and $\lambda^T r^j = 0$. Thus x_j is a nonbasic variable with reduced cost 0 in LP(λ). This means that the reduced costs of LP(λ) do not change after a pivot with x_j entering. Let $\hat{\mathcal{B}}$ be the resulting basis with any feasible pivot and entering variable x_j. Then $\lambda^T R \geqq 0$ and $\lambda^T r^j = 0$ at $\hat{\mathcal{B}}$, i.e. $\hat{\mathcal{B}}$ is an optimal basis for LP(λ) and therefore an adjacent efficient basis. □

We need a method to check whether a nonbasic variable x_j at an efficient basis $\mathcal{B}$ is efficient. This can be done by performing a test that consists in solving an LP.

Theorem 7.8 (Evans and Steuer (1973)). *Let $\mathcal{B}$ be an efficient basis and let x_j be a nonbasic variable. Variable x_j is an efficient nonbasic variable if and only if the LP*

$$\begin{aligned} \max \quad & e^t v \\ \text{subject to } & Rz - r^j\delta + Iv = 0 \\ & z, \delta, v \geqq 0 \end{aligned} \tag{7.3}$$

has an optimal value of 0.

Proof. By Definition 7.4 x_j is an efficient nonbasic variable if the LP

$$\begin{aligned} \min \quad & 0^T\lambda = 0 \\ \text{subject to} \quad & R^T\lambda \geqq 0 \\ & (r^j)^T\lambda = 0 \\ & I\lambda \geqq e \\ & \lambda \geqq 0 \end{aligned} \tag{7.4}$$

has an optimal objective value of 0, i.e. if it is feasible. The first two constraints of (7.4) together are equivalent to $R^T\lambda \geqq 0, (r^j)^T\lambda \leq 0$, or $R^T\lambda \geqq 0, (-r^j)^T\lambda \geq 0$, which gives the LP

$$\begin{aligned} \min \quad & 0^T\lambda = 0 \\ \text{subject to} \quad & R^T\lambda \geqq 0 \\ & -(r^j)^T\lambda \geq 0 \\ & I\lambda \geqq e \\ & \lambda \geqq 0. \end{aligned} \tag{7.5}$$

The dual of (7.5) is

$$\begin{aligned} \max \quad & e^T v \\ \text{subject to} \quad & Rz - r^j\delta + Iv + It = 0 \\ & z, \delta, v, t \geqq 0. \end{aligned} \tag{7.6}$$

Since an optimal solution of (7.6) will always contain t at value zero, this is equivalent to

$$\begin{aligned} \max \quad & e^T v \\ \text{subject to} \quad & Rz - r^j\delta + Iv = 0 \\ & z, \delta, v \geqq 0, \end{aligned}$$

which is (7.3). □

It is important to note that the test problem (7.3) is always feasible since $(z, \delta, v) = 0$ can be chosen. The proof also 7.8 also shows that (7.3) can only have either an optimal solution with $v = 0$ (the objective value of (7.4) is zero), or be unbounded. With this observation we conclude that

- x_j is an efficient nonbasic variable if and only if (7.3) is bounded and has optimal value 0,
- x_j is an "inefficient" nonbasic variable if and only if (7.3) is unbounded.

The Simplex algorithm works by moving along adjacent bases until an optimal one is found. We want to make use of this principle to identify all efficient bases, i.e. we want to move from efficient basis to efficient basis. Therefore we must prove that it is indeed possible to restrict ourselves to adjacent bases only, i.e. that the efficient bases are connected in terms of adjacency.

Definition 7.9. *Two efficient bases $\mathcal{B}$ and $\hat{\mathcal{B}}$ are called* connected *if one can be obtained from the other by performing only efficient pivots.*

We prove that all efficient bases are connected using parametric programming. Note that single objective optimal pivots (i.e. the entering variable is

x_s with $\bar{c}_s = 0$) as well as parametric pivots are efficient pivots (one of the two reduced costs is negative, the other positive) according to (6.23). These cases are also covered by Proposition 7.5. Theorem 7.10 is the foundation for the multicriteria Simplex algorithm. We present a proof by Steuer (1985).

Theorem 7.10 (Steuer (1985)). *All efficient bases are connected.*

Proof. Let $\mathcal{B}$ and $\hat{\mathcal{B}}$ be two efficient bases. Let $\lambda, \hat{\lambda} \in \mathbb{R}^p_>$ be the positive weighting vectors for which $\mathcal{B}$ and $\hat{\mathcal{B}}$ are optimal bases for LP(λ) and LP($\hat{\lambda}$), respectively. We consider the parametric LP with objective function

$$c(\Phi) = \Phi\hat{\lambda}^T C + (1 - \Phi)\lambda^T C \tag{7.7}$$

with $\Phi \in [0, 1]$.

Let $\hat{B}$ be the first basis (for $\Phi = 1$). After several parametric programming or optimal pivots we get a basis $\tilde{\mathcal{B}}$ which is optimal for LP(λ). Since $\lambda^* = \Phi\hat{\lambda} + (1 - \Phi)\lambda \in \mathbb{R}^p_>$ for all $\Phi \in [0, 1]$ all intermediate bases are optimal for LP(λ^*) for some $\lambda^* \in \mathbb{R}^p_>$, i.e. they are efficient bases. All parametric and optimal pivots are efficient pivots as explained above. If $\tilde{\mathcal{B}} = \mathcal{B}$ we are done. Otherwise $\mathcal{B}$ can be obtained from $\tilde{\mathcal{B}}$ by efficient pivots (i.e. optimal pivots for LP(λ)), because both $\mathcal{B}$ and $\tilde{\mathcal{B}}$ are optimal bases for this LP. □

It is now possible to explain why the nontriviality assumption is necessary. Without it, the existence of efficient nonbasic variables is not guaranteed, and therefore Theorem 7.10 may fail. Example 7.11 also demonstrates a problem with degenerate MOLPs.

Example 7.11 (Steuer (2002)). We want to solve the following MOLP

$$\begin{array}{llllll}
\min & & - & 2x_2 & + & x_3 \\
\min & -x_1 & + & 2x_2 & + & x_3 \\
\text{subject to} & & & x_2 & + & 4x_3 \leq 8 \\
 & x_1 & + & x_2 & & \leq 8 \\
x_1, & & & x_2, & & x_3 \geq 0.
\end{array}$$

We introduce slack variables x_4, x_5 to write the LP in equality form. It is clear that both objective functions are minimized at the same solution, $\hat{x} = (0, 4, 0, 0, 0)$. Thus $\mathcal{X}_E = \{\hat{x}\}$. Because the only nonzero variable at $\hat{x}$ is $\hat{x}_2 = 2$, there are four different bases that define the same efficient basic feasible solution, namely $\{1, 2\}, \{2, 3\}, \{2, 4\}$, and $\{2, 5\}$ (the problem is degenerate). Below we show the Simplex tableaus for these four bases.

$\bar{c}^1$	0	0	7	2	0	16
$\bar{c}^2$	0	0	8	$\frac{9}{4}$	-$\frac{1}{4}$	16
x_1	1	0	-1	-$\frac{1}{4}$	$\frac{1}{4}$	0
x_2	0	1	4	1	0	8

$\bar{c}^1$	7	0	0	$\frac{1}{4}$	$\frac{7}{4}$	16
$\bar{c}^2$	8	0	0	$\frac{1}{4}$	$\frac{7}{4}$	16
x_3	-1	0	1	$\frac{1}{4}$	-$\frac{1}{4}$	0
x_2	4	1	0	0	1	8

$\bar{c}^1$	8	0	-1	0	2	16
$\bar{c}^2$	9	0	-1	0	2	16
x_4	-4	0	4	1	-1	0
x_2	4	1	0	0	1	8

$\bar{c}^1$	0	0	7	2	0	16
$\bar{c}^2$	1	0	7	2	0	16
x_2	0	1	4	1	0	8
x_5	4	0	-4	-1	1	0

Bases $\{1, 2\}$ and $\{2, 4\}$ are not efficient according to Definition 7.2 because R contains columns that do not have positive entries. This is due to degeneracy, which makes the negative reduced cost values possible, despite the BFS being efficient/optimal.

Furthermore, bases $\{2, 3\}$ and $\{2, 5\}$ are efficient. The definition is satisfied for all $\lambda \in \mathbb{R}^2_>$. However, for these bases R has no negative entries at all, hence no efficient nonbasic variable according to Definition 7.3 exist. The example therefore shows, that the assumption (6.2) is necessary to guarantee existence of efficient nonbasic variables, and the validity of Theorem 7.10. □

From Theorem 7.8 we know that we must consider negative pivot elements, i.e. $\tilde{A}_{rj} < 0$. What happens if nonbasic variable x_j is efficient and column j of $\tilde{A}$ contains no positive elements at all? Then the increase of x_j is unbounded, a fact that indicated an unbounded LP in the single objective case. However, since $\lambda^T r^j = 0$ this is not the case in the multiobjective LP. Rather, unboundedness of $\mathcal{X}_E$ is detected in direction d given by the vector with components $-\tilde{b}_i/\tilde{A}_{ij}, i \in \mathcal{B}, x_j = 1$. Of course, this is not a feasible pivot, as it does not lead to another basis.

The results so far allow us to move from efficient basis to efficient basis. To formulate a multiobjective Simplex algorithm we now need an efficient basis to start with.

For the MOLP

$$\min\{Cx : Ax = b, x \geqq 0\}$$

one and only one of the following cases can occur:

- The MOLP is infeasible, i.e. $\mathcal{X} = \emptyset$,
- it is feasible ($\mathcal{X} \neq \emptyset$) but has no efficient solutions ($\mathcal{X}_E = \emptyset$), or
- it is feasible and has efficient solutions, i.e. $\mathcal{X}_E \neq \emptyset$.

The multicriteria Simplex algorithm deals with these situations in three phases as follows.

Phase I: Determine an initial basic feasible solution or stop with the conclusion that $\mathcal{X} = \emptyset$. This phase does not involve the objective function matrix C, and the usual auxiliary LP (6.18) can be used.

Phase II: Determine an initial efficient efficient basis or stop with the conclusion that $\mathcal{X}_E = \emptyset$.

Phase III: Pivot among efficient bases to determine all efficient bases and directions of unboundedness of $\mathcal{X}_E$.

In Phase II, the solution of a weighted sum LP(λ) with $\lambda > 0$ will yield an efficient basis, provided LP(λ) is bounded. If we do not know that in advance it is necessary to come up with a procedure that either concludes that $\mathcal{X}_E = \emptyset$ or returns an appropriate λ for which LP(λ) has an optimal solution. Assuming that $\mathcal{X} \neq \emptyset$ Phase I returns a basic feasible solution $x^0 \in \mathcal{X}$, which may or may not be efficient. We proceed in two steps: First, the auxiliary LP (6.8) is solved to check whether $\mathcal{X}_E = \emptyset$. Proposition 6.12 and duality imply that $\mathcal{X}_\mathcal{E} \neq \emptyset$ if and only if (6.8) has an optimal solution. In this case the optimal solution of (6.8) returns an appropriate weighting vector $\hat{w}$, analogously to the argument we have used in the proof of Theorem 6.11.

From Proposition 6.12 the MOLP $\min\{Cx : Ax = b, x \geqq 0\}$ has an efficient solution if and only if the LP 7.8

$$\max\left\{e^T z : Ax = b, Cx + Iz = Cx^0, x, z \geqq 0\right\} \tag{7.8}$$

has an optimal solution. Moreover $\hat{x}$ in an optimal solution of (6.7) is efficient. However, we do not know if $\hat{x}$ is a basic feasible solution of the MOLP and we can in general not choose $\hat{x}$ to start Phase III of the algorithm.

Instead we apply linear programming duality (Theorem 6.8): (7.8) has an optimal solution if and only if its dual (7.9)

$$\min\left\{u^T b + w^T Cx^0 : u^T A + w^T C \geqq 0, w \geqq e\right\} \tag{7.9}$$

has an optimal solution $(\hat{u}, \hat{w})$ with $\hat{u}^T b + \hat{w}^T Cx^0 = e^T \hat{z}$. Then $\hat{u}$ is also an optimal solution of the LP (7.10)

$$\min\left\{u^T b : u^T A, \geqq -\hat{w}TC\right\} \tag{7.10}$$

which is just (7.9) for $w = \hat{w}$ fixed. As in the proof of Theorem 6.11 the dual of (7.10) has an optimal solution, and therefore an optimal basic feasible solution, which is efficient. The dual of (7.10) is equivalent to the weighted sum LP($\hat{w}$)

$$\min\left\{\hat{w}^T Cx : Ax = b, x \geqq 0.\right\}$$

It follows that the LPs (7.9) and LP($\hat{w}$) are the necessary tools in Phase II. If (7.9) is infeasible, $\mathcal{X}_E = \emptyset$. Otherwise an optimal solution of (7.9) yields an appropriate weighting vector $\lambda = \hat{w}$ for which LP(λ) has an optimal basic feasible solution, which is an initial efficient basic feasible solution of the MOLP.

In the following description of the multiobjective Simplex algorithm, which finds all efficient bases and all efficient basic feasible solutions, we need to store a list $\mathcal{L}_1$ of efficient bases to be processed and a list of efficient bases $\mathcal{L}_2$ for output, as well as a list $\mathcal{EN}$ of efficient nonbasic variables.

Algorithm 7.1 (Multicriteria Simplex algorithm.)

Input: *Data* A, b, C *of an MOLP.*

Initialization: *Set* $\mathcal{L}_1 := \emptyset$, $\mathcal{L}_2 := \emptyset$.

Phase I: *Solve the LP* $\min\{e^T z : Ax + Iz = b, x, z \geqq 0\}$. *If the optimal value of this LP is nonzero, STOP,* $\mathcal{X} = \emptyset$. *Otherwise let* x^0 *be a basic feasible solution* x^0 *of the MOLP.*

Phase II: *Solve the LP* $\min\{u^T b + w^T Cx^0 : u^T A + w^T C \geqq 0, w \geqq e\}$. *If this problem is infeasible, STOP,* $\mathcal{X}_E = \emptyset$. *Otherwise let* $(\hat{u}, \hat{w})$ *be an optimal solution.*

Find an optimal basis $\mathcal{B}$ *of the LP* $\min\{\hat{w}^T Cx : Ax = b, x \geqq 0\}$.

$\mathcal{L}_1 := \{\mathcal{B}\}$, $\mathcal{L}_2 := \emptyset$.

Phase III:

While $\mathcal{L}_1 \neq \emptyset$

Choose $\mathcal{B}$ *in* $\mathcal{L}_1$, *set* $\mathcal{L}_1 := \mathcal{L}_1 \setminus \{\mathcal{B}\}$, $\mathcal{L}_2 := \mathcal{L}_2 \cup \{\mathcal{B}\}$.

Compute $\tilde{A}$, $\tilde{b}$, *and* R *according to* $\mathcal{B}$.

$\mathcal{EN} := \mathcal{N}$.

For all $j \in \mathcal{N}$.

Solve the LP $\max\{e^T v : Ry - r^j\delta + Iv = 0; y, \delta, v \geqq 0\}$.

If this LP is unbounded$\mathcal{EN} := \mathcal{EN} \setminus \{j\}$.

End for

For all $j \in \mathcal{EN}$.

For all $i \in \mathcal{B}$.

If $\mathcal{B}' = (\mathcal{B} \setminus \{i\}) \cup \{j\}$ *is feasible and* $\mathcal{B}' \notin \mathcal{L}_1 \cup \mathcal{L}_2$

then $\mathcal{L}_1 := \mathcal{L}_1 \cup \mathcal{B}'$.

End for.

End for.

End while.
Output: $\mathcal{L}_2$.

We have formulated the algorithm only using bases. It is clear that efficient basic feasible solutions can be computed form the list $\mathcal{L}_2$ after completion of the algorithm, or during the algorithm. It is of course necessary to update $\tilde{A}$ and $\tilde{b}$ when moving from one basis to the next. Since this has been described in Algorithm 6.1, we omitted details. It is also possible to get directions in which $\mathcal{X}_E$ is unbounded. As mentioned before, these are characterized by columns of $\tilde{A}$ that do not contain positive entries.

Is Algorithm 7.1 an efficient algorithm? While we only introduce computational complexity in Section 8.1 we comment on the performance of multicriteria Simplex algorithms here. Because the (single objective) Simplex algorithm may require an exponential number of pivot steps (in terms of problem size m, n, p, see e.g. Dantzig and Thapa (1997) for a famous example), the same is true for our multicriteria Simplex algorithm.

The question, whether a polynomial time algorithm for multicriteria linear programming (e.g. a generalization of Karmarkar's interior point algorithm Karmarkar (1984)) is possible depends on the number of efficient extreme points. Unfortunately, it is easy to construct examples with exponentially many.

Example 7.12. Consider a multicriteria linear program, the feasible set of which is a hypercube in $\mathbb{R}^n$, i.e. $\mathcal{X} = [0,1]^n$ and which has objectives to minimize x_i as well as $-x_i$. Formally,

$$\begin{array}{lrl} \min & x_i & i = 1, \ldots, n \\ \min & -x_i & i = 1, \ldots, n \\ \text{subject to} & x_i \leq 1 & i = 1, \ldots, n \\ & -x_i \leq 1 & i = 1, \ldots, n. \end{array}$$

This problem has n variables, $m = 2n$ constraints and $p = 2n$ objective functions. It is obvious, that all 2^n extreme points of the feasible set are efficient. □

Some investigations show that the average number of efficient extreme points can be huge. Benson (1998c) reports on such numerical tests. Results on three problem classes (with inequality constraints) with 10 random examples each are summarized in Table 7.1.

However, Küfer (1998) did a probabilistic analysis and found that the expected number of efficient extreme points for a certain family of randomly generated MOLPs is polynomial in n, m, and p.

Table 7.1. Number of efficient extreme points.

n	m	Q	Number of efficient extreme points
30	25	4	7,245.9 on average
50	50	4	83,780.6 on average
60	50	4	more than 200,000 in each problem

We close this section with an example for the multicriteria Simplex algorithm.

Example 7.13 (Wiecek (1995)). We solve an MOLP with three objectives, three variables, and three constraints:

$$\begin{array}{llllll}
\min & -x_1 & -2x_2 & & & \\
\min & -x_1 & & +2x_3 & & \\
\min & x_1 & & -x_3 & & \\
\text{subject to} & x_1 & +x_2 & & \leq & 1 \\
 & & x_2 & & \leq & 2 \\
 & x_1 & -x_2 & +x_3 & \leq & 4.
\end{array}$$

Slack variables x_4, x_5, x_6 are introduced to write the constraints in equality form $Ax = b$.

Phase I: It is clear that $\mathcal{B} = \{4, 5, 6\}$ is a feasible basis and $x^0 = (0, 0, 0, 1, 2, 4)$ is a basic feasible solution.

Phase II: We solve (7.9) with x^0 from Phase I:

$$\begin{array}{ll}
\min & u_1 + 2u_2 + 4u_3 \\
\text{subject to} & u^T \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{pmatrix} + w^T \begin{pmatrix} -1 & -2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \geqq 0 \\
 & \hfill w \geqq e
\end{array}$$

The w component of the optimal solution is $\hat{w} = (1, 1, 1)$.
We now solve $\min\{\hat{w}^T Cx : x \in \mathcal{X}\}$. x^0 is an initial basic feasible solution for this problem. An optimal basis is $\mathcal{B}^1 = \{2, 5, 6\}$ with optimal basic feasible solution $x^1 = (0, 1, 0, 0, 1, 5)$. Therefore we initialize $\mathcal{L}_1 = \{\{2, 5, 6\}\}$ and move to Phase III.

Phase III:

Iteration 1: We choose basis $\mathcal{B}^1 = \{2, 5, 6\}$ and set $\mathcal{L}_1 = \emptyset$, $\mathcal{L}_2 = \{\{2, 5, 6\}\}$. The tableau for this basis is given below.

$\bar{c}^1$	1	0	0	2	0	0	2
$\bar{c}^2$	-1	0	2	0	0	0	0
$\bar{c}^3$	1	0	-1	0	0	0	0
x_2	1	1	0	1	0	0	1
x_5	-1	0	0	-1	1	0	1
x_6	2	0	1	1	0	1	5

$\mathcal{EN} := \{1, 3, 4\}$.
The LP to check if x_1 is an efficient nonbasic variable is given in tableu form, where the objective coefficients of 1 for variables v have been eliminated by subtracting all constraint rows from the objective row to obtain a basic feasible solution with basic variables $v = 0$. This LP does have an optimal solution that is found after only one pivot. Pivot elements are highlighted by square frames.

1	1	2	-1	0	0	0	0
1	0	2	-1	1	0	0	0
-1	2	0	$\boxed{1}$	0	1	0	0
1	-1	0	-1	0	0	1	0

The LP to check if variable x_3 is an efficient nonbasic variable is shown below. The problem has an optimal solution, proved by performing the indicated pivot.

1	1	2	-1	0	0	0	0
1	0	2	0	1	0	0	0
-1	2	0	-2	0	1	0	0
1	-1	0	$\boxed{1}$	0	0	1	0

Finally, we check nonbasic variable x_4. In the tableau displayed below, column three indicates that the LP is unbounded, and x_4 is not efficient.

1	1	2	-2	0	0	0	0
1	0	2	-2	1	0	0	0
-1	2	0	0	0	1	0	0
1	-1	0	0	0	0	1	0

As a result of these checks we have that $\mathcal{EN} = \{1, 3\}$. Checking in the tableau for $\mathcal{B}^1 = \{2, 5, 6\}$, we find that the feasible pivots are 1) x_1 enters and x_2 leaves, giving basis $\mathcal{B}^2 = \{1, 5, 6\}$ and 2) x_3 enters and x_6 leaves, yielding basis $\mathcal{B}^3 = \{2, 3, 5\}$.
$\mathcal{L}_1 := \{\{1, 5, 6\}, \{2, 3, 5\}\}$.

Iteration 2: Choose $\mathcal{B}^2 = \{1, 5, 6\}$ with BFS $x^2 = (1, 0, 0, 0, 2, 3)$.

$\mathcal{L}_1 = \{\{2, 3, 5\}\}$, $\mathcal{L}_2 = \{\{2, 5, 6\}, \{2, 3, 5\}\}$.

The tableau for the basis is as follows.

$\bar{c}^1$	0	-1	0	1	0	0	1
$\bar{c}^2$	0	1	2	1	0	0	1
$\bar{c}^3$	0	-1	-1	-1	0	0	-1
x_2	1	1	0	1	0	0	1
x_5	0	1	0	0	1	0	2
x_6	0	-2	1	-1	0	1	3

$\mathcal{EN} = \{2, 3, 4\}$.

If x_2 enters the basis, x_1 leaves, which leads to basis $(2, 5, 6)$ which is the previous one. Therefore x_2 need not be checked.

The tableau for checking x_3 is displayed below. After one pivot column 3 shows that the LP is unbounded and x_3 is not efficient.

-1	1	1	-1	0	0	0	0
-1	0	1	0	1	0	0	0
1	2	1	-2	0	1	0	0
-1	-1	-1	1	0	0	1	0

We check nonbasic variable x_4. One iteration is again enough to exhibit unboundedness, and x_4, too, is not efficient.

-1	1	1	-1	0	0	0	0
-1	0	1	-1	1	0	0	0
1	2	1	-1	0	1	0	0
-1	-1	-1	1	0	0	1	0

These checks show that there are no new bases and BFSs to add, therefore $\mathcal{EN} = \emptyset$ and we proceed to the next iteration.

Iteration 3: We choose $\mathcal{B}^3 = \{2, 3, 5\}$ with BFS $x^3 = (0, 1, 5, 0, 1, 0)$.

$\mathcal{L}_1 = \emptyset$, $\mathcal{L}_2 = \{\{2, 5, 6\}, \{1, 5, 6\}, \{2, 3, 5\}\}$.

The tableau for the basis is shown below.

$\bar{c}^1$	1	0	0	2	0	0	2
$\bar{c}^2$	-5	0	0	-2	0	-2	-10
$\bar{c}^3$	3	0	0	1	0	1	5
x_2	1	1	0	1	0	0	1
x_5	-1	0	0	-1	1	0	1
x_3	2	0	1	1	0	1	5

$\mathcal{EN} = \{1, 4, 6\}$.

We test nonbasic variable x_1. After one pivot column 4 in the tableau shows that the LP is unbounded.

-1	1	-1	1	0	0	0	0
[1]	2	0	-1	1	0	0	0
-5	-2	-2	5	0	1	0	0
3	1	1	-3	0	0	1	0

The test of nonbasic variable x_4 yields the following tableau, and again one pivot is enough to determine unboundedness.

-1	1	-1	-1	0	0	0	0
1	2	0	-2	1	0	0	0
-5	-2	-2	2	0	1	0	0
3	1	[1]	-1	0	0	1	0

Since $\mathcal{EN} = \emptyset$ the iteration is finished.

Iteration 4: Since $\mathcal{L}_1 = \emptyset$ the algorithm terminates.

Output: List of efficient bases $\mathcal{B}^1 = \{2,5,6\}, \mathcal{B}^2 = \{1,5,6\}, \mathcal{B}^3 = \{2,3,5\}$.

During the course of the algorithm, we identified three efficient bases and three corresponding efficient basic feasible solutions. Their adjacency structure is shown in Figure 7.1. A line indicates that bases are adjacent. Note that bases $\{1,5,6\}$ and $\{2,3,5\}$ are not adjacent, because at least two pivots are needed to obtain one from the other. They are, however, connected via basis $\{2,5,6\}$

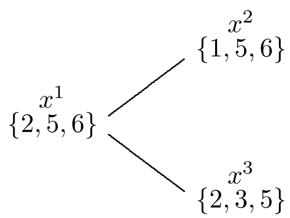

Fig. 7.1. Adjacency of efficient bases and corresponding BFSs.

The problem is displayed in decision space in Figure 7.2. The efficient set consists of the edges connecting x^1 and x^2 and x^1 and x^3.

□

In the following section we study the geometry of multiobjective linear programming. Amongst other things we shall see, how the results from the

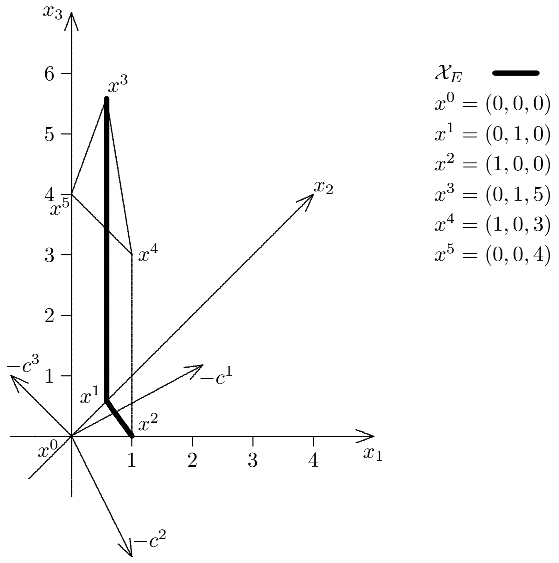

Fig. 7.2. Feasible and efficient set in Example 7.13.

multicriteria Simplex algorithm (i.e. the list of efficient bases and their adjacency structure) is exploited to identify the maximal efficient faces.

7.2 The Geometry of Multiobjective Linear Programming

First we observe that efficient BFS correspond to extreme points of $\mathcal{X}_E$.

Lemma 7.14. *1. Let $\mathcal{B}$ be an efficient basis and $(x_\mathcal{B}, 0)$ be the corresponding basic feasible solution. Then $(x_\mathcal{B}, 0)$ is an extreme point of $\mathcal{X}_E$.*
2. Let $x \in \mathcal{X}_E$ be an extreme point. Then there is an efficient basis $\mathcal{B}$ such that $x = (x_\mathcal{B}, 0)$.

Proof. This result follows from Theorem 6.11, the definition of an efficient basis and the single objective counterpart in Theorem 6.17. □

Note that, as in the single objective case, several efficient bases might identify the same efficient extreme point, if the MOLP is degenerate.

If $(x_\mathcal{B}, 0)$ and $(x_{\hat{\mathcal{B}}}, 0)$ are the efficient basic feasible solutions defined by adjacent efficient bases $\mathcal{B}$ and $\hat{\mathcal{B}}$, we see from the proof of Lemma 7.7 that both $(x_{\mathcal{B}}, 0)$ and $(x_{\hat{\mathcal{B}}}, 0)$ are optimal solutions of the same LP(λ). Therefore, due to linearity, the edge $\mathrm{conv}((x_{\mathcal{B}}, 0), (x_{\hat{\mathcal{B}}}, 0))$ is contained in $\mathcal{X}_E$.

Lemma 7.15. *Let $\mathcal{B}$ and $\hat{\mathcal{B}}$ be optimal bases for LP(λ). Then the edge* $\mathrm{conv}((x_{\mathcal{B}}, 0), (x_{\hat{\mathcal{B}}}, 0))$ *is contained in $\mathcal{X}_E$.*

We also have to take care of efficient unbounded edges: $\mathcal{X}_E$ may contain some unbounded edges $\mathcal{E} = \{x : x = x^i + \mu d^j,\ \mu \geq 0\}$, where d^j is an extreme ray and x^i is an extreme point of $\mathcal{X}$. This can happen even if the LP(λ) is bounded if $c(\lambda)$ is parallel to d^j. An unbounded edge always starts at an extreme point, which must therefore be efficient.

Let $\mathcal{B}$ be an efficient basis associated with that extreme point. Then the unbounded efficient edge is detected by an efficient nonbasic variable, in which the column $\tilde{A}^j$ contains only nonpositive elements, showing that $\mathcal{X}$ is unbounded in that direction. Because $\lambda^T r^j = 0$ this does not constitute unboundedness of the objective function.

Definition 7.16. *Let $\mathcal{F} \subset \mathcal{X}$ be a face of $\mathcal{X}$. $\mathcal{F}$ is called* efficient face*, if $\mathcal{F} \subset \mathcal{X}_E$. It is called* maximal efficient face*, if there is no efficient face $\mathcal{F}'$ of higher dimension with $\mathcal{F} \subset \mathcal{F}'$.*

Lemma 7.17. *If there is a $\lambda \in \mathbb{R}^p_>$ such that $\lambda^T Cx = \gamma$ is constant for all $x \in \mathcal{X}$ then $\mathcal{X}_E = \mathcal{X}$. Otherwise*

$$\mathcal{X}_E \subset \bigcup_{t=1}^{T} \mathcal{F}_t, \tag{7.11}$$

where $\{\mathcal{F}_t : t = 1, \ldots, T\}$ is the set of all proper faces of $\mathcal{X}$ and T is the number of proper faces of $\mathcal{X}$.

Proof. The first case is obvious, because if $\lambda^T Cx = \gamma$ for all $x \in \mathcal{X}$ then the whole feasible set is optimal for this particular LP(λ). Then from Theorem 6.6 $\mathcal{X} \subset \mathcal{X}_E$.

The second part follows from the fact that optimal solutions of LP(λ) are on the boundary of $\mathcal{X}$ (Theorem 6.17) and, once more, Theorem 6.11. Of course $\mathrm{bd}\,\mathcal{X} = \cup_{t=1}^{T} \mathcal{F}_t$. □

Thus, in order to describe the complete efficient set $\mathcal{X}_E$, we need to identify the maximally efficient faces of $\mathcal{X}$. We will need the representation of a point x in a face $\mathcal{F}$ as a convex combination of the extreme points and a nonnegative combination of the extreme rays of $\mathcal{F}$. This result is know as Minkowski's theorem. A proof can be found in Nemhauser and Wolsey (1999, Chapter I.4, Theorem 4.8).

Theorem 7.18 (Minkowski's Theorem). *Let $\mathcal{X}$ be a polyhedron and $x \in \mathcal{X}$. Let $x^1, \ldots, x^k$ be the extreme points and let $d^1, \ldots, d^l$ be the extreme rays of $\mathcal{X}$, then there are nonnegative real numbers $\alpha_i, i = 1, \ldots, k$ and $\mu_j, j = 1, \ldots, l$ such that $0 \leq \alpha_i \leq 1, i = 1, \ldots, k$, $\sum_{i=1}^k \alpha_i = 1$, and*

$$x = \sum_{i=1}^{k} \alpha_i x^i + \sum_{j=1}^{l} \mu_j d^j. \tag{7.12}$$

Furthermore, if $x \in \operatorname{ri} \mathcal{X}$ the numbers α_i and μ_j can be chosen to be positive.

Example 7.19. Consider the polyhedron $\mathcal{X}$ defined as follows:

$$\mathcal{X} := \{x \in \mathbb{R}^2 : x \geqq 0, 2x_1 + x_2 \geq 2, -x_1 + x_2 \leq 2\}$$

shown in Figure 7.3. Clearly, $\mathcal{X}$ has two extreme points $x^1 = (0, 2)$ and $x^2 = (1, 0)$. The two extreme rays are $d^1 = (1, 1)$ and $d^2 = (1, 0)$.

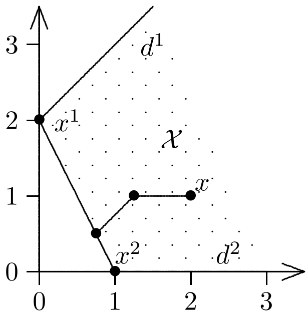

Fig. 7.3. A polyhedron with extreme points and extreme rays.

The point $x = (2, 1) \in \operatorname{ri} \mathcal{X}$ can be written as

$$x = \frac{1}{4}\begin{pmatrix}0\\2\end{pmatrix} + \frac{3}{4}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} + \frac{3}{4}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}.$$

□

Suppose that $\emptyset \neq \mathcal{X}_E \neq \mathcal{X}$. Using Minkowski's theorem, applied to a face $\mathcal{F}$, we prove that the whole face is efficient if and only if it contains an efficient solution in its relative interior.

Theorem 7.20. *A face $\mathcal{F} \subset \mathcal{X}$ is an efficient face if and only if it has an efficient solution $\hat{x}$ in its relative interior.*

Proof. "$\Longrightarrow$" If $\mathcal{F}$ is an efficient face all its relative interior points are efficient by definition.

"$\Longleftarrow$" Let $\hat{x} \in \mathcal{X}_E$ belong to the relative interior of $\mathcal{F}$. We show that there is a $\hat{\lambda} \in \mathbb{R}^p_>$ such that the whole face $\mathcal{F}$ is optimal for LP($\hat{\lambda}$).

First, by Theorem 6.11 we can find a $\hat{\lambda} \in \mathbb{R}^p_>$ such that $\hat{x}$ is an optimal solution of LP($\hat{\lambda}$). In particular LP($\hat{\lambda}$) is bounded. Therefore

$$\hat{\lambda}^T C x^i \geq \hat{\lambda}^T C \hat{x} \tag{7.13}$$

for all extreme points $x^i, i = 1, \ldots, k$ of $\mathcal{F}$ and

$$\hat{\lambda}^T C d^j \geq 0 \tag{7.14}$$

for all extreme rays $d^j, j = 1, \ldots, l$ of $\mathcal{F}$. Note that whenever $\hat{\lambda}^T C d^j < 0$ for some extreme ray d^j LP($\hat{\lambda}$) will be unbounded. Assume there is an extreme point x^i, $i \in \{1, \ldots, k\}$ which is not optimal for LP($\hat{\lambda}$), i.e.

$$\hat{\lambda}^T C x^i > \hat{\lambda}^T C x^0. \tag{7.15}$$

Then from Theorem 7.18 there are positive α_i and μ_j such that with (7.12) $\hat{x} = \sum_{i=1}^k \alpha_i x^i + \sum_{j=1}^l \mu_j d^j$ and

$$\begin{aligned} \hat{\lambda}^T C \hat{x} &= \sum_{i=1}^k \alpha_i \hat{\lambda}^T C x^i + \sum_{j=1}^l \mu_j \hat{\lambda}^T C d^j \\ &> \sum_{i=1}^k \alpha_i \hat{\lambda}^T C x^0 = \hat{\lambda}^T C \hat{x}. \end{aligned} \tag{7.16}$$

We have used positivity of α_i, nonnegativity of μ_i, (7.13), (7.14), and (7.15) for the inequality, and $\sum_{i=1}^k \alpha_i = 1$ for the second equality. The impossibility (7.16) means that

$$\hat{\lambda}^T C x^i = {\lambda^0}^T C x^0. \tag{7.17}$$

for all extreme points x^i, which are thus optimal solutions of LP($\hat{\lambda}$). To complete the proof, consider (7.16) again, using (7.17) this time to get that ${\lambda^0}^T C d^j = 0$ for all extreme rays d^j, because $\mu_j > 0$ since $\hat{x}$ is a relative interior point of $\mathcal{F}$. □

We state to further results about efficient edges and efficient faces, which we leave as exercises for the reader, see Exercises 7.5 and 7.6.

Proposition 7.21. *Assume that the MOLP is not degenerate. Let x^1 and x^2 be efficient extreme points of $\mathcal{X}$ and assume that the corresponding bases are adjacent (i.e. one can be obtained from the other by an efficient pivot). Then* $\text{conv}(x^1, x^2)$ *is an efficient edge.*

Theorem 7.22. *A face $\mathcal{F}$ of $\mathcal{X}$ is efficient if and only if there is a $\lambda > 0$ such that all extreme points of $\mathcal{F}$ are optimal solutions of LP(λ) and $\lambda^T d = 0$ for all extreme rays of $\mathcal{F}$.*

With Theorem 7.20 and Lemma 7.17 we know that $\mathcal{X}_E$ is the union of maximally efficient faces, each of which is the set of optimal solutions of LP(λ), for some $\lambda \in \mathbb{R}^p_>$. If we combine this with the fact that the set of efficient extreme points is connected by efficient edges, (as follows again from Theorem 7.10 and Theorem 7.20, see also page 185) we get the connectedness result for the efficient set of multicriteria linear programs.

Theorem 7.23. *$\mathcal{X}_E$ is connected and, therefore, $\mathcal{Y}_N$ is connected.*

Proof. The result for $\mathcal{X}_E$ follows from Theorem 7.10 and Lemma 7.17 together with Theorem 7.20. Thus, $\mathcal{Y}_N$ is connected because $\mathcal{X}_E$ is and C is linear, i.e. continuous. □

Example 7.24. Figure 7.4 shows the feasible set of a biobjective LP, with the two maximal efficient faces indicated by bold lines.

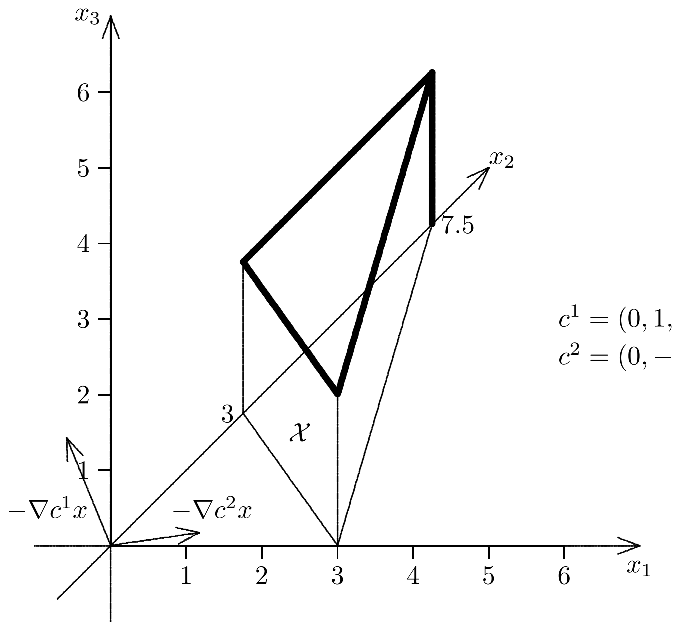

Fig. 7.4. $\mathcal{X}_E$ in Example 7.24.

To check that $\mathcal{X}_E$ is correct we can use Theorem 6.11, i.e $x \in \mathcal{X}_E$ if and only if there is $c(\lambda) = \lambda c^1 + (1-\lambda)c^2$ such that x is an optimal solution of LP(λ), and apply it graphically in this case. The negative gradient of the objective $c(\lambda)$ for different values of λ can be used to graphically determine the optimal faces. In this example, $\mathcal{X}_E$ has a 2-dimensional face and a 1-dimensional face as the only maximal efficient faces. However, the three edges of the efficient triangle and the four efficient extreme points are not maximal efficient faces. The example clearly shows that – even for linear multicriteria optimization problems – the efficient set is in general not convex. □

In the proof of Theorem 7.20 we have seen that for each efficient face $\mathcal{F}$ there exists a $\lambda \in \mathbb{R}^p_>$ such that $\mathcal{F}$ is the set of optimal solutions of LP(λ). Suppose we know efficient face $\mathcal{F}$, how can we find all λ with that property?

Essentially, we want to subdivide the set $\Lambda = \{\lambda \in \mathbb{R}^p_> : \sum_{k=1}^p \lambda_k = 1\}$ into regions that correspond to those weighting vectors λ, which make a certain face efficient. That is, for each efficient face $\mathcal{F}$ we want to find $\Lambda_{\mathcal{F}} \subset \Lambda$ such that $\mathcal{F}$ is optimal for LP(λ) for all $\lambda \in \Lambda_{\mathcal{F}}$.

Let us first assume that $\mathcal{X}$ is nonempty and bounded, so that in particular $\mathcal{X}_E$ is nonempty. Let $\mathcal{F}$ be an efficient face, and $x^i,\ i = 1, \ldots, k$ be the set of all extreme points of $\mathcal{F}$. Because $\mathcal{F}$ is an efficient face, from the proof of Theorem 7.20 there is some $\lambda_{\mathcal{F}} \in \Lambda$ such that $\mathcal{F} = \text{conv}(x^1, \ldots, x^k)$ is optimal for LP($\lambda_{\mathcal{F}}$). In particular, $x^1, \ldots, x^k$ are optimal solutions of LP($\lambda_{\mathcal{F}}$).

Hence we can apply the optimality condition for linear programs. Let R^i be the reduced cost matrix of a basis associated with x^i. Then x^i is optimal if and only if $\lambda^T R^i \geqq 0$ (note that we assume nondegeneracy here, see Lemma 6.14). Therefore, the face $\mathcal{F}$ is optimal if and only if $\lambda^T R^i \geqq 0,\ i = 1, \ldots, k$.

Proposition 7.25. *The set of all λ for which efficient face $\mathcal{F}$ is the optimal solution set of LP(λ) is defined by the linear system*

$$\begin{aligned} \lambda^T e &= 1 \\ \lambda^T R^i &\geqq 0 \quad i = 1, \ldots, k \\ \lambda &\geqq 0, \end{aligned}$$

where R^i is the reduced cost matrix of a basis associated with extreme point x^i of $\mathcal{F}$.

Example 7.26. Let us consider the efficient face $\text{conv}(x^1, x^2)$ in Example 7.13. Extreme point x^1 corresponds to basis $\{2, 5, 6\}$ with

$$R^1 = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ 1 & -1 & 0 \end{pmatrix}$$

and extreme point x^2 corresponds to basis $\{1,5,6\}$

$$R^2 = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & -1 \end{pmatrix}.$$

The linear system of Proposition 7.25 is $\lambda^T R^1 \geqq 0,\ \lambda^T R^2 \geqq 0, \lambda^T e = 1, \lambda \geq 0$, which we write as

$$\begin{aligned} \lambda_1 - \lambda_2 + \lambda_3 &\geq 0 \\ 2\lambda_2 - \lambda_3 &\geq 0 \\ 2\lambda_1 \qquad\qquad &\geq 0 \\ -\lambda_1 + \lambda_2 - \lambda_3 &\geq 0 \\ 2\lambda_2 - \lambda_3 &\geq 0 \\ \lambda_1 + \lambda_2 - \lambda_3 &\geq 0 \\ \lambda_1 + \lambda_2 + \lambda_3 &= 1 \\ \lambda_1, \lambda_2, \lambda_3 &\geq 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} \lambda_1 - \lambda_2 + \lambda_3 &= 0 \\ 2\lambda_2 - \lambda_3 &\geq 0 \\ \lambda_1 + \lambda_2 - \lambda_3 &\geq 0 \\ \lambda_1 + \lambda_2 + \lambda_3 &= 1 \\ \lambda_1, \lambda_2, \lambda_3 &\geq 0. \end{aligned}$$

Eliminating λ_3 we obtain $\lambda_2 = 0.5$, $0 \leq \lambda_1 \leq 0.5$. Proceeding in the same way for the efficient face $\text{conv}(x^1, x^2)$ and the efficient extreme points, we obtain the subdivision of Λ depicted in Figure 7.5. For the efficient extreme points x^i there are two-dimensional regions, for the edges, there are line segments that yield the respective face as optimal solutions of $\text{LP}(\lambda)$.

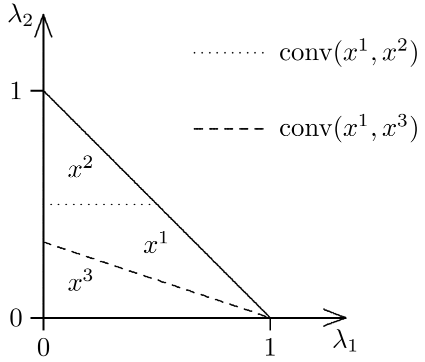

Fig. 7.5. Weights to obtain efficient faces in Example 7.13.

□

If $\mathcal{X}$ is unbounded, it may happen that $\mathcal{X}_E$ contains unbounded efficient faces . In this case an efficient face $\mathcal{F}$ contains unbounded edges, i.e. we must take care of extreme rays in the linear system of Proposition 7.25. We extend it by $\lambda^t C d^j = 0$ for the extreme rays $d^1, \ldots, d^l$ of face $\mathcal{F}$. The proof of Theorem 7.20 shows that this condition has to be satisfied.

If there is some $\lambda \in \Lambda$ such that LP(λ) is unbounded, there is, in addition to the sets $\Lambda_{\mathcal{F}} \subset \Lambda$ for all efficient faces $\mathcal{F}$, a subset $\Lambda_0 \subset= \{\lambda \in \Lambda :$ LP(λ) is unbounded$\}$. This set is the remainder of Λ, which is not associated with any of the efficient faces. Note that this case can only occur if there is a $\lambda > 0$ and an extreme ray d of $\mathcal{X}$ such that $\lambda^T C d < 0$.

Let us finally turn to the determination of maximal efficient faces. The method we present is from Isermann (1977). Let $\mathcal{B}$ be an efficient basis and let $\mathcal{N}^f \subset \mathcal{N}$ be the set of nonbasic variables, which allow feasible pivots. Let $\mathcal{J} \subset \mathcal{N}^f$. Then we have the following proposition.

Proposition 7.27. *All variables in $\mathcal{J}$ are efficient nonbasic variables if and only if the LP*

$$\begin{aligned} \max \quad & e^T v \\ \text{subject to } & Rz - R^{\mathcal{J}}\delta + Iv = e \\ & z, \delta, v \geqq 0 \end{aligned} \tag{7.18}$$

has an optimal solution. Here $R^{\mathcal{J}}$ denotes the columns of R pertaining to variables in $\mathcal{J}$.

Proof. The proof is similar to the proof of Theorem 7.8 and is left to the reader, see Exercise 7.1. □

Let us call $\mathcal{J} \subset \mathcal{N}^f$ a maximal set of efficient nonbasic variables , if there is no $\mathcal{J}' \subset \mathcal{N}^f$ such that $\mathcal{J} \subset J'$ and (7.18) has an optimal solution for $\mathcal{J}'$. Now let $\mathcal{B}^\tau, \tau = 1, \ldots, t$ be the efficient bases and $\mathcal{J}^{\tau,\rho}, \tau = 1, \ldots, t, \rho = 1, \ldots, r$ be all maximal index sets of efficient nonbasic variables at efficient basis $\mathcal{B}^\tau$. Furthermore, let $\mathcal{E}^\nu = (\mathcal{B}^\tau, d^\nu), \nu = 1, \ldots, v$ denote unbounded efficient edges, where d^ν is an extreme ray of $\mathcal{X}$.

We define $\mathcal{Q}^{\tau,\rho} := \mathcal{B}^\tau \cup \mathcal{J}^{\tau,\rho}$. $\mathcal{Q}^{\tau,\rho}$ contains bases adjacent to $\mathcal{B}^\tau$, and the convex hull of the extreme points associated with all bases found in $\mathcal{Q}^{\tau,\rho}$ plus the conical hull of any unbounded edges attached to any of these bases constitutes a candidate for an efficient face.

As we are only interested in identifying maximal efficient faces, we select a minimal number of index sets representing all $\mathcal{Q}^{\tau,\rho}$, i.e. we choose index sets $\mathcal{U}^1, \ldots, \mathcal{U}^o$ with the following properties:

1. For each $\mathcal{Q}^{\tau,\rho}$ there is a set $\mathcal{U}^s$ such that $\mathcal{Q}^{\tau,\rho} \subset \mathcal{U}^s$.
2. For each $\mathcal{U}^s$ there is a set $\mathcal{Q}^{\tau,\rho}$ such that $\mathcal{U}^s = \mathcal{Q}^{\tau,\rho}$.
3. There are no two sets $\mathcal{U}^s, \mathcal{U}^{s'}$ with $s \neq s'$ and $\mathcal{U}^s \subset \mathcal{U}^{s'}$.

Now we determine which extreme points and which unbounded edges are associated with bases in the sets $\mathcal{U}^s$. For $s \in \{1, \ldots, o\}$ let

$$\mathcal{I}_b^s := \{\tau \in \{1, \ldots, t\} : \mathcal{B}^\tau \subset \mathcal{U}^s\},$$
$$\mathcal{I}_u^s := \{\nu \in \{1, \ldots, v\} : \mathcal{B}^\tau \subset U^s\}$$

and define

$$\mathcal{X}_s = \left\{ x \in \mathcal{X} : x = \sum_{\tau \in I_b^s} \alpha_\tau x^\tau + \sum_{\nu \in I_u^s} \mu_\nu d^\nu, \sum_{\tau \in I_b^s} \alpha_\tau = 1, \alpha_\tau \geq 0, \mu_\nu \geq 0 \right\}. \tag{7.19}$$

The sets $\mathcal{X}_s$ are faces of $\mathcal{X}$ and efficient (Theorem 7.28) and in fact they are the maximal efficient faces (Theorem 7.29), if the MOLP is not degenerate.

Theorem 7.28 (Isermann (1977)). *$\mathcal{X}_s \subset \mathcal{X}_E$ for $s = 1, \ldots, o$.*

Proof. By definition of $\mathcal{U}^s$ there is a set $\mathcal{Q}^{\tau,\rho}$ such that $\mathcal{Q}^{\tau,\rho} = \mathcal{U}^s$. Therefore the linear program (7.18) with $\mathcal{J} = \mathcal{Q}^{\tau,\rho} \setminus \mathcal{B}^\tau$ in Proposition 7.27 has an optimal solution. Thus, the dual of this LP

$$\begin{array}{lr} \min & e^T\lambda \\ \text{subject to} & R^T\lambda \geqq 0 \\ & (-R^J)^T\lambda \geqq 0 \\ & \lambda \geqq e \end{array}$$

has an optimal solution $\hat{\lambda}$. But the constraints of the LP above are the optimality conditions for $LP(\lambda)$, where in particular $(R^J)^T\lambda = 0$. Therefore all $x \in \mathcal{X}_s$ are optimal solutions of $\text{LP}(\hat{\lambda})$ and $\mathcal{X}_s \subset \mathcal{X}_E$. □

Theorem 7.29 (Isermann (1977)). *If $x \in \mathcal{X}_E$ there is an $s \in \{1, \ldots, o\}$ such that $x \in \mathcal{X}_s$.*

Proof. Let $x \in \mathcal{X}_E$. Then x is contained in a maximal efficient face $\mathcal{F}$, which is optimal for some $\text{LP}(\lambda)$. Let $\mathcal{I}_b$ be the index set of efficient bases corresponding to the extreme points of $\mathcal{F}$ and $\mathcal{I}_u$ be the index set of extreme rays of face $\mathcal{F}$. Then, according to (7.12), x can be written as

$$x = \sum_{i \in \mathcal{I}_b} \alpha_i x^i + \sum_{j \in I_u} \mu_j d^j.$$

We choose any extreme point x^i of $\mathcal{F}$ and let $\mathcal{B}^i$ be a corresponding basis. Furthermore, we let $\mathcal{J}^0 := \{\cup_{\tau \in \mathcal{I}_b} \mathcal{B}^\tau\} \setminus \mathcal{B}^i$. Because all $\mathcal{B}^\tau$ are efficient, $\mathcal{J}^0$ is a set of efficient nonbasic variables at $\mathcal{B}^i$.

Therefore (7.18) has an optimal solution and there exists a maximal index set of efficient nonbasic variables $\mathcal{J}$ with $\mathcal{J}^0 \subset \mathcal{J}$. During the further construction of index sets, none of the indices of extreme points in $\mathcal{J}^0$ is lost, and $\mathcal{B}^i \cup \mathcal{J}^0 \subset \mathcal{U}^s$ for some s. Therefore $x \in \mathcal{X}_s$ for some $s \in \{1, \ldots, o\}$. □

The proofs show that if all efficient bases are nondegenerate, $\mathcal{X}_s$ are exactly the maximal efficient faces of $\mathcal{X}$. Otherwise some $\mathcal{X}_s$ may not be maximal, because there is a choice of bases representing an efficient extreme point, and the maximal sets of efficient nonbasic variables need not be the same for all of them.

Example 7.30. We apply this method to Example 7.13. $\mathcal{X}$ does not contain unbounded edges. The computation of the index sets is summarized in Table 7.2.

Table 7.2. Criteria and alternatives in Example 7.30.

Efficient basis $\mathcal{B}^\tau$	Maximal index set $\mathcal{J}^{\tau,\rho}$	$\mathcal{Q}^{\tau,\rho}$
$\mathcal{B}^1 = \{2,5,6\}$	$\mathcal{J}^{1,1} = \{1\}$	$\mathcal{Q}^{1,1} = \{1,2,5,6\}$
	$\mathcal{J}^{1,2} = \{3\}$	$\mathcal{Q}^{1,2} = \{3,2,5,6\}$
$\mathcal{B}^2 = \{1,5,6\}$	$\mathcal{J}^{2,1} = \{2\}$	$\mathcal{Q}^{2,1} = \{1,2,5,6\}$
$\mathcal{B}^3 = \{2,3,5\}$	$\mathcal{J}^{3,1} = \{6\}$	$\mathcal{Q}^{3,1} = \{2,3,5,6\}$

The sets $\mathcal{U}^s$ are $\mathcal{U}^1 = \{1,2,5,6\}$ and $\mathcal{U}^2 = \{2,3,5,6\}$ and checking, which bases are contained in these sets, we get $\mathcal{I}_b^1 = \{1,2\}$ and $\mathcal{I}_b^2 = \{1,3\}$. From (7.19) we get

$$\mathcal{X}_1 = \{x = \alpha_1 x^1 + \alpha_2 x^2 : \alpha_1 + \alpha_2 = 1,\ \alpha_i \geq 0\} = \operatorname{conv}(x^1, x^2),$$
$$\mathcal{X}_2 = \{x = \alpha_1 x^1 + \alpha_2 x^3 : \alpha_1 + \alpha_2 = 1,\ \alpha_i \geq 0\} = \operatorname{conv}(x^1, x^3),$$

and confirm $\mathcal{X}_E = \mathcal{X}_1 \cup \mathcal{X}_2$, as expected. □

7.3 Notes

A number of multicriteria Simplex algorithms have been published. Their general structure follows the three phase scheme presented above. For pivoting among efficient bases it is necessary to identify efficient nonbasic variables. Other than those of Theorems 7.8 and 7.27 tests for nonbasic variable efficiency have been proposed by Ecker and Kouada (1978) and Zionts and Wallenius (1980). An alternative method to find an initial efficient extreme point is given in Benson (1981). Several proofs of the connectedness result of Theorem 7.10 are known, see e.g. Zeleny (1974), Yu and Zeleny (1975), and Isermann (1977). More on connectedness of efficient basic feasible solutions for degenerate MOLPs can be found in Schechter and Steuer (2005).

Algorithms based on the Simplex method are proposed by Armand (1993); Armand and Malivert (1991), Evans and Steuer (1973), Ecker *et al.* (1980); Ecker and Kouada (1978), Isermann (1977), Gal (1977), Philip (1972, 1977), Schönfeld (1964), Strijbosch *et al.* (1991), Yu and Zeleny (1975, 1976), Zeleny (1974). The algorithm by Steuer (1985) is implemented in the ADBASE Steuer (2000) code.

While all these algorithms identify efficient bases and extreme points, an algorithm by Sayin (1996) a top-down approach instead, that starts by finding the highest dimensional efficient faces first and then proceeds down to extreme points (zero dimensional faces).

In Proposition 7.25 we have shown how to decompose the weight space Λ to identify those weighting vectors that have an efficient face as optimal solutions of LP(λ). Such a partition can be attempted with respect to efficient bases of the MOLP or with respect to extreme points of $\mathcal{X}_E$ or $\mathcal{Y}_N$. Benson and Sun (2000) investigates the decomposition of the weight space according to the extreme points of $\mathcal{Y}_N$.

Interior point methods have revolutionized linear programming since the 1980's. However, they are not easily adaptable to multiobjective linear programming. Most methods proposed in the literature find one efficient solution, and involve the the elicitation of the decision makers preferences in an interactive fashion, see the work of Arbel (1997) and references therein. The only interior point method that is not interactive is Abhyankar *et al.* (1990).

The observation that the feasible set in objective space $\mathcal{Y}$ is usually of much smaller dimension than $\mathcal{X}$ has lead to a stream of research work on solving MOLPs in objective space. Publications on this topic include Dauer and Liu (1990); Dauer and Saleh (1990); Dauer (1993); Dauer and Gallagher (1990) and Benson (1998c,a,b).

Exercises

7.1 (Isermann (1977)). Let $\mathcal{J} \subset \mathcal{N}$ be an index set of nonbasic variables at efficient basis $\mathcal{B}$. Show that each variable $x_j, j \in \mathcal{J}$ is efficient if and only if the linear program

$$\begin{array}{ll} \max & e^T v \\ \text{subject to} & Rz - R^{\mathcal{J}}\delta + Iv = e \\ & z, \delta, v \geqq 0 \end{array}$$

has an optimal solution. Here $R^{\mathcal{J}}$ is the part of R pertaining to variables $x_j, j \in \mathcal{J}$. Hint: Use the definition of efficient nonbasic variable and look at the dual of the above LP.

7.2. A basis $\mathcal{B}$ is called weakly efficient, if $\mathcal{B}$ is an optimal basis of LP(λ) for some $\lambda \in \mathbb{R}^p_{\geq}$. A feasible pivot with nonbasic variable x_j entering the basis is called weakly efficient if the basis obtained is weakly efficient. Prove the following theorem.

Let x_j be nonbasic at weakly efficient basis $\mathcal{B}$. Then all feasible pivots with x_j as entering variable are weakly efficient if and only if the linear program

$$\begin{array}{ll} \max & v \\ \text{subject to} & Rz - r^j\delta + ev \geqq 0 \\ & z, \delta, v \geqq 0 \end{array}$$

has an optimal objective value of zero.

7.3. Solve the MOLP

$$\begin{array}{lrcrl} \min & -3x_1 & - & x_2 & \\ \min & x_1 & - & 2x_2 & \\ \text{subject to} & 3x_1 & + & 2x_2 & \geq 6 \\ & x_1 & & & \leq 10 \\ & & & x_2 & \leq 3 \\ & x_1, & & x_2 & \geq 0 \end{array}$$

using the multicriteria Simplex algorithm 7.1.

7.4. Determine, for each efficient extreme point x^i of the MOLP in Exercise 7.3, the set of all λ for which x^i is an optimal solution of LP(λ) and determine all maximal efficient faces.

7.5. Assume that the MOLP is not degenerate. Let x^1 and x^2 be efficient extreme points of $\mathcal{X}$ and assume that the corresponding bases are adjacent (i.e. one can be obtained from the other by an efficient pivot). Show that $\text{conv}(x^1, x^2)$ is an efficient edge.

7.6. Prove that a face $\mathcal{F}$ of $\mathcal{X}$ is efficient if and only if there is a $\lambda > 0$ such that all extreme points of $\mathcal{F}$ are optimal solutions of LP(λ) and $\lambda^T d = 0$ for all extreme rays of $\mathcal{F}$.

7.7. Let $\mathcal{X} = \{x \in \mathbb{R}^n : \ Ax = b, \ x \geqq 0\}$ and consider the MOLP $\min_{x \in \mathcal{X}} Cx$. An *improving direction* d direction at $x^0 \in \mathcal{X}$ is a vector $d \in \mathbb{R}^n$ such that $Cd \leq 0$ and there is some $t > 0$ such that $x^0 + \tau d \in \mathcal{X}$ for all $\tau \in [0, t]$.

Let $D := \{d \in \mathbb{R}^n \ : \ Cd \leq 0\}$ and $x^0 \in \mathcal{X}$. Prove that $x^0 \in \mathcal{X}_E$ if and only if $(x^0 + D) \cap \mathcal{X} = \{x_0\}$, i.e. if there is no improving direction at x^0. Illustrate the result for the problem of Exercise 7.3.