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Introduction to residues and resultants

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Summary. This chapter is an expanded version of the lecture notes prepared by the second-named author for her introductory course at the CIMPA Graduate School on Systems of Polynomial Equations held in Buenos Aires, Argentina, in July 2003. We present an elementary introduction to residues and resultants and outline some of their multivariate generalizations. Throughout we emphasize the application of these ideas to polynomial system solving.

1.0 Introduction

This chapter is an introduction to the theory of residues and of resultants. These are very classical topics with a long and distinguished history. It is not our goal to present a full historical account of their development but rather to introduce the basic notions in the one-dimensional case, to discuss some of their applications -in particular, those related to polynomial system solving- and present their multivariate generalizations. We emphasize in particular the applications of residues to duality theory and the explicit computation of resultants which, in turn, results in the explicit elimination of variables.

Most readers are probably familiar with the classical theory of local residues which was introduced by Augustin-Louis Cauchy in 1825 as a powerful tool for the computation of integrals and for the summation of infinite series. Perhaps less familiar is the fact that given a meromorphic form $(H(z)/P(z))dz$ on the complex plane, its global residue, i.e. the sum of local residues at the zeros of P, defines an easily computable linear functional on the quotient algebra $\mathcal{A} := \mathbb{C}[z]/\langle P(z)\rangle$ whose properties encode many important features of this algebra. As in Chapters 2 and 3, it is through the study of this algebra, and its multivariate generalization, that we make the connection with the roots of the associated polynomial system.

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The basic definitions and properties of the univariate residue are reviewed in Section 1.1 and we discuss some nice applications in Section 1.2. Although there are many different possible definitions of the residue, we have chosen to follow the classical integral approach for the definition of the local residue. Alternatively, one could define the global residue by its algebraic properties and use ring localization to define the local residue. We indicate how this is done in a particular case.

In Section 1.5 we study multidimensional residues. Although, as coefficients of certain Laurent expansions, they are already present in the work of Jacobi [Jac30], the first systematic treatment of bivariate residue integrals is the 1887 memoir of Poincaré [Poi87], more than 60 years after the introduction of univariate residues. He makes the very interesting observation that geometers were long stopped from extending the one-dimensional theory because of the lack of geometric intuition in 4 dimensions (referring to $\mathbb{C}^2$). The modern theory of residues and the duality in algebraic geometry is due to Leray and Grothendieck. There have been many developments since the late 70's: in the algebro-geometric side with the work of Grothendieck (cf. [Har66]); in analytic geometry where we may mention the books by Griffiths and Harris [GH78] and Arnold, Varchenko and Guseĭn -Zadé [AGZV85]; in commutative algebra with the work of Scheja and Storch [SS75, SS79], Kunz [Kun86], and Lipman [Lip87]; and in the analytic side with the residual currents approach pioneered by Coleff and Herrera [CH78]. In the 90's the possibility of implementing symbolic computations brought about another important expansion in the theory and computation of multidimensional residues and its applications to elimination theory as pioneered by the Krasnoyarsk school [AY83, BKL98, Tsi92]. It would, of course, be impossible to fully present all these approaches to the theory of residues or to give a complete account of all of its applications. Indeed, even a rigorous definition of multivariate residues would take us very far afield. Instead we will attempt to give an intuitive idea of this notion, explain some of its consequences, and describe a few of its applications. In analogy with the one-variable case we will begin with an "integral" definition of local residue from which we will define the total residue as a sum of local ones. The reader who is not comfortable with integration of differential forms should not despair since, as in the univariate case, we soon show how one can give a purely algebraic definition of global, and then local, residues using Bezoutians. We also touch upon the geometric definition of Arnold, Varchenko and Guseĭn-Zadé.

In Sections 1.3 and 1.4 we discuss the definition and application of the univariate resultant. This is, again, a very classical concept which goes back to the work of Euler, Bézout, Sylvester and Cayley. It was directly motivated by the problem of elimination of variables in systems of polynomial equations. While the idea behind the notion of the resultant is very simple, its computation leads to very interesting problems such as the search for determinantal formulas. We recall the classical Sylvester and Bezoutian matrices in Section 1.4.

The rebirth of the classical theory of elimination in the last decade owes much to the work of Jouanolou [Jou79, Jou91, Jou97] and of Gelfand, Kapranov and Zelevinsky [GKZ94], as well as to the possibility of using resultants not only as a computational tool to solve polynomial systems but also to study their complexity aspects. In particular, homogeneous and multi-homogeneous resultants are essential tools in the implicitization of surfaces. We discuss the basic constructions and properties in Section 1.6. We refer to [Stu93, Stu98], [Stu02, Ch. 4] and to Chapters 2, 3, and 7 in this book for further background and applications. A new theoretical tool in elimination theory yet to be fully explored is the use of exterior algebra methods in commutative algebra (starting with Eisenbud and Schreyer [ESW03] and Khetan [Khe03, Khe]).

In the last section of this chapter we recall how the resultant appears naturally as the denominator of the residue and apply this to obtain a normal form algorithm for the computation of resultants which, as far as we know, has not been noted before.

Although many of the results in this chapter, including those in the last section, are valid in much greater generality, we have chosen to restrict most of the exposition to the affine and projective cases. We have tried to direct the reader to the appropriate references.

For further reading we refer to a number of excellent books on the topics treated here: [AY83, AGZV85, CLO98, GKZ94, GH78, EM, Tsi92].

1.1 Residues in one variable

1.1.1 Local analytic residue

We recall that, given a holomorphic function $h(z)$ with an isolated singularity at a point ξ in $\mathbb{C}$, we may consider its Laurent expansion

$$h(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-\xi)^n} + \overline{h}(z),$$

where $\overline{h}$ is holomorphic in a neighborhood of ξ, and define the *residue* of h at ξ as

$$\mathrm{res}_\xi(h) = b_1. \tag{1.1}$$

The classical Residue Theorem tells us that the residue is "what remains after integrating" the differential form $(1/2\pi \mathrm{i})\, h(z)\, dz$ on a small circle around ξ. Precisely:

$$\mathrm{res}_\xi(h) = \frac{1}{2\pi \mathrm{i}} \int_{|z-\xi|=\delta} h(z) dz,$$

for any sufficiently small positive δ and where the circle $\{|z-\xi| = \delta\}$ is oriented counter-clockwise.

Remark 1.1.1. As defined in (1.1), the residue depends on the choice of local coordinate z. Associating the residue to the meromorphic 1-form $h(z)\,dz$ makes it invariant under local change of coordinates. We will, however, maintain the classical notation, $\mathrm{res}_\xi(h)$ rather than write $\mathrm{res}_\xi(h(z)dz)$.

We can also think of the residue of a holomorphic function h at ξ as a linear operator $\mathrm{res}_\xi[h] : \mathcal{O}_\xi \to \mathbb{C}$, which assigns to any holomorphic function f defined near ξ the complex number

$$\mathrm{res}_\xi[h](f) := \mathrm{res}_\xi(f \cdot h).$$

Suppose h has a pole at ξ of order m, Then, the action of $\mathrm{res}_\xi[h]$ maps

$$\begin{aligned} 1 &\longmapsto b_1 \\ z-\xi &\longmapsto b_2 \\ \vdots \quad &\;\vdots \quad \vdots \\ (z-\xi)^{m-1} &\longmapsto b_m \end{aligned}$$

and for any $k \geq m$, $(z-\xi)^k \mapsto 0$ since $(z-\xi)^k \cdot h$ is holomorphic at ξ. These values suffice to characterize the residue map $\mathrm{res}_\xi[h]$ in this case: indeed, given f holomorphic near ξ, we write

$$f(z) = \sum_{j=0}^{m-1} \frac{f^{(j)}(\xi)}{j!}(z-\xi)^j \;+\; (z-\xi)^m g(z),$$

with g holomorphic in a neighborhood of ξ. Therefore

$$\mathrm{res}_\xi[h](f) = \sum_{j=0}^{m-1} \frac{f^{(j)}(\xi)}{j!}\mathrm{res}_\xi[h]((z-\xi)^j) = \sum_{j=0}^{m-1} \frac{b_{j+1}}{j!} f^{(j)}(\xi). \tag{1.2}$$

Note, in particular, that the residue map $\mathrm{res}_\xi[h]$ is then the evaluation at ξ of a constant coefficient differential operator and that it carries the information of the principal part of h at ξ.

1.1.2 Residues associated to polynomials

In this notes we will be interested in the algebraic and computational aspects of residues and therefore we shall restrict ourselves to the case when $h(z)$ is a rational function $h(z) = H(z)/P(z)$, $H, P \in \mathbb{C}[z]$. Clearly, $\mathrm{res}_\xi(h) = 0$ unless $P(\xi) = 0$. It is straightforward to check the following basic properties of residues:

- If ξ is a simple zero of P, then

$$\mathrm{res}_\xi\left(\frac{H(z)}{P(z)}\right) = \frac{H(\xi)}{P'(\xi)}. \tag{1.3}$$

- If ξ is a root of P of multiplicity m, then

$$\mathrm{res}_\xi\left(\frac{H(z)P'(z)}{P(z)}\right) = m \cdot H(\xi). \tag{1.4}$$

Since $(P'(z)/P(z))dz = d(\ln P(z))$ wherever a logarithm $\ln P$ of P is defined, the expression above is often called the (local) logarithmic residue.

Given a polynomial $P \in \mathbb{C}[z]$, its polar set $Z_P := \{\xi \in \mathbb{C} : P(\xi) = 0\}$ is finite and we can consider the total sum of local residues

$$\mathrm{res}\left(\frac{H}{P}\right) \ = \ \sum_{\xi \in Z_P} \mathrm{res}_\xi(H/P),$$

where $H \in \mathbb{C}[z]$. We will be particularly interested in the global residue operator.

Definition 1.1.2. *The global residue* $\mathrm{res}_P : \mathbb{C}[z] \to \mathbb{C}$ *is the sum of local residues:*

$$\mathrm{res}_P(H) = \sum_{\xi \in Z_P} \mathrm{res}_\xi(H/P)$$

Remark 1.1.3. We may define the sum of local residues over the zero set of P for any rational function h which is regular on Z_P. Moreover, if we write $h = H/Q$, with $Z_P \cap Z_Q = \emptyset$, then by the Nullstellensatz, there exist polynomials R, S such that $1 = RP + SQ$. It follows that the total sum of local residues

$$\sum_{\xi \in Z_P} \mathrm{res}_\xi(h/P) = \mathrm{res}_P(HS),$$

coincides with the global residue of the polynomial HS.

Let $R > 0$ be large enough so that Z_P be contained in the open disk $\{|z| < R\}$. Then, for any polynomial H the rational function $h = H/P$ is holomorphic for $|z| > R$ and has a Laurent expansion $\sum_{n\in\mathbb{Z}} e_n z^n$ valid for $|z| > R$. The residue of h at infinity is defined as

$$\mathrm{res}_\infty(h) := -e_{-1}. \tag{1.5}$$

Note that integrating term by term the Laurent expansion, we get

$$\mathrm{res}_\infty(h) = -\frac{1}{2\pi \mathrm{i}} \int_{|z|=R} h(z)dz.$$

Since by the Residue Theorem,

$$\mathrm{res}_P(H) = \frac{1}{2\pi \mathrm{i}} \int_{|z|=R} \frac{H(z)}{P(z)} dz,$$

we easily deduce

Proposition 1.1.4. *Let $P, H \in \mathbb{C}[z]$. Then* $\mathrm{res}_P(H) = -\mathrm{res}_\infty(H/P)$.

Remark 1.1.5. We note that the choice of sign in (1.5) is consistent with Remark 1.1.1: If $h = H/P$ is holomorphic for $|z| > R$, then we may regard h as being holomorphic in a punctured neighborhood of the point at infinity in the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$. Taking $w = 1/z$ as local coordinate at infinity we have: $h(z)dz = -(h(1/w)/w^2)dw$ and

$$\mathrm{res}_0(-(h(1/w)/w^2)) \;=\; -e_{-1}. \tag{1.6}$$

Note also that Proposition 1.1.4 means that the sum of the local residues of the extension of the meromorphic form $(H(z)/P(z))\,dz$ to the Riemann sphere is zero.

Proposition 1.1.6. *Given $P, H \in \mathbb{C}[z]$, $\mathrm{res}_P(H)$ is linear in H and is a rational function of the coefficients of P with coefficients in $\mathbb{Q}$.*

Proof. The first statement follows from the definition of $\mathrm{res}_P(H)$ and the linearity of the local residue. Thus, in order to prove the second statement it suffices to consider $\mathrm{res}_P(z^k)$, $k \in \mathbb{N}$. Let $d = \deg P$, $P(z) = \sum_{j=0}^d a_j z^j$, $a_d \neq 0$. Then, if follows from Proposition 1.1.4 and (1.6) that

$$\mathrm{res}_P(z^k) \;=\; \mathrm{res}_0\left(\frac{(1/w)^k}{w^2\,P(1/w)}\right) \;=\; \mathrm{res}_0\left(\frac{1}{w^{k+2-d}\,P_1(w)}\right),$$

where $P_1(w) = \sum_{j=0}^d a_j w^{d-j}$. Note that $P_1(0) = a_d \neq 0$ and therefore $1/P_1(w)$ is holomorphic near 0. Hence

$$\mathrm{res}_P(z^k) = \begin{cases} 0 & \text{if } k+2-d \leq 0 \\ \frac{1}{\ell!}\,\frac{d^\ell}{dw^\ell}\left(\frac{1}{P_1}\right)(0) & \text{if } \ell := k+1-d \geq 0 \end{cases} \tag{1.7}$$

Now, writing $P_1 = a_d(1 + \sum_{j=0}^{d-1} \frac{a_j}{a_d} w^{d-j})$, the expression $\frac{1}{\ell!}\frac{d^\ell}{dw^\ell}\left(\frac{1}{P_1}\right)(0)$ may be computed as the w^ℓ coefficient of the geometric series

$$\frac{1}{a_d}\sum_{r=0}^{\infty}\left(-\sum_{j=0}^{d-1}\frac{a_j}{a_d}w^{d-j}\right)^r \tag{1.8}$$

and the result follows.

In fact, we can extract from (1.7) and (1.8) the following more precise dependence of the global residue on the coefficients of P.

Corollary 1.1.7. *Given a polynomial $P = \sum_{j=0}^d a_j z^j \in \mathbb{C}[z]$ of degree d and $k \geq d-1$, there exists a polynomial with integer coefficients C_k such that*

$$\mathrm{res}_P(z^k) \;=\; \frac{C_k(a_0, \ldots, a_d)}{a_d^{k-d+2}}.$$

In particular, when P, H have coefficients in a subfield $\mathbf{k}$*, it holds that* $\mathrm{res}_P(H) \in \mathbf{k}$.

We also deduce from (1.7) a very important vanishing result:

Theorem 1.1.8. (Euler-Jacobi vanishing conditions) *Given polynomials* $P, H \in \mathbb{C}[z]$ *satisfying* $\deg(H) \leq deg(P) - 2$, *the global residue*

$$\mathrm{res}_P(H) \ = \ 0.$$

We note that, in view of (1.3), when all the roots of P are simple, Theorem 1.1.8 reduces to the following algebraic statement: For every polynomial $H \in \mathbb{C}[z]$, with $\deg H < \deg P - 1$,

$$\sum_{\xi \in Z_P} \frac{H(\xi)}{P'(\xi)} \ = \ 0. \tag{1.9}$$

The following direct proof of this statement was suggested to us by Askold Khovanskii. Let $d = \deg(P)$, $Z_P = \{\xi_1, \ldots, \xi_d\}$, and $P(z) = a_d \prod_{i=1}^d (z - \xi_i)$. Let L_i be the Lagrange interpolating polynomial

$$L_i(z) \ = \ \frac{\prod_{j \neq i}(z - \xi_j)}{\prod_{j \neq i}(\xi_i - \xi_j)}.$$

For any polynomial H with $\deg(H) \leq d - 1$,

$$H(z) = \sum_{i=1}^{d} H(\xi_i)\, L_i(z).$$

So, if $\deg(H) < d - 1$, the coefficient of z^{d-1} in this sum should be 0. But this coefficient is precisely

$$\sum_{i=1}^{d} H(\xi_i) \frac{1}{\prod_{j \neq i}(\xi_i - \xi_j)} \ = a_d \sum_{i=1}^{d} \frac{H(\xi_i)}{P'(\xi_i)}.$$

Since $a_d \neq 0$, statement (1.9) follows.

Since, clearly, $\mathrm{res}_P(G.P) = 0$, for all $G \in \mathbb{C}[z]$, the global residue map res_P descends to $\mathcal{A} := \mathbb{C}[z]/\langle P \rangle$, the quotient algebra by the ideal generated by P. On the other hand, if $\deg P = d$, then $\mathcal{A}$ is a finite dimensional $\mathbb{C}$-vector space of dimension $\deg(P)$, and a basis is given by the classes of $1, z, \ldots, z^{d-1}$. As in 2 we will denote by $[H]$ the class of H in the quotient $\mathcal{A}$. It follows from (1.7) and (1.8) that, as a linear map,

$$\mathrm{res}_P \colon \mathcal{A} \to \mathbb{C}$$

is particularly simple:

$$\mathrm{res}_P([z^k]) = \begin{cases} 0 & \text{if } 0 \leq k \leq d-2, \\ \frac{1}{a_d} & \text{if } k = d-1. \end{cases} \tag{1.10}$$

The above observations suggest the following "normal form algorithm" for the computation of the global residue $\mathrm{res}_P(H)$ for any $H \in \mathbb{C}[z]$:

1) Compute the remainder $r(z) = r_{d-1}z^{d-1} + \cdots + r_1 z + r_0$ in the Euclidean division of H by $P = a_d z^d + \cdots + a_0$.
2) Then, $\mathrm{res}_P(H) = \frac{r_{d-1}}{a_d}$.

We may also use (1.10) to reverse the local-global direction in the definition of the residue obtaining, in the process, an algebraic definition which extends to polynomials with coefficients in an arbitrary algebraically-closed field $\mathbb{K}$ of characteristic zero. We illustrate this construction in the case of a polynomial $P(z) = \sum_{j=0}^{d} a_j z^j \in \mathbb{K}[z]$ with simple zeros. Define a linear map $L\colon \mathbb{K}[z]/\langle P\rangle \to \mathbb{K}$ as in (1.10). Let $Z_P = \{\xi_1, \ldots, \xi_d\} \subset \mathbb{K}$ be the zeros of P and $L_1, \ldots, L_d$ be the interpolating polynomials. For any $H \in \mathbb{K}[z]$ we set:

$$\mathrm{res}_{\xi_i}(H/P) \;:=\; L([H.L_i])\,.$$

One can then check that the defining property (1.3) is satisfied. We will discuss another algebraic definition of the univariate residue in Section 1.2.1 and we will discuss the general passage from the global to the local residue in Section 1.5.3. We conclude this section by remarking on another consequence of Theorem 1.1.8. Suppose $P_1, P_2 \in \mathbb{C}[z]$ are such that their set of zeros Z_1, Z_2 are disjoint. Then, for any $H \in \mathbb{C}[z]$ such that

$$\deg H \le \deg P_1 + \deg P_2 - 2$$

we have that

$$\sum_{\xi \in Z_1 \cup Z_2} \mathrm{res}_\xi\left(\frac{H}{P_1\,P_2}\right) \;=\; 0$$

and, therefore

$$\mathrm{res}_{P_1}(H/P_2) = \sum_{\xi \in Z_1} \mathrm{res}_\xi\left(\frac{H}{P_1\,P_2}\right) = -\sum_{\xi \in Z_2} \mathrm{res}_\xi\left(\frac{H}{P_1\,P_2}\right) = -\mathrm{res}_{P_2}(H/P_1). \tag{1.11}$$

We denote the common value by $\mathrm{res}_{\{P_1,P_2\}}(H)$. Note that it is skew-symmetric on P_1, P_2. This is the simplest manifestation of a *toric* residue ([Cox96, CCD97]). We will discuss a multivariate generalization in Section 1.5.6.

1.2 Some applications of residues

1.2.1 Duality and Bezoutian

The global residue may be used to define a dualizing form in the algebra $\mathcal{A}$. We give, first of all, a proof of this result based on the local properties of the residue and, after defining the notion of the Bezoutian, we will give an algebraic construction of the dual basis.

Theorem 1.2.1. *For $P \in \mathbb{C}[z]$, let $\mathcal{A} = \mathbb{C}[z]/\langle P\rangle$. The pairing $\mathcal{A} \times \mathcal{A} \to \mathbb{C}$*

$$([H_1],[H_2]) \mapsto \mathrm{res}_P(H_1 \cdot H_2)$$

is non degenerate, i.e.

$$\mathrm{res}_P(H_1 \cdot H_2) = 0 \quad \textit{for all} \;\; H_2 \quad \textit{if and only if} \quad H_1 \in \langle P\rangle.$$

Proof. Let $d = \deg P$ and denote by $\xi_1,\ldots,\xi_r$ the roots of P, with respective multiplicities $m_1,\ldots,m_r$. Assume, for simplicity, that P is monic. Suppose $\mathrm{res}_P(H_1 \cdot H_2) = 0$ for all H_2. Given $i = 1\ldots,r$, let $G_i = \prod_{j\neq i}(z-\xi_j)^{m_j}$. Then, for any $\ell \leq m_i$,

$$0 = \mathrm{res}_P(H_1 \cdot (z-\xi_i)^\ell G_i) = \mathrm{res}_{\xi_i}(H_1/(z-\xi_i)^{m_i-\ell})$$

which, in view of (1.1.1), implies that $(z-\xi_i)^{m_i}$ divides H_1. Since these factors of P are pairwise coprime, it follows that $H_1 \in \langle P\rangle$, as desired.

As before, we denote by $\mathbb{K}$ an algebraically-closed field of characteristic zero.

Definition 1.2.2. *Let $P \in \mathbb{K}[z]$ be a polynomial of degree d. The Bezoutian associated to P is the bivariate polynomial*

$$\Delta_P(z,w) \;:=\; \frac{P(z)-P(w)}{z-w} \;=\; \sum_{i=0}^{d-1} \Delta_i(z) w^i \;\in\; \mathbb{K}[z,w].$$

Proposition 1.2.3. *The classes $[\Delta_0(z)],\ldots,[\Delta_{d-1}(z)] \in \mathcal{A} = \mathbb{K}[z]/\langle P\rangle$ give the dual basis of the standard basis $[1],[z],\ldots,[z^{d-1}]$, relative to the non-degenerate pairing defined by the global residue.*

Proof. We note, first of all, that

$$P(z)-P(w) = \left(\sum_{i=0}^{d-1} \Delta_i(z)w^i\right)(z-w) = \sum_{i=0}^{d} \left(z\Delta_i(z) - \Delta_{i-1}(z)\right) w^i,$$

where it is understood that $\Delta_{-1}(z) = \Delta_d(z) = 0$. Writing $P(w) = \sum_{i=0}^d a_i w^i$ and comparing coefficients we get the following recursive definition of $\Delta_i(z)$:

$$z\Delta_i(z) = \Delta_{i-1}(z) - a_i, \tag{1.12}$$

with initial step: $z\Delta_0(z) = P(z) - a_0$. We now compute $\mathrm{res}_P([z^j]\cdot[\Delta_i(z)])$. Since $\deg \Delta_i = d-1-i$, $\deg(z^j\Delta_i(z)) = d-1-i+j$. Hence, if $i > j$, $\deg(z^j\Delta_i(z)) \leq d-2$ and, by Theorem 1.1.8,

$$\mathrm{res}_P([z^j]\cdot[\Delta_i(z)]) \;=\; 0 \text{ for } \quad i > j\,.$$

If $i = j$, then $\deg(z^j \Delta_j) = d-1$ and it is easy to check from (1.12) that its leading coefficient is a_d, the leading coefficient of P. Hence

$$\mathrm{res}_P([z^j]\cdot[\Delta_j(z)]) \;=\; \mathrm{res}_P(a_d z^{d-1}) = 1.$$

Finally, we consider the case $i < j$. The relations (1.12) give:

$$z^j \Delta_i(z) \;=\; z^{j-1} z \Delta_i(z) \;=\; z^{j-1}(\Delta_{i-1}(z) - a_i)$$

and, therefore

$$\mathrm{res}_P(z^j \Delta_i(z)) = \mathrm{res}_P(z^{j-1}\Delta_{i-1}(z))$$

given that $\mathrm{res}_P(a_i z^{j-1}) = 0$ since $j-1 \le d-2$. Continuing in this manner we obtain

$$\mathrm{res}_P(z^j \Delta_i(z)) = \cdots = \mathrm{res}_P(z^{j-i}\Delta_0(z)) = \mathrm{res}_P(z^{j-i-1}P(z)) = 0.$$

Remark 1.2.4. Note that Proposition 1.2.3 provides an algebraic proof of Theorem 1.2.1. Indeed, we have shown that Theorem 1.2.1 only depends on the conditions (1.10) that we used in the algebraic characterization of the global residue. We may also use Proposition 1.2.3 to give an alternative algebraic definition of the global residue. Let $\Phi\colon \mathcal{A}\times\mathcal{A}\to\mathcal{A}$ denote the bilinear symmetric form defined by the requirement that $\Phi([z^i],[\Delta^j]) = \delta_{ij}$. Then, the global residue map $\mathrm{res}\colon \mathcal{A}\to\mathbb{K}$ is defined as the unique linear map such that $\Phi(\alpha,\beta) = \mathrm{res}(\alpha\cdot\beta)$, for $\alpha,\beta\in\mathcal{A}$.

Remark 1.2.5. The recursive relations (1.12) are exactly those defining the classical *Horner polynomials* $H_{d-i}(z) = a_d z^{i-1} + a_{d-1}z^{i-2} + \cdots + a_{d-i+1}$, associated to the polynomial $P(z) = \sum_{j=0}^{d} a_j z^j$.

1.2.2 Interpolation

Definition 1.2.6. *Let $Z := \{\xi_1,\ldots,\xi_r\}\subset\mathbb{K}$ be a finite set of points together with multiplicities $m_1,\ldots,m_r\in\mathbb{N}$. Let $d = m_1+\cdots+m_r$ and $h\in\mathbb{K}[z]$. A polynomial $H\in\mathbb{K}[z]$ is said to interpolate h over Z if* $\deg H \le d-1$ *and $H^{(j)}(\xi_i) = h^{(j)}(\xi_i)$ for all $j = 1,\ldots,m_i-1$.*

Proposition 1.2.7. *Let $Z\subset\mathbb{K}$ and $h\in\mathbb{K}[z]$ be as above. Let $P(z) := \prod_{i=1}^{r}(z-\xi_i)^{m_i}$. Then H interpolates h over Z if and only if $[H] = [h]$ in $\mathcal{A} = \mathbb{K}[z]/\langle P\rangle$, i.e. if H is the remainder of dividing h by P.*

Proof. If we write $h = Q\cdot P + H$, with $\deg H < d$, then

$$h^{(j)}(\xi_i) \;=\; \sum_{k=0}^{j} c_k Q^{(k)}(\xi_i)P^{(k-j)}(\xi_i) + H^{(j)}(\xi_i),$$

for suitable coefficients $c_k\in\mathbb{K}$. Since $P^{(\ell)}(\xi_i) = 0$ for $\ell = 0,\ldots,m_i-1$, it follows that H interpolates h. On the other hand, it is easy to check that the interpolating polynomial is unique and the result follows.

Lemma 1.2.8. *With notation as above, given $h \in \mathbb{K}[z]$, the interpolating polynomial H of h over Z equals*

$$H(w) = \sum_{i=1}^{d-1} c_i(h) w^i \quad \text{where}\, c_i(h) = \text{res}_P(h \cdot \Delta_i).$$

Proof. This is a straightforward consequence of the fact that $\text{res}_p(z^j \cdot \Delta_i(z)) = \delta_{ij}$. For the sake of completeness, we sketch a proof for the complex case using the integral representation of the residue.

For any $\epsilon > 0$ and any w with $|P(w)| < \epsilon$, we have by the Cauchy integral formula

$$h(w) = \frac{1}{2\pi \text{i}} \int_{|P(z)|=\epsilon} \frac{h(z)}{z-w}\, dz = \frac{1}{2\pi \text{i}} \int_{|P(z)|=\epsilon} \frac{h(z)}{P(z)-P(w)} \Delta_P(z,w) dz.$$

Denote $\Gamma := \{|P(z)| = \epsilon\}$; for any $z \in \Gamma$ we have the expansion

$$\frac{1}{P(z)-P(w)} = \frac{1}{P(z)} \frac{1}{1-\frac{P(w)}{P(z)}} = \sum_{n\geq 0} \frac{P(w)^n}{P(z)^{n+1}},$$

which is uniformly convergent over Γ. Then,

$$h(w) = \sum_{n\geq 0} \left(\frac{1}{2\pi \text{i}} \int_{\Gamma} \frac{h(z)\, \Delta_P(z,w)}{P(z)^{n+1}} \right) P(w)^n, \tag{1.13}$$

and so, isolating the first summand we get

$$h(w) = \text{res}_P(h(z)\, \Delta_P(z,w)) + Q(w)\, P(w). \tag{1.14}$$

Finally, call $H(w) := \text{res}_P(h(z)\, \Delta_P(z,w))$. It is easy to check that $H = 0$ or $\deg(H) \leq d-1$, and by linearity of the residue operator, $H(w) = \sum_{i=1}^{d-1} c_i(h)\, w^i$, as desired.

1.2.3 Ideal membership

Let again $P(z) = \sum_{i=0}^{d} a_i z^i \in \mathbb{C}[z]$. While in the univariate case is trivial, it is useful to observe that Theorem 1.2.1 allows us to derive a *residual* system of d linear equations in the coefficients of all polynomials $H(z) = \sum_{j=1}^{m} h_j z^j$ of degree less than or equal to m, whose vanishing is equivalent to the condition that $H \in \langle P \rangle$.

Such a system can be deduced from any basis $B = \{\beta_0, \ldots, \beta_{d-1}\}$ of $\mathcal{A} = \mathbb{C}[z]/\langle P \rangle$. We can choose for instance the canonical basis of monomials $\{[z^j], j = 0, \ldots, d-1\}$, or the dual basis $\{[\Delta_k(z)], k = 0, \ldots, d-1\}$. Theorem 1.2.1 means that $H \in \langle P \rangle$, i.e. $[H] = 0$ if and only if

$$\mathrm{res}_P([H]\cdot\beta_i) \;=\; \sum_{j=0}^{m} h_j \mathrm{res}_P([z^j]\,\beta_i) \;=\; 0 \quad \forall\, i=0,\ldots,d-1.$$

Suppose $m \geq d$, when B is the monomial basis, the first $d\times d$ minor of the $d\times m$ matrix of the system is triangular, while if B is the dual basis given by the Bezoutian, this minor is simply the identity.

If $H \in \langle P\rangle$, we can obtain the quotient $Q(z) \;=\; H(z)/P(Z) \in \mathbb{C}[z]$ from equations (1.13), (1.14). Indeed, we have:

$$Q(w) \;=\; \sum_{n\geq 1} \mathrm{res}[P^{n+1}](H(z)\,\Delta_P(z,w))P(w)^{n-1}.$$

By Theorem 1.1.8, the terms in this sum vanish when $n \geq \dfrac{\deg(H)+1}{d}$.

1.2.4 Partial fraction decomposition

We recall the *partial fraction decomposition* of univariate rational functions. This is a very important classical result because of its usefulness in the computation of integrals of rational functions.

Let $P, H \in \mathbb{K}[z]$ with $\deg(H)+1 \leq \deg(P) = d$. Let $\{\xi_1,\ldots,\xi_r\}$ be the zeros of P and let $m_1,\ldots,m_r$ denote their multiplicities. Then the rational function $H(z)/P(z)$ may be written as:

$$\frac{H(z)}{P(z)} \;=\; \sum_{i=1}^{r}\left(\frac{A_{i1}}{(z-\xi_i)} + \cdots + \frac{A_{im_i}}{(z-\xi_i)^{m_i}}\right) \tag{1.15}$$

for appropriate constants $A_{ij} \in \mathbb{K}$.

There are, of course, many elementary proofs of this result. Here we would like to show how it follows from the Euler-Jacobi vanishing Theorem 1.1.8. The argument below also gives a simple formula for the coefficients in (1.15) when P has only simple zeros.

For any $z \notin \{\xi_1,\ldots,\xi_r\}$ we consider the auxiliary polynomial $P_1(w) = (z-w)P(w) \in \mathbb{K}[w]$. Its zeros are ξ_i, with multiplicity m_i, $i=1,\ldots,r$, and z with multiplicity one. On the other hand, $\deg H \leq \deg P_1 - 2$, and therefore Theorem 1.1.8 gives:

$$0 \;=\; \mathrm{res}_{P_1}(H) \;=\; \mathrm{res}_z(H/P_1) + \sum_{i=1}^{r} \mathrm{res}_{\xi_i}(H/P_1)\,.$$

Since P_1 has a simple zero at z, we have $\mathrm{res}_z(H/P_1) \;=\; H(z)/P_1'(z) \;=\; -H(z)/P(z)$ and, therefore

$$\frac{H(z)}{P(z)} \;=\; \sum_{i=1}^{r} \mathrm{res}_{\xi_i}\left(\frac{H(w)}{(z-w)P(w)}\right).$$

In case P has simple zeros we have $\mathrm{res}_{\xi_i}(H/P_1) = H(\xi_i)/P_1'(\xi_i)$ which gives:

$$\frac{H(z)}{P(z)} = \sum_{i=1}^{r} \frac{(H(\xi_i)/P'(\xi_i))}{(z-\xi_i)} .$$

In the general case, it follows from (1.2) that

$$\mathrm{res}_{\xi_i}(H/P_1) = \sum_{j=0}^{m_i-1} k_j \frac{d^j(H(w)/(z-w))}{dw^j}(\xi_i) = \sum_{j=0}^{m_i-1} \frac{a_j}{(z-\xi_i)^{j+1}}$$

for suitable constants k_j and a_j.

We leave it as an exercise for the reader to compute explicit formulas for the coefficients A_{ij} in (1.15).

1.2.5 Computation of traces and Newton sums

Let $P(z) = \sum_{i=0}^{d} a_i z^i \in \mathbb{C}[z]$ be a polynomial of degree d, $\{\xi_1, \ldots, \xi_r\}$ the set of zeros of P, and $m_1, \ldots, m_r$ their multiplicities. As always, we denote by $\mathcal{A}$ the $\mathbb{C}$-algebra $\mathcal{A} = \mathbb{C}[z]/\langle P \rangle$. We recall (cf. Theorem 2.1.4 in Chapter 2) that for any polynomial $Q \in \mathbb{C}[z]$, the eigenvalues of the multiplication map

$$M_Q \colon \mathcal{A} \to \mathcal{A} ; \quad [H] \mapsto [Q \cdot H]$$

are the values $Q(\xi_i)$. In particular, using (1.4), the trace of M_Q may be expressed in terms of global residues:

$$\mathrm{tr}(M_Q) = \sum_i m_i Q(\xi_i) = \mathrm{Res}_P(Q \cdot P') .$$

Theorem 1.2.9. *The pairing* $\mathcal{A} \times \mathcal{A} \to \mathbb{C}$

$$([g_1],[g_2]) \mapsto \mathrm{tr}\,(M_{g_1 g_2}) = \mathrm{Res}_P(g_1 \cdot g_2 \cdot P')$$

is non degenerate only when all zeros of P are simple. More generally, the trace $\mathrm{tr}\,(M_{g_1 g_2}) = 0$ *for all g_2 if and only if $g_1(\xi_i) = 0$, for all $i = 1, \ldots, r$ or, equivalently, if and only if $g_1 \in \sqrt{\langle P \rangle}$.*

Proof. Fix $g_1 \in \mathbb{C}[z]$. As $\mathrm{tr}\,(M_{g_1 g_2}) = \mathrm{res}_P(g_1 \cdot P' \cdot g_2)$, it follows from Theorem 1.2.1 that the trace of $g_1 \cdot g_2$ vanishes for all g_2 if and only if $g_1\, P' \in \langle P \rangle$. But this happens if and only if g_1 vanishes over Z_P, since the multiplicity of P' at any zero p of P is one less than the multiplicity of P at p.

As $\mathrm{tr}_P(Q)$ is linear in Q, all traces can be computed from those corresponding to the monomials z^k; i.e. the *power sums* of the roots:

$$S_k := \sum_{i=1}^{r} m_i \xi_i^k = \mathrm{res}_P(z^k \cdot P'(z)).$$

It is well known that the S_k's are rational functions of the elementary symmetric functions on the zeros of P, i.e. the coefficients of P, and conversely (up to the choice of a_d). Indeed, the classical *Newton identities* give recursive relations to obtain one from the other. It is interesting to remark that not only the power sums S_k can be expressed in terms of residues, but that we can also use residues to obtain the Newton identities. The proof below is an adaptation to the one-variable case of the approach followed by Aĭzenberg and Kytmanov [AK81] to study the multivariate analogues.

Lemma 1.2.10. (Newton identities) *For all* $\ell = 0, \ldots, d-1$,

$$(d-\ell)a_\ell = -\sum_{j>\ell}^{d} a_j S_{j-\ell} \tag{1.16}$$

Proof. The formula (1.16) follows from computing:

$$\operatorname{res}\left(\frac{P'(z)}{z^\ell\, P(z)}\, P(z)\right) \quad ; \quad \ell \in \mathbb{N}$$

in two different ways:

i) As $\operatorname{res}\left(\dfrac{P'(z)}{z^\ell}\right) = \operatorname{res}_0\left(\dfrac{P'(z)}{z^\ell}\right) = \ell a_\ell.$

ii) Expanding it as a sum:

$$\sum_{j=0}^{d} a_j \operatorname{res}\left(\frac{P'(z)\, z^j}{z^\ell\, P(z)}\right) = \sum_{j<\ell} a_j \operatorname{res}\left(\frac{P'(z)}{z^{\ell-j} P(z)}\right) + \sum_{j\geq\ell} a_j \operatorname{res}\left(\frac{P'(z) z^{j-\ell}}{P(z)}\right)$$

The terms in the first sum vanish by Theorem 1.1.8 since $\deg(z^{\ell-j}P(z)) \geq \deg(P'(z))+2$, while the second sum may be expressed as $\sum_{j\geq\ell} a_j S_{j-\ell}$. Since $S_0 = d$, the identity (1.16) follows.

1.2.6 Counting integer points in lattice tetrahedra

Let $\mathcal{P} \subset \mathbb{R}^n$ be a polytope with integral vertices and let $\mathcal{P}^\circ$ denote its interior. For any $t \in \mathbb{N}$, call

$$L(\mathcal{P}, t) := \#(t \cdot \mathcal{P}) \cap \mathbb{Z}^n \quad ; \quad L(\mathcal{P}^\circ, t) := \#(t \cdot \mathcal{P}^\circ) \cap \mathbb{Z}^n,$$

the number of the lattice points in the dilated polyhedron $t \cdot \mathcal{P}$ and in its dilated interior. Ehrhart [Ehr67] proved that these are polynomial functions of degree n. They are known as the Ehrhart polynomials associated to $\mathcal{P}$ and $\mathcal{P}^\circ$. Moreover, he determined the two leading coefficients and the constant term in terms of the volume of the polytope, the normalized volume of its boundary and its Euler characteristic. The other coefficients are not as easily accessible, and a method of computing these coefficients was unknown until quite recently (cf. [Bar94, KK93, Pom93]). There is a remarkable relation between these two polynomials, the Ehrhart-Macdonald reciprocity law:

$$L(\mathcal{P}^\circ, t) \;=\; (-1)^n\, L(\mathcal{P}, t).$$

In [Bec00], Matthias Beck shows how to express these polynomials in terms of (multidimensional) residues. In the particular case when $\mathcal{P}$ is a tetrahedron, this is just a rational one-dimensional residue. We illustrate Beck's approach by sketching a proof of Ehrhart-Macdonald reciprocity in the case of a tetrahedron.

Fix $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$ and consider the tetrahedron with vertices at the origin and at the points $(0, \ldots, \alpha_i, \ldots, 0)$:

$$\Sigma \;=\; \{(x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0} \;:\; \sum_{k=1}^{n} \frac{x_k}{\alpha_k} \;\leq 1\,\}.$$

Clearly,

$$\Sigma^\circ \;=\; \{(x_1, \ldots, x_n) \in \mathbb{R}^n_{> 0} \;:\; \sum_{k=1}^{n} \frac{x_k}{\alpha_k} \;< 1\,\}.$$

Let $A := \prod_{i=1}^n \alpha_i$, $A_k := \prod_{i \neq k} \alpha_i$, $k = 1, \ldots, n$. Then,

$$\begin{aligned}
L(\Sigma, t) &= \#\{m \in \mathbb{Z}^n_{\geq 0} \;:\; \sum_{k=1}^{n} \frac{m_k}{\alpha_k} \;\leq t\,\} \\
&= \#\{m \in \mathbb{Z}^n_{\geq 0} \;:\; \sum_{k=1}^{n} m_k A_k \;\leq\; t\, A\} \\
&= \#\{m \in \mathbb{Z}^{n+1}_{\geq 0} \;:\; \sum_{k=1}^{n} m_k A_k \;+ m_{n+1} \;=\; t\, A\}.
\end{aligned}$$

So, we can interpret $L(\Sigma, t)$ as the coefficient of z^{tA} in the series product:

$$(1 + z^{A_1} + z^{2A_1} + \ldots) \ldots (1 + z^{A_n} + z^{2A_n} + \ldots)(1 + z + z^2 + \ldots),$$

i.e. as the coefficient of z^{tA} in the Taylor expansion at the origin of

$$\frac{1}{(1 - z^{A_1}) \ldots (1 - z^{A_n})(1 - z)}$$

Thus,

$$\begin{aligned}
L(\Sigma, t) \;&=\; \mathrm{res}_0 \left(\frac{z^{-tA-1}}{(1 - z) \cdot \prod_{i=1}^n (1 - z^{A_i})} \right) \\
&=\; 1 + \mathrm{res}_0 \left(\frac{z^{-tA} - 1}{z \cdot (1 - z) \cdot \prod_{i=1}^n (1 - z^{A_i})} \right).
\end{aligned}$$

For $t \in \mathbb{Z}$, let us denote by $f_t(z)$ the rational function

$$f_t(z) \;:=\; \frac{z^{tA}-1}{z\cdot(1-z)\cdot\prod_{i=1}^n(1-z^{A_i})}\,.$$

Note that for $t>0$, $\mathrm{res}_0(f_t)=-1$, while for $t<0$, $\mathrm{res}_\infty(f_t)=0$. In particular, denoting by Z the set of non-zero, finite poles of f_t, we have for $t>0$:

$$L(\Sigma,t) \;=\; 1+\mathrm{res}_0(f_{-t}(z)) \;=\; 1-\sum_{\xi\in Z}\mathrm{res}_\xi(f_{-t}(z))\,. \tag{1.17}$$

Since $L(\Sigma,t)$ is a polynomial, this identity now holds for every t.

Similarly, we compute that

$$L(\Sigma^\circ,t) \;=\; \#\{m\in\mathbb{Z}^{n+1}_{>0} \;:\; \sum_{k=1}^n m_kA_k\,+m_{n+1} \;=\; t\,A\}.$$

That means that $L(\Sigma^\circ,t)$ is the coefficient of w^{tA} in the series product:

$$(w^{A_1}+w^{2A_1}+\dots)\dots(w^{A_n}+w^{2A_n}+\dots)(w+w^2+\dots)$$

or, in terms of residues:

$$L(\Sigma^\circ,t)=\mathrm{res}_0\left(\frac{w^{A_1}\dots\,w^{A_n}\,(w^{-tA}-1)}{(1-w^{A_1})\dots(1-w^{A_n})(1-w)}\right).$$

The change of variables $z=1/w$ now yields

$$\begin{aligned} L(\Sigma^\circ,t) \;&=\; (-1)^n\,\mathrm{res}_\infty\left(\frac{z^{tA}-1}{z\,(1-z^{A_1})\dots(1-z^{A_n})\,(1-z)}\right)\\ &=\; (-1)^n\mathrm{res}_\infty(f_t(z))\,. \end{aligned} \tag{1.18}$$

The Ehrhart-Macdonald reciprocity law now follows from comparing (1.17) and (1.18), and using the fact that for $t>0$, $\mathrm{res}_0(f_t)=-1$.

1.3 Resultants in one variable

1.3.1 Definition

Fix two natural numbers d_1,d_2 and consider generic univariate polynomials of these degrees and coefficients in a field $\mathbf{k}$:

$$P(z)=\sum_{i=0}^{d_1}a_iz^i\,,\;\;Q(z)=\sum_{i=0}^{d_2}b_iz^i\,. \tag{1.19}$$

The system $P(z) = Q(z) = 0$ is, in general, overdetermined and has no solutions. The following result is classical:

Theorem 1.3.1. *There exists a unique (up to sign) irreducible polynomial*

$$\mathrm{Res}_{d_1,d_2}(P,Q) = \mathrm{Res}_{d_1,d_2}(a_0,\ldots,a_{d_1},b_0,\ldots,b_{d_2}) \in \mathbb{Z}[a_0,\ldots,a_{d_1},b_0,\ldots,b_{d_2}],$$

called the resultant of P and Q, which verifies that for any specialization of the coefficients a_i, b_i in $\mathbf{k}$ with $a_{d_1} \neq 0, b_{d_2} \neq 0$, the resultant vanishes if and only if the polynomials P and Q have a common root in any algebraically closed field $\mathbb{K}$ containing $\mathbf{k}$.

Geometrically, the hypersurface $\{(a,b) \in \mathbb{K}^{d_1+d_2+2} : \mathrm{Res}_{d_1,d_2}(a,b) = 0\}$ is the projection of the incidence variety $\{(a,b,z) \in \mathbb{K}^{d_1+d_2+3} : \sum_{i=0}^{d_1} a_i z^i = \sum_{i=0}^{d_2} b_i z^i = 0\}$; that is to say, the variable z is eliminated. Here, and in what follows, $\mathbb{K}$ denotes an algebraically closed field.

A well known theorem of Sylvester allows us to compute the resultant as the determinant of a matrix of size d_1+d_2, whose entries are 0 or a coefficient of either P or Q. For instance, when $d_1 = d_2 = 2$, the resultant is the following polynomial in 6 variables $(a_0,a_1,a_2,b_0,b_1,b_2)$:

$$b_2^2a_0^2 - 2b_2a_0a_2b_0 + a_2^2b_0^2 - b_1b_2a_1a_0 - b_1a_1a_2b_0 + a_2b_1^2a_0 + b_0b_2a_1^2$$

and can be computed as the determinant of the 4×4 matrix:

$$M_{2,2} := \begin{pmatrix} a_0 & 0 & b_0 & 0 \\ a_1 & a_0 & b_1 & b_0 \\ a_2 & a_1 & b_2 & b_1 \\ 0 & a_2 & 0 & b_2 \end{pmatrix}. \tag{1.20}$$

Let us explain how one gets this result. The basic idea is to linearize the problem in order to use the eliminant polynomial par excellence: the determinant. Note that the determinant of a square homogeneous linear system $A \cdot x = 0$ allows to eliminate x: the existence of a non trivial solution $x \neq 0$ of the system, is equivalent to the fact that the determinant of A (a polynomial in the entries of A) vanishes.

Assume $\deg(P) = d_1$, $\deg(Q) = d_2$. A first observation is that P and Q have a common root if and only if they have a common factor of positive degree (since $P(z_0) = 0$ if and only if $z - z_0$ divides P). Moreover, the existence of such a common factor is equivalent to the existence of polynomials g_1, g_2 with $\deg(g_1) \leq d_2 - 1, \deg(g_2) \leq d_1 - 1$, such that $g_1P + g_2Q = 0$. Denote by S_ℓ the space of polynomials of degree ℓ and consider the map

$$\begin{array}{ccc} S_{d_2-1} \times S_{d_1-1} & \longrightarrow & S_{d_1+d_2-1} \\ (g_1,g_2) & \longmapsto & g_1P + g_2Q \end{array} \tag{1.21}$$

This defines a $\mathbb{K}$-linear map between two finite dimensional $\mathbb{K}$-vector spaces of the same dimension d_1+d_2, which is surjective (and therefore an isomorphism) if and only if P and Q do not have any common root in $\mathbb{K}$. Denote by M_{d_1,d_2}

the matrix of this linear map in the monomial bases. It is called the *Sylvester matrix* associated to P and Q. Then

$$\operatorname{Res}_{d_1,d_2}(P,Q) = \pm\det(M_{d_1,d_2}). \tag{1.22}$$

The sign in this last equality cannot be determined, but the positive sign is taken by convention.

Note that for $d_1 = d_2 = 2$ we obtain the matrix $M_{2,2}$ in (1.20). The general shape of the Sylvester matrix is:

$$\begin{pmatrix}
a_0 & & & b_0 & & \\
a_1 & a_0 & & b_1 & b_0 & \\
 & a_1 & \ddots & & b_1 & \ddots \\
\vdots & \ddots & a_0 & \vdots & \ddots & b_0 \\
 & \vdots & a_1 & & \vdots & b_1 \\
a_{d_1} & & & b_{d_2} & & \\
 & a_{d_1} & \vdots & & b_{d_2} & \vdots \\
 & & \ddots & & & \ddots \\
 & & a_{d_1} & & & b_{d_2}
\end{pmatrix}$$

where the blank spaces are filled with zeros.

Note that setting $a_{d_1} = 0$ but $b_{d_2} \neq 0$, the determinant of the Sylvester matrix equals b_{d_2} times the determinant of the Sylvester matrix M_{d_1-1,d_2} (in $a_0, \dots, a_{d_1-1}, b_0, \dots, b_{d_2}$). We deduce that when $\deg(P) = d_1' < d_1$ and $\deg(Q) = d_2$, the restriction of the (d_1, d_2) resultant polynomial to the closed set $(a_{d_1} = \cdots = a_{d_1'+1} = 0)$ of polynomials of degrees d_1', d_2 factorizes as $\operatorname{Res}_{d_1,d_2}(P,Q) = b_{d_2}^{d_1-d_1'} \operatorname{Res}_{d_1',d_2}(P,Q)$.

What happens if we specialize both P and Q to polynomials of respective degrees smaller than d_1 and d_2? Then, the last row of the Sylvester matrix is zero and so the resultant vanishes, but in principle P and Q do not need to have a common root in K. One way to recover the equivalence between the vanishing of the resultant and the existence of a common root is the following.

Given P, Q as in (1.19), consider the homogenizations P^h, Q^h defined by

$$P^h(z,w) = \sum_{i=0}^{d_1} a_i z^i w^{d_1-i}, \quad Q^h = \sum_{i=0}^{d_2} b_i z^i w^{d_2-i}.$$

Then, P, Q can be recovered by evaluating at $w = 1$ and $(z_0, 1)$ is a common root of P^h, Q^h if and only if $P(z_0) = Q(z_0) = 0$. But also, on one hand $P^h(0,0) = Q^h(0,0) = 0$ for any choice of coefficients, and on the other P^h, Q^h have the common root $(1,0)$ when $a_{d_1} = b_{d_2} = 0$. The space obtained as the classes of pairs $(z,w) \neq (0,0)$ after identification of (z,w) with $(\lambda z, \lambda w)$ for any $\lambda \in \mathbb{K} - \{0\}$, denoted $\mathbb{P}^1(\mathbb{K})$, is called the projective line over $\mathbb{K}$. Since for homogeneous polynomials as P^h it holds that $P^h(\lambda z, \lambda w) = \lambda^{d_1} P^h(z,w)$ (and similarly for Q^h) , it makes sense to speak of their zeros in $\mathbb{P}^1(\mathbb{K})$. So, we could restate Theorem 1.3.1 saying that for any specialization of the coefficients of P and Q, the resultant vanishes if and only if their homogenizations have a common root in $\mathbb{P}^1(\mathbb{K})$. As we have already remarked, when $\mathbb{K} = \mathbb{C}$, the projective space $\mathbb{P}^1(\mathbb{C})$ can be identified with the Riemann sphere, a compactification of the complex plane, where the class of the point $(1, 0)$ is identified with the point at infinity.

1.3.2 Main properties

It is interesting to realize that many properties of the resultant can be derived from its expression (1.22) as the determinant of the Sylvester matrix:

i) The resultant Res_{d_1,d_2} is homogeneous in the coefficients of P and Q separately, with respective degrees d_2, d_1. So, the degree of the resultant in the coefficients of P is the number of roots of Q, and vice-versa.
ii) The resultants Res_{d_1,d_2} and Res_{d_2,d_1} coincide up to sign.
iii) There exist polynomials $A_1, A_2 \in \mathbb{Z}[a_0, \ldots, b_{d_2}][z]$ with $\deg(A_1) = d_2 - 1, \deg(A_2) = d_1 - 1$ such that

$$\mathrm{Res}_{d_1,d_2}(P,Q) = A_1 P + A_2 Q. \tag{1.23}$$

Let us sketch the proof of property iii). If we add to the first row in the Sylvester matrix z times the second row, plus z^2 times the third row, and so on, the first row becomes

$$P(z) \quad zP(z) \quad \ldots \quad z^{d_2-1}P(z) \quad Q(z) \quad zQ(z) \quad z^{d_1-1}Q(z)$$

but the determinant is unchanged. Expanding along this modified first row, we obtain the desired result.

Another important classical property of the resultant $\mathrm{Res}_{d_1,d_2}(P,Q)$ is that it can be written as a product over the zeros of P or Q:

Proposition 1.3.2. (Poisson formula) *Let P, Q polynomials with respective degrees d_1, d_2 and write $P(z) = a_{d_1} \prod_{i=1}^r (z - p_i)^{m_i}$, $Q(z) = b_{d_2} \prod_{j=1}^s (z - q_j)^{n_j}$. Then*

$$\mathrm{Res}_{d_1,d_2}(P,Q) = a_{d_1}^{d_2} \prod_{i=1}^{r} Q(p_i)^{m_i} = (-1)^{d_1 d_2}\, b_{d_2}^{d_1} \prod_{j=1}^{s} P(q_i)^{n_i}$$

Proof. Again, one possible way of proving the Poisson formula is by showing that

$$\mathrm{Res}_{d_1,d_2}((z-p)P_1, Q) = Q(p)\mathrm{Res}_{d_1-1,d_2}(P_1, Q),$$

using the expression of the resultant as the determinant of the Sylvester matrix, and standard properties of determinants. The proof would be completed by induction, and the homogeneity of the resultant.

Alternatively, one could observe that $R'(a,b) := a_{d_1}^{d_2} \prod_{i=1}^{r} Q(p_i)^{m_i}$ depends polynomially on the coefficients of Q and, given the equalities

$$R'(a,b) = a_{d_1}^{d_2} b_{d_2}^{d_1} \prod_{i,j} (p_i - q_j)^{m_i n_j} \;=\; (-1)^{d_1 d_2}\, b_{d_2}^{d_1} \prod_{j=1}^{s} P(q_i)^{n_i}\,,$$

on the coefficients of P as well. Since the roots are unchanged by dilation of the coefficients, we see that, as Res_{d_1,d_2}, the polynomial R' has degree $d_1 + d_2$ in the coefficients $(a,b) = (a_0, \dots, b_{d_2})$. Moreover, $R'(a,b) = 0$ if and only if there exists a common root, i.e. if and only if $\mathrm{Res}_{d_1,d_2}(a,b) = 0$. This holds in principle over the open set $(a_{d_1} \neq 0, b_{d_2} \neq 0)$ but this implies that the loci $\{R' = 0\}$ and $\{\mathrm{Res}_{d_1,d_2} = 0\}$ in $\mathbb{K}^{d_1+d_2+2}$ agree. Then, the irreducibility of Res_{d_1,d_2} implies the existence of a constant $c \in \mathbb{K}$ such that $\mathrm{Res}_{d_1,d_2} = c \cdot R'$. Evaluating at $P(z) = 1, Q(z) = z^{d_2}$, the Sylvester matrix M_{d_1,d_2} reduces to the identity $I_{d_1+d_2}$ and we get $c = 1$.

We immediately deduce

Corollary 1.3.3. *Assume $P = P_1 \cdot P_2$ with deg$(P_1) = d'_1$, deg$(P_2) = d''_1$ and deg$(Q) = d_2$. Then,*

$$\mathrm{Res}_{d'_1+d''_1,d_2}(P,Q) = \mathrm{Res}_{d'_1,d_2}(P_1,Q)\,\mathrm{Res}_{d''_1,d_2}(P_2,Q).$$

There are other determinantal formulas to compute the resultant, coming from suitable generalizations of the map (1.21), which are for instance described in [DD01]. In case $d_1 = d_2 = 3$, the Sylvester matrix $M_{3,3}$ is 6×6. Denote $[ij] := a_i b_j - a_j b_i$, for all $i,j = 0, \dots, 3$. The resultant $\mathrm{Res}_{3,3}$ can also be computed as the determinant of the following 3×3 matrix:

$$B_{3,3} := \begin{pmatrix} [03] & [02] & [01] \\ [13] & [03]+[12] & [02] \\ [23] & [13] & [03] \end{pmatrix}, \tag{1.24}$$

or as minus the determinant of the 5×5 matrix

$$\begin{pmatrix} a_0 & 0 & b_0 & 0 & [01] \\ a_1 & a_0 & b_1 & b_0 & [02] \\ a_2 & a_1 & b_2 & b_1 & [03] \\ a_3 & a_2 & b_3 & b_3 & 0 \\ 0 & a_3 & 0 & b_3 & 0 \end{pmatrix}.$$

Let us explain how the matrix $B_{3,3}$ was constructed and why $\mathrm{Res}_{3,3} = \det(B_{3,3})$. We assume, more generally, that $d_1 = d_2 = d$.

Definition 1.3.4. *Let P, Q polynomials of degree d as in (1.19). The Bezoutian polynomial associated to P and Q is the bivariate polynomial*

$$\Delta_{P,Q}(z,y) = \frac{P(z)Q(y) - P(y)Q(z)}{z-y} = \sum_{i,j=0}^{d-1} c_{ij} z^i y^j .$$

The $d \times d$ matrix $B_{P,Q} = (c_{ij})$ is called the Bezoutian matrix associated to P and Q.

Note that $\Delta_{P,1} = \Delta_P$ defined in (1.2.2) and that each coefficient c_{ij} is a linear combination with integer coefficients of the brackets $[k,\ell] = a_k b_\ell - a_\ell b_k$.

Proposition 1.3.5. *With the above notations,*

$$\mathrm{Res}_{d,d}(a,b) = \det(B_{P,Q}). \tag{1.25}$$

Proof. The argument is very similar to the one presented in the proof of Poisson's formula. Call $R' := \det(B_{P,Q})$. This is a homogeneous polynomial in the coefficients (a,b) of the same degree $2d = d+d$ as the resultant. Moreover, if $\mathrm{Res}_{d,d}(a,b) = 0$, there exists $z_0 \in \mathbb{K}$ such that $P(z_0) = Q(z_0) = 0$, and so, $\Delta_{P,Q}(y,z_0) = \sum_{i=0}^{d-1} \left(\sum_{j=0}^{d-1} c_{ij} z_0^j\right) y^i$ is the zero polynomial. This shows that $R'(a,b) = 0$ since the non trivial vector $(1, z_0, \ldots, z_0^{d-1})$ lies in the kernel of the Bezoutian matrix $B_{P,Q}$. By Hilbert's Nullstellensatz, the resultant divides a power of R'. Using the irreducibility of $\mathrm{Res}_{d,d}$ plus a particular specialization to adjust the constant, we get the desired result.

The Bezoutian matrices are more compact and practical experience seems to indicate that these matrices are numerically more stable than the Sylvester matrices.

1.4 Some applications of resultants

1.4.1 Systems of equations in two variables

Suppose that we want to solve a polynomial system in two variables $f(z,y) = g(z,y) = 0$ with $f, g \in \mathbb{K}[z,y]$. We can "hide the variable y in the coefficients" and think of $f, g \in \mathbb{K}[y][z]$. Denote by d_1, d_2 the respective degrees in the variable z. Then, the resultant $\mathrm{Res}_{d_1,d_2}(f,g)$ with respect to the variable z will give us back a polynomial (with integer coefficients) in the coefficients, i.e. we will have a polynomial in y, which vanishes on every y_0 for which there exists z_0 with $f(z_0,y_0) = g(z_0,y_0) = 0$. So, we can eliminate the variable z from the system, detect the second coordinates y_0 of the solutions, and then try to recover the full solutions (z_0, y_0).

Assume for instance that $f(z,y) = z^2 + y^2 - 10$, $g(z,y) = z^2 + 2y^2 + zy - 16$. We write

$$f(z,y) = z^2 + 0z + (y^2 - 10), \quad g(z,y) = z^2 + yz + (2y^2 - 16).$$

Then, $\mathrm{Res}_{2,2}(f,g)$ equals

$$\mathrm{Res}_{2,2}((1,0,y^2-10),(1,y,2y^2-16)) = -22y^2+2y^4+36 = 2(y+3)(y-3)(y^2-2).$$

For each of the four roots $y_0 = -3, 3, \sqrt{2}, -\sqrt{2}$, we replace $g(z,y_0)=0$ and we need to solve $z = \frac{y_0^2-6}{y_0}$. Note that $f(z,y_0)=0$ will also be satisfied due to the vanishing of the resultant. So, there is precisely one solution z_0 for each y_0. The system has $4 = 2\times 2$ real solutions.

It is easy to deduce from the results and observations made in Section 1.3 the following extension theorem.

Theorem 1.4.1. *Write* $f(z,y) = \sum_{i=1}^{d_1} f_i(y)z^i$, $g(z,y) = \sum_{i=1}^{d_2} g_i(y)z^i$, *with* $f_i, g_i \in \mathbb{K}[y]$, *and* f_{d_1}, g_{d_2} *non zero. Let* y_0 *be a root of the resultant with respect to* z, $\mathrm{Res}_{d_1,d_2}(f,g) \in \mathbb{K}[y]$. *If either* $f_{d_1}(y_0) \neq 0$ *or* $g_{d_2}(y_0) \neq 0$, *there exists* $z_0 \in \mathbb{K}$ *such that* $f(z_0,y_0) = g(z_0,y_0) = 0$.

Assume now that $f(z,y) = yz-1$, $g(z,y) = y^3-y$. It is immediate to check that they have two common roots, namely $\{f=g=0\} = \{(1,1),(-1,-1)\}$. Replace g by the polynomial $\tilde{g} := g+f$. Then, $\{f=\tilde{g}=0\} = \{f=g=0\}$ but now both $f, \tilde{g}$ have positive degree 1 with respect to the variable z. The resultant with respect to z equals

$$\mathrm{Res}_{1,1}(f,\tilde{g}) = \det\begin{pmatrix} y & -1 \\ y & y^3-y-1 \end{pmatrix} = y^2(y^2-1).$$

Since both leading coefficients with respect to z are equal to the polynomial y, Theorem 1.4.1 asserts that the two roots $y_0 = \pm 1$ can be extended. On the contrary, the root $y_0 = 0$ cannot be extended.

Consider now $f(z,y) = yz^2+z-1$, $g(z,y) = y^3-y$ and let us again consider f and $\tilde{g} := g+f$, which have positive degree 2 with respect to z. In this case, $y_0 = 0$ is a root of $\mathrm{Res}_{2,2}(f,\tilde{g}) = y^4(y^2-1)^2$. Again, $y_0 = 0$ annihilates both leading coefficients with respect to z. But nevertheless it can be extended to the solution $(0,1)$.

So, two comments should be made. The first one is that finding roots of univariate polynomials is in general not an algorithmic task! One can try to detect the rational solutions or to approximate the roots numerically if working with polynomials with complex coefficients. The second one is that even if we can obtain the second coordinates explicitly, we have in general a sufficient but not necessary condition to ensure that a given partial solution y_0 can be extended to a solution (z_0,y_0) of the system, and an ad hoc study may be needed.

1.4.2 Implicit equations of curves

Consider a parametric plane curve $\mathcal{C}$ given by $z = f(t), y = g(t)$, where $f, g \in \mathbb{K}[t]$, or more precisely,

$$\mathcal{C} \;=\; \{(z,y) \in \mathbb{K}^2 :\; z = f(t)\,,\; y = g(t)\,\text{for some}\, t \in \mathbb{K}\}.$$

Having this parametric expression allows one to “follow” or “travel along” the curve, but it is hard to detect if a given point in the plane is in $\mathcal{C}$. One can instead find an implicit equation $f \in \mathbb{K}[z,y]$, i.e. a bivariate polynomial f such that $\mathcal{C} = \{f = 0\}$. This amounts to eliminating t from the equations $z - f(t) = y - g(t) = 0$ and can thus be done by computing the resultant with respect to t of these polynomials.

This task could also be solved by a Gröbner basis computation. But we propose the reader to try in any computer algebra system the following example suggested to us by Ralf Fröberg. Consider the curve $\mathcal{C}$ defined by $z = t^{32}$, $y = t^{48} - t^{56} - t^{60} - t^{62} - t^{63}$. Then the resultant $\mathrm{Res}_{32,63}(t^{32} - z, t^{48} - t^{56} - t^{60} - t^{62} - t^{63} - y)$ with respect to t can be computed in a few seconds, giving the answer $f(z,y)$ we are looking for. It is a polynomial of degree 63 in z and degree 32 in y with 257 terms. On the other side, a Gröbner basis computation seems to be infeasible.

For a plane curve $\mathcal{C}$ with a rational parametrization; i.e.

$$\mathcal{C} \;=\; \{(p_1(t)/q_1(t), p_2(t)/q_2(t)) \;: q_1(t) \neq 0, q_2(t) \neq 0\}\;,$$

where $p_i, q_i \in \mathbb{K}[t]$, the elimination ideal

$$I_1 \;:=\; \langle q_1(t)z - p_1(t), q_2(t)y - p_2(t)\rangle \cap \mathbb{K}[z,y]$$

defines the Zariski closure of $\mathcal{C}$ in $\mathbb{K}^2$. We can obtain a generator of I_1 with a resultant computation that eliminates t. For example, let

$$\mathcal{C} \;=\; \left\{\left(\frac{t^2-1}{(1+2t)^2}, \frac{t+1}{(1+2t)(1-t)}\right)\;,\; t \neq 1\,,\; -1/2\right\}.$$

Then $\overline{\mathcal{C}} = \mathcal{V}(I_1)$ is the zero locus of

$$f(z,y) = \mathrm{Res}_{2,2}((1+2t)^2 z - (t^2-1), (1+2t)(1-t))y - (t+1))$$

which equals

$$27y^2z - 18yz + 4y + 4z^2 - z.$$

We leave it to the reader to verify that $\mathcal{C}$ is not Zariski closed.

One could also try to implicitize non planar curves. We show a general classical trick in the case of the space curve $\mathcal{C}$ with parametrization $x = t^2, y = t^3, z = t^5$. We have 3 polynomials $x - t^2, y - t^3, z - t^5$ from which we want to eliminate t. Add two new indeterminates u, v and compute the resultant

$$\mathrm{Res}_{2,5}(x - t^2, u(y - t^3) + v(z - t^5)) = (-y^2 + x^3)u^2 + (2x^4 - 2yz)uv + (-z^2 + x^5)v^2.$$

Then, since the resultant must vanish for all specializations of u and v, we deduce that

$$\mathcal{C} = \{-y^2 + x^3 = 2x^4 - 2yz = -z^2 + x^5 = 0\}\,.$$

1.4.3 Bézout's theorem in two variables

Similarly to the construction of $\mathbb{P}^1(\mathbb{K})$, one can define the projective plane $\mathbb{P}^2(\mathbb{K})$ (and in general projective n-space) as the complete variety whose points are identified with lines through the origin in $\mathbb{K}^3$. We may embed $\mathbb{K}^2$ in $\mathbb{P}^2(\mathbb{K})$ as the set of lines through the points $(x, y, 1)$. Again, it makes sense to speak of the zero set in $\mathbb{P}^2(\mathbb{K})$ of homogeneous polynomials (i.e. polynomials $f(x, y, z)$ such that $f(\lambda x, \lambda y, \lambda z) = \lambda^d f(x, y, z)$, for $d = \deg(f)$).

Given two homogeneous polynomials $f, g \in \mathbb{K}[x, y, z]$ without common factors, with $\deg(f) = d_1$, $\deg(\mathrm{g}) = d_2$, a classical theorem of Bézout asserts that they have $d_1 \cdot d_2$ common points of intersection in $\mathbb{P}^2(\mathbb{K})$, counted with appropriate intersection multiplicities. A proof of this theorem using resultants is given for instance in [CLO97]. The following weaker version suffices to obtain such nice consequences as Pascal's Mystic Hexagon theorem [CLO97, Sect. 8.7] (see Corollary 1.5.15 for a proof using multivariable residues).

Theorem 1.4.2. *Let $f, g \in \mathbb{K}[x, y, z]$ be homogeneous polynomials, without common factors, and of respective degrees d_1, d_2. Then $(f = 0) \cap (g = 0)$ is finite and has at most $d_1 \cdot d_2$ points.*

Proof. Assume $(f = 0) \cap (g = 0)$ have more than $d_1 \cdot d_2$ points, which we label $p_0, \ldots, p_{d_1 d_2}$. Let L_{ij} be the line through p_i and p_j for $i, j = 0, \ldots, d_1 d_2$. Making a linear change of coordinates, we can assume that $(0, 0, 1) \notin (f = 0) \cup (g = 0) \cup (\cup_{ij} L_{ij})$. Write $f = \sum_{i=0}^{d_1} a_i z^i$, $g = \sum_{j=0}^{d_2} b_j z^j$, as polynomials in z with coefficients $a_i, b_j \in \mathbb{K}[x, y]$. Since $f(0, 0, 1) \neq 0$, $g(0, 0, 1) \neq 0$ and f and g do not have any common factor, it is straightforward to verify from the expression of the resultant as the determinant of the Sylvester matrix, that the resultant $\mathrm{Res}_{d_1,d_2}(f, g)$ with respect to z is a non zero homogeneous polynomial in x, y of total degree $d_1 \cdot d_2$. Write $p_i = (x_i, y_i, z_i)$. Then, $\mathrm{Res}_{d_1,d_2}(f, g)(x_i, y_i) = 0$ for all $i = 0, \ldots, d_1 \cdot d_2$. The fact that $(0, 0, 1)$ does not lie in any of the lines L_{ij} implies that the $(d_1 d_2 + 1)$ points (x_i, y_i) are distinct, and we get a contradiction.

1.4.4 GCD computations and Bézout identities

Let P, Q be two univariate polynomials with coefficients in a field $\mathbf{k}$. Assume they are coprime, i.e. that their greatest common divisor $\mathrm{GCD}(P, Q) = 1$. We can then find polynomials $h_1, h_2 \in \mathbf{k}[z]$ such that the *Bézout identity* $1 = h_1 P + h_2 Q$ is satisfied, by means of the Euclidean algorithm to compute $\mathrm{GCD}(P, Q)$. A we have already remarked, $\mathrm{GCD}(P, Q) = 1$ if and only if P and Q do not have any common root in any algebraically field $\mathbb{K}$ containing $\mathbf{k}$. If d_1, d_2 denote the respective degrees, this happens precisely when $\mathrm{Res}_{d_1,d_2}(P, Q) \neq 0$. Note that since the resultant is an integer polynomial in the coefficients, $\mathrm{Res}_{d_1,d_2}(P, Q)$ also lies in $\mathbf{k}$. Moreover, by property iii) in Section 1.3.2, one deduces that

$$1 = \frac{A_1}{\mathrm{Res}_{d_1,d_2}(P,Q)} P + \frac{A_2}{\mathrm{Res}_{d_1,d_2}(P,Q)} Q. \tag{1.26}$$

So, it is possible to find h_1, h_2 whose coefficients are rational functions with integer coefficients evaluated in the coefficients of the input polynomials P, Q, and denominators equal to the resultant. Moreover, these polynomials can be explicitly obtained from the proof of (1.23). In particular, the coefficients of A_1, A_2 are particular minors of the Sylvester matrix M_{d_1,d_2}.

This has also been extended to compute $\mathrm{GCD}(P,Q)$ even when P and Q are not coprime (and the resultant vanishes), based on the so called *subresultants*, which are again obtained from particular minors of M_{d_1,d_2}. Note that $\mathrm{GCD}(P,Q)$ is the (monic polynomial) of least degree in the ideal generated by P and Q (i.e. among the polynomial linear combinations $h_1 P + h_2 Q$). So one is led to study non surjective specializations of the linear map (1.21). In fact, the dimension of its kernel equals the degree of $\mathrm{GCD}(P,Q)$, i.e. the number of common roots of P and Q, counted with multiplicity.

Note that if $1 \leq d_2 \leq d_1$ and $C = \sum_{i=0}^{d_1-d_2} c_i z^i$ is the quotient of P in the Euclidean division by Q, the remainder equals

$$R = P - \sum_{i=0}^{d_1-d_2} c_i (z^i Q).$$

Thus, subtracting from the first column of M_{d_1,d_2} the linear combination of the columns corresponding to $z^i Q, i = 0, \ldots, d_1 - d_2$, with respective coefficients c_i, we do not change the determinant but we get the coefficients of R in the first column. In fact, it holds that

$$R_{d_1,d_2}(P,Q) = a_{d_1}^{d_2 - \deg(R)} R_{\deg(R),d_2}(R,Q).$$

So, one could describe an algorithm for computing resultants similar to the Euclidean algorithm. However, the Euclidean remainder sequence to compute greatest common divisors has a relatively bad numerical behavior. Moreover, it has bad specialization properties when the coefficients depend on parameters. Collins [Col67] studied the connections between subresultants and Euclidean remainders, and he proved in particular that the polynomials in the two sequences are pairwise proportional. But the subresultant sequence has a good behavior under specializations and well controlled growth of the size of the coefficients. Several efficient algorithms have been developed to compute subresultants [LRD00].

1.4.5 Algebraic numbers

A complex number α is said to be algebraic if there exists a polynomial $P \in \mathbb{Q}[z]$ such that $P(\alpha) = 0$. The algebraic numbers form a subfield of $\mathbb{C}$. This can be easily proved using resultants.

Lemma 1.4.3. *Let $P, Q \in \mathbb{Q}[z]$ with degrees d_1, d_2 and let $\alpha, \beta \in \mathbb{C}$ such that $P(\alpha) = Q(\beta) = 0$. Then,*

i) $\alpha + \beta$ is a root of the polynomial $u_+(z) = \mathrm{Res}_{d_1,d_2}(P(z-y), Q(y)) = 0$,
ii) $\alpha \cdot \beta$ is a root of the polynomial $u_\times(z) = \mathrm{Res}_{d_1,d_2}(y^{d_1}P(z/y), Q(y))$,
iii) for $\alpha \neq 0$, α^{-1} is a root of the polynomial $u_{-1}(z) = \mathrm{Res}_{d_1,d_2}(zy-1, P(y))$,

where the resultants are taken with respect to y.

The proof of Lemma 1.4.3 is immediate. Note that even if P (resp. Q) is the minimal polynomial annihilating α (resp. β), i.e. the monic polynomial with minimal degree having α (resp. β) as a root, the roots of the polynomial $u_\times$ are all the products $\alpha_i \cdot \beta_j$ where α_i (resp. β_j) is any root of P (resp. Q), which need not be all different, and so $u_\times$ need not be the minimal polynomial annihilating $\alpha \cdot \beta$. This happens for instance in case $\alpha = \sqrt{2}, P(z) = z^2 - 2, \beta = \sqrt{3}, Q(z) = z^2 - 3$, where $u_\times(z) = (z^2 - 6)^2$.

1.4.6 Discriminants

Given a generic univariate polynomial of degree d, $P(z) = a_0 + a_1 z + \cdots + a_d z^d$, $a_d \neq 0$, it is also classical the existence of an irreducible polynomial $D_d(P) = D_d(a_0, \ldots, a_d) \in \mathbb{Z}[a_0, \ldots, a_d]$, called the *discriminant* (or d-discriminant) whose value at a particular set of coefficients (with $a_d \neq 0$) is non-zero if and only if the corresponding polynomial of degree d has only simple roots. Equivalently, $D_d(a_0, \ldots, a_n) = 0$ if and only if there exists $z \in \mathbb{C}$ with $P(z) = P'(z) = 0$.

Geometrically, the discriminantal hypersurface

$$\{a = (a_0, \ldots, a_d) \in \mathbb{C}^{d+1} : D_d(a) = 0\}$$

is the projection over the first $(d+1)$ coordinates of the intersection of the hypersurfaces $\{(a, z) \in \mathbb{C}^{d+2} : a_0 + a_1 z + \cdots + a_d z^d = 0\}$ and $\{(a, z) \in \mathbb{C}^{d+2} : a_1 + 2a_2 z + \cdots + d a_d z^{d-1} = 0\}$, i.e. the variable z is eliminated.

The first guess would be that $D_d(P)$ equals the resultant $\mathrm{Res}_{d,d-1}(P, P')$, but it is easy to see that in fact $\mathrm{Res}_{d,d-1}(P, P') = (-1)^{d(d-1)/2}\, a_d D_d(P)$. In case $d = 2$, $P(z) = az^2 + bz + c$, $D_2(a, b, c)$ is the well known discriminant $b^2 - 4ac$. When $d = 6$ for instance, D_6 is an irreducible polynomial of degree 10 in the coefficients $(a_0, \ldots, a_6)$ with 246 terms.

The extremal monomials and coefficients of the discriminant have very interesting combinatorial descriptions. This notion has important applications in singularity theory and number theory. The distance of the coefficients of a given polynomial to the discriminantal hypersurface is also related to the numerical stability of the computation of its roots. For instance, consider the Wilkinson polynomial $P(z) = (z+1)(z+2)\ldots(z+19)(z+20)$, which clearly has 20 real roots at distance at least 1 from the others, and is known to be numerically unstable. The coefficients of P are very close to the coefficients of a polynomial with a multiple root. The polynomial $Q(z) = P(z) + 10^{-9} z^{19}$, obtained by a "small perturbation" of one of the coefficients of P, has only 12 real roots and 4 pairs of imaginary roots, one of which has imaginary part close to $\pm 0.88i$. Consider then the parametric family of polynomials $P_\lambda(z) = P(z) + \lambda z^{19}$ and note that $P(z) = P_0$ and $Q(z) = P_{10^{-9}}$. Thus, for some intermediate value of λ, two complex roots merge to give a double real root and therefore that value of the parameter is a zero of the discriminant $D(\lambda) = D_{20}(P_\lambda)$.

1.5 Multidimensional residues

In this section we will extend the theory of residues to the several variables case. As in the one-dimensional case we will begin with an "integral" definition of local residue from which we will define the total residue as a sum of local ones. We will also indicate how one can give a purely algebraic definition of global, and then local, residues using Bezoutians. We shall also touch upon the geometric definition of Arnold, Varchenko and Guseĭn-Zadé [AGZV85].

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and let $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be a zero-dimensional ideal. We denote by $Z(I) = \{\xi_1, \ldots, \xi_s\} \subset \mathbb{K}^n$ the variety of zeros of I. We will assume, moreover, that I is a *complete intersection* ideal, i.e. that it has a presentation of the form $I = \langle P_1, \ldots, P_n \rangle$, $P_i \in \mathbb{K}[x_1, \ldots, x_n]$. For simplicity, we will denote by $\langle P \rangle$ the ordered n-tuple $\{P_1, \ldots, P_n\}$. As before, let $\mathcal{A}$ be the finite dimensional commutative algebra $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]/I$. Our goal is to define a linear map

$$\mathrm{res}_{\langle P \rangle} : \mathcal{A} \to \mathbb{K}$$

whose properties are similar to the univariate residue map. In particular, we would like it to be dualizing in the sense of Theorem 1.2.1 and to be compatible with local maps $\mathrm{res}_{\langle P \rangle, \xi} : \mathcal{A}_\xi \to \mathbb{K}$, $\xi \in Z(I)$.

1.5.1 Integral definition

In case $\mathbb{K} = \mathbb{C}$, given $\xi \in Z(I)$, let $\mathcal{U} \subset \mathbb{C}^n$ be an open neighborhood of ξ containing no other points of $Z(I)$, and let $H \in \mathbb{C}[x_1, \ldots, x_n]$. We define the local *Grothendieck* residue

$$\mathrm{res}_{\langle P \rangle, \xi}(H) \;=\; \frac{1}{(2\pi i)^n} \int_{\Gamma_\xi(\epsilon)} \frac{H(x)}{P_1(x) \cdots P_n(x)}\, dx_1 \wedge \cdots \wedge dx_n\,, \qquad (1.27)$$

where $\Gamma_\xi(\epsilon)$ is the real n-cycle $\Gamma_\xi(\epsilon) = \{x \in \mathcal{U} : |P_i(x)| = \epsilon_i\}$ oriented by the n-form $d(\arg(P_1)) \wedge \cdots \wedge d(\arg(P_n))$. For almost every $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ in a neighborhood of the origin, $\Gamma_\xi(\epsilon)$ is smooth and by Stokes' Theorem the integral (1.27) is independent of ϵ. The choice of the orientation form implies that $\mathrm{res}_{\langle P \rangle, \xi}(H)$ is skew-symmetric on $P_1, \ldots, P_n$. We note that this definition makes sense as long as H is holomorphic in a neighborhood of ξ. If $\xi \in Z(I)$ is a point of multiplicity one then the Jacobian

$$J_{\langle P\rangle}(\xi) \;:=\; \det\left(\frac{\partial P_i}{\partial x_j}(\xi)\right)$$

is non-zero, and

$$\mathrm{res}_{\langle P\rangle,\xi}(H) \;=\; \frac{H(\xi)}{J_{\langle P\rangle}(\xi)}\,. \tag{1.28}$$

This identity follows from making a change of coordinates $y_i = P_i(x)$ and iterated integration.

It follows from Stokes's theorem that if $H \in I_\xi$, the ideal defined by I in the local ring defined by ξ (cf. Section 2.1.3 in Chapter 2), then $\mathrm{res}_{\langle P\rangle,\xi}(H) = 0$ and therefore the local residue defines a map $\mathrm{res}_{\langle P\rangle,\xi}\colon \mathcal{A}_\xi \to \mathbb{C}$. We then define the global residue map as the sum of local residues

$$\mathrm{res}_{\langle P\rangle}(H) \;:=\; \sum_{\xi\in Z(I)} \mathrm{res}_{\langle P\rangle,\xi}(H)$$

which we may view as a map $\mathrm{res}_{\langle P\rangle}\colon \mathcal{A} \to \mathbb{C}$. We may also define the global residue $\mathrm{res}_{\langle P\rangle}(H_1/H_2)$ of a rational function regular on $Z(I)$, i.e. such that H_2 does not vanish on $Z(I)$. At this point one may be tempted to replace the local cycles $\Gamma_\xi(\epsilon)$ by a global cycle

$$\Gamma(\epsilon) \;:=\; \{x \in \mathbb{C}^n : |P_i(x)| = \epsilon_i\}$$

but $\Gamma(\epsilon)$ need not be compact and integration might not converge. However, if the map

$$(P_1,\ldots,P_n)\colon\, \mathbb{C}^n \to \mathbb{C}^n$$

is proper, then $\Gamma(\epsilon)$ is compact and we can write

$$\mathrm{res}_{\langle P\rangle}(H) \;:=\; \frac{1}{(2\pi i)^n}\int_{\Gamma(\epsilon)} \frac{H(x)}{P_1(x)\cdots P_n(x)}\,dx_1\wedge\cdots\wedge dx_n\,.$$

The following two theorems summarize basic properties of the local and global residue map.

Theorem 1.5.1 (Local and Global Duality). *Let* $I = \langle P_1,\ldots,P_n\rangle \subset \mathbb{C}[x_1,\ldots,x_n]$ *be a complete intersection ideal and* $\mathcal{A} = \mathbb{C}[x_1,\ldots,x_n]/I$. *Let* $\mathcal{A}_\xi$ *be the localization at* $\xi \in Z(I)$. *The pairings*

$$\mathcal{A}_\xi \times \mathcal{A}_\xi \to \mathbb{C} \quad;\quad ([H_1],[H_2]) \mapsto \mathrm{res}_{\langle P\rangle,\xi}(H_1\cdot H_2)$$

and

$$\mathcal{A} \times \mathcal{A} \to \mathbb{C} \quad;\quad ([H_1],[H_2]) \mapsto \mathrm{res}_{\langle P\rangle}(H_1\cdot H_2)$$

are non-degenerate.

Theorem 1.5.2 (Local and Global Transformation Laws). *Let* $I = \langle P_1, \ldots, P_n \rangle$ *and* $J = \langle Q_1, \ldots, Q_n \rangle$ *be zero-dimensional ideals such that* $J \subset I$. *Let*

$$Q_j(x) \;=\; \sum_{i=1}^{n} a_{ij}(x) P_i(x)\,.$$

Denote by $A(x)$ *the* $n \times n$*-matrix* $(a_{ij}(x))$, *then for any* $\xi \in Z(I)$,

$$\mathrm{res}_{\langle P \rangle, \xi}(H) \;=\; \mathrm{res}_{\langle Q \rangle, \xi}(H \cdot \det(A))\,. \tag{1.29}$$

Moreover, a similar formula holds for global residues

$$\mathrm{res}_{\langle P \rangle}(H) \;=\; \mathrm{res}_{\langle Q \rangle}(H \cdot \det(A))\,.$$

Remark 1.5.3. We refer the reader to [Tsi92, Sect. 5.6 and 8.4] for a proof of the duality theorems and to [Tsi92, Sect. 5.5 and 8.3] for full proofs of the transformation laws. The local theorems are proved in [GH78, Sect. 5.1] and extended to the global case in [TY84]; a General Global Duality Law is discussed in [GH78, Sect. 5.4] Here we will just make a few remarks about Theorem 1.5.2.

Suppose that $\xi \in Z(I)$ is a simple zero and that $\det(A(\xi)) \neq 0$. Then, since

$$J_{\langle Q \rangle}(\xi) \;:=\; J_{\langle P \rangle}(\xi) \cdot \det(A(\xi))$$

we have

$$\mathrm{res}_{\langle P \rangle, \xi}(H) \;=\; \frac{H(\xi)}{J_{\langle P \rangle}(\xi)} \;=\; \frac{H(\xi) \cdot \det(A(\xi))}{J_{\langle Q \rangle}(\xi)} \;=\; \mathrm{res}_{\langle Q \rangle, \xi}(H \cdot \det(A))\,,$$

as asserted by (1.29). The case of non-simple zeros which are common to both I and J is dealt-with using a perturbation technique after showing that when the input $\{P_1, \ldots, P_n\}$ depends smoothly on a parameter so does the residue. Finally, one shows that if $\xi \in Z(J) \backslash Z(I)$, then $\det(A) \in J_\xi$ and the local residue $\mathrm{res}_{\langle Q \rangle, \xi}(H \cdot \det(A))$ vanishes.

1.5.2 Geometric definition

For the sake of completeness, we include a few comments about the geometric definition of the residue of Arnold, Varchenko and Guseĭn-Zadé [AGZV85]. Here, the starting point is the definition of the residue at a simple zero $\xi \in Z(I)$ as in (1.28). Suppose now that $\xi \in Z(I)$ has multiplicity μ. In a sufficiently small neighborhood $\mathcal{U}$ of ξ in $\mathbb{C}^n$ we can consider the map

$$P = (P_1, \ldots, P_n) : \mathcal{U} \to \mathbb{C}^n\,.$$

By Sard's theorem, almost all values $y \in P(\mathcal{U})$ are regular and at such points the equation $P(x) - y = 0$ has exactly μ simple roots $\eta_1(y), \ldots, \eta_\mu(y)$. Consider the map

$$\phi(y) \; := \; \sum_{i=1}^{\mu} \frac{H(\eta_i(y))}{J_{\langle P \rangle}(\eta_i(y))} \, .$$

It is shown in [AGZV85, Sect. 5.18] that $\phi(y)$ extends holomorphically to $0 \in \mathbb{C}^n$. We can then define the local residue $\mathrm{res}_{\langle P \rangle, \xi}(H)$ as the value $\phi(0)$. A continuity argument shows that both definitions agree.

1.5.3 Residue from Bezoutian

In this section we generalize to the multivariable case the univariate approach discussed in Section 1.2.1. This topic is also discussed in Section 3.3 of Chapter 3. We will follow the presentation of [BCRS96] and [RS98] to which we refer the reader for details and proofs. We note that other purely algebraic definitions of the residue may also be found in [KK87, Kun86, SS75, SS79].

Let $\mathbb{K}$ be an algebraically closed field $\mathbb{K}$ of characteristic zero and let $\mathcal{A}$ be a finite-dimensional commutative $\mathbb{K}$ algebra. Recall that $\mathcal{A}$ is said to be a *Gorenstein* algebra if there exists a linear form $\ell \in \widehat{\mathcal{A}} := \mathrm{Hom}_{\mathbb{K}}(\mathcal{A}, \mathbb{K})$ such that the bilinear form

$$\phi_\ell : \mathcal{A} \times \mathcal{A} \to \mathbb{K} \quad ; \quad \phi_\ell(a, b) \; := \; \ell(a \cdot b)$$

is non-degenerate. Given such a dualizing linear form ℓ, let $\{a_1, \ldots, a_r\}$ and $\{b_1, \ldots, b_r\}$ be ϕ_ℓ-dual bases of $\mathcal{A}$, and set

$$B_\ell \; := \; \sum_{i=1}^{r} a_i \otimes b_i \; \in \; \mathcal{A} \otimes \mathcal{A} \, .$$

B_ℓ is independent of the choice of dual bases and is called a *generalized Bezoutian*. It is characterized by the following two properties:

- $(a \otimes 1) B_\ell \; = \; (1 \otimes a) B_\ell$, for all $a \in \mathcal{A}$, and
- If $\{a_1, \ldots, a_r\}$ is a basis of $\mathcal{A}$ and $B_\ell = \sum_i a_i \otimes b_i$, then $\{b_1, \ldots, b_r\}$ is a basis of $\mathcal{A}$ as well.

It is shown in [BCRS96, Th. 2.10] that the correspondence $\ell \mapsto B_\ell$ is an equivalence between dualizing linear forms on $\mathcal{A}$ and generalized Bezoutians in $\mathcal{A} \otimes \mathcal{A}$.

As in Section 1.2.5 we can relate the dualizing form, the Bezoutian and the computation of traces. The dual $\widehat{\mathcal{A}}$ may be viewed as a module over $\mathcal{A}$ by $a \cdot \lambda(b) := \lambda(a \cdot b)$, $a, b \in \mathcal{A}, \lambda \in \widehat{\mathcal{A}}$. A dualizing form $\ell \in \widehat{\mathcal{A}}$ generates $\widehat{\mathcal{A}}$ as an $\mathcal{A}$-module. Moreover, it defines an isomorphism $\mathcal{A} \to \widehat{\mathcal{A}}$, $a \mapsto \ell(a \bullet)$. In particular there exists a unique element $J_\ell \in \mathcal{A}$ such that $\mathrm{tr}(M_q) = \ell(J_\ell \cdot q)$, where $M_q \colon \mathcal{A} \to \mathcal{A}$ denotes multiplication by $q \in \mathcal{A}$. On the other hand, if $\{a_1, \ldots, a_r\}$ and $\{b_1, \ldots, b_r\}$ are ϕ_ℓ-dual bases of $\mathcal{A}$, then

$$M_q(a_j) = q \cdot a_j = \sum_{i=1}^{r} \phi_\ell(q \cdot a_j, b_i) a_i$$

and therefore

$$\mathrm{tr}(M_q) = \sum_{i=1}^{r} \phi_\ell(q \cdot a_i, b_i) = \sum_{i=1}^{r} \ell(q \cdot a_i \cdot b_i) = \ell\big(q \cdot \sum_{i=1}^{r} a_i b_i\big)$$

from which it follows that

$$J_\ell \ = \ \sum_{i=1}^{r} a_i \cdot b_i\,. \tag{1.30}$$

Note that, in particular,

$$\ell(J_\ell) \ = \ \sum_{i=1}^{r} \ell(a_i \cdot b_i) \ = \ r \ = \ \dim(\mathcal{A})\,. \tag{1.31}$$

Suppose now that $I \subset \mathbb{K}[x_1, \ldots, x_n]$ is a zero-dimensional complete intersection ideal. We may assume without loss of generality that I is generated by a regular sequence $\{P_1, \ldots, P_n\}$. The quotient algebra $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]/I$ is a Gorenstein algebra. This can be done by defining directly a dualizing linear form (global residue or Kronecker symbol) or by defining an explicit Bezoutian as in [BCRS96, Sect. 3]:

Let

$$\partial_j P_i \ := \ \frac{P_i(y_1, \ldots, y_{j-1}, x_j, \ldots, x_n) - P_i(y_1, \ldots, y_j, x_{j+1}, \ldots, x_n)}{x_j - y_j} \tag{1.32}$$

and set

$$\Delta_{\langle P \rangle}(x, y) \ = \ \det(\partial_j P_i) \ \in \ \mathbb{K}[x, y]\,. \tag{1.33}$$

We shall also denote by $\Delta_{\langle P \rangle}(x, y)$ its image in the tensor algebra

$$\mathcal{A} \otimes \mathcal{A} \ \cong \ \mathbb{K}[x, y]/\langle P_1(x), \ldots, P_n(x), P_1(y), \ldots, P_n(y)\rangle\,. \tag{1.34}$$

Remark 1.5.4. In the analytic context, the polynomials $\partial_j P_i$ are the coefficients of the so called Hefer expansion of P_i. We refer to [TY84] for the relationship between Hefer expansions and residues.

Theorem 1.5.5. *The element $\Delta_{\langle P \rangle}(x, y) \in \mathcal{A} \otimes \mathcal{A}$ is a generalized Bezoutian.*

This is Theorem 3.2 in [BCRS96]. It is easy to check that $\Delta_{\langle P \rangle}$ satisfies the first condition characterizing generalized Bezoutians. Indeed, given the identification (1.34), it suffices to show that $[f(x)] \cdot \Delta_{\langle P \rangle}(x, y) = [f(y)] \cdot \Delta_{\langle P \rangle}(x, y)$ for all $[f] \in \mathcal{A}$. This follows directly from the definition of $\Delta_{\langle P \rangle}$. The proof of the second property is much harder. Becker et al. show it by reduction to the local case where it is obtained through a deformation technique somewhat similar to that used in the geometric case in [AGZV85].

We denote by τ the Kronecker symbol; that is, the dualizing linear form associated with the Bezoutian $\Delta_{\langle P\rangle}$. As we shall see below, for $\mathbb{K} = \mathbb{C}$, the Kronecker symbol agrees with the global residue. In order to keep the context clear, we will continue to use the expression Kronecker symbol throughout this section.

If H_1/H_2 is a rational function such that H_2 does not vanish on $Z(I)$, then $[H_2]$ has an inverse $[G_2]$ in $\mathcal{A}$ and we define $\tau(H_1/H_2) := \tau([H_1]\cdot[G_2])$.

If $\{[x^\alpha]\}$ is a monomial basis of $\mathcal{A}$ and we write

$$\Delta_{\langle P\rangle}(x,y) \;=\; \sum x^\alpha \Delta_\alpha(y)$$

then $\{[x^\alpha]\}$ and $\{[\Delta_\alpha(x)]\}$ are dual basis and it follows from (1.30) and (1.34) that

$$J_{\langle P\rangle}(x) \;:=\; J_\tau(x) \;=\; \sum_\alpha x^\alpha \Delta_\alpha(x) \;=\; \Delta_{\langle P\rangle}(x,x)\,.$$

Since $\lim_{y\to x} \partial_j P_i(x,y) = \dfrac{\partial P_i}{\partial x_j}$ it follows that $J_{\langle P\rangle}(x)$ agrees with the standard Jacobian of the polynomials $P_1,\ldots,P_n$. As we did in Section 1.1.2 for univariate residues, we can go from the global Kronecker symbol to local operators. Let $Z(I) = \{\xi_1,\ldots,\xi_s\}$ and let

$$I \;=\; \cap_{\xi\in Z(I)} I_\xi$$

be the primary decomposition of I as in Section 2.1.3 of Chapter 2. Let $\mathcal{A}_\xi = \mathbb{K}[x_1,\ldots,x_n]/I_\xi$, we have an isomorphism:

$$\mathcal{A} \;\cong\; \prod_{\xi\in Z(I)} \mathcal{A}_\xi\,.$$

We recall (cf. [CLO98, Sect. 4.2]) that there exist idempotents $e_\xi \in \mathbb{K}[x_1,\ldots,x_n]$ such that, in $\mathcal{A}$, $\sum_{\xi\in Z(I)} e_\xi = 1$, $e_{\xi_i}e_{\xi_j} = 0$ if $i\neq j$, and $e_\xi^2 = 1$. These generalize the interpolating polynomials we discussed in Section 1.1.2. We can now define

$$\tau_\xi([H]) \;:=\; \tau(e_\xi\cdot[H])$$

and it follows easily that the global Kronecker symbol is the sum of the local ones. In analogy with the global case, we may define the local Kronecker symbol $\tau_\xi([H_1/H_2])$ of a rational function H_1/H_2, regular at ξ as $\tau_\xi([H_1]\cdot[G_2])$, where $[G_2]$ is the inverse of $[H_2]$ in the algebra $\mathcal{A}_\xi$. The following proposition shows that in the case of simple zeros and $\mathbb{K} = \mathbb{C}$, the Kronecker symbol agrees with the global residue defined in Section 1.5.1.

Proposition 1.5.6. *Suppose that* $J_{\langle P\rangle}(\xi)\neq 0$ *for all* $\xi\in Z(I)$. *Then*

$$\tau(H) \;=\; \sum_{\xi\in Z(I)} \frac{H(\xi)}{J_{\langle P\rangle}(\xi)} \tag{1.35}$$

for all $H\in\mathbb{K}[x_1,\ldots,x_n]$.

Proof. Recall that the assumption that $J_{\langle P\rangle}(\xi) \neq 0$ for all $\xi \in Z(I)$ implies that $[J_{\langle P\rangle}]$ is invertible in $\mathcal{A}$. Indeed, since $J_{\langle P\rangle}, P_1, \ldots, P_n$ have no common zeros in $\mathbb{K}^n$, the Nullstellensatz implies that there exists $G \in \mathbb{K}[x_1, \ldots, x_n]$ such that

$$G.J_{\langle P\rangle} = 1 \mod I\,.$$

Given $H \in \mathbb{K}[x_1, \ldots, x_n]$, consider the trace of the multiplication map $M_{H\cdot G}\colon \mathcal{A} \to \mathcal{A}$. On the one hand, we have from Theorem 2.1.4 in Chapter 2 that

$$\mathrm{tr}(M_{H\cdot G}) \;=\; \sum_{\xi \in Z(I)} H(\xi)G(\xi) \;=\; \sum_{\xi \in Z(I)} \frac{H(\xi)}{J_{\langle P\rangle}(\xi)}\,.$$

But, recalling the definition of the Jacobian we also have

$$\mathrm{tr}(M_{H\cdot G}) \;=\; \tau(J_{\langle P\rangle} \cdot G \cdot H) \;=\; \tau(H)$$

and (1.35) follows.

Remark 1.5.7. As in the geometric case discussed in Section 1.5.2 one can use continuity arguments to show that the identification between the Kronecker symbol and the global residue extends to the general case. We refer the reader to [RS98] for a proof of this fact as well as for a proof of the Transformation Laws in this context. In particular, Theorem 1.5.2 holds over any algebraically closed field of characteristic zero.

1.5.4 Computation of residues

In this section we would like to discuss briefly some methods for the computation of global residues; a further method is discussed in Section 3.3.1 in Chapter 3. Of course, if the zero-dimensional ideal $I = \langle P_1, \ldots, P_n\rangle$ is radical and we can compute the zeros $Z(I)$, then we can use (1.28) to compute the local and global residue. We also point out that the transformation law gives a general, though not very efficient, algorithm to compute local and global residues. Indeed, since I is a zero dimensional ideal there exist univariate polynomials $f_1(x_1), f_2(x_2), \ldots, f_n(x_n)$ in the ideal I. In particular we can write

$$f_j(x_j) \;=\; \sum_{i=1}^{n} a_{ij}(x) P_i(x)$$

and for any $H \in \mathbb{K}[x_1, \ldots, x_n]$,

$$\mathrm{res}_{\langle P\rangle}(H) \;=\; \mathrm{res}_{\langle f\rangle}(H \cdot \det(a_{ij}))\,. \tag{1.36}$$

Moreover, the right hand side of the above equation may be computed as an iterated sequence of univariate residues. What makes this a less than desirable computational method is that even if the polynomials $P_1, \ldots, P_n$ and

$f_1, \ldots, f_n$ are very simple, the coefficients $a_{ij}(x)$ need not be so. The following example illustrates this.

Consider the polynomials

$$\begin{aligned} P_1 &= x_1^2 - x_3 \\ P_2 &= x_2 - x_1 x_3^2 \\ P_3 &= x_3^2 - x_1^3 \end{aligned} \tag{1.37}$$

The ideal $I = \langle P_1, P_2, P_3 \rangle$ is a zero-dimensional ideal; the algebra $\mathcal{A}$ has dimension four, and the zero-locus $Z(I)$ consists of two points, the origin, which has multiplicity three, and the point $(1, 1, 1)$. Gröbner basis computations with respect to lexicographic orders give the following univariate polynomials in the ideal I:

$$\begin{aligned} f_1 &= x_1^4 - x_1^3 \\ f_2 &= x_2^2 - x_2 \\ f_3 &= x_3^3 - x_3^2 \,. \end{aligned} \tag{1.38}$$

We observe that we could also have used iterated resultants to find univariate polynomials in I. However, this will generally yield higher degree polynomials. For instance, for our example (1.37) a Singular [GPS01] computation gives:

```
>resultant(resultant(P_1,P_2,x_3),resultant(P_2,P_3,x_3),x_2);
x_1^10-2*x_1^9+x_1^8
```

Returning to the polynomials (1.38), we can obtain, using the Singular command "division", a coefficient matrix $A = (a_{ij}(x))$:

$$\begin{pmatrix} x_1^2 + x_3 & x_3^3 + (x_1^2 + x_1 + 1)x_3^2 + (x_1^2 + x_1 + x_2)x_3 + x_1^2 x_2 & (x_1 + 1)x_3 + x_1^2 \\ 0 & x_1^3 + x_2 - 1 & 0 \\ 1 & (x_1 + 1)(x_2 + x_3) + x_3^2 & x_1 + x_3 \end{pmatrix}$$

So that

$$\begin{aligned} \det(A) &= (x_2 + x_1^3 - 1)x_3^2 + (x_1^2 x_2 + x_1^5 - x_1^3 - x_1^2 - x_2 + 1)x_3 + \\ &= x_1^3 x_2 + x_1^6 - x_1^5 - x_1^3 + x_1^2 - x_1^2 x_2 \,. \end{aligned}$$

Rather than continuing with the computation of a global residue $\mathrm{res}_{\langle P \rangle}(H)$ using (1.36) and iterated univariate residues or Bezoutians, we will refer the reader to Chapter 3 where improved versions are presented and discuss instead how we can use the multivariate Bezoutian in computations. The Bezoutian matrix $(\partial_j P_i)$ is given by

$$\begin{pmatrix} x_1 + y_1 & -x_3^2 & -(x_1^2 + x_1 y_1 + y_1^2) \\ 0 & 1 & 0 \\ -1 & -y_1(x_3 + y_3) & x_3 + y_3 \end{pmatrix}$$

and therefore

$$\Delta_{\langle P\rangle}(x,y) \;=\; x_1x_3 + x_1y_3 + x_3y_1 + y_1y_3 - x_1^2 - x_1y_1 - y_1^2\,.$$

Computing a Gröbner basis relative to grevlex gives a monomial basis of $\mathcal{A}$ of the form $\{1, x_1, x_2, x_3\}$. Reducing $\Delta_{\langle P\rangle}(x,y)$ relative to the corresponding basis of $\mathcal{A}\otimes\mathcal{A}$ we obtain:

$$\Delta_{\langle P\rangle}(x,y) \;=\; (y_2 - y_3) + (y_3 - y_1)x_1 + x_2 + (y_1 - 1)x_3\,.$$

Hence the dual basis of $\{1, x_1, x_2, x_3\}$ is the basis $\{x_2 - x_3, x_3 - x_1, 1, x_1 - 1\}$.

We now claim that given $H \in \mathbb{K}[x_1,\dots,x_n]$, if we compute the grevlex normal form:

$$N(H) \;=\; \lambda_0 + \lambda_1[x_1] + \lambda_2[x_2] + \lambda_3[x_3]$$

then, $\mathrm{res}_{\langle P\rangle}(H) \;=\; \lambda_2$. More generally, suppose that $\{[x^\alpha]\}$ is a monomial basis of $\mathcal{A}$ and that $\{[\Delta_\alpha(x)]\}$ is the dual basis given by the Bezoutian, then if $[H] = \sum_\alpha \lambda_\alpha[x^\alpha]$ and $1 = \sum_\alpha \mu_\alpha[\Delta_\alpha]$,

$$\mathrm{res}_{\langle P\rangle}(H) \;=\; \sum_\alpha \lambda_\alpha\mu_\alpha\,. \tag{1.39}$$

Indeed, we have

$$\begin{aligned}\mathrm{res}_{\langle P\rangle}(H) \;&=\; \mathrm{res}_{\langle P\rangle}(H\cdot 1) \;=\; \mathrm{res}_{\langle P\rangle}\Big(\sum_\alpha \lambda_\alpha x^\alpha \cdot \sum_\beta \mu_\beta \Delta_\beta\Big)\\ &=\; \sum_{\alpha,\beta} \lambda_\alpha\mu_\beta \mathrm{res}_{\langle P\rangle}(x^\alpha\cdot\Delta_\beta) \;=\; \sum_\alpha \lambda_\alpha\mu_\alpha\,.\end{aligned}$$

Although the computational method based on the Bezoutian allows us to compute $\mathrm{res}_{\langle P\rangle}(H)$ as a linear combination of normal form coefficients of H, it would be nice to have a method that computes the global residue as a single normal form coefficient, generalizing the univariate algorithm based on the identities (1.10). This can be done if we make some further assumptions on the generators of the ideal I. We will discuss here one such case which has been extensively studied both analytically and algebraically, following the treatment in [CDS96]. A more general algorithm will be presented in Section 1.5.6. Assume the generators $P_1,\dots,P_n$ satisfy:

Assumption: $P_1,\dots,P_n$ are a Gröbner basis for some term order $\prec$.

Since we can always find a weight $w\in\mathbb{N}^n$ such that $\mathrm{in}_w(P_i) \;=\; \mathrm{in}_\prec(P_i)$, $i = 1,\dots,n$, and given that I is a zero dimensional ideal, it follows that, up to reordering the generators, our assumption is equivalent to the existence of a weight w such that:

$$\mathrm{in}_w(P_i) = c_i\, x_i^{r_i+1} \tag{1.40}$$

It is clear that in this case $\dim_{\mathbb{K}}(\mathcal{A}) = r_1\cdots r_n$, and a monomial basis of $\mathcal{A}$ is given by $\{[x^\alpha] : 0 \le \alpha_i \le r_i\}$.

We point out that, for appropriately chosen term orders, our assumption leads to interesting examples.

- Suppose $\prec$ is lexicographic order with $x_n \prec \cdots \prec x_1$. In this case
$$P_i \;=\; c_i x_i^{r_i+1} + P_i'(x_i, \ldots, x_n)$$
and $\deg_{x_i}(P_i') \le r$. This case was considered in [DS91].
- Let $\prec$ be degree lexicographic order with $x_1 \prec \cdots \prec x_n$. Then
$$P_i(x) \;=\; c_i x_i^{r_i+1} \;+\; \sum_{j=1}^{i-1} z_j\, \phi_{ij}(x) \;+\; \psi_i(x)\,,$$
where $\deg(\phi_{ij}) = r_i$ and $\deg(\psi_i(x)) \le r_i$. This case has been extensively studied by the Krasnoyarsk School (see, for example, [AY83, Ch. 21] and [Tsi92, II.8.2]) using integral methods. Some of their results have been transcribed to the algebraic setting in [BGV02] under the name of Pham systems of type II.

Note also that the polynomials in (1.37) satisfy these conditions. Indeed, for $w = (3, 14, 5)$ we have:
$$\mathrm{in}_w(P_1) = x_1^2\ , \quad \mathrm{in}_w(P_2) = x_2\ , \quad \mathrm{in}_w(P_3) = x_3^2 \tag{1.41}$$

The following theorem, which may be viewed as a generalization of the basic univariate definition (1.1), is due to Aĭzenberg and Tsikh. Its proof may be found in [AY83, Ch. 21] and [CDS96, Th. 2.3].

Theorem 1.5.8. *Let $P_1, \ldots, P_n \in \mathbb{C}[x_1, \ldots, x_n]$ satisfy (1.40). Then for any $H \in \mathbb{C}[x_1, \ldots, x_n]$ $\mathrm{res}_{\langle P\rangle}(H)$ is equal to the $\frac{1}{x_1\cdots x_n}$-coefficient of the Laurent series expansion of:*
$$\frac{H(x)}{\prod_i c_i x_i^{r_i+1}}\ \prod_i \left(\frac{1}{1 \,+\, P_i'(x)/(c_i x_i^{r_i+1})}\right)\,, \tag{1.42}$$
obtained through geometric expansions.

The following simple consequence of Theorem 1.5.8 generalizes (1.10) and is the basis for its algorithmic applications.

Corollary 1.5.9. *Let $P_1, \ldots, P_n \in \mathbb{C}[x_1, \ldots, x_n]$ satisfy (1.40) and let $\{[x^\alpha] : 0 \le \alpha_i \le r_i\}$ be the corresponding monomial basis of $\mathcal{A}$. Let $\mu = (r_1, \ldots, r_n)$, then*
$$\mathrm{res}_{\langle P\rangle}([x^\alpha]) = \begin{cases} 0 & \text{if } \alpha \ne \mu \\ \frac{1}{c_1\cdots c_n} & \text{if } \alpha = \mu \end{cases} \tag{1.43}$$

Remark 1.5.10. A proof of (1.43) using the Bezoutian approach may be found in [BCRS96]. Hence, Corollary 1.5.9 may be used in the algebraic setting as well.

As in the univariate case, we are led to the following algorithm for computing residues when $P_1, \ldots, P_n$ satisfy (1.40).

Algorithm 1: Compute the normal form $N(H)$ of $H \in \mathbb{K}[x_1,\ldots,x_n]$ relative to any term order which refines w-degree. Then,

$$\operatorname{res}_{\langle P\rangle}(H) \;=\; \frac{a_\mu}{c_1\cdots c_n}\,, \tag{1.44}$$

where a_μ is the coefficient of x^μ in $N(H)$.

Remark 1.5.11. Given a weight w for which (1.40) holds it is easy to carry the computations in the above algorithm using the weighted orders *wp* (weighted grevlex) and *Wp* (weighted deglex) in Singular [GPS01]. For example, for the polynomials in (1.37), the Jacobian $J_{\langle P\rangle}(x) = 4x_1x_3 - 3x_1^2$ and we get:

```
> ring R = 0, (x1, x2, x3), wp(3,14,5);
> ideal I = x1^2-x3, x2-x1*x3^2, x3^2 - x1^3;
> reduce(4*x1*x3 - 3*x1^2,std(I));
                          4*x1*x3-3*x3
```

Thus, the x_1x_3 coefficient of the normal form of $J_{\langle P\rangle}(x)$ is 4, i.e. $\dim_{\mathbb{K}}(\mathcal{A})$ as asserted by (1.30).

1.5.5 The Euler-Jacobi vanishing theorem

We will now discuss the multivariate extension of Theorem 1.1.8. The basic geometric assumption that we need to make is that if we embed $\mathbb{C}^n$ in a suitable compactification then the ideal we are considering has all its zeros in $\mathbb{C}^n$. Here we will restrict ourselves to the case when the chosen compactification is weighted projective space. The more general vanishing theorems are stated in terms of global residues in the torus and toric compactifications as in [Kho78a].

Let $w \in \mathbb{N}^n$ and denote by $\deg_w$ the weighted degree defined by w. We set $|w| = \sum_i w_i$. Let $I = \langle P_1,\ldots,P_n\rangle$ be a zero-dimensional complete intersection ideal and write

$$P_i(x) \;=\; Q_i(x) + P_i'(x)\,,$$

where $Q_i(x)$ is weighted homogeneous of w-degree d_i and $\deg_w(P_i') < d_i$. We call Q_i the *leading form* of P_i. We say that I has no zeros at infinity in weighted projective space if and only if

$$Q_1(x) \;=\; \cdots \;=\; Q_n(x) \;=\; 0 \quad \text{if and only if} \quad x \;=\; 0. \tag{1.45}$$

In the algebraic context an ideal which has a presentation by generators satisfying (1.45) is called a *strict complete intersection* [KK87].

Theorem 1.5.12 (Euler-Jacobi vanishing). *Let $I = \langle P_1,\ldots,P_n\rangle$ be a zero-dimensional complete intersection ideal with no zeros at infinity in weighted projective space. Then,*

$$\operatorname{res}_{\langle P\rangle}(H) \;=\; 0 \quad \textit{if} \quad \deg_w(H) < \sum_{i=1}^n \deg_w(P_i) - |w|.$$

Proof. We begin by proving the assertion in the particular case when $Q_i(x) = x_i^{N+1}$. By linearity it suffices to prove that if x^α is a monomial with $\langle w, \alpha \rangle < N|w|$, then $\mathrm{res}_{\langle P \rangle}(x^\alpha) = 0$. We prove this by induction on $\delta = \langle w, \alpha \rangle$. If $\delta = 0$ then $x^\alpha = 1$ and the result follows from Corollary 1.5.9. Suppose then that the result holds for any monomial of degree less than $\delta = \langle w, \alpha \rangle$, if every $\alpha_i \leq N$ then the result follows, again, from Corollary 1.5.9. If, on the other hand, some $\alpha_i > N$ then we can write

$$x^\alpha \;=\; x^\beta \cdot P_i - x^\beta \cdot P_i' \,,$$

where $\beta = \alpha - (N+1)e_i$. It then follows that $\mathrm{res}_{\langle P \rangle}(x^\alpha) = -\mathrm{res}_{\langle P \rangle}(x^\beta \cdot P_i')$, but all the monomials appearing in the right-hand side have weighted degree less than δ and therefore the residue vanishes.

Consider now the general case. In view of (1.45) and the Nullstellensatz there exists N sufficiently large such that

$$x_i^{N+1} \;\in\; \langle Q_1(x), \ldots, Q_n(x) \rangle \,.$$

In particular, we can write

$$x_j^{N+1} \;=\; \sum_{i=1}^{n} a_{ij}(x) Q_i(x) \,,$$

where $a_{ij}(x)$ is w-homogeneous of degree $(N+1)w_j - d_i$. Let now

$$F_j(x) \;=\; \sum_{i=1}^{n} a_{ij}(x) P_i(x) \;=\; x_j^{N+1} + F_j'(x) \,,$$

and $\deg_w(F_j') < (N+1)w_j$. Given now $H \in \mathbb{K}[x_1, \ldots, x_n]$ with $\deg_w(H) < \sum_i d_i - |w|$, we have by the Global Transformation Law:

$$\mathrm{res}_{\langle P \rangle}(H) \;=\; \mathrm{res}_{\langle F \rangle}(\det(a_{ij}) \cdot H) \,.$$

But, $\deg_w(\det(a_{ij})) \leq (N+1)|w| - \sum_i d_i$ and therefore

$$\deg_w(det(a_{ij}) \cdot H) \leq \deg_w(\det(a_{ij})) + \deg_w(H) < N|w| \,,$$

and the result follows from the previous case.

Remark 1.5.13. The Euler-Jacobi vanishing theorem is intimately connected to the continuity of the residue. The following argument from [AGZV85, Ch. 1, Sect. 5] makes the link evident. Suppose $P_1, \ldots, P_n$ have only simple zeros and satisfy (1.45). For simplicity we take $w = (1, \ldots, 1)$, the general case is completely analogous. Consider the family of polynomials

$$\tilde{P}_i(x; t) \;:=\; t^{d_i} P_i(t^{-1}x_1, \ldots, t^{-1}x_n) \,. \tag{1.46}$$

Note that $\tilde{P}_i(t\cdot x, t) = t^{d_i} P_i(x)$. In particular if $P_i(\xi) = 0$, $\tilde{P}_i(t\xi;t) = 0$ as well. Suppose now that $\deg(H) < \sum_i d_i - n$ and let $\tilde{H}(x;t)$ be defined as in (1.46). Then

$$\operatorname{res}_{\langle\tilde{P}\rangle}(\tilde{H}) \;=\; \sum_{\xi\in Z(I)} \frac{\tilde{H}(t\xi;t)}{\operatorname{Jac}_{\langle\tilde{P}\rangle}(t\xi)} \;=\; t^a \sum_{\xi\in Z(I)} \frac{H(\xi)}{\operatorname{Jac}_{\langle P\rangle}(\xi)} \;=\; t^a \operatorname{res}_{\langle P\rangle}(H)\,,$$

where $a = \deg(H) - \deg(\operatorname{Jac}_{\langle P\rangle}(x)) = \deg(H) - (\sum_i d_i - n)$. Hence, if $a < 0$, the limit

$$\lim_{t\to 0} \operatorname{res}_{\langle\tilde{P}\rangle}(\tilde{H})$$

may exist only if $\operatorname{res}_{\langle P\rangle}(H) = 0$ as asserted by the Euler-Jacobi theorem.

We conclude this subsection with some applications of Theorem 1.5.12 to plane projective geometry (cf. [GH78, 5.2]). The following theorem is usually referred to as the Cayley-Bacharach Theorem though, as Eisenbud, Green, and Harris point out in [EGH96], it should be attributed to Chasles.

Theorem 1.5.14 (Chasles). *Let C_1 and C_2 be curves in $\mathbb{P}^2$, of respective degrees d_1 and d_2, intersecting in $d_1 d_2$ distinct points. Then, any curve of degree $d = d_1 + d_2 - 3$ that passes through all but one of the points in $C_1 \cap C_2$ must pass through the remaining point as well.*

Proof. After a linear change of coordinates, if necessary, we may assume that no point in $C_1 \cap C_2$ lies in the line $x_3 = 0$. Let $C_i = \{\tilde{P}_i(x_1, x_2, x_3) = 0\}$, $\deg P_i = d_i$. Set $P_i(x_1, x_2) = \tilde{P}_i(x_1, x_2, 1)$. Given $\tilde{H} \in \mathbb{K}[x_1, x_2, x_3]$, homogeneous of degree d, let $H \in \mathbb{K}[x_1, x_2]$ be similarly defined. We can naturally identify the points in $C_1 \cap C_2$ with the set of common zeros

$$Z = \{\xi \in \mathbb{K}^2 \,:\, P_1(\xi) = P_2(\xi) = 0\}\,.$$

Since $\deg H < \deg P_1 + \deg P_2 - 2$, Theorem 1.5.12 implies that $\operatorname{res}_{\langle P\rangle}(H) = 0$, but then

$$0 \;=\; \operatorname{res}_{\langle P\rangle}(H) = \sum_{\xi\in Z} \frac{H(\xi)}{\operatorname{Jac}_{\langle P\rangle}(\xi)}$$

which implies that if H vanishes at all but one of the points in Z it must vanish on the remaining one as well.

Corollary 1.5.15 (Pascal's Mystic Hexagon). *Consider a hexagon inscribed in a conic curve of $\mathbb{P}^2$. Then, the pairs of opposite sides meet in collinear points.*

Proof. Let $L_1 \dots L_6$ denote the hexagon inscribed in the conic $Q \subset \mathbb{P}^2$, where L_i is a line in $\mathbb{P}^2$. Let ξ_{ij} denote the intersection point $L_i \cap L_j$. Consider the cubic curves $C_1 = L_1 + L_3 + L_5$ and $C_2 = L_2 + L_4 + L_6$. The intersection $C_1 \cap C_2$ consists of the nine points ξ_{ij} with i odd and j even. The cubic $Q + L(\xi_{14}\xi_{36})$, where $L(\xi_{14}\xi_{36})$ denotes the line joining the two points, passes through eight of the points in $C_1 \cap C_2$ hence must pass through the ninth point ξ_{52}. For degree reasons this is only possible if $\xi_{52} \in L(\xi_{14}\xi_{36})$ and therefore the three points are collinear.

1.5.6 Homogeneous (projective) residues

In this section we would like to indicate how the notion or residue may be extended to meromorphic forms in projective space. This is a special instance of a much more general theory of residues in toric varieties. A full discussion of this topic is beyond the scope of these notes so we will restrict ourselves to a presentation of the basic ideas, in the case $\mathbb{K} = \mathbb{C}$, and refer the reader to [GH78, TY84, PS83, Cox96, CCD97] for details and proofs.

Suppose $F_0, \ldots, F_n \in \mathbb{C}[x_0, \ldots, x_n]$ are homogeneous polynomials of degrees $d_0, \ldots, d_n$, respectively. Let $V_i = \{x \in \mathbb{P}^n : F_i(x) = 0\}$ and assume that

$$V_0 \cap V_1 \cdots \cap V_n \;=\; \emptyset\,. \tag{1.47}$$

This means that the zero locus of the ideal $I = \langle F_0, \ldots, F_n \rangle$ is the origin $0 \in \mathbb{C}^{n+1}$. Given any homogeneous polynomial $H \in \mathbb{C}[x_0, \ldots, x_n]$ we can define the *projective residue* of H relative to the $n{+}1$-tuple $\langle F \rangle = \{F_0, \ldots, F_n\}$ as:

$$\mathrm{res}^{\mathbb{P}^n}_{\langle F \rangle}(H) \;:=\; \mathrm{res}_{\langle F \rangle}(H) \;=\; \mathrm{res}_{\langle F \rangle, 0}(H).$$

It is clear from the integral definition of the Grothendieck residue, that the local residue at 0 is invariant under the change of coordinates $x_i \mapsto \lambda x_i$, $\lambda \in \mathbb{C}^*$. On the other hand, if $\deg(H) = d$ we see that, for

$$\rho \;:=\; \sum_{i=0}^{n} (d_i - 1)\,,$$

$$\frac{H(\lambda \cdot x)}{F_0(\lambda \cdot x) \cdots F_n(\lambda \cdot x)} d(\lambda x_0) \wedge \cdots \wedge d(\lambda x_n) = \frac{\lambda^{d-\rho}\; H(x)}{F_0(x) \cdots F_n(x)} dx_0 \wedge \cdots \wedge dx_n\,.$$

Hence,

$$\mathrm{res}^{\mathbb{P}^n}_{\langle F \rangle}(H) \;=\; 0 \quad \text{if} \quad \deg(H) \neq \rho\,.$$

Being a global (and local) residue, the projective residue is a dualizing form in the algebra $\mathcal{A} = \mathbb{C}[x_0, \ldots, x_n]/I$. Moreover, since I is a homogeneous ideal, $\mathcal{A}$ is a graded algebra and the projective residue is compatible with the grading. These dualities properties are summarized in the following theorem.

Theorem 1.5.16. *The graded algebra* $\mathcal{A} = \oplus \mathcal{A}_d$ *satisfies:*

a) $\mathcal{A}_d = 0$ *for* $d > \rho := d_0 + \cdots + d_n - (n+1)$.
b) $\mathcal{A}_\rho \cong \mathbb{C}$.

c) *For* $0 \leq d \leq \rho$, *the bilinear pairing*

$$\mathcal{A}_d \times \mathcal{A}_{\rho-d} \to \mathbb{C}\ ; \quad ([H_1],[H_2]) \mapsto \mathrm{res}^{\mathbb{P}^n}_{\langle F\rangle}(H_1 \cdot H_2)$$

is non-degenerate.

Proof. The assumption (1.47) implies that $F_0,\dots,F_n$ are a regular sequence in the ring $\mathbb{C}[x_0,\dots,x_n]$. Computing the Poincaré series for $\mathcal{A}$ using the exactness of the Koszul sequence yields the first two assertions. See [PS83, Sect. 12] for details. A proof using residues may be found in [Tsi92, Sect. 20]. The last assertion follows from Theorem 1.5.1.

An important application of Theorem 1.5.16 arises in the study of smooth hypersurfaces $X_F = \{x \in \mathbb{P}^n : F(x) = 0\}$, of degree d, in projective space [CG80]. In this case we take $F_i = \partial F/\partial x_i$, the smoothness condition means that $\{F_0,\dots,F_n\}$ satisfy (1.47), and the Hodge structure of X may be described in terms of the *Jacobian ideal* generated by $\{\partial F/\partial x_i\}$. Indeed, $\rho = (n+1)(d-2)$ and setting, for $0 \leq p \leq n-1$, $\delta(p) := d(p+1)-(n+1)$, we have $\delta(p)+\delta(n-1-p) = \rho$, and

$$H^{p,n-1-p}(X) \;\cong\; A_{\delta(p)}\,.$$

Moreover, the pairing

$$\mathrm{res}^{\mathbb{P}^n}_{\langle F\rangle} : A_{\delta(p)} \times A_{\delta(n-1-p)} \to \mathbb{C}$$

corresponds to the intersection pairing

$$H^{p,n-1-p}(X) \times H^{n-1-p,p}(X) \to \mathbb{C}\,.$$

The projective residue may be related to affine residues in a different way. If we identify $\mathbb{C}^n \cong \{x \in \mathbb{P}^n : x_0 \neq 0\}$, then after a linear change of coordinates, if necessary, we may assume that for every $i = 0,\dots,n$,

$$Z_i \;:=\; V_0 \cap \cdots \cap \widehat{V_i} \cap \cdots \cap V_n \;\subset\; \mathbb{C}^n\,. \tag{1.48}$$

Let $P_i \in \mathbb{C}[x_1,\dots,x_n]$ be the polynomial $P_i(x_1,\dots,x_n) = F_i(1,x_1,\dots,x_n)$ and let us denote by $\langle P^{\hat{i}}\rangle$ the n-tuple of polynomials $P_0,\dots,P_{i-1},P_{i+1},\dots,P_n$.

Theorem 1.5.17. *For any homogeneous polynomial* $H \in \mathbb{C}[x_0,\dots,x_n]$ *with* $\deg(H) \leq \rho$,

$$\mathrm{res}^{\mathbb{P}^n}_{\langle F\rangle}(H) \;:=\; (-1)^i\,\mathrm{res}_{\langle P^{\hat{i}}\rangle}(h/P_i)\,, \tag{1.49}$$

where $h(x_1,\dots,x_n) = H(1,x_1,\dots,x_n)$.

Proof. We will only prove the second, implicit, assertion that the right-hand side of (1.49) is independent of i. This statement, which generalizes the identity (1.11), is essentially Theorem 5 in [TY84]. For the main assertion we refer to [CCD97, Sect. 4], where it is proved in the more general setting of simplicial toric varieties.

Note that the assumption (1.47) implies that the rational function h/P_i is regular on Z_i and hence it makes sense to compute $\mathrm{res}_{\langle P^{\hat{i}}\rangle}(h/P_i)$. For each $i = 0,\ldots,n$, consider the n-tuple of polynomials in $\mathbb{K}[x_1,\ldots,x_n]$: $\langle Q_i\rangle = \{P_0,\ldots,(P_i \cdot P_{i+1}),\ldots,P_n\}$, if $i < n$ and $\langle Q_n\rangle = \{P_1,\ldots,P_{n-1},(P_n \cdot P_0)\}$. The set of common zeros of the polynomials in Q_i is $Z(Q_i) = Z_i \cup Z_{i+1}$. Hence, it follows from (1.48) that the ideal generated by the n-tuple Q_i is zero-dimensional and has no zeros at infinity. Hence, given that $\deg(H) \leq \rho$, the Euler-Jacobi vanishing theorem implies that

$$\begin{aligned} 0 &= \mathrm{res}_{\langle Q_i\rangle}(h) = \sum_{\xi\in Z_i} \mathrm{res}_{\langle Q_i\rangle,\xi}(h) \;+\; \sum_{\xi\in Z_{i+1}} \mathrm{res}_{\langle Q_i\rangle,\xi}(h) \\ &= \sum_{\xi\in Z_i} \mathrm{res}_{\langle P^{\hat{i}}\rangle,\xi}(h/P_i) \;+\; \sum_{\xi\in Z_{i+1}} \mathrm{res}_{\langle P^{\widehat{i+1}}\rangle,\xi}(h/P_{i+1}) \\ &= \mathrm{res}_{\langle P^{\hat{i}}\rangle}(h/P_i) \;+\; \mathrm{res}_{\langle P^{\widehat{i+1}}\rangle}(h/P_{i+1}) \end{aligned}$$

and, consequently, the theorem follows. We should point out that the equality $\mathrm{res}_{\langle Q_i\rangle,\xi}(h) = \mathrm{res}_{\langle P^{\hat{i}}\rangle,\xi}(h/P_i)$, which is clear from the integral definition of the local residue, may be obtained in the general case from the Local Transformation Law and the fact that $\mathrm{res}_{\langle P^{\hat{i}}\rangle,\xi}(h/P_i)$ was defined as $\mathrm{res}_{\langle P^{\hat{i}}\rangle,\xi}(h \cdot Q_i)$, where Q_i inverts P_i in the local algebra $A_\xi^{\hat{i}}$ and, consequently, the statement holds over any algebraically closed field of characteristic zero.

We can use the transformation law to exhibit a polynomial $\Delta(x)$ of degree ρ with non-zero residue. Write

$$F_j \;=\; \sum_{i=0}^{n} a_{ij}(x)\, x_i \;; \quad j = 0,\ldots,n\,,$$

and set $\Delta(x) = \det(a_{ij}(x))$. Then, $\deg(\Delta) = \rho$, and

$$\mathrm{res}^{\mathbb{P}^n}_{\langle F\rangle}(\Delta) \;=\; 1 \tag{1.50}$$

Indeed, let $\langle G\rangle$ denote the $n+1$-tuple $G = \{x_0,\ldots,x_n\}$. Then by the transformation law

$$\mathrm{res}^{\mathbb{P}^n}_{\langle G\rangle}(1) \;=\; \mathrm{res}^{\mathbb{P}^n}_{\langle F\rangle}(\Delta)$$

and a direct computation shows that the left-hand side of the above identity is equal to 1.

Putting together part b) of Theorem 1.5.16 with (1.50) we obtain the following normal form algorithm for computing the projective residue $\mathrm{res}^{\mathbb{P}^n}_{\langle F\rangle}(H)$:

Algorithm 2: 1. Compute a Gröbner basis of the ideal $\langle F_0,\ldots,F_n\rangle$.

2. Compute the normal form $N(H)$ of H and the normal form $N(\Delta)$ of Δ, with respect to the Gröbner basis.
3. The projective residue $\mathrm{res}^{\mathbb{P}^n}_{\langle F\rangle}(H) = \dfrac{N(H)}{N(\Delta)}$.

Remark 1.5.18. There is a straightforward variant of this algorithm valid for weighted homogeneous polynomials. This more general algorithm has been used by Batyrev and Materov [BM02], to compute the Yukawa 3-point function of the generic hypersurface in weighted projective $\mathbb{P}^4_w$, $w = (1,1,2,2,2)$. This function, originally computed in [CdlOF+94] has a series expansion whose coefficients have enumerative meaning. We refer to [BM02, 10.3] and [CK99, 5.6.2.1] for more details.

We can combine Theorem 1.5.17 and Algorithm 2 to compute the global (affine) residue with respect to a zero-dimensional complete intersection ideal with no zeros at infinity in projective space. The construction below is a special case of a much more general algorithm described in [CD97] and it applies, in particular, to the weighted case as well. It also holds over any algebraically closed field $\mathbb{K}$ of characteristic zero.

Let $I = \{P_1, \ldots, P_n\} \in \mathbb{K}[x_1, \ldots, x_n]$ be polynomials satisfying (1.45). Let $d_i = \deg(P_i)$ and denote by

$$F_i(x_0, x_1, \ldots, x_n) \;:=\; x_0^{d_i}\, P(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})$$

the homogenization of P_i. Let $h(x_1, \ldots, x_n) \in \mathbb{K}[x_1, \ldots, x_n]$. If $d = \deg(h) < \sum_i (d_i - 1)$, then $\mathrm{res}_{\langle P\rangle}(h) = 0$ by the Euler-Jacobi theorem. Suppose, then that $d \geq \sum_i (d_i - 1)$, let $H \in \mathbb{K}[x_0, \ldots, x_n]$ be its homogenization, and let

$$F_0 \;=\; x_0^{d_0}\;; \quad d_0 = d - \sum_{i=1}^{n} (d_i - 1) + 1\,.$$

Then, $d = \sum_{i=0}^{n} (\deg(F_i) - 1)$ and it follows from Theorem 1.5.17 that

$$\mathrm{res}^{\mathbb{P}^n}_{\langle F\rangle}(H) \;=\; \mathrm{res}_{\langle P\hat{0}\rangle}(h/P_0) \;=\; \mathrm{res}_{\langle P\rangle}(h)\,.$$

1.5.7 Residues and elimination

One of the basic applications of residues is to elimination theory. The key idea is very simple (see also Section 3.3.1 in Chapter 3). Let $I = \langle P_1, \ldots, P_n\rangle \subset \mathbb{K}[x_1, \ldots, x_n]$ be a zero-dimensional, complete intersection ideal. Let $\xi_i = (\xi_{i1}, \ldots, \xi_{in}) \in \mathbb{K}^n$, $i = 1, \ldots, r$, be the zeros of I. Let $\mu_1, \ldots, \mu_r$ denote their respective multiplicities. Then the power sum

$$S_j^{(k)} \;:=\; \sum_{i=1}^{r} \xi_{ij}^k$$

is the trace of the multiplication map $M_{x_j^k} : \mathcal{A} \to \mathcal{A}$ and, therefore, it may be expressed as a global residue:

$$S_j^{(k)} \;=\; \mathrm{tr}(M_{x_j^k}) \;=\; \mathrm{res}_{\langle P\rangle}(x_j^k \cdot J_{\langle P\rangle}(x)).$$

The univariate Newton identities of Section 1.2.5 now allow us to compute inductively the coefficients of a polynomial in the variable x_j with roots at $\xi_{1j}, \dots, \xi_{rj} \in \mathbb{K}$ and respective multiplicities $\mu_1, \dots, \mu_r$.

We illustrate the method with the following example. Let

$$I \;=\; \langle x_1^3 + x_1^2 - x_2, x_1^3 - x_2^2 + x_1 x_2 \rangle \,.$$

It is easy to check that the given polynomials are a Gröbner basis for any term order that refines the weight order defined by $w = (5, 9)$. The leading terms are $x_1^3, -x_2^2$. A normal form computation following Algorithm 1 in Section 1.5.4 yields:

$$S_1^{(1)} = -2 \;;\; S_1^{(2)} = 4 \;;\; S_1^{(3)} = -2 \;;\; S_1^{(4)} = 0 \;;\; S_1^{(5)} = 8 \;;\; S_1^{(6)} = -20.$$

For example, the following Singular [GPS01] computation shows how the values $S_1^{(3)}$ and $S_1^{(4)}$ were obtained:

```
> ring R = 0, (x1,x2), wp(5,9);
> ideal I = x1^3 + x1^2 - x2, x1^3 - x2^2 + x1*x2;
> poly J = -6*x1^2*x2+3*x1^3-4*x1*x2+5*x1^2+x2;
>  reduce(x1^3*J,std(I));
2*x1^2*x2+2*x1*x2+10*x1^2-10*x2
>  reduce(x1^4*J,std(I));
-8*x1*x2-12*x1^2+12*x2
```

Now, using the Newton identities (1.16) we may compute the coefficients of a monic polynomial of degree 6 on the variable x_1 lying on the ideal:

$$a_5 = 2 \;;\; a_4 = 0 \;;\; a_3 = -2 \;;\; a_2 = 0 \;;\; a_1 = 0 \;;\; a_0 = 0.$$

Hence, $f_1(x_1) = x_1^6 + 2x_1^5 - 2x_1^3 \in I$.

We refer the reader to [AY83, BKL98] for a fuller account of this elimination procedure. Note also that in Section 3.6 of Chapter3 there is an application of residues to the implicitization problem.

1.6 Multivariate resultants

In this section we will extend the notion of the resultant to multivariate systems. We will begin by defining the resultant of $n+1$ homogeneous polynomials in $n+1$ variables and discussing some formulas to compute it. We will also discuss some special examples of the so-called sparse or toric resultant.

1.6.1 Homogeneous resultants

When trying to generalize resultants associated to polynomials in any number of variables, the first problem one faces is which families of polynomials one is going to study, i.e. which will be the variables of the resultant. For example, in the univariate case, fixing the degrees d_1, d_2 amounts to setting $(a_0, \dots, a_{d_1}, b_0, \dots, b_{d_2})$ as the input variables for the resultant Res_{d_1,d_2}. One obvious, and classical choice, in the multivariable case is again, to fix the degrees $d_0, \dots, d_n$ of $n+1$ polynomials in n variables, which will generally define an overdetermined system. If one wants the vanishing of the resultant $\mathrm{Res}_{d_0,\dots,d_n}$ to be equivalent to the existence of a common root, one realizes that a compactification of affine space naturally comes into the picture, in this case projective n-space.

Consider, for instance, a bivariate linear system

$$\begin{cases} f_0(x,y) = a_{00}x + a_{01}y + a_{02} \\ f_1(x,y) = a_{10}x + a_{11}y + a_{12} \\ f_2(x,y) = a_{20}x + a_{21}y + a_{22} \end{cases} \tag{1.51}$$

We fix the three degrees equal to 1, i.e. we have nine variables a_{ij} $(i, j = 0, 1, 2)$, and we look for an irreducible polynomial $\mathrm{Res}_{1,1,1} \in \mathbb{Z}[a_{ij}, \, i, j = 0, 1, 2]$ which vanishes if and only the system has a solution (x, y). If such a solution (x, y) exists, then $(x, y, 1)$ would be a non-trivial solution of the augmented 3×3-linear system and consequently the determinant of the matrix (a_{ij}) must vanish. However, as the following example easily shows, the vanishing of the determinant does not imply that (1.51) has a solution. Let

$$\begin{cases} f_0(x,y) = x + 2y + 1 \\ f_1(x,y) = x + 2y + 2 \\ f_2(x,y) = x + 2y + 3 \end{cases}$$

The determinant vanishes but the system is incompatible in $\mathbb{C}^2$. On the other hand, the lines defined by $f_i(x,y) = 0$ are parallel and therefore we may view them as having a common point at infinity in projective space. We can make this precise by passing to the homogenized system

$$\begin{cases} F_0(x,y,z) = x + 2y + z \\ F_1(x,y,z) = x + 2y + 2z \\ F_2(x,y,z) = x + 2y + 3z, \end{cases}$$

which has non zero solutions of the form $(-2y, y, 0)$, i.e. the homogenized system has a solution in the projective plane $\mathbb{P}^2(\mathbb{C})$, a compactification of the affine plane $\mathbb{C}^2$.

We denote $x = (x_0, \dots, x_n)$ and for any $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$, $|\alpha| = \alpha_0 + \dots + \alpha_n$, $x^\alpha = x_0^{\alpha_0} \dots x_n^{\alpha_n}$. Recall that $f = \sum_\alpha a_\alpha x^\alpha \in \mathbf{k}[x_0, \dots, x_n]$ is called homogeneous (of degree $\deg(f) = d$) if $|\alpha| = d$ for all $|\alpha|$ with $a_\alpha \neq 0$, or equivalently, if for all $\lambda \in \mathbf{k}$, it holds that $f(\lambda\, x) = \lambda^d\, f(x)$, for all $x \in \mathbf{k}^{n+1}$.

As we already remarked in Section 1.3.1, the variety of zeros of a homogeneous polynomial is well defined over $\mathbb{P}^n(\mathbf{k}) = \left(\mathbf{k}^{n+1}\backslash\{0\}\right) / \sim$, where we identify $x \sim \lambda x$, for all $\lambda \in \mathbf{k}\backslash\{0\}$. As before, $\mathbb{K}$ denotes the algebraic closure of $\mathbf{k}$.

The following result is classical.

Theorem 1.6.1. *Fix* $d_0, \ldots, d_n \in \mathbb{N}$ *and write* $F_i = \sum_{|\alpha|=d_i} a_{i\alpha} x^\alpha$, $i = 1, \ldots, n$. *There exists a unique irreducible polynomial*

$$\mathrm{Res}_{d_0,\ldots,d_n}(F_0, \ldots, F_n) \in \mathbb{Z}[a_{i\alpha}\,;\, i = 0, \ldots, n, |\alpha| = d_i]$$

which verifies:

(i) $\mathrm{Res}_{d_0,\ldots,d_n}(F_0, \ldots, F_n) = 0$ *for a given specialization of the coefficients in* $\mathbf{k}$ *if and only if there exists* $x \in \mathbb{P}^n(\mathbb{K})$ *such that* $F_0(x) = \cdots = F_n(x) = 0$.
(ii) $\mathrm{Res}_{d_0,\ldots,d_n}(x_0^{d_0}, \ldots, x_n^{d_n}) = 1$.

The resultant $\mathrm{Res}_{d_0,\ldots,d_n}$ depends on N variables, where $N = \sum_{i=0}^n \binom{n+d_i}{d_i}$. A geometric proof of this theorem, which is widely generalizable, can be found for instance in [Stu98]. It is based on the consideration of the incidence variety

$$\mathcal{Z} = \{((a_{i\alpha}), x) \in \mathbb{K}^N \times \mathbb{P}^n(\mathbb{K}) : \sum_{|\alpha|=d_i} a_{i\alpha} x^\alpha,\ i = 1, \ldots, n\},$$

and its two projections to $\mathbb{K}^N$ and $\mathbb{P}^n(\mathbb{K})$. In fact, $\mathcal{Z}$ is an irreducible variety of dimension $N-1$ and the fibers of the first projection is generically $1-1$ onto its image.

As we noticed above, in the linear case $d_0 = \cdots = d_n = 1$, the resultant is the determinant of the linear system. We now state the main properties of multivariate homogeneous resultants, which generalize the properties of determinants and of the univariate resultant (or bivariate homogeneous resultant) in Section 1.3.2. The proofs require more background, and we will omit them.

Main properties

i) The resultant $\mathrm{Res}_{d_0,\ldots,d_n}$ is homogeneous in the coefficients of F_i of degree $d_0 \ldots d_{i-1} d_{i+1} \ldots d_n$, i.e. by Bézout's theorem, the number of generic common roots of $F_0 = \cdots = F_{i-1} = F_{i+1} = \cdots = F_n = 0$.
ii) The resultants $\mathrm{Res}_{d_0,\ldots,d_i\ldots,d_j,\ldots,d_n}$ and $\mathrm{Res}_{d_0,\ldots,d_j\ldots,d_i,\ldots,d_n}$ coincide up to sign.
iii) For any monomial x^γ of degree $|\gamma|$ greater than the *critical degree* $\rho := \sum_{i=0}^n (d_i - 1)$, there exist homogeneous polynomials $A_0, \ldots, A_n$ in the variables $x_0, \ldots, x_n$ with coefficients in $\mathbb{Z}[(a_{i\alpha})]$ and $\deg(A_i) = |\gamma| - d_i$, such that

$$\mathrm{Res}_{d_0,\ldots,d_n}(F_0, \ldots, F_n) \cdot x^\gamma = A_0 F_0 + \cdots + A_n F_n. \tag{1.52}$$

Call $f_i(x_1,\dots,x_n) = F_i(1,x_1,\dots,x_n) \in \mathbf{k}[x_1,\dots x_n]$ the dehomogenizations of $F_0,\dots,F_n$. One can define the resultant

$$\mathrm{Res}_{d_0,\dots,d_n}(f_0,\dots,f_n) := \mathrm{Res}_{d_0,\dots,d_n}(F_0,\dots,F_n)$$

and try to translate to the affine setting these properties of the homogeneous resultant. We point out the following direct consequence of (1.52). Taking $\gamma = (\rho+1,0,\dots,0)$ and then specializing $x_0 = 1$, we deduce that there exist polynomials $A_0,\dots,A_n \in \mathbb{Z}[(a_{i\alpha})][x_1,\dots,x_n]$, with $\deg(A_i)$ bounded by $\rho+1-d_i = \sum_{j\neq i} d_i - n$, and such that

$$\mathrm{Res}_{d_0,\dots,d_n}(f_0,\dots,f_n) = A_0 f_0 + \dots + A_n f_n. \tag{1.53}$$

As we remarked in the linear case, the resultant $\mathrm{Res}_{d_0,\dots,d_n}(f_0,\dots,f_n)$ can vanish even if $f_0,\dots,f_n$ do not have any common root in $\mathbb{K}^n$ if their homogenizations $F_0,\dots,F_n$ have a nonzero common root with $x_0 = 0$. Denote by $f_{i,d_i} = F_i(0,x_1,\dots,x_n)$ the homogeneous component of top degree of each f_i. The corresponding version of Proposition 1.3.2 is as follows.

Proposition 1.6.2. (Homogeneous Poisson formula) *Let $F_0,\dots,F_n$ be homogeneous polynomials with degrees $d_0,\dots,d_n$ and let $f_i(x_1,\dots,x_n)$ and $f_{i,d_i}(x_1,\dots,x_n)$ as above. Then*

$$\mathrm{Res}_{d_0,\dots,d_n}(F_0,\dots,F_n)) = \mathrm{Res}_{d_1,\dots,d_n}(f_{1,d_1},\dots,f_{n,d_n})^{d_0} \prod_{\xi\in V} f_0(\xi)^{m_\xi},$$

where V is the common zero set of $f_1,\dots,f_n$, and m_ξ denotes the multiplicity of $\xi \in V$.

This factorization holds in the field of rational functions over the coefficients $(a_{i\alpha})$. Stated differently, the product $\prod_{\xi\in V} f_0(\xi)^{m_\xi}$ is a rational function of the coefficients, whose numerator is the irreducible polynomial $\mathrm{Res}_{d_0,\dots,d_n}(F_0,\dots,F_n)$ and whose denominator is the d_0 power of the irreducible polynomial $\mathrm{Res}_{d_1,\dots,d_n}(f_{1,d_1},\dots,f_{n,d_n})$, which only depends on the coefficients of the monomials of highest degree $d_1,\dots,d_n$ of $f_1,\dots,f_n$. Note that taking $F_0 = x_0$ we get, in particular, the expected formula

$$\mathrm{Res}_{1,d_1,\dots,d_n}(x_0,F_1,\dots,F_n) = \mathrm{Res}_{d_1,\dots,d_n}(f_{1,d_1},\dots,f_{n,d_n}). \tag{1.54}$$

Another direct consequence of Proposition 1.6.2 is the multiplicative property:

$$\mathrm{Res}_{d_0'.d_0'',d_1,\dots,d_n}(F_0'\cdot F_0'',F_1,\dots,F_n) = \tag{1.55}$$
$$\mathrm{Res}_{d_0',d_1,\dots,d_n}(F_0',F_1,\dots,F_n)\cdot \mathrm{Res}_{d_0'',d_1,\dots,d_n}(F_0'',F_1,\dots,F_n),$$

where F_0', F_0'' are homogeneous polynomials of respective degrees d_0', d_0''. More details and applications of the homogeneous resultant to study V and the quotient ring by the ideal $\langle f_1,\dots,f_n\rangle$ can be found in 2, Section 2.3.2.

Some words on the computation of homogeneous resultants

When trying to find explicit formulas for multivariate resultants like the Sylvester or Bézout formulas (1.22) (1.25), one starts searching for maps as (1.21) which are an isomorphism if and only if the resultant does not vanish. But this is possible only in very special cases or low dimensions, and higher linear algebra techniques are needed, in particular the notion of the determinant of a complex [GKZ94]. Given $d_0, \dots, d_n$, the first idea to find a linear map whose determinant equals the resultant $\mathrm{Res}_{d_0,\dots,d_n}(F_0, \dots, F_n)$, is to consider the application

$$\begin{aligned} S_{\rho+1-d_0} \times \cdots \times S_{\rho+1-d_n} &\longrightarrow S_{\rho+1} \\ (G_0, \dots, G_n) &\longmapsto G_0F_0 + \cdots + G_nF_n, \end{aligned} \tag{1.56}$$

where we denote by S_ℓ the space of homogeneous polynomials of degree ℓ and we recall that $\rho + 1 = d_0 + \cdots + d_n - n$.

For any specialization in $\mathbb{K}$ of the coefficients of $F_0, \dots, F_n$ (with respective degrees $d_0, \dots, d_n$), we get a $\mathbb{K}$-linear map between finite dimensional $\mathbb{K}$-vector spaces which is surjective if and only if $F_0, \dots, F_n$ do not have any common root in $\mathbb{K}^{n+1} \setminus \{0\}$. But it is easy to realize that the dimensions are not equal, except if $n = 1$ or $d_0 = \cdots = d_n = 1$. Macaulay [Mac02, Mac94] then proposed a choice of a generically non zero maximal minor of the corresponding rectangular matrix in the standard bases of monomials, which exhibits the multivariate resultant not as a determinant but as a quotient of two determinants. More details on this can be found in Chapters 2 and 3; see also [CLO98].

We now recall the multivariate Bezoutian defined in Section 1.5 (cf. also Chapter 3).

Let $F_0, \dots, F_n$ polynomials of degrees $d_0, \dots, d_n$. Write $x = (x_0, \dots, x_n)$, $y = (y_0, \dots, y_n)$ and let $F_i(x) - F_i(y) = \sum_{j=0}^n F_{ij}(x,y)(x_j - y_j)$, where F_{ij} are homogeneous polynomials in $2(n+1)$ variables of degree $d_i - 1$. The Bezoutian polynomial $\Delta_{\langle F\rangle}$ is defined as the determinant

$$\Delta_{\langle F\rangle}(x,y) = \det((F_{ij}(x,y))) = \sum_{|\alpha| \le \rho} \Delta_\alpha(x) y^\alpha.$$

For instance, we can take as in (1.32)

$$F_{ij}(x,y) = \left(F_i(y_0, \dots, y_{j-1}, x_j, \dots, x_n) - F_i(y_0, \dots, y_j, x_{j+1}, \dots, x_n)\right)/(x_j - y_j).$$

This polynomial is well defined modulo $\langle F_0(x) - F_0(y), \dots, F_n(x) - F_n(y)\rangle$. Note that the sum of the degrees $\deg(\Delta_\alpha) + |\alpha|$ equals the critical degree $\rho = \sum_i (d_i - i)$. In fact, for any specialization of the coefficients in $\mathbf{k}$ such that $R_{d_0,\dots,d_n}(F_0, \dots, F_n)$ is non zero, the specialized polynomials $\{\Delta_\alpha, |\alpha| = m\}$ give a system of generators (over $\mathbf{k}$) of the classes of homogeneous polynomials of degree m in the quotient $\mathbf{k}[x_0, \dots, x_n]/\langle F_0(x), \dots, F_n(x)\rangle$, for any $m \le \rho$.

In particular, according to Theorem 1.5.16, the graded piece of degree ρ of the quotient has dimension one and a basis is given by the coefficient

$$\Delta_0(x) = \Delta_{\langle F\rangle}(x, 0). \tag{1.57}$$

On the other side, by (1.52), any homogeneous polynomial of degree at least $\rho + 1$ lies in the ideal $\langle F_0(x), \ldots, F_n(x)\rangle$.

There is a determinantal formula for the resultant $\mathrm{Res}_{d_0,\ldots,d_n}$ (as the determinant of a matrix involving coefficients of the given polynomials and coefficients of their Bezoutian $\Delta_{\langle F\rangle}$) only when $d_2 + \cdots + d_n < d_0 + d_1 + n$. In general, it is possible to find smaller Macaulay formulas than those arising from (1.56), as the quotient of the determinants of two such explicit matrices (c.f. [Jou97], [WZ94], [DD01]).

Assume for example that $n = 2$, $(d_0, d_1, d_2) = (1, 1, 2)$, and let

$$\begin{aligned} F_0 &= a_0x_0 + a_1x_1 + a_2x_2 \\ F_1 &= b_0x_0 + b_1x_1 + b_2x_2 \\ F_2 &= c_1x_0^2 + c_2x_1^2 + c_3x_2^2 + c_4x_0x_1 + c_5x_0x_2 + c_6x_1x_2 \end{aligned}$$

be generic polynomials of respective degrees $1, 1, 2$. Macaulay's classical matrix looks as follows:

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & c_1 \\ 0 & a_1 & 0 & b_1 & 0 & c_2 \\ 0 & 0 & a_2 & 0 & b_2 & c_3 \\ a_1 & a_0 & 0 & b_0 & 0 & c_4 \\ a_2 & 0 & a_0 & 0 & b_0 & c_5 \\ 0 & a_2 & a_1 & b_2 & b_1 & c_6 \end{pmatrix}$$

and its determinant equals $-a_0\mathrm{Res}_{1,1,2}$. In this case, the extraneous factor a_0 is the 1×1 minor formed by the element in the fourth row, second column. On the other hand, we can exhibit a determinantal formula for $\pm\mathrm{Res}_{1,1,2}$, given by the determinant of

$$\begin{pmatrix} \Delta_{(1,0,0)} & a_0 & b_0 \\ \Delta_{(0,1,0)} & a_1 & b_1 \\ \Delta_{(0,0,1)} & a_2 & b_2 \end{pmatrix},$$

where the coefficients Δ_γ of the Bezoutian $\Delta_{\langle F\rangle}$ are given by

$$\Delta_{(1,0,0)} = c_1(a_1b_2 - a_2b_1) - c_4(a_0b_2 - a_2b_0) + c_5(a_0b_1 - a_1b_0),$$

$$\Delta_{(0,1,0)} = c_6(a_0b_1 - a_1b_0) - c_2(a_0b_2 - b_0a_2)$$

and

$$\Delta_{(0,0,1)} = c_3(a_0b_1 - b_0a_1).$$

In fact, in this case the resultant can be also computed as follows. The generic space of solutions of the linear system $f_0 = f_1 = 0$ is generated by the vector of minors $(a_1b_2 - a_2b_1, -(a_0b_2 - a_2b_0), a_1b_2 - a_2b_1)$. Then

$$\mathrm{Res}_{1,1,2}(F_0, F_1, F_2) = F_2(a_1b_2 - a_2b_1, -(a_0b_2 - a_2b_0), a_1b_2 - a_2b_1).$$

Suppose now that $F_0 = \sum_{i=0}^{n} a_i x_i$ is a linear form. As in expression (1.54) one gets, using the homogeneity of the resultant, that

$$\mathrm{Res}_{1,d_1,\ldots,d_n}(F_0, F_1, \ldots, F_n) = a_0^{d_1\cdots d_n} \mathrm{Res}_{1,d_1,\ldots,d_n}(x_0 + \sum_{i=1}^{n} \frac{a_i}{a_0} x_i, F_1, \ldots, F_n)$$

$$= a_0^{d_1\cdots d_n} \mathrm{Res}_{d_1,\ldots,d_n}(F_1(-\sum_{i=1}^{n} \frac{a_i}{a_0}\, x_i, x_1, \ldots, x_n), \ldots, F_n(-\sum_{i=1}^{n} \frac{a_i}{a_0}\, x_i, x_1, \ldots, x_n)).$$

More generally, let $\ell_0, \ldots, \ell_{r-1}$ be generic linear forms and $F_r, \ldots, F_n$ be homogeneous polynomials of degree $d_r, \ldots, d_n$ on the variables $x_0, \ldots, x_n$. Write $\ell_i = \sum_{j=0}^{n} a_j^i x_j$ and for any subset J of $\{0, \ldots, n\}$, $|J| = r$, denote by δ_J the determinant of the square submatrix $A_J := (a_j^i), j \in J$. Obviously, $\delta_J \in \mathbb{Z}[a_j^i, j \in J]$ vanishes if and only if $\ell_0 = \cdots = \ell_{r-1} = 0$ cannot be parametrized by the variables $(x_j)_{j \notin J}$.

Assume for simplicity that $J = \{0, \ldots, r-1\}$ and let $\delta_J \neq 0$. Left multiplying by the inverse matrix of A_J, the equality $A.x^t = 0$ is equivalent to $x_k =$ k-th coordinate of $-(A_J)^{-1}.(a_j^i)_{j \notin J}(x_r, \ldots, x_n)^t$, for all $k \in J$. Call $F_j^J(x_r, \ldots, x_n), j = r, \ldots, n$, the homogeneous polynomials of degrees $d_r, \ldots, d_n$ respectively gotten from $F_j, j = r, \ldots, n$ after this substitution. Using standard properties of Chow forms (defined below), we then have

Proposition 1.6.3. *Up to sign,*

$$\mathrm{Res}_{1,\ldots,1,d_r,\ldots,d_n}(\ell_0, \ldots, \ell_{r-1}, F_r, \ldots, F_n) = \delta_J^{d_r\cdots d_n} \mathrm{Res}_{d_r,\ldots,d_n}(F_r^J, \ldots, F_n^J).$$

In case $r = n$ we moreover have

$$\mathrm{Res}_{1,\ldots,1,d_n}(\ell_0, \ldots, \ell_{n-1}, F_n) = F_n(\delta_{\{1,\ldots,n\}}, -\delta_{\{0,2,\ldots,n\}}, \ldots, (-1)^n \delta_{\{0,\ldots,n-1\}}).$$

As we have already remarked in the univariate case, resultants can, in principle, be obtained by a Gröbner basis computation using an elimination order. However, this is often not feasible in practice, while using geometric information contained in the system of equations to build the resultant matrices may make it possible to obtain the result. These matrices may easily become huge (c.f. [DD01] for instance), but they are structured. For some recent implementations of resultant computations in `Macaulay2` and Maple, together with examples and applications, we also refer to [Bus03].

The unmixed case

Assume we have an unmixed system, i.e. all degrees are equal. Call $d_0 = \cdots = d_n = d$ and write $F_i(x) = \sum_{|\gamma|=d} a_{i\gamma} x^\gamma$. Then, the coefficients of each Δ_α are linear combinations with integer coefficients of the brackets $[\gamma_0, \ldots, \gamma_n] := \det(a_{i\gamma_j}, i, j = 0, \ldots, n)$, for any subset $\{\gamma_0, \ldots, \gamma_n\}$ of multi-indices of degree

d. In fact, in this equal-degree case, if $F_0, \dots, F_n$ and $G_0, \dots, G_n$ are homogeneous polynomials of degree d, and $G_i = \sum_{j=0}^{n} m_{ij} F_j$, $i = 0, \dots, n$, where $M = (m_{ij}) \in \mathbf{k}^{(n+1)\times(n+1)}$, then,

$$\mathrm{Res}_{d,\dots,d}(G_0, \dots, G_n) = \det(M)^{d^n} \mathrm{Res}_{d,\dots,d}(F_0, \dots, F_n).$$

In particular, the resultant $\mathrm{Res}_{d,\dots,d}$ is invariant under the action of the group $SL(n, \mathbf{k})$ of matrices with determinant 1, and by the Fundamental Theorem of Invariant Theory, there exists a (non unique) polynomial P in the brackets such that $\mathrm{Res}_{d,\dots,d}(F_0, \dots, F_n) = P([\gamma_0, \dots, \gamma_n], |\gamma_i| = d)$. There exists a determinantal formula in terms of the coefficients of the Bezoutian as in (1.24) only if $n = 1$ or $d = 1$. In the "simple" case $n = 2$, $d = 2$, $\mathrm{Res}_{2,2,2}$ is a degree 12 polynomial with more than 20,000 terms in the 18 coefficients of F_0, F_1, F_2, while it has degree 4 in the 20 brackets with considerably fewer terms.

Given a projective variety $X \in \mathbb{P}^N(\mathbb{K})$, of dimension n, and n generic linear forms $\ell_1, \dots, \ell_n$, the intersection $X \cap (\ell_1 = 0) \cap \dots \cap (\ell_n = 0)$ is finite of cardinal equal to the degree of the variety $\deg(X)$. If we take instead $(n+1)$ generic linear forms $\ell_0, \dots, \ell_n$ in $\mathbb{P}^N(\mathbb{K})$, the intersection $X_\ell := X \cap (\ell_0 = 0) \cap \dots \cap (\ell_n = 0)$ is empty. The Chow form $\mathcal{C}_X$ of X is an irreducible polynomial in the coefficients of $\ell_0, \dots, \ell_n$ verifying

$$\mathcal{C}_X(\ell_0, \dots, \ell_n) = 0 \iff X_\ell \neq \emptyset.$$

Consider for example the twisted cubic, i.e the curve V defined as the closure in $\mathbb{P}^3(\mathbb{K})$ of the points parametrized by $(1 : t : t^2 : t^3)$, $t \in \mathbb{K}$. It can also be presented as

$$V = \{(\xi_0 : \xi_1 : \xi_2 : \xi_3) \in \mathbb{P}^3(\mathbb{K}) : \ : \xi_1^2 - \xi_0\xi_2 = \xi_2^2 - \xi_1\xi_3 = \xi_0\xi_3 - \xi_1\xi_2 = 0\}.$$

Given a linear form $\ell_0 = a_0\xi_0 + a_1\xi_1 + a_2\xi_2 + a_3\xi_3$ (resp. $\ell_1 = b_0\xi_0 + b_1\xi_1 + b_2\xi_2 + b_3\xi_3$), a point in V of the form $(1 : t : t^2 : t^3)$ is annihilated by ℓ_0 (resp. ℓ_1) if and only if t is a root of the cubic polynomial $f_0 = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ (resp. $f_1 = b_0 + b_1 t + b_2 t^2 + b_3 t^3$). It follows that

$$\mathcal{C}_V(\ell_0, \ell_1) = \mathrm{Res}_{3,3}(f_0, f_1).$$

In general, given n and d, denote $N = \binom{n+d}{d}$ and consider the Veronese variety $V_{n,d}$ in $\mathbb{P}^{N-1}(\mathbb{K})$ defined as the image of the Veronese map

$$\begin{aligned} \mathbb{P}^n(\mathbb{K}) &\longrightarrow \mathbb{P}^{N-1}(\mathbb{K}) \\ (t_0 : \dots : t_n) &\longmapsto (t^\alpha)_{|\alpha|=d}. \end{aligned}$$

Given coefficients $(a_{i\alpha}, i = 0, \dots, n, |\alpha| = d)$, denote by $\ell_i = \sum_{|\alpha|=d} a_{i\alpha}\xi_\alpha$ and $f_i = \sum_{|\alpha|=d} a_{i\alpha} t^\alpha$, $i = 0, \dots, n$, the corresponding linear forms in the N variables ξ_α and degree d polynomials in the n variables t_i. Then,

$$\mathcal{C}_{V_{n,d}}(\ell_0, \dots, \ell_n) = \mathrm{Res}_{d,\dots,d}(f_0, \dots, f_n).$$

For the use of exterior algebra methods to compute Chow forms, and a fortiori unmixed resultants, we refer to [ESW03].

1.6.2 A glimpse of other multivariate resultants

Resultants behave quite badly with respect to specializations or give no information, and so different notions of resultants tailored for special families of polynomials are needed, together with appropriate different algebraic compactifications.

Suppose we want to define a resultant which describes the existence of a common root of three degree 2 polynomials of the form

$$f_i(x_1, x_2) \;=\; a_i\, x_1 x_2 + b_i\, x_1 + c_i\, x_2 + d_i\,;\; a_i, b_i, c_i, d_i \in \mathbb{K},\; i = 0, 1, 2, \quad (1.58)$$

i.e. ranging in the subvariety of the degree 2 polynomials with zero coefficients in the monomials x_1^2, x_2^2. Note that the homogenized polynomials

$$F_i(x_0, x_1, x_2) == a_i\, x_1 x_2 \,+\, b_i\, x_0\, x_1 \,+\, c_i\, x_0\, x_2 \,+\, d_i\, x_0^2, \quad i = 0, 1, 2,$$

vanish at $(0,1,0)$ and $(0,0,1)$ for any choice of coefficients a_i, b_i, c_i, d_i. Therefore the homogeneous resultant $\mathrm{Res}_{2,2,2}(f_0, f_1, f_2)$ is meaningless because it is identically zero. Nevertheless, the closure in the 12 dimensional parameter space $\mathbb{K}^{12}$ with coordinates $(a_0, \ldots, d_2)$ of the vectors of coefficients for which f_0, f_1, f_2 have a common root in $\mathbb{K}^2$, is an irreducible hypersurface, whose equation is the following polynomial with 66 terms:

$$\begin{aligned}
\mathrm{Res}_{(1,1),(1,1),(1,1)}(f_0, f_1, f_2) \;=\; & -c_2a_0d_1^2a_2b_0 - a_1c_2^2b_0^2d_1 - a_1c_0^2b_2^2d_1 + a_2^2c_1d_0^2b_1 \\
& +2a_0c_1b_2c_2b_1d_0 - a_1c_2b_0c_0b_1d_2 - a_0c_1^2b_2^2d_0 + c_2a_0^2d_1^2b_2 - c_2^2a_0b_1^2d_0 + a_1c_2d_0a_0b_1d_2 \\
& +c_0a_2^2d_1^2b_0 + 2c_0a_2b_1c_1b_0d_2 - 2c_2a_0d_1b_2a_1d_0 + a_2c_1^2b_0b_2d_0 + a_1c_2d_0a_2b_0d_1 + a_1^2c_2d_0^2b_2 \\
& +a_2c_1d_0a_0b_2d_1 - a_2^2c_1d_0b_0d_1 + a_2c_1d_0a_1b_0d_2 - a_2c_1d_0^2b_2a_1 + c_0a_2d_1b_2a_1d_0 - a_1c_2d_0^2b_1a_2 \\
& +c_2a_0d_1b_1a_2d_0 + c_2a_0d_1a_1b_0d_2 - a_1c_0d_2^2a_0b_1 - c_0a_2b_1b_0c_2d_1 - a_2c_1b_0b_2c_0d_1 - c_0^2a_2b_1^2d_2 \\
& -a_1c_2b_0c_1b_2d_0 + c_2^2a_0b_1b_0d_1 + a_1c_2b_0^2c_1d_2 - a_0c_1b_2c_0b_1d_2 + a_0c_1b_2^2c_0d_1 - 2a_1c_0d_2a_2b_0d_1 \\
& +a_1c_0d_2a_0b_2d_1 - c_0a_2d_1^2a_0b_2 - a_0^2c_1d_2b_2d_1 - a_1^2c_2d_0b_0d_2 - 2a_0c_1d_2b_1a_2d_0 + c_0a_2d_1a_0b_1d_2 \\
& -c_0a_2^2d_1b_1d_0 + c_0^2a_2b_1b_2d_1 + a_1c_0^2b_2b_1d_2 + a_0c_1d_2a_2b_0d_1 - a_0c_1d_2^2a_1b_0 + a_2c_1b_0^2c_2d_1 \\
& +c_0a_2b_1^2c_2d_0 + a_1c_0d_2b_1a_2d_0 + a_0c_1d_2b_2a_1d_0 + c_2a_0b_1^2c_0d_2 - c_2a_0b_1b_2c_0d_1 - c_0a_2b_1c_1b_2d_0 \\
& -a_1c_0b_2c_1b_0d_2 + 2a_1c_0b_2b_0c_2d_1 - a_2c_1b_0c_2b_1d_0 - a_1c_0b_2c_2b_1d_0 + a_1c_0b_2^2c_1d_0 + a_0c_1^2b_2b_0d_2 \\
& -a_0c_1b_2b_0c_2d_1 - c_2a_0b_1c_1b_0d_2 - c_2a_0^2d_1b_1d_2 - a_1^2c_0d_2b_2d_0 + a_1^2c_0d_2^2b_0 + a_1c_2^2b_0b_1d_0 \\
& -a_2c_1^2b_0^2d_2 + a_0^2c_1d_2^2b_1. \qquad (1.59)
\end{aligned}$$

This polynomial is called the multihomogeneous resultant (associated to bidegrees $(1,1)$). In Section 1.7 we will describe a method to compute it.

There are also determinantal formulas to compute this resultant, i.e. formulas that present $\mathrm{Res}_{(1,1),(1,1),(1,1)}(f_0, f_1, f_2)$ as the determinant of a matrix whose entries are coefficients of the given polynomials or of an adequate version of their Bezoutian. The smallest such formula gives the resultant as the determinant of a 2×2 matrix, as follows. Given f_0, f_1, f_2, as in (1.58) introduce two new variables y_1, y_2 and let B be the matrix:

$$B = \begin{pmatrix} f_0(x_1,x_2) & f_1(x_1,x_2) & f_2(x_1,x_2) \\ f_0(y_1,x_2) & f_1(y_1,x_2) & f_2(y_1,x_2) \\ f_0(y_1,y_2) & f_1(y_1,x_2) & f_2(y_1,x_2) \end{pmatrix}$$

Compute the Bezoutian polynomial

$$\frac{1}{(x_1-y_1)(x_2-y_2)}\det(B) = B_{11} + B_{12}x_2 + B_{21}y_1 + B_{22}x_2y_1,$$

where the coefficients B_{ij} are homogeneous polynomials of degree 3 in the coefficients $(a_0,\ldots,b_2)$ with tridegree $(1,1,1)$ with respect to the coefficients of f_0, f_1 and f_2. Moreover, they are brackets in the coefficient vectors; for instance, $B_{11} = c_1b_0d_2 - b_0c_2d_1 - c_0b_1d_2 + c_2b_1d_0 + b_2c_0d_1 - c_1b_2d_0$ is the determinant of the matrix with rows $(b_0,c_0,d_0),(b_1,c_1,d_1),(b_2,c_2,d_2)$. Finally,

$$\mathrm{Res}_{(1,1),(1,1),(1,1)}(f_0,f_1,f_2) = \det(B_{ij}).$$

These formulas go back to the pioneering work of Dixon [Dix08]. For a modern account of determinantal formulas for multihomogeneous resultants see [DE03].

Multihomogeneous resultants are special instances of *sparse (or toric) resultants*. We refer to 7 for the computation and applications of sparse resultants. The setting is as follows (cf. [GKZ94, Stu93]). We fix $n+1$ finite subsets $A_0,\ldots,A_n$ of $\mathbb{Z}^n$. To each $\alpha \in \mathbb{Z}^n$ we associate the Laurent monomial $x_1^{\alpha_1}\ldots x_n^{\alpha_n}$ and consider consider

$$f_i = \sum_{\alpha\in A_i} a_{i\alpha}x^\alpha, \quad i = 0,\ldots,n.$$

For instance, one could fix lattice polytopes $P_0,\ldots,P_n$ and take $A_i = P_i \cap \mathbb{Z}^n$. In general A_i is a subset of the lattice points in its convex hull P_i. For generic choices of the coefficients $a_{i\alpha}$, the polynomials $f_0,\ldots,f_n$ have no common root. We consider then, the closure H_A of the set of coefficients for which $f_0,\ldots,f_n$ have a common root in the torus $(\mathbb{K}\backslash\{0\})^n$. If H_A is a hypersurface, it is irreducible, and its defining equation, which has integer coefficients (defined up to sign by the requirement that its content be 1), is called the sparse resultant $\mathrm{Res}_{A_0,\ldots,A_n}$. The hypersurface condition is fulfilled if the family of polytopes $P_0,\ldots,P_n$ is *essential*, i.e. if for any proper subset I of $\{0,\ldots,n\}$, the dimension of the Minkowski sum $\sum_{i\in I} P_i$ is at least $|I|$. In this case, the sparse resultant depends on the coefficients of all the polytopes; this is the case of the homogeneous resultant. When the codimension of H_A is greater than 1, the sparse resultant is defined to be the constant 1. For example, set $n=4$ and consider polynomials of the form

$$\begin{cases} f_0 = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5 \\ f_1 = b_1x_1 + b_2x_2 \\ f_2 = c_1x_1 + c_2x_2 \\ f_3 = b_3x_3 + b_4x_4 \\ f_4 = c_3x_3 + c_4x_4. \end{cases}$$

Then, the existence of a common root in the torus implies the vanishing of both determinants $b_1c_2 - b_2c_1$ and $b_3c_4 - b_4c_3$, i.e. the variety H_A has codimension two. In this case, the sparse resultant is defined to be 1 and it does not vanish for those vectors of coefficients for which there is a common root. Another unexpected example is the following, which corresponds to a non essential family. Set $n = 2$ and let

$$\begin{cases} f_0 = a_1x_1 + a_2x_2 + a_3 \\ f_1 = b_1x_1 + b_2x_2 \\ f_2 = c_1x_1 + c_2x_2. \end{cases}$$

In this case, the sparse resultant equals the determinant $b_1c_2 - b_2c_1$ which does not depend on the coefficients of f_0.

There are also arithmetic issues that come into the picture, as in the following simple example. Set $n = 1$ and consider two univariate polynomials of degree 2 of the form $f_0 = a_0 + b_0x^2$, $f_1 = a_1 + b_1x^2$. In this case, the sparse resultant equals the determinant $D := a_0b_1 - b_0a_1$. But if we think of f_0, f_1 as being degree 2 polynomials with vanishing x-coefficient, and we compute its univariate resultant $\mathrm{Res}_{2,2}(f_0, f_1)$, the answer is D^2. The exponent 2 is precisely the rank of the quotient of the lattice $\mathbb{Z}$ by the lattice $2\mathbb{Z}$ generated by the exponents in f_0, f_1. As in the case of the projective resultant, there is an associated algebraic compactification $X_{A_n,\dots,A_n}$ of the n-torus, called the toric variety associated to the family of supports, which contains $(\mathbb{K} \setminus \{0\})^n$ as a dense open set. For essential families, the sparse resultant vanishes at a vector of coefficients if and only if the closures of the hypersurfaces $(f_i = 0), i = 0, \dots, n$, have a common point of intersection in $X_{A_n,\dots,A_n}$. In the bihomogeneous example (1.58) that we considered, $A_i = \{(0,0), (1,0), (0,1), (1,1)\}$ are the vertices of the unit square in the plane for $i = 0, 1, 2$, and the corresponding toric variety is the product variety $\mathbb{P}^1(\mathbb{K}) \times \mathbb{P}^1(\mathbb{K})$.

Sparse resultants are in turn a special case of *residual resultants*. Roughly speaking, we have families of polynomials which generically have some fixed common points of intersection, and we want to find the condition under which these are the only common roots. Look for instance at the homogeneous case: for any choice of positive degrees $d_0, \dots, d_n$, generic polynomials $F_0, \dots, F_n$ with these degrees will all vanish at the origin $0 \in \mathbb{K}^{n+1}$, and the homogeneous resultant $\mathrm{Res}_{d_0,\dots,d_n}(F_0, \dots, F_n)$ is non zero if and only if the origin is the only common solution. This problem arises naturally when trying to find implicit equations for families of parametric surfaces with base points of codimension greater than 1. We refer to Chapter 3 and to [Bus03, BEM03] for more background and applications.

1.7 Residues and resultants

In this section we would like to discuss some of the connections between residues and resultants. We will also sketch a method, based on residues, to compute multidimensional resultants which, as far as we know, has not been made explicit before.

Suppose $P(z), Q(z)$ are univariate polynomials of respective degrees d_1, d_2 as in (1.19) and let $Z_P = \{\xi_1, \dots, \xi_r\}$ be the zero locus of P. If Q is regular on Z_P, equivalently $\mathrm{Res}_{d_1,d_2}(P,Q) \neq 0$, then the global residue $\mathrm{res}_P(1/Q)$ is defined and the result will be a rational function on the coefficients (a,b) of P and Q. Thus, it is reasonable to expect that the denominator of this rational function (in a minimal expression) will be the resultant. This is the content of the following proposition:

Proposition 1.7.1. *For any* $k = 0, \dots, d_1+d_2-2$, *the residue* $\mathrm{res}_P(z^k/Q)$ *is a rational function of the coefficients* (a,b) *of* P, Q, *and there exists a polynomial* $C_k \in \mathbb{Z}[a,b]$ *such that*

$$\mathrm{res}_P\left(z^k/Q\right) \;=\; \frac{C_k(a,b)}{\mathrm{Res}_{d_1,d_2}(P,Q)}.$$

Proof. We have from (1.26) that

$$1 = \frac{A_1}{\mathrm{Res}_{d_1,d_2}(P,Q)} P + \frac{A_2}{\mathrm{Res}_{d_1,d_2}(P,Q)} Q,$$

with $A_1, A_2 \in \mathbb{Z}[a,b][z]$, $\deg(A_1) = d_2 - 1$, and $\deg(A_2) = d_1 - 1$. Then,

$$\mathrm{res}_P\left(z^k/Q\right) \;=\; \mathrm{res}_P\left(z^k \frac{A_2}{\mathrm{Res}_{d_1,d_2}(P,Q)}\right),$$

and we deduce from Corollary 1.1.7 that there exists a polynomial $C'_k(a,b) \in \mathbb{Z}[a,b][z]$ such that

$$\mathrm{res}_P\left(z^k/Q\right) = \frac{C'_k(a,b)}{\mathrm{Res}_{d_1,d_2}(P,Q)\, a_{d_1}^{k+1}}.$$

Thus, it suffices to show that $a_{d_1}^{k+1}$ divides $C'_k(a,b)$. But, since $k \leq d_1 + d_2 - 2$ we know from (1.11) that

$$\mathrm{res}_P\left(z^k/Q\right) \;=\; -\mathrm{res}_Q\left(z^k/P\right) \;=\; \frac{C''_k(a,b)}{\mathrm{Res}_{d_1,d_2}(P,Q)\, b_{d_2}^{k+1}},$$

for a suitable polynomial $C''_k \in \mathbb{Z}[a,b][z]$. Since $\mathrm{Res}_{d_1,d_2}(P,Q)$ is irreducible, the result follows.

Note that according to Theorem 1.5.17, we have

$$\mathrm{res}^{\mathbb{P}^1}_{\tilde P,\tilde Q}(z^k) \;=\; \mathrm{res}_P\left(z^k/Q\right) \;=\; -\mathrm{res}_Q\left(z^k/P\right),$$

where $\tilde P, \tilde Q$ denote the homogenization of P and Q, respectively. This is the basis for the generalization of Proposition 1.7.1 to the multidimensional case. The following is a special case of [CDS98, Th. 1.4].

Theorem 1.7.2. *Let $F_i(x) = \sum_{|\alpha|=d_i} a_{i\alpha}x^\alpha \in \mathbb{C}[x_0,\ldots,x_n]$, $i = 0,\ldots,n$, be homogeneous polynomials of degrees $d_0,\ldots,d_n$. Then, for any monomial x^β with $|\beta| = \rho = \sum_i(d_i-1)$, the homogeneous residue $\mathrm{res}^{\mathbb{P}^n}_{\langle F\rangle}(x^\beta)$ is a rational function on the coefficients $\{a_{i\alpha}\}$ which can be written as*

$$\mathrm{res}^{\mathbb{P}^n}_{\langle F\rangle}(x^\beta) \;=\; \frac{C_\beta(a_{i\alpha})}{\mathrm{Res}_{d_0,\ldots,d_n}(F_0,\ldots,F_n)}$$

for a suitable polynomial $C_\beta \in \mathbb{Z}[a_{i\alpha}]$.

We sketch a proof of this result, based on [Jou97, CDS98] and the notion of the determinant of a complex [GKZ94].

Proof. We retrieve the notations in (1.56), but we consider now the application "at level ρ"

$$\begin{array}{ccc} S_{\rho-d_0}\times\cdots\times S_{\rho-d_n}\times S_0 & \longrightarrow & S_\rho \\ (G_0,\ldots,G_n,\lambda) & \longmapsto & G_0F_0+\cdots+G_nF_n+\lambda\Delta_0, \end{array} \tag{1.60}$$

where Δ_0 is defined in (1.57). For any specialization in $\mathbb{K}$ of the coefficients of $F_0,\ldots,F_n$ (with respective degrees $d_0,\ldots,d_n$), we get a $\mathbb{K}$-linear map between finite dimensional $\mathbb{K}$-vector spaces which is surjective if and only if $F_0,\ldots,F_n$ do not have a common root in $\mathbb{K}^{n+1}\setminus\{0\}$, or equivalently, if and only if the resultant $\mathrm{Res}_{d_0,\ldots,d_n}(F_0,\ldots,F_n)$ is non zero. Moreover, it holds that the resultant equals the greatest common divisor of all maximal minors of the above map. Let $\mathcal{U}$ be the intersection of Zariski open set in the space of coefficients $a = (a_{i\alpha})$ of the given polynomials where all (non identically zero) maximal minors do not vanish. For $a \in \mathcal{U}$, the specialized $\mathbb{K}$-linear map is surjective and for any monomial x^β of degree ρ we can write

$$x^\beta \;=\; \sum_{i=0}^{n} A_i(a;x)\,F_i(a;x) \;+\; \lambda(a)\,\Delta_0(a;x)\,,$$

where λ depends rationally on a. Since the residue vanishes on the first sum and takes the value 1 on Δ_0, we have that

$$\mathrm{res}^{\mathbb{P}^n}_{\langle F\rangle}(x^\beta) \;=\; \lambda(a)\,,$$

This implies that every maximal minor which is not identically zero must involve the last column and that $\lambda(a)$ is unique. Thus, it follows from Cramer's rule that $\mathrm{res}^{\mathbb{P}^n}_{\langle F\rangle}(x^\beta)$ may be written as a rational function with denominator M for all non-identically zero maximal minors M. Consequently it may also be written as a rational function with denominator $\mathrm{Res}_{d_0,\ldots,d_n}(F_0,\ldots,F_n)$.

In fact, (1.60) can be extended to a generically exact complex

$$0 \to S_{d_0-(n+1)} \times \cdots \times S_{d_n-(n+1)} \to \cdots \to S_{\rho-d_0} \times \cdots \times S_{\rho-d_n} \times S_0 \to S_\rho \to 0,$$

which is a graded piece of the Koszul complex associated to $F_0, \ldots, F_n$, which is exact if and only if $\mathrm{Res}_{d_0,\ldots,d_n}(F_0, \ldots, F_n) \neq 0$. Moreover, the resultant equals (once we index appropriately the terms and choose monomial bases for them) the determinant of the complex. This concept goes back to Cayley [Cay48] and generalizes the determinant of a linear map between two vector spaces of the same dimension with chosen bases. For short exact sequences of finitely dimensional vector spaces V_{-1}, V_0, V_1 with respective chosen bases, the determinant of the based complex is defined as follows [GKZ94, Appendix A]. Call d_{-1} and d_0 the linear maps

$$0 \longrightarrow V_{-1} \xrightarrow{d_{-1}} V_0 \xrightarrow{d_0} V_1 \longrightarrow 0\,,$$

and let $\ell_i = \dim V_i$, $i = -1, 0, 1$. Thus, $\ell_0 = \ell_{-1} + \ell_1$. Denote by M_{-1} and M_0 the respective matrices of d_{-1} and d_0 in the chosen bases. Choose any subset I of $\{0, \ldots, \ell_0\}$ with $|I| = \ell_{-1}$ and let M_{-1}^I be the submatrix of M_{-1} given by all the ℓ_{-1} rows and the ℓ_{-1} columns corresponding to the index set I. Similarly, denote by M_0^I the submatrix of M_0 given by the ℓ_1 rows indexed by the complement of I and all the ℓ_1 columns. Then, it can be easily checked that $\det(M_{-1}^I) \neq 0 \iff \det(M_0^I) \neq 0$. Moreover, up to (an explicit) sign, it holds that whenever they are non zero, the quotient of determinants

$$\frac{\det(M_{-1}^I)}{\det(M_0^I)}$$

is independent of the choice of I. The determinant of the based complex is then defined to be this common value. In the case of the complex given by a graded piece of the Koszul complex we are considering, the hypotheses of [GKZ94, Appendix A, Th. 34] are fulfilled, and its determinant equals the greatest common divisor of the rightmost map (1.60) we considered in the proof of Theorem 1.7.2.

We recall that, by b) in Theorem 1.5.16, the graded piece of degree ρ in the graded algebra $\mathcal{A} = \mathbb{C}[x_0, \ldots, x_n]/\langle F_0, \ldots, F_n \rangle$, is one-dimensional. We can exploit this fact together with the relation between residues and resultants to propose a new algorithm for the computation of resultants. Given a term order $\prec$, there will be a unique standard monomial of degree ρ, the smallest monomial x^{β_0}, relative to $\prec$, not in the ideal $\langle F_0, \ldots, F_n \rangle$. Consequently, for any $H \in \mathbb{C}[x_0, \ldots, x_n]_\rho$, its normal form $N(H)$ relative to the reduced Gröbner basis for $\prec$, will be a multiple of x^{β_0}.

In particular, let $\Delta \in \mathbb{C}[x_0, \ldots, x_n]$ be the element of degree ρ and homogeneous residue 1 constructed in Section 1.5.6. We can write

$$N(\Delta) \;=\; \frac{P(a_{i\alpha})}{Q(a_{i\alpha})} \cdot x^{\beta_0}\,.$$

Theorem 1.7.3. *With notation as above, if $P(a_{i\alpha})$, and $Q(a_{i\alpha})$ are relatively prime*

$$\mathrm{Res}_{d_0,\dots,d_n}(F_0,\dots,F_n) \;=\; P(a_{i\alpha}).$$

Proof. We have:

$$1 \;=\; \mathrm{res}^{\mathbb{P}^n}_{\langle F\rangle}(\Delta) \;=\; \mathrm{res}^{\mathbb{P}^n}_{\langle F\rangle}\left(\frac{P(a_{i\alpha})}{Q(a_{i\alpha})}\cdot x^{\beta_0}\right) \;=\; \frac{P(a_{i\alpha})}{Q(a_{i\alpha})}\,\frac{C_{\beta_0}(a_{i\alpha})}{\mathrm{Res}_{d_0,\dots,d_n}(F_0,\dots,F_n)}.$$

Therefore

$$\mathrm{Res}_{d_0,\dots,d_n}(F_0,\dots,F_n)Q(a_{i\alpha}) \;=\; P(a_{i\alpha})C_{\beta_0}(a_{i\alpha}),$$

but since $\mathrm{Res}_{d_0,\dots,d_n}(F_0,\dots,F_n)$ is irreducible and coprime with $C_{\beta_0}(a_{i\alpha})$ this implies the assertion.

Remark 1.7.4. Note that Theorem 1.7.3 holds even if the polynomials F_i are not densely supported as long as the resultant $\mathrm{Res}_{d_0,\dots,d_n}(F_0,\dots,F_n)$ is not identically zero.

Consider the example from Section 1.6.1:

$$\begin{aligned} F_0 &= a_0x_0 + a_1x_1 + a_2x_2 \\ F_1 &= b_0x_0 + b_1x_1 + b_2x_2 \\ F_2 &= c_1x_0^2 + c_2x_1^2 + c_3x_2^2 + c_4x_0x_1 + c_5x_0x_2 + c_6x_1x_2 \end{aligned}$$

Then $\rho = 1$ and

$$\Delta \;=\; \det\begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_1x_0 + c_4x_1 + c_5x_2 & c_2x_1 + c_6x_2 & c_3x_2 \end{pmatrix}.$$

We can now read off the resultant $\mathrm{Res}_{1,1,2}(F_0,F_1,F_2)$ from the normal form of Δ relative to any Gröbner basis of $I = \langle F_0,F_1,F_2\rangle$. For example computing relative to grevlex with $x_0 > x_1 > x_2$, we have:

$$\begin{aligned} N(\Delta) = \big(&(a_0^2b_1^2c_3 - a_0^2b_1b_2c_6 + a_0^2b_2^2c_2 + a_0a_1b_0b_2c_6 - a_0a_2b_1^2c_5 + \\ &a_0a_1b_1b_2c_5 - a_0a_1b_2^2c_4 + a_0a_2b_0b_1c_6 - a_0a_2b_1b_2c_4 - 2a_0a_1b_0b_1c_3 + a_1^2b_0^2c_3 - \\ &a_1^2b_0b_2c_5 + a_1^2b_2^2c_1 - a_1a_2b_0^2c_6 + a_1a_2b_0b_1c_5 + a_1a_2b_0b_2c_4 + 2a_0a_2b_0b_2c_2 - \\ &2a_1a_2b_1b_2c_1 + a_2^2b_0^2c_2 - a_2^2b_0b_1c_4 + a_2^2b_1^2c_1)/(a_0b_1 - a_1b_0)\big)x_2 \end{aligned}$$

and the numerator of the coefficient of x_2 in this expression is the resultant. Its denominator is the *s*ubresultant polynomial in the sense of [Cha95], whose vanishing is equivalent to the condition $x_2 \in I$

Theorem 1.7.3 is a special case of a more general result which holds in the context of toric varieties [CD]. We will not delve into this general setup here but will conclude this section by illustrating this computational method in the case of the sparse polynomials described in (1.58). As noted in Section 1.6.2, the homogeneous resultant of these three polynomials is identically zero. We may however view them as three polynomials with support in the unit square $\mathcal{P} \subset \mathbb{R}^2$ and consider their homogenization relative to $\mathcal{P}$. This is equivalent to compactifying the torus $(\mathbb{C}^*)^2$ as $\mathbb{P}^1 \times \mathbb{P}^1$ and considering the natural homogenizations of our polynomials in the homogeneous coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$, i.e. the ring of polynomials $\mathbb{C}[x_1, y_1, x_2, y_2]$ bigraded by $(\deg_{x_1,y_1}, \deg_{x_2,y_2})$. We have:

$$F_i(x_1, x_2, y_1, y_2) = a_i x_1 x_2 + b_i x_1 y_2 + c_i x_2 y_1 + d_i y_1 y_2, \quad a_i, b_i, c_i, d_i \in \mathbb{K}.$$

These polynomials have the property that

$$F_i(\lambda_1 x_1, \lambda_1 y_1, \lambda_2 x_2, \lambda_2 y_2) = \lambda_1\, \lambda_2\, F_i(x_1, x_2, y_1, y_2),$$

for all non zero λ_1, λ_2.

Notice that $\langle F_0, F_1, F_2\rangle \subset \langle x_1, x_2, y_1 y_2\rangle$ and we can take as Δ the determinant of any matrix that expresses the F_j in terms of those monomials. For example

$$\Delta \;=\; \det \begin{pmatrix} a_0 x_2 + b_0 y_2 & c_0 y_1 & d_0 \\ a_1 x_2 + b_1 y_2 & c_1 y_1 & d_1 \\ a_2 x_2 + b_2 y_2 & c_2 y_1 & d_2 \end{pmatrix}$$

We point out that in this case $\rho = (1,1) = 3(1,1) - (2,2)$, which is the bidegree of Δ. If we consider for instance the reverse lexicographic term order with $y_2 \prec y_1 \prec x_2 \prec x_1$, the least monomial of degree ρ is $y_1 y_2$. The normal form of Δ modulo a Gröbner basis of the bi-homogeneous ideal $\langle F_0, F_1, F_2\rangle$ equals a coefficient times $y_1 y_2$. This coefficient is a rational function of $(a_0, \ldots, d_2)$ whose numerator is the $\mathbb{P}^1 \times \mathbb{P}^1$ resultant of F_0, F_1, F_2 displayed in (1.59). We invite the reader to check that its denominator equals the determinant of the 3×3 square submatrix of the matrix of coefficients of the given polynomials

$$\begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}.$$

Again, this is precisely the subresultant polynomial whose vanishing is equivalent to $y_1 y_2 \in \langle F_0, F_1, F_2\rangle$ (c.f. also [DK]).

As a final remark, we mention briefly the relation between residues, resultants and rational A-hypergeometric functions in the sense of Gel'fand, Kapranov and Zelevinsky [GZK89]. Recall that given a configuration

$$A \;=\; \{a_1, \ldots, a_n\} \subset \mathbb{Z}^p$$

or, equivalently an integral $p \times n$ matrix A, a function F, holomorphic in an open set $\mathcal{U} \subset \mathbb{C}^n$, is said to be A-hypergeometric of degree $\beta \in \mathbb{C}^p$ if and only if it satisfies the differential equations:

$$\partial^u F - \partial^v F = 0\,,$$

for all $u, v \in \mathbb{N}^n$ such that $A \cdot u = A \cdot v$, where $\partial^u = \dfrac{\partial^{|u|}}{\partial z_1^{u_1} \dots z_n^{u_n}}$, and

$$\sum_{j=1}^{n} a_{ij} z_j \frac{\partial F}{\partial z_j} = \beta_i F$$

for all $i = 1, \dots, p$. The study of A-hypergeometric functions is a very active area of current research with many connections to computational and commutative algebra. We refer the reader to [SST00] for a comprehensive introduction and restrict ourselves to the discussion of a simple example.

Let $\Sigma(d)$ denote the set of integer points in the m-simplex

$$\{u \in \mathbb{R}^m_{\geq 0} \colon \sum_{j=1}^{m} u_j \leq d\}\,.$$

Let $A \subset \mathbb{Z}^{2m+1}$ be the Cayley configuration

$$A = (\{e_0\} \times \Sigma(d)) \cup \dots \cup (\{e_m\} \times \Sigma(d)).$$

Let $f_i(t) = \sum_{\alpha \in \Sigma(d)} z_{i\alpha} t^\alpha$, $i = 0, \dots, d$ be an $m+1$-tuple of generic polynomials supported in $\Sigma(d)$. Denote by $F_i(x_0, \dots, x_d)$ the homogenization of f_i. Given an $m+1$-tuple of positive integers $a = (a_0, \dots, a_m)$ let $\langle F^a \rangle$ be the collection $\langle F_0^{a_0}, \dots, F_m^{a_m} \rangle$. The following result is a special case of a more general result (see [AS96, CD97, CDS01]) involving the Cayley product of a general family of configurations $A_i \subset \mathbb{Z}^m$, $i = 0, \dots, m$.

Theorem 1.7.5. *For any $b \in \mathbb{N}^{m+1}$ with $|b| = d|a| - (n+1)$, the homogeneous residue $\mathrm{res}^{\mathbb{P}^2}_{\langle F^a \rangle}(x^b)$, viewed as a function of the coefficients $x_{i\alpha}$, is a rational A-hypergeometric function of degree $\beta = (-a_0, \dots, -a_m, -b_1 - 1, \dots, -b_m - 1)$.*

Suppose, for example, that $m = 2$ and $d = 1$. Then, we have

$$A = \begin{pmatrix} 1\,1\,1\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,1\,1\,1\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,1\,1\,1 \\ 0\,1\,0\,0\,1\,0\,0\,1\,0 \\ 0\,0\,1\,0\,0\,1\,0\,0\,1 \end{pmatrix}$$

and $F_i(x_0, x_1, x_2) = a_{i0}x_0 + a_{i1}x_1 + a_{i2}x_2$. Let $a = (2, 1, 1)$ and $b = (0, 1, 0)$. Then the residue $\mathrm{res}^{\mathbb{P}^2}_{\langle F^a \rangle}(x_1)$ might be computed using Algorithm 2 in Section 1.5.6 to obtain the following rational function

$$(a_{20}a_{12} - a_{10}a_{22}) / \det(a_{ij})^2.$$

Note that, according to Theorem 1.7.2 and (1.55), the denominator of the above expression is the homogeneous resultant

$$\mathrm{Res}_{2,1,1}(F_0^2, F_1, F_2) = \mathrm{Res}_{1,1,1}(F_0, F_1, F_2)^2 \,.$$

Indeed, as

$$\frac{x_1}{F_0^2 F_1 F_2} = -\frac{\partial}{\partial a_{01}} \left(\frac{1}{F_0 F_1, F_2} \right),$$

differentiation “under the integral sign” gives the equality

$$\mathrm{res}^{\mathbb{P}^2}_{\langle F^a \rangle}(x_1) = -\frac{\partial}{\partial a_{01}} \left(\frac{1}{\det(a_{ij})} \right).$$

One can also show that the determinant $\det(a_{ij})$ agrees with the discriminant of the configuration A. We should point out that Gel’fand, Kapranov and Zelevinsky have shown that the irreducible components of the singular locus of the A-hypergeometric system for any degree β have as defining equations the discriminant of A and of its facial subsets, which in this case correspond to all minors of (a_{ij}) .

In [CDS01] it is conjectured that essentially all rational A-hypergeometric functions whose denominators are a multiple of the A-discriminant arise as the toric residues of Cayley configurations. We refer to [CDS02, CD04] for further discussion of this conjecture.

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