GRACE Gradiometer

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Abstract. Improving the accuracy of the spherical harmonic coefficients of the Earth's gravity field and its temporal variations at long and medium spatial-scales with unprecedented accuracy is the primary science objective of the GRACE mission. The line of sight (LOS) acceleration difference between the satellite pair is the most frequently utilized form of the observable. It is the simplest form of the observable which can be easily employed. Nevertheless, the observable is a two-point function and has no direct relationship with the field geometry at the evaluation point.

In this paper, as the alternative, *gradiometry approach* is proposed. Being a one-point function and having a direct relation with the field geometry (curvature of the field at the point) are two noteworthy achievements of the alternative formulation. Besides, using an observation quantity that is related to the second instead of the first-order derivatives of the gravitational potential amplifies the high-frequency part of the signal.

Complexity of the derived mathematical model and its proper treatment is the severe problem for the gradiometry approach. Herein, mixed gravitational acceleration-gradient model and also use of the available Earth' gravity model as a priori information on the low-degree harmonics are addressed.

The first recently released EIGEN2 CHAMP-only Earth's gravity model was employed for numerical analysis. Error analysis showed that the residuals of the estimated degree variances were of about 10^{-4} for $n \leq 90$. Also, the gravity anomaly residuals were less than 5 mGal for most points on the Earth.

Keywords. GRACE Gradiometer, Gradiometry, Sequential solution

1 Introduction

The line of sight (LOS) acceleration difference between a satellite pair has been frequently used for mapping the field globally (e.g. Hajela , 1974; Rummel , 1980; Garcia , 2002; Han et al. , 2003). The idea can also be applied to the GRACE observable as the first realization of the LL-SST mode.

Moreover, the GRACE configuration can be viewed as a huge one-component gradiometer with an arm length of 250 km. Rummel (2003) showed that the accuracy of this virtual one-dimensional gradiometer is about $10^{-6}\,\mathrm{E}/\sqrt{\mathrm{Hz}}$. Consequently, the GRACE configuration can be considered as potentially a precise one-dimensional virtual gradiometer. The advantage of looking GRACE observations as gradiometer data over looking at them as satellite-to satellite tracking data is that in the gradiometry mode the gravity field recovery can be done in a space-wise approach. The space-wise approach leads to an inversion-free recovery algorithm, which makes all the measures obsolete, which have to be taken to stabilize the linear system of equations in the satellite to satellite mode.

The paper starts with the mathematical description of gravitational field recovery by satellite-to-satellite tracking in the low-low mode as it has already been worked out in the contribution Rummel et al. (1978) and Rummel (1980) . This traditional approach relates the observation to a two-point function of the potential namely the gravitational acceleration difference of the satellites, projected onto the inter-satellite unit vector. The gravitational acceleration is proportional to the first-order derivative of the potential. For a high-resolution determination of the gravitational potential, an observation would be desirable which is a one-point function and relates to higher order derivatives of the potential, since for a one-point functional of the gravity field recovery in the space-wise model becomes possible.

Hence, we consider the GRACE mission as a one-axis gradiometer and formulate the problem in terms of the gravitational acceleration tensor components. Besides some remainder terms which can be modelled with sufficient accuracy, the gravitational acceleration gradient is related to the second order derivatives of the gravitational potential at the mid-point of the satellites configuration. In this way, the two-point first order problem is replaced by a one-point second order problem, promising a higher resolution of gravitational potential recovery.

Due to the complexity, the developed mathematical model can not be fully coded. Therefore, for ease of computation, we have to approximate the model with numerically applicable forms. Including the higher order terms beyond the linear term of the Taylor expansion makes the numerical computation very complicated. From the numerical point of view, the only possible form would be the linear approximation of the equation. On the other hand, excluding the higher order terms of the expansion results in truncation error whose contribution to the observation equation is considerable. Hence, using either a higher order approximation of the equation or modified linear approximation is inevitable.

Herein, we introduce two modified gradiometry algorithms which result in simple practical mathematical models. Mixed gravitational acceleration-gradient model and use of the available Earth' gravity model as a priori information on the low-degree harmonics are formulated. In both cases, increment to the low-degree harmonics and the higher degree coefficients are simultaneously estimated.

The final part of the paper is dedicated to recovery of the gravitational field and the analysis of the results. As a priori information, we will employ the first recently released EIGEN2 CHAMP-only Earth's gravity model to show the practical performance of the modified formulation. The article will end in some conclusions and recommendations.

2 Mathematical Formulation

The key observables of the GRACE mission are the inter-satellite distance ρ, and its first and second time derivatives $\dot{\rho}$ and $\ddot{\rho}$. These principal scalar quantities measured by the K-band Ranging system (KBR) are considered as the LL-SST information. They can be related to the Earth's gravitational field if the absolute positions of the spacecrafts, i.e. $\mathbf{r}_1$ and $\mathbf{r}_2$ are known. Therefore, the two satellites have been equipped with dual-frequency Blackjack GPS receivers to provide the HL-SST information. Eq. (1) connects the LL-SST observations to those of the HL-SST at each evaluation point (Rummel et al. , 1978):

$$\Delta\ddot{\mathbf{r}} \cdot \mathbf{e} = \ddot{\rho} + \rho^{-1}(\dot{\rho}^2 - \|\Delta\dot{\mathbf{r}}\|^2) \tag{1}$$

where, $\mathbf{e} = \rho^{-1}(\mathbf{r}_2 - \mathbf{r}_1)$ is the unit vector along the LOS. $\Delta\dot{\mathbf{r}} = \dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_1$ and $\Delta\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1$ are the difference of the velocities and accelerations of the two satellites here expressed in an inertial frame. We consider Eq. (1) as the basic equation of the $HL{+}LL$ combination observable and modify it in each case accordingly.

2.1 Gravitational Acceleration Difference

In the absence of non-gravitational forces, the left-hand-side of Eq. (1) can be considered as the Earth's gravitational acceleration difference along LOS, $\Delta\Gamma^{\mathrm{LOS}}$.

$$\Delta\Gamma^{\mathrm{LOS}} = \ddot{\rho} + \frac{\dot{\rho}^2}{\rho} - \frac{\|\Delta\dot{\mathbf{r}}\|^2}{\rho} \tag{2}$$

The left-hand-side of Eq. (2) is a function of the gravitational potential partial derivatives and the inter-satellite unit vector $\mathbf{e}$, whereas the right-hand-side is the observational quantity. Using a sequence with an adequate number of observations, we set up the linear system of observation equations and recover the spherical harmonic coefficients. More details can be found in Keller and Sharifi (2004) and the references cited therein.

2.2 Gravitational Acceleration Gradient

As mentioned earlier, combining the two SST concepts, as shown in Fig. 1, makes the twin satellites to appear as a very accurate one-component gradiometer. Rummel (2003) showed that the accuracy of this virtual one-dimensional gradiometer is about 10^{-6} E/$\sqrt{\mathrm{Hz}}$. This unique characteristic of GRACE is a motivation to switch from the first derivatives of the gravitational potential to the second derivatives of the field. In other words, we write the observation equation (Eq. 2) as a function of the gravitational acceleration gradient components instead of the gravitational potential gradient.

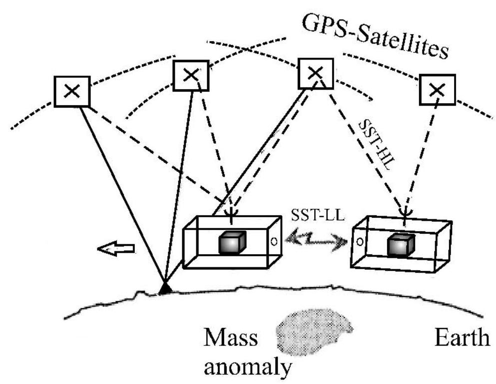

Fig. 1. Gradiometry with the GRACE twin satellites (from Rummel et al. (2002))

To derive the respective mathematical formulae, we expand the gravitational acceleration at the two satellites' respective positions around the mid-point using Taylor expansion. Subtracting the resultant expression yields (Keller and Sharifi , 2004):

$$\Delta\Gamma = \sum_{j=1:2:\infty} \frac{2^{1-j}}{j!}(\nabla^j \otimes \Gamma^T_{\mathrm{mid}}) \cdot \Delta\mathbf{r}^j, \tag{3}$$

where, $\otimes$ is Kronecker product symbol and Γ_{mid} is the gravitational acceleration at the mid-point of the satellites configuration. The left-hand-side of Eq. (3) is a two-point first order quantity, whereas the right-hand-side is a one-point higher order (at least second order) one. Consequently, inserting Eq. (3) into Eq. (2) results in the sought-after formulation.

Obviously, the expansion (Eq. 3) contains partial derivatives of the Earth's gravitational potential higher than the second order. Including the partial derivatives beyond the linear term makes the mathematical model rather complicated. Therefore, we will consider the linear term of the expansion and modify the equations to minimize the linearization error. Herein, we also assume the Earth's gravitational force as the only governing force field.

2.2.1 Linear Approximation

Assuming $j = 1$ in Eq. (3) dismisses the summation out and makes the equation as simple as possible:

$$\Delta\Gamma \doteq (\nabla \otimes \Gamma^T) \cdot \Delta\mathbf{r} = \mathbf{G}\Delta\mathbf{r}, \tag{4}$$

where, $\mathbf{G}$ is the Earth's gravitational gradient tensor. Inserting Eq. (4) into Eq. (2) and dividing both sides of the equation by ρ results in Eq. (5), which is called *linear gradiometry equation*:

$$\mathbf{e}^T\mathbf{G}\mathbf{e} \doteq \frac{\ddot{\rho}}{\rho} + \frac{\dot{\rho}^2}{\rho^2} - \frac{\|\Delta\dot{\mathbf{r}}\|^2}{\rho^2} \tag{5}$$

Right-hand-side of Eq. (5) is a linear function of the Earth's gravitational gradient tensor elements. Comparing Eqs. (2) and (5), leads to a criteria for evaluation of the linearization error:

$$|\mathbf{e}^T\mathbf{G}\mathbf{e} - \frac{1}{\rho}\Delta\Gamma^{\text{LOS}}| \leq \epsilon \tag{6}$$

The linear approximation is valid as long as the linearization error is negligible. Otherwise, the error degrades the model and the linear approximation of the grtadiometry equation will collapse.

Keller and Sharifi (2004) investigated the linearization error for the GRACE configuration and showed that the error is at the level of $0.55\,\text{E}(1\,\text{E} = 1\,\textit{Eötvös Unit} = 10^{-9}\,\text{s}^{-2})$, which can not be neglected. Thus, we should either include at least the cubic term or modify the model to lower the linearization error. Due to the complexity of the cubic approximation, we prefer to retain the linear approximation and apply the remove-restore technique to reduce influence of the cubic term.

2.2.2 Mixed Mathematical Model

The largest orbit perturbation for all satellite orbits is the so-called J_2-effect caused by the flattening of the Earth; and the effect of the next three zonal harmonics in the expansion of the Earth's gravitational field is about two orders of magnitude smaller than the perturbation from J_2.

Keller and Heß (1998) and Keller and Sharifi (2004) showed that the linearization error reduced to few ten mE by introducing an ellipsoidal reference field. Nevertheless, the estimation process leads to unacceptable solution at the presence of the linearization error residual corresponding to the gravitational disturbing potential. Thus, instead of introducing an ellipsoidal reference field, we split the gravitational potential (V) into a low-degree spheroidal reference field (V_l) and an incremental one (V^l). Accordingly, we consider observation equation of Eqs. (2) and (5) types for the first l and the higher degree ($> l$) terms of the gravitational potential harmonic expansion respectively. Therefore, Eq. (5) can be recast into:

$$\frac{1}{\rho}\Delta\Gamma_l^{\text{LOS}} + \mathbf{e}^T \cdot \mathbf{G}^l \cdot \mathbf{e} \doteq \frac{\ddot{\rho}}{\rho} + \frac{\dot{\rho}^2}{\rho^2} - \frac{\|\Delta\dot{\mathbf{r}}\|^2}{\rho^2} \tag{7}$$

where, $\mathbf{G}^l$ stands for the gradient tensor corresponding to the higher degree harmonics of the gravitational potential expansion. Analogously, we redefine the linearization error criteria:

$$|\mathbf{e}^T\mathbf{G}^l\mathbf{e} - \frac{1}{\rho}\Delta\Gamma^{l\,\text{LOS}}| \leq \epsilon. \tag{8}$$

An appropriate choice of l leads to some negligible linearization error residual. For instance, for $l = 10$, it is at the level of few mE (Keller and Sharifi , 2004). Consequently, all the spherical harmonic coefficients are estimated all together in a linear system of equations with reasonable accuracy.

Compared with Eq. (5), Eq. (7) contains less systematic error. In contrast, it is partially a two-point first order problem. In the following subsection, we improve this deficiency by introducing the sequential solution.

2.2.3 Sequential Estimation

Up to now, different global gravity models of the Earth have been released to public and many more may be developed later on. Combining the existing models with any new set of observations, carried out on the Earth's gravity field, is of particular interest to geoscientists. In other words, hybrid solution would be without doubt one of the most interesting challenges of the coming years.

As already discussed, the linearization error is the greatest single obstacle to the linear gradiometry equation. On the other hand, the low-degree harmonics' contribution is the most dominant one. Therefore, we consider one of the available Earth's gravity models and utilize the low-degree coefficients of the model as a priori information. Accordingly, any quantity corresponding to the low-degree harmonics can be split into the approximate value plus the respective correction. The first term on the left-hand-side of Eq. (7), for instance, can be written as:

$$\frac{1}{\rho}\Delta\Gamma_l^{\mathrm{LOS}} = \underbrace{\frac{1}{\rho}\Delta\Gamma_{0l}^{\mathrm{LOS}}}_{\text{approximate value}} + \underbrace{\frac{1}{\rho}\delta\Delta\Gamma_l^{\mathrm{LOS}}}_{\text{correction}}. \quad (9)$$

Replacing the correction term by the corresponding expression of the gradiometry type yields:

$$\frac{1}{\rho}\Delta\Gamma_l^{\mathrm{LOS}} = \mathbf{e}^T\delta\mathbf{G}_l\mathbf{e} + \frac{1}{\rho}\Delta\Gamma_{0l}^{\mathrm{LOS}}. \quad (10)$$

Inserting Eq. (10) into Eq. (7) results in Eq. (11), which is called *sequential gradiometry observation equation*:

$$\mathbf{e}^T(\delta\mathbf{G}_l + \mathbf{G}^l)\mathbf{e} \doteq \frac{\ddot{\rho}}{\rho} + \frac{\dot{\rho}^2}{\rho^2} - \frac{\|\Delta\dot{\mathbf{r}}\|^2}{\rho^2} - \frac{1}{\rho}\Delta\Gamma_{0l}^{\mathrm{LOS}} \quad (11)$$

Eq. (11) results from application of standard sequential adjustment to Eq. (7). The linearization error criteria is modified as:

$$|\mathbf{e}^T(\delta\mathbf{G}_l+\mathbf{G}^l)\mathbf{e} - \frac{1}{\rho}(\Delta\Gamma^{\mathrm{LOS}} - \Delta\Gamma_{0l}^{\mathrm{LOS}})| \leq \epsilon. \quad (12)$$

We implemented the idea based on some simulated data and the achieved results will be presented in section. (3). So, in brief:

- the two-point first order problem is replaced by a one-point second order problem,
- the linearization error reduces to an acceptable level,
- both the low- and the high-degree harmonic coefficients are estimated.

Eventually, It should be noted that the first two terms of the right-hand-side of Eqs. (2), (5), (7) and (11) are computed by means of the GRACE ranging data. For the third term, GPS and Doppler observations have to be used. The modelling of this term from GPS and Doppler observations was investigated by Keller and Heß (1998). They showed that this term can be modelled with an accuracy of about $10^{-13}\,\mathrm{s}^{-2}$ under realistic assumptions, which is sufficient for the purpose of the presented study.

3 Numerical Analysis

Numerical studies are based on the IAG simulated data of the Earth's gravity field dedicated satellite missions (Ilk et al. , 2003). As the pseudo-real gravity field of the Earth, *EGM96* (Lemoine et al. , 1998) complete to degree 300, has been considered. Moreover, we utilize EIGEN2 *CHAMP*-only (Reigber et al. , 2003) as a priori information.
First, the sequential gradiometry equation's truncation error and random error of the approximate value are evaluated. Finally, we will recover some low-degree coefficients of the gravitational potential based on the sequential gradiometry approach. In both cases, a one-month span of the GRACE observations is considered.

3.1 Evaluation of the Truncation Error and Random error of the Approximate Value

Using Eq. (12), we can determine the truncation error of the sequential observation equation. The evaluation was done for a few low degrees of the spherical harmonics and the results of a one-day span of the mission were shown in Fig. (2). As shown in Fig. (2),

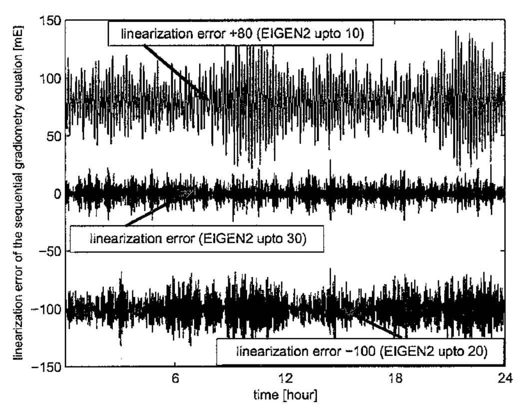

Fig. 2. Linearization error of the sequential gradiometry equation, Eq. 12, (EGM96 upto 90 as the pseudo-real field)

increasing the degree l decreases the linearization error. On the other hand, stepping the degree up increases random error of the approximate value. As a representative example, using the variance components provided in EIGEN2 data file, the error was estimated and the results were depicted for one revolution of the mission for $l = 30$ and $l = 50$ in Figs. (3) and (4) respectively. The variance-covariance matrix has a dominant block diagonal structure in both cases. However, the diagonal elements correspond-

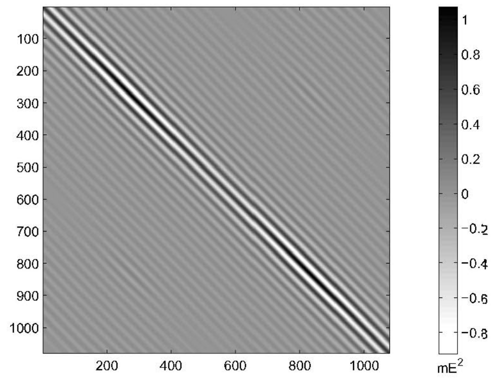

Fig. 3. Variance-covariance matrix structure of the approximate value corresponding to one revolution of the GRACE mission (pseudo-real field: EIGEN2 upto $l = 30$).

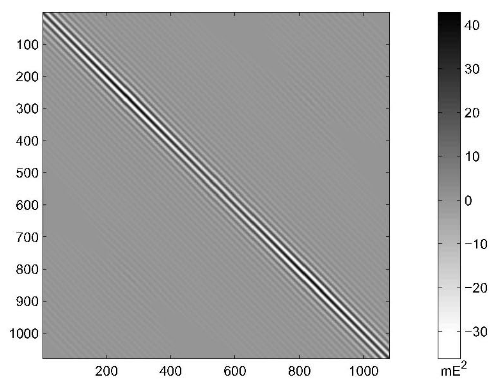

Fig. 4. Variance-covariance matrix structure of the approximate value corresponding to one revolution of the GRACE mission (pseudo-real field: EIGEN2 upto $l = 50$).

ing to $l = 30$ are about $1\,\mathrm{mE}^2$, whereas they exceed $40\,\mathrm{mE}^2$ for $l = 50$. Compared with the linearization error of the sequential gradiometry equation (Eq. 12), random error of the approximate value is negligible for $l = 30$. Nevertheless, subtracting the approximate value corresponding to $l = 50$ will double the diagonal elements of the variance-covariance matrix of the reduced observations. Hence, we employ the approximate value corresponding to $l = 30$ whose respective uncertainties are really negligible.

3.2 Recovery of the Spherical Harmonic Coefficients

In this subsection, we analyze the sequential gradiometry approach performance. We considered $n = 90$ as the maximum degree of the sought-after spherical harmonics. Therefore, to avoid the *omission* errors, the simulated observations only contain the respective signals (n, m upto 90). We plugged the sequence of the simulated observations in Eq. (11) to estimate the spherical harmonic coefficients. The achieved results, as well as the original coefficients (EGM96), were plotted in Fig. (5) in terms of degree variances. Besides the degree variances, the figure shows estimation error of the coefficient . As

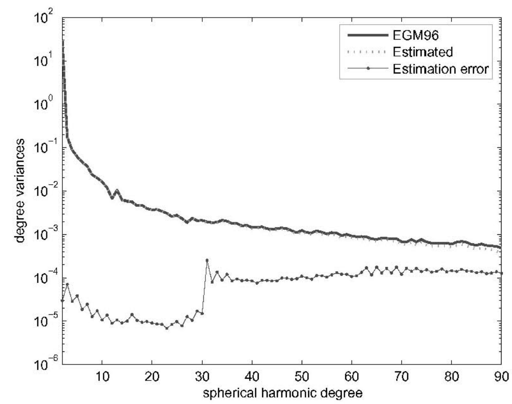

Fig. 5. Estimated degree variances upto 90 based on EIGEN2 upto 30 as a priori information.

seen in Fig. (5), estimation error is lower than 10^{-4} for $l \leq 30$. The error steps up to 10^{-4} for $l > 30$. This jump is in accordance with the split point of the Earth's gravitational potential in the sequential formulation. Therefore, it indicates that the higher value of l leads to a better accuracy, at least for $n \leq l$. However, as mentioned earlier, the high-degree of l will dramatically increase uncertainty of the reduced observations. Then, a medium degree of l would be an optimal choice.

Moveover, the gravity anomaly is computed on a regular $2^\circ \times 2^\circ$ grid on the mean sphere using both the estimated coefficients and EGM96's. As we see in Fig. (6), the gravity anomaly errors are less than 5 mGal for most points on the Earth. The error does not exceed 15 mGal.

4 Conclusion

The GRACE mission is the first mission that has realized satellite-to-satellite tracking concept in LL-mode. Despite the mission realization, the idea has been investigated theoretically since 1970 (e.g. Wolff , 1969). Consequently, different approaches have been introduced by many authors and researchers.

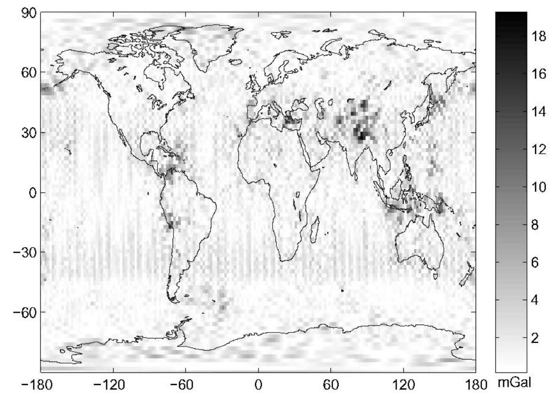

Fig. 6. Gravity anomaly residuals (absolute values).

Among them, LOS acceleration difference between the two satellites has been of particular interest. It is a two-point first-order problem. However, a one-point second-order formulation is far preferable to a two-point first-order formulation. For instance, having direct relationship with the gravity field geometry and promising improved gravitational field resolution are two noteworthy achievements of the sought-after formulation. Moreover, a relatively long inter-satellite range is a motivation to consider the spacecrafts as one-dimensional virtual gradiometer.

In this regards, we derived the desired formulation simply by expanding the LOS acceleration differences around the mid-point of the satellite configuration. We utilized linear term of the expansion called linear gradiometry equation, because of its simplicity. However, because of considerable linearization error, we introduced the sequential gradiometry equation as an intermediate solution to lower the linearization error.

EGM96 upto 90 and EIGEN2 CHAMP-only model upto 30 were respectively employed as the pseudo-real field and a priori information on the low-degree harmonics of the gravitational potential. The spherical harmonic degree variances estimated with an accuracy of about 10^{-4}. Also, Gravity anomaly residuals were less than 5 mGal for most points on the Earth's surface. To sum up, the estimated results indicate the high level performance of the proposed method.

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