Effect of geopotential model errors on the projection of GOCE gradiometer observables

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Abstract The forthcoming GOCE mission will provide gravity gradient observations along its orbit at varying altitude. It is necessary for certain data processing strategies to project the GOCE gravity gradients to a mean reference sphere. In the present simulation study the radial distance of the projection is in the order of 10 km, and can be done using the Taylor expansion of the gravity gradients.

In this paper we present an error analysis of such a projection. The omission of higher-order terms of the Taylor expansion and commission errors of the geopotential model are discussed.

The paper presents an error analysis study based on simulated GOCE gradiometry. The results are validated with stringent accuracy requirements of the GOCE mission.

Keywords. gravity gradient tensor, geopotential model, space gradiometry, projection error

1 Introduction

Observables of Satellite Gravity Gradiometry (SGG) are the elements of the Eötvös tensor. Four components, three diagonal and one off-diagonal, of the SGG tensor will be observed with high accuracy in the measurement bandwidth (MBW) of 5 to 100 mHz in the Gradiometer Reference Frame (GRF). In this reference frame the *x*-axis on average is in the velocity direction, the *z*-axis approximately radially outward and the *y*-axis complements the right-handed frame. Simulations of SGG measurement errors range from 6 mE/√Hz (1E = 10^{-9} s^{-2}) in the high frequency part of the MBW to 15÷79 mE/√Hz in the low frequency part for the diagonal gravity gradients (V_{xx}, V_{yy}, V_{zz}). See for reference Cesare (2002) and Floberhagen et al. (2004).

There are different data processing strategies for GOCE gradiometry. The time-wise approach describes the gravity gradients with Keplerian elements, showing the explicit time dependence of gravity gradient measurements via the time coordinate (mean anomaly) along the quasi-fixed orbit defined by the other 5 elements (cf. Colombo 1986; Koop 1993). In case of the space-wise approach the time dependency is implicit via the time dependence of the orbit. Certain data processing strategies would require a pre-processing stage, e.g. rotation into another frame, interpolation, reduction to a sphere, etc. For example, combination with ground data may require integration over a certain region on a sphere (Haagmans et al., 2003). The application of such approximations introduces errors into the solution. The question is whether these errors are negligible compared to measurement errors.

Several authors analyzed the accuracies of GOCE gradients in various reference frames and error assessment using along track interpolation (Müller, 2003; Bouman and Koop, 2003). The present paper deals with the error analysis of the projection of gravity gradient observations along an orbit to a mean sphere, using the Taylor expansion of the gravity gradients. In the first part of the paper we overview the theoretical background. Subsequently, geopotential model induced omission and commission errors of the projection in the natural geographical frame (the Earth-fixed reference frame (ERF) defined by *x*-axis pointing towards North, *y*-axis to East and the *z* upwards) are discussed.

A GOCE-like orbit has been simulated using the orbit integrator at IAPG Munich. The simulated orbit spans 90 days. The orbit data sampled at 1 Hz.

2 Radial projection of GOCE gravity gradients in the geographical frame

The projection of gravity gradients along an orbit (i.e. GOCE) onto a sphere means a change along the radial direction, *r*. Therefore it is done by the Taylor expansion of the gravity gradients by *r*.

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$$
V_{ij}^{\text{sphere}}(\varphi,\lambda,r^{\text{sphere}}) = V_{ij}^{\text{obs}}(\varphi,\lambda,r) + \frac{\partial V_{ij}^{\text{obs}}(\varphi,\lambda,r)}{\partial r}dr + \frac{1}{2}\frac{\partial^2 V_{ij}^{\text{obs}}(\varphi,\lambda,r)}{\partial r^2}dr^2 + \ldots \tag{1}
$$

where $i,j=x,y,z$ in the ERF. If the projection distance dr is small, i.e. of order 10 km, then higher order terms of the Taylor expansion can be neglected. The magnitude of the second order term is the most important in this respect, but it can be neglected, as we will see later on. This is true for a nearly circular orbit where the deviations of the satellite positions from a mean sphere remain relatively small. The contribution of the second order term will be investigated in this study as well.

We make use of the spherical harmonic representation of the gravity gradients and of its radial derivatives in eq. (1) of the gravity gradient observations are expressed by spherical harmonics. We derive all the third derivatives of the gravitational potential, since no complete derivatives in the literature was found. However, due to the space limit of the present paper, we do not show the derivations here. For the radial derivatives (cf. eq. 2a-j), on the other hand, we can refer e.g. to Rummel (1997), where all six second derivatives of the geopotential can be found in the ERF. Since the z-axis is equivalent to the radial direction, we need six third derivatives of the geopotential V with respect to z in this recent study. It is straightforward to compute from the above mentioned six second derivatives all the required radial derivatives, therefore they can readily be checked. Since every component of degree ℓ of any element of the Eötvös tensor, i.e. V_{ij} is a homogeneous function of degree $-(\ell+3)$, its first and second radial derivatives can be computed by multiplying all degree ℓ terms by the factors $-(\ell+3)/r$ and $-(\ell+3)(\ell+4)/r^2$, respectively (Rummel et. al, 1993). We consider especially important the third radial derivative V_{zzz}, since this is the largest term in the context of GOCE Any gravity field functional computed from a geopotential model has two main sources of error:

- propagation of errors of geopotential coefficients (*commission error*)
- errors introduced by neglecting coefficients higher than the maximum degree of the model (*omission error*)

These errors propagate into errors of the radial projection of gravity gradients using eq. (1).

The computation of the commission error of the radial derivatives in eq. (1) is straightforward if the variance-covariance matrix of geopotential coefficients is available. Strict error propagation using the full variance-covariance matrix is a computationally demanding task, however. Therefore we have used only the error standard deviations of the geopotential model and the errors of the coefficients are assumed to be uncorrelated.

$$
\begin{aligned}
V_{xxx} &= \frac{1}{r^3\cos^3\varphi}V_{\lambda\lambda\lambda} - \frac{3\tan\varphi}{r^3\cos\varphi}V_{\varphi\lambda} + \frac{3}{r^2\cos\varphi}V_{r\lambda} - \frac{2}{r^3\cos^3\varphi}V_{\lambda}\\
V_{xxy} &= \frac{1}{r^3\cos^2\varphi}V_{\varphi\lambda\lambda} + \frac{2\tan\varphi}{r^3\cos^2\varphi}V_{\lambda\lambda} - \frac{\tan\varphi}{r^3}V_{\varphi\varphi} + \frac{1}{r^2}V_{r\varphi} - \frac{1}{r^3\cos^2\varphi}V_{\varphi}\\
V_{xxz} &= \frac{1}{r^2\cos^2\varphi}V_{r\lambda\lambda} - \frac{2}{r^3\cos^2\varphi}V_{\lambda\lambda} - \frac{\tan\varphi}{r^3}V_{r\varphi} + \frac{1}{r}V_{rr} + \frac{2\tan\varphi}{r^3}V_{\varphi} - \frac{1}{r^2}V_{r}\\
V_{xyy} &= \frac{1}{r^3\cos\varphi}V_{\varphi\varphi\lambda} + \frac{2\tan\varphi}{r^3\cos\varphi}V_{\varphi\lambda} + \frac{1}{r^2\cos\varphi}V_{r\lambda} + \frac{2\tan^2\varphi}{r^3\cos\varphi}V_{\lambda}\\
V_{xyz} &= \frac{1}{r^2\cos\varphi}V_{r\varphi\lambda} - \frac{2}{r^3\cos\varphi}V_{\varphi\lambda} + \frac{\tan\varphi}{r^2\cos\varphi}V_{r\lambda} - \frac{2\tan\varphi}{r^3\cos\varphi}V_{\lambda}\\
V_{yyy} &= \frac{1}{r^3}V_{\varphi\varphi\varphi} + \frac{3}{r^2}V_{r\varphi} - \frac{2}{r^3}V_{\varphi}\\
V_{yyz} &= \frac{1}{r^2}V_{r\varphi\varphi} - \frac{2}{r^3}V_{\varphi\varphi} + \frac{1}{r}V_{rr} - \frac{1}{r^2}V_{r}\\
V_{yzz} &= \frac{1}{r}V_{rr\varphi} - \frac{2}{r^2}V_{r\varphi} + \frac{2}{r^3}V_{\varphi}\\
V_{zzx} &= \frac{1}{r\cos\varphi}V_{rr\lambda} - \frac{2}{r^2\cos\varphi}V_{r\lambda} + \frac{2}{r^3\cos\varphi}V_{\lambda}\\
V_{zzz} &= V_{rrr}
\end{aligned}
\tag{2a-j}
$$

Since we are interested in the error PSD, which is the error variance at various frequencies, it is unclear, at least to us, how to derive easily this error PSD from the process error variances. These (spatial) error variances of course can be derived easily through error propagation from error standard deviations of the geopotential coefficients, but we need the PSD, not the (spatial) error variances of the signal. Therefore, we followed a more pragmatic approach here, i.e. to simulate the actual errors and then it is straightforward to estimate the error PSD.

First we generated a non-correlated normally distributed random noise model of geopotential coefficients within the error variances of the geopotential model. Only one realization of this model was considered for each spherical harmonic expansion degree. Then this error model served the purpose of commission error computations, so all subsequent commission error tests were performed by subsequent realizations of these pseudo-random geopotential models.

The omission error in principle can be derived from a suitably high degree reference model, i.e. higher than the geopotential model used for projection, for evaluating the error effect of the unused degrees. The practical solution to the problem of finding a suitable degree of the reference model depends on many factors. These are the required gravity field functional, computation altitude and error tolerance for the omission error. The chosen maximal degree of the omission error reference model was 720, since the signal degree variance of the third radial derivative, V_{zzz}, falls off rapidly at the altitude of GOCE (250 km), and reaches the level of about 10^{-10} mE/km at this degree (cf. Fig. 3)

3 Computations and results

The mean reference sphere was defined by the mean height of the GOCE orbit. The distribution of

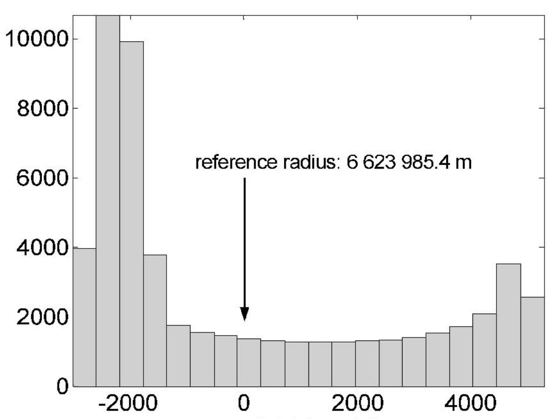

Fig. 1 Histogram of the radial distances of 55 000 simulated GOCE orbit points with respect to a mean reference sphere of radius 6 623 985.4 m.

radial positions with respect to this mean reference sphere is shown in Fig. 1. The distribution is obviously not normal, but condensed around two peaks. This is due to the elliptical orbit geometry.

In all our tests we have examined extensively the most critical term, the radial projection of the V_{zz} gravity gradient, but also tested other components. Our computations were performed in the ERF geographical reference frame and not in the GRF. Though we feel that results in GRF may give useful hints on the projection of GOCE observables, especially for the tested V_{zz} component.

The computations of the commission error of the radial projection were done with a combined model. It is defined by a linear transition of the coefficients from GRACE GGM01C to EGM96 models between spherical harmonic degrees 81-90. This way strong contribution of a state-of-the-art satellite-only gravity model on long-wavelength has been combined with the more reasonable short-wavelength information of a combined model containing terrestrial data as well. Though a combined model was used in this study, we should keep in mind that it is not an inconsistent geopotential model, created with the purpose of simulating a better geopotential model as recently available. The error propagation of the projection using this model was determined and power spectra were computed at different spherical harmonic degrees (Fig. 2). In the low frequency part of the spectrum the models above degree 120 gave almost a constant commission error level of about 2 mE/√Hz. It means that commission errors of the radial projection mostly contributed by coefficients up to degree 120, and the higher degree coefficients altering the characteristics of the falloff of the error in the 0.01 – 0.03 Hz frequency band (harmonic degrees 50-160).

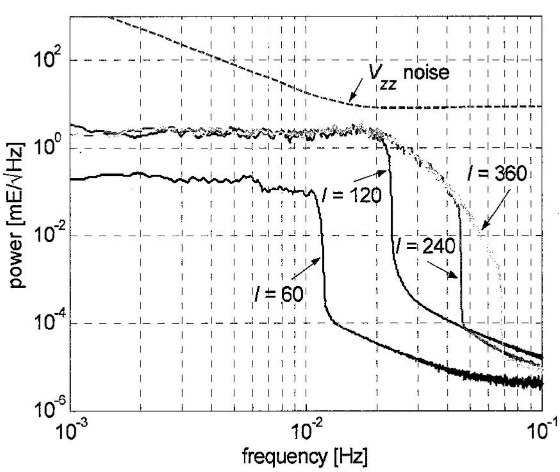

Fig. 2 Commission error PSD of the radial projection of the V_{zz} component from the combined GRACE GGM01C – EGM96 geopotential model at different spherical harmonic degrees. In total 55000 consecutive points of the 1 sec simulation (i.e. ~10 orbit revolutions) were projected to a mean sphere. The line labeled 'V_{zz} noise' shows the estimated gradiometer noise level.

Degrees close to degree 100 are the weakest part of the current geopotential models with respect to the projection procedure. We mean that in the vicinity of this wavelength the error spectrum approaches the gradiometer noise level most closely (cf. figure 2).

Next, omission errors were computed at 55000 points by comparison to the degree 720 solution (Fig. 5). According to the expectations, the omission error PSD from the projection is dependent on the maximum degree of the geopotential model used. The omission error within the MBW (i.e. between 0.005 mHz and 0.1 mHz) is well below the gradiometer noise level with a maximum degree of 240 or higher. For degree 240 expansion or higher the power spectral density of omission errors of the radial projection is below 0.1 mE/√Hz across the whole spectrum (cf. Fig. 5).

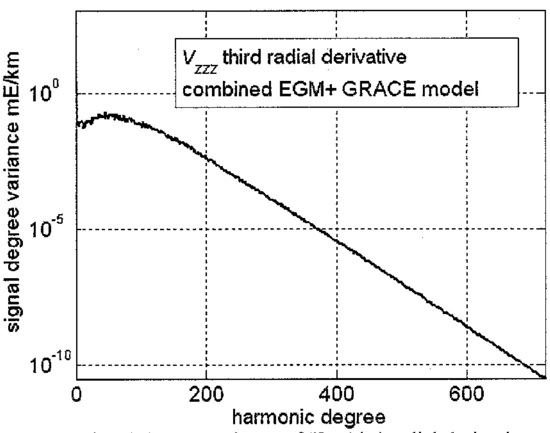

Fig. 3 Signal degree variance of V_{zzz} third radial derivative at the altitude of GOCE. The combined GRACE GGM01C+EGM96 geopotential model was used to maximum degree 720.

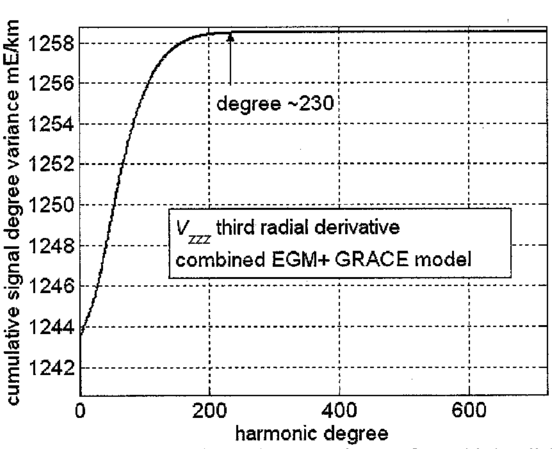

Fig. 4 Cumulative signal degree variance of V_{zzz} third radial derivative at the altitude of GOCE. The combined GRACE GGM01C+EGM96 geopotential model was used to maximum spherical harmonic expansion degree 720.

The cumulative signal degree variances of the third radial derivative (Fig. 4) confirm that most of the power in the third radial derivative is contained below the spherical harmonic degree of about 230 at the altitude of GOCE. Therefore it is permitted to use a high degree reference model (in this study the maximum degree of the reference model was 720) to evaluate the omission error spectrum of the projection.

From figures 2 and 5 it is obvious, that projection errors with a use of a geopotential model up to degree and order 240 introduces almost an order of magnitude smaller error than that of observations.

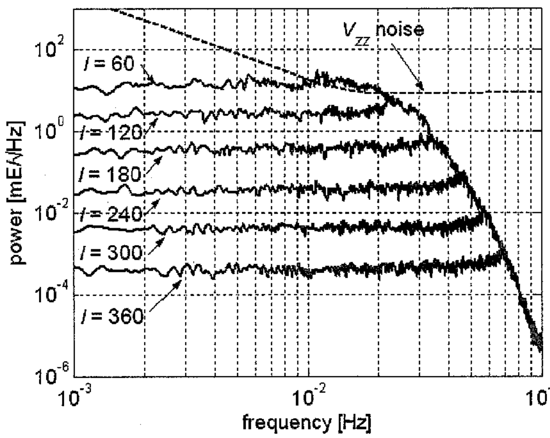

Fig. 5 Omission error PSD of projection of V_{zz} gravity gradients to a mean sphere in GRF. The combined GRACE GGM01C+EGM96 geopotential model was used to maximum degree 720 as reference (zero omission error).

The actual commission error of the projection depends somehow on the accuracy of the chosen geopotential model, at least below harmonic degree of about 90 (cf. Fig . 6). Therefore it is recommended to use a more accurate gravity model than recent geopotential solutions.

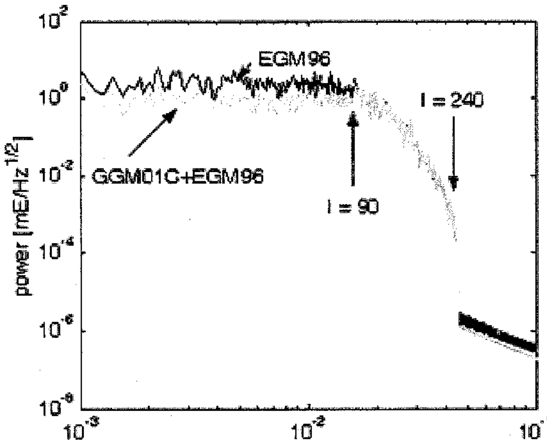

Fig. 6 Comparison of commission error PSD of projection of V_{zz} gravity gradients to a mean sphere in GRF. The combined GRACE GGM01C+EGM96 geopotential model was compared to the EGM96 model up to maximum spherical harmonic expansion degree 240. Below harmonic degree $l = 90$ (15 mHz), the combined GRACE model is clearly superior to EGM96.

Finally the effect of the quadratic term in eq. (1) was investigated. The results for the V_{zz} component in the ERF can be seen in Fig. 7. The quadratic errors are much below the observation errors at every frequency, therefore the use of this term is unnecessary.

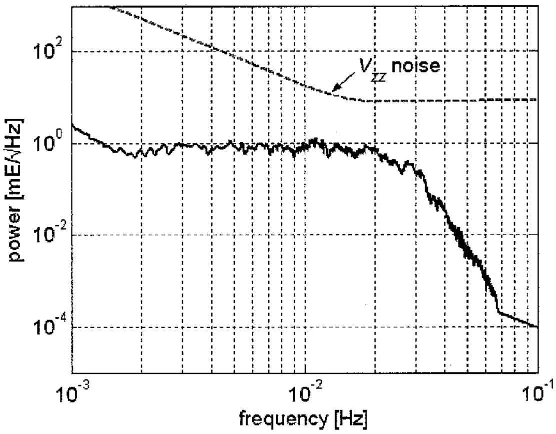

Fig. 7 PSD of the quadratic term of V_{zz} in Eq. (1) for radial projection to a mean sphere. The combined GRACE GGM01C and EGM96 geopotential model to degree and order 360 was used for the computation.

4 Conclusions and recommendations

Our main conclusion from the above study is that a radial projection procedure of GOCE observables distributed globally using a geopotential model is reasonable only if an error level of about 2-3 mE/√Hz is tolerable on the most critical 10-30 mHz part of the gradiometer spectrum. This error level is only by a factor of 2-4 smaller than the current predictions on the gradiometer performance. The error itself comes mainly from the errors of the geopotential coefficients above about degree 80 (cf. Fig. 2). In any case the expansion degree of the used model should be at least about 240 to reduce the omission error to be negligible compared to the gradiometer errors.

The analysis performed is relevant only to the case when gravity gradients are in the Earth-fixed reference frame. The real observations will be in the gradiometer reference frame, and therefore the proper orientation of the gradiometer axes should be taken into account. It depends mainly on the performance of the star sensors. The orbit errors, on the other hand, will play a much smaller role in gradiometry errors, since displacement of gravity gradients with some centimeters (typical accuracy of precise orbit determination) is much below the measurement accuracy.

It is sometimes desirable to project the observations to other locations both horizontally and vertically, e.g. to look at crossovers (Bouman and Koop, 2003). This is another interesting topic, which may be addressed in a forthcoming paper.

Acknowledgements

The above investigations were funded by OTKA projects T037929, T046418 and Hungarian Space Office (project TP 205).

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