Regional Geoid Undulations from CHAMP, Represented by Locally Supported Basis Functions

Rainer Mautz[1,2], Burkhard Schaffrin[1], C. K. Shum[1], Shin-Chan Han[1]

[1] Laboratory for Space Geodesy and Remote Sensing, The Ohio State University, Columbus, Ohio 43210, U.S.A. *schaffrin.1@osu.edu*
[2] Institute of Astronomical, Physical and Mathematical Geodesy, Technical University of Berlin, Germany. *rainer@mca.bv.tu-berlin.de*

Summary. Regional geoid undulations are determined from CHAMP data using various locally supported basis functions to assess their respective efficiency, accuracy and multi-resolution representation properties. These functions include (biharmonic) B-spline tensor wavelets (with or without compression), multiquadrics (with or without flexible centering and predetermined smoothing) and radially symmetric truncated polynomials.
It is concluded that the B-spline wavelet model is the computationally most efficient approach. The non-periodic variation of the B-spline wavelets allows one to handle data on a bounded domain with small edge effects, and the piecewise linear version allows one to model the geoid using a patch-wise approach. The use of multiquadrics without centering in the data points and predetermined smoothing constant allows handling of heterogeneously distributed data using global optimization. The linear multiquadrics model fits the data best when comparing the residuals of different models with a fixed number of unknowns. For an efficient data synthesis the nonlinear models are best suited due to their far smaller number of basis functions. The smoothest surface was obtained using the nonlinear polynomial approach, whereas the multiquadrics show peaks and the wavelet models show horizontal and vertical edges in their representations. The linear B-spline wavelets are biharmonic, and the approach is capable of an efficient multi-resolution representation of regional gravity field models combining satellite (CHAMP, GRACE, GOCE) and in-situ data.

Key words: geoid undulations, B-spline wavelets, multi-quadrics, nonlinear models

1 Introduction

Recent global gravity models are based on spherical harmonic functions which are excellent for representing the geopotential up to a certain degree of detail or resolution. Even though the global spherical harmonic representation is adequate for low degree global gravity modeling using satellite data (e.g. CHAMP), but for a detailed regional representation of the earth gravity field determined by satellite as well as terrestrial observations, spherical harmonics may not be the best basis functions. In this context, see Schmidt et al. (2002) who have been using spherical wavelets to represent the finer details of the gravity field.

Our goal is to use harmonic or biharmonic locally supported basis functions in order to construct patch-wise models to enhance the high frequency parts of the signals on the sphere leading to a multi-resolution representation of the gravity field. Towards selecting the most suitable functional model, some approaches are compared numerically using regional undulations derived from CHAMP disturbing potential data. For more details regarding CHAMP data processing, we refer to Han et al. (2002; 2003).

The first functional model in this investigation is the B-spline wavelet; see Chui and Quak (1992). These wavelets have become well known due to their useful properties such as compact support, semi-orthogonality and simplicity. Algorithms and applications for computer graphics can be found in Stollnitz et al. (1996).

The second model is based on the multiquadric method which fits a set of quadric (e.g., hyperbolic or conical) functions to the observations. It was introduced by Hardy (1971) and further developed by Hardy and Göpfert (1975) in order to interpolate gravity anomalies.

The comparison also includes polynomial radial symmetric basis functions with local support, which have been used successfully in Mautz et al. (2003).

Furthermore, the linear models described above are compared with their nonlinear counterparts having flexible positioning. Geodetic global models have been studied by Mautz (2001; 2002) while nonlinear models for surface data were discussed by Kaschenz (2002; 2003).

For a comparison of different surface representations on a large scale, see Franke (1982). Here, we focus on the data-fit, the computational effort, the number of basis functions and the smoothness/roughness of the surface as criteria.

2 Functional Models

The models discussed here can be classified as linear or nonlinear. The systems for solving the unknowns in capital letters (e.g., the amplitudes A, B) are linear if these are the only parameters. In this case the estimated parameters, the residuals, and the estimated variance component can be obtained using BLUUE (Best Linear Uniformly Unbiased Estimate) and BIQUUE (Best Invariant Quadratic Uniformly Unbiased Estimate). A detailed discussion of linear models is provided by Grafarend and Schaffrin (1993).

Less common are models where the positions or scaling coefficients are considered as unknown. With the models becoming nonlinear due to the flexible centering and scaling, the solving techniques require global optimization methods. Gradient methods like the Gauss-Newton iteration are not applicable as reliable starting values cannot usually be provided. Thus, we stick to global optimization techniques such as heuristic methods, interval strategies or genetic algorithms. The idea of optimized centering dates back to Barthelmes (1986).

(a) *2-dimensional B-spline wavelets:* If the 2-D signal is given by $f(x, y)$ the model function reads

$$f(x,y)=\sum_{k_x=1}^{2^{j_{\min}}+g}\sum_{k_y=1}^{2^{j_{\min}}+g} A_{j_{\min},k_x,k_y}\tilde{\phi}_{j_{\min},k_x,k_y}(x,y)+\sum_{j=j_{\min}}^{J}\sum_{\eta=1}^{3}\sum_{k_x=1}^{2^j+g}\sum_{k_y=1}^{2^j+g} B_{j,\eta,k_x,k_y}\tilde{\psi}_{j,\eta,k_x,k_y}(x,y), \quad (1)$$

where $\phi(x, y)$ is the 2-D scaling function, $\tilde{\phi}(x, y)$ its dual; $\psi(x, y)$ are the 2-D wavelet functions, and $\tilde{\psi}(x, y)$ their duals. Their polynomial degree is expressed by g. The different levels of detail are denoted by the index j. Wavelet coefficients with a larger j indicate higher detail levels, essentially representing the high-frequency part. The index η denotes the three directional components (horizontal, vertical and diagonal), and the indices $k_x, k_y \in \ _0$ denote the shift of the wavelets to different locations on the (x, y)-patch. The variables A and B denote the unknown coefficients for the scaling function and the wavelet functions. The problem of estimating A and B is linear; due to orthogonal subspaces, it is not necessary to solve one big system with linear equations of problem-size, but a sequence of smaller systems. This model has been discussed in more detail by Mautz et al. (2002) and Schaffrin et al. (2003).

(b) *Compressed 2-D B-spline wavelets:* In contrast to model (a), the hierarchical structure is now developed to the maximum level $J_{\max}$, fulfilling the condition

$$J_{\max} \le \ln\left(\sqrt{n} - g\right) / \ln 2, \tag{2}$$

where n is the number of data points. The number of coefficients is then reduced by neglecting terms with coefficient values smaller than a predetermined bound. Figure 1 shows the shape of a linear B-wavelet according to model (a) and (b).

(c) *Multiquadric basis functions:* The multiquadric functional model, resp. its inverse, reads

$$f(x, y) = \sum\nolimits_{k=1}^{n} A_k K\left(r_k\right), \tag{3}$$

where the kernel functions $K(r_k)$ are given by

$$K\left(r_k\right) = \left(r_k^2 + c^2\right)^t, \tag{4}$$

with typically $t = ½$, resp. $t = -½$ for the inverse case. The radial distances r_k between the evaluation point (x, y) and a fixed center position (x_k, y_k), which could be chosen from the locations of the observations, are given by

$$r_k(x, y) = \sqrt{\left(x - x_k\right)^2 + \left(y - y_k\right)^2}. \tag{5}$$

The planar distance r_k may be replaced with the spherical distance ψ_k using the spherical coordinates ϕ and λ, along with the relation

$$\cos\left(\psi_k\right) = \sin\left(\phi\right)\sin\left(\phi_k\right) + \cos\left(\phi\right)\cos\left(\phi_k\right)\cos\left(\lambda - \lambda_k\right). \tag{6}$$

A_k are the unknown parameters and $c \in$ is a predefined constant. The unknown parameters A_k are estimated by solving a linear system. See Figure 1 for a graph of the radial multiquadric function according to model (c) and (d).

(d) *Multiquadric basis functions, with flexible positioning:* We now introduce $x_k \in$ and $y_k \in$ as unknowns for every $k \in \{1, 2, ..., n\}$. The resulting model becomes nonlinear, and the solving technique requires global optimization.

(e) *Locally supported radial functions, linear model:* Local support allows function values other than zero only within a certain distance from the center point location. Thus, the continuous model function needs to be truncated. The model

$$f(x, y) = \begin{cases} \sum\nolimits_{k=1}^{n} A_k \left(\frac{r^2}{c_k^2} - 1\right)^2, & r < c_k, \\ 0, & r \ge c_k. \end{cases} \tag{7}$$

avoids a discontinuity in the function and in its first derivative at the cut-off location. With fixed center positions (x_k, y_k) and parameters c_k, the model is linear.

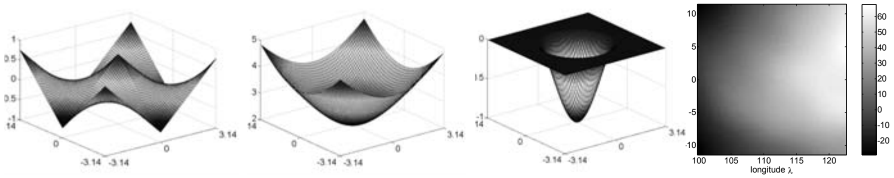


Fig. 1. Left panel: Diagonal linear B-spline wavelet of level 0. Mid left: Multiquadric function with $c = 2$. Mid right: polynomial function with $c = 2$. Right panel: Geoid undulations in [m] from CHAMP only solution, between [100°; 122.5°] longitude and [-11.25°; +11.25°] latitude; sampled onto a 65 × 65 grid. The range of the data values is [–29.0 m; +67.8 m].

(f) *Locally supported radial functions, nonlinear model:* Introducing the center positions (x_k, y_k) as unknowns for every $k \in \{1, 2,..., n\}$ the model becomes nonlinear. In addition to flexible centering, the parameters c_k, serving as scaling parameters, are also considered as unknown. Figure 1 shows a graph for $c = 2$.

3 Model Comparison Based on CHAMP Geoid

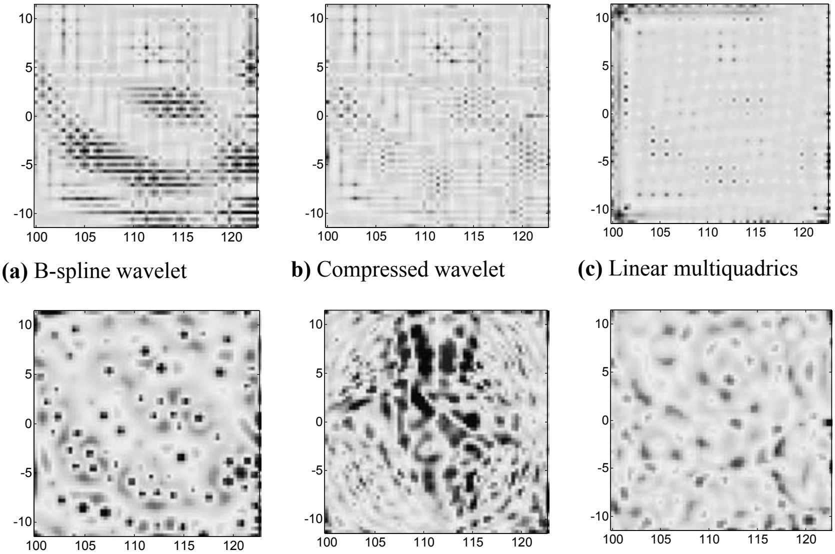
In order to make proper comparisons the models' special requirements have to be taken into account. The multiresolution representations (a) and (b) necessitate observations in form of a 2^j by 2^j, ($j \in$) grid for efficient handling. Their application requires a prior adjustment to the grid. The compressed wavelet model (b) needs extra memory for storing the locations of the remaining terms. All comparisons are based on the specific dataset shown in Figure 1.

As shown in Table 1, the residual information is used to rate the models numerically. The criteria are the standard variation, the maximum deviation, the squared sum of the residuals and the average deviation. The number of unknowns is kept fixed.

Generally, linear models involve a normal equations system of problem size. Setting up the system requires a complexity of $(n\, m^2)$ and its inversion (m^3), where n is the number of observations and m the number of parameters. However, properties like semi-orthogonality and local support causing banded matrix struc-

Model	No. of un-- knowns	No. of basis func- tions	$m_0 = \sqrt{\hat{\sigma}_0^2} = \sqrt{\tilde{\mathbf{e}}^T\tilde{\mathbf{e}}/(n-m)}$ [m]	$\max\{\lvert\tilde{\mathbf{e}}_j\rvert\}$ $j = 1, 2...n$ [m]	$\tilde{\mathbf{e}}^T\tilde{\mathbf{e}}$ [m^2]	$\frac{\sum_{j=1}^{n}\lvert\tilde{\mathbf{e}}_j\rvert}{n}$ [m]	CPU on a 200 Mhz PC
(a) Wavelet, 4 levels	289	289	0.08	0.41	31	0.06	**1s**
(b) Wavelet, compression	289	289	0.06	0.26	12	0.04	**1s**
(c) Multiquadric (linear)	289	289	**0.05**	0.36	**9**	**0.03**	3 min
(d) Multiquadric (nonlin.)	288	96	0.09	0.58	34	0.07	10 h
(e) Local radial fct. (linear)	289	289	0.12	0.56	61	0.12	3 min
(f) Local radial fct. (nonlin.)	288	**72**	0.07	**0.25**	19	0.05	10 h

Table 1. Comparison of various surface representation models with a constant number of unknowns resp. 93% redundancy; units: [m] or [m^2]. The models fulfilling a criterion best are highlighted in bold.



(a) B-spline wavelet **b)** Compressed wavelet **(c)** Linear multiquadrics

(d) Nonlinear multiquadrics **(e)** Linear polynomials **(f)** Nonlinear polynomials

Fig. 2. Residuals [m] of 6 different models. All models have 288 or 289 parameters. The number of basis functions is 289 for the linear and 72 for the nonlinear models. The unit for the colorbar is [m] and for the axes it is [°] latitude and longitude.

tures as it is the case for the B-spline wavelet model. The astonishing result in Table 2 for the wavelet model is due to a reduction of a factor $m^{1/2}$ by usage of the tensor product. The multiquadrics, having global support are of complexity ($n\ m^2$). If a hierarchical data thinning algorithm is used according to Hales and Levesley (2000), the algorithm may be of linear complexity, but only if the data are spaced equidistantly. However, the computational burden decreases drastically for locally supported functions when handling very large m and n, assuming that an optimized algorithm is implemented.

The computing time for nonlinear models may be higher in general, since a sequential series of inversions has to be performed. Nevertheless, making use of heuristic strategies and adaptation of the algorithm to the problem at hand, the computational effort can be reduced drastically.

The behavior of the six different models (a)-(f) is visualized in Figure 2, for a patch between 100° and 122.5° longitude and -11.25° to +11.25° latitude respectively. All models have 288 (or 289) parameters. The wavelet model shows

complexity	B-splines	multiquadrics	locally supported fcts.	nonlinear models
linear system:	($n\ m^{1/2}$)	($n\ m^2$)	< ($n\ m^2$)	involve solutions of
inversion:	< ($m^{3/2}$)	(m^3)	< (m^3)	many linear systems

Table 2. Computational effort of surface models with the number of observations n and the number of parameters m.

distinct horizontal features in the residual plot. At the center point locations of the multiquadric models, some peaks can be seen. The locally supported radial basis functions show a smooth surface throughout, particularly in the nonlinear case.

4 Conclusions

Due to unequal premises and different rating at various criteria, a strict ranking of the models is not feasible. Nevertheless, it has been verified, that the B-spline wavelet model is computationally the most efficient approach. The linear multiquadrics model fits the data best when comparing the residuals of different models with a fixed number of unknowns. For an efficient data synthesis the nonlinear models are best suited due to their far smaller number of basis functions. The smoothest surface was obtained using the nonlinear polynomial approach, whereas the multiquadrics show peaks and the wavelet models show horizontal and vertical edges in their representations.

Towards modeling the geopotential it can be outlined that the B-spline wavelets are biharmonic for the linear case and the multiquadric functions fulfill conditions for a harmonic upwards continuation in the case $c = 0$ and exponent $t = -0.5$.

The CHAMP only data do not have significant detail information for the multiresolution analysis discussed here. Nevertheless, this approach could be quite useful for the determination of a regional high-resolution gravity field model by combining CHAMP, GRACE, GOCE (N_{max}= 300) and in-situ terrestrial gravity data.

Acknowledgements. R. Mautz acknowledges the partial support of a Feodor-Lynen scholarship of the Alexander-von-Humboldt Foundation (Germany). R. Mautz, C. Shum, and S. Han acknowledge the support of a grant from NIMA's University Research Initiative program.

References

Barthelmes F (1986) On the approximation of the outer earth gravity field by point masses with optimised positions (in German). Veröffentlichungen des Zentralinstitut für Physik der Erde, *92*, Potsdam, Germany.

Chui CK, Quak E (1992) Wavelets on a bounded interval. in: Braess D, Schumaker L, eds, Numerical Methods in Approximation Theory*, Vol 9*: 53-77.

Franke R (1982) Scattered data interpolation: Tests of some methods. Math Comps *38 (157)*: 181-199.

Grafarend E, Schaffrin B (1993) Adjustment Computations in Linear Models (in German), Bibliograph. Inst., Mannheim, Germany.

Hales S, Levesley J (2000) Multi-level approximation to scattered data using inverse multiquadrics. in: Cohen A, Rabut C, Schumaker L, eds, Curve and Surface Fitting: Saint-Malo 1999, Vanderbilt University Press, Nashville/TN: 247-254.

Han S-C, Jekeli C, Shum CK (2002) Efficient gravity field recovery using in-situ disturbing potential observables from CHAMP. Geophys Res Letters *29 (16)*: 36-41.

Han S, Shum CK, Jekeli C, Braun A, Chen Y, Kuo C (2003) CHAMP mean and temporal gravity field solutions and geophysical constraint studies. 2nd CHAMP Sci Meeting, GeoForschungsZentrum (GFZ), Potsdam, Sept 2003.

Hardy R (1971) Multiquadric equations of topography and other irregular surfaces. J Geophys Res *76*: 1905-1915.
Hardy R, Göpfert W (1975) Least-squares prediction of gravity anomalies, geoidal undulations, and deflections of the vertical by multiquadric harmonic functions. Geophys Res Letters *2 (10)*: 423-426.
Kaschenz J (2002) Multidimensional spectral analysis with problem-oriented frequencies (in German). Diploma Thesis; Inst of Astronomical, Physical and Math Geodesy, Technical University of Berlin, Germany.
Kaschenz J (2003) Data modelling using various frequency series (in German). Z für Vermessungswesen (ZfV) *128 (4)*: 260-265.
Mautz R (2001) On the determination of frequencies in time series solving nonlinear adjustment problems (in German). German Geodetic Comm, C-532, Munich, Germany.
Mautz R (2002) Solving nonlinear adjustment problems by global optimization. Bollettino di Geodesia e Scienze Affini *61 (2)*: 123-134.
Mautz R, Schaffrin B, Schmidt M, Shum CK (2002) Representation of spatial functions in geodesy using B-spline wavelets with compact support. Proc of the Heiskanen Symposium in Geodesy (Jekeli C, Shum CK, eds), Columbus/Ohio, Oct 2002.
Mautz R, Schaffrin B, Kaschenz J (2003) Biharmonic spline wavelets versus generalized multi-quadrics for continuous surface representations. IUGG General Assembly, IAG-Symp G4, Sapporo, Japan, July 2003.
Schaffrin B, Mautz R, Shum CK, Tseng HZ (2003) Towards a spherical pseudo-wavelet basis for geodetic applications. Computer-Aided Civil and Infrastructure Engineering *18 (5)*: 369-378.
Schmidt M, Fabert O, Shum CK (2002) Multi-resolution representation of the gravity field using spherical wavelets. Proc of the Heiskanen Symposium in Geodesy (Jekeli C, Shum CK, eds.), Columbus/Ohio, Oct 2002.
Stollnitz E, DeRose T, Salesin D (1996) Wavelets for Computer Graphics. M Kaufmann, San Francisco/CA.

Earth Magnetic Field

Main Field Studies, Lithospheric Magnetisation, Induction Studies and Ionospheric Current Systems, Instruments