Multiscale Modeling from EIGEN-1S, EIGEN-2, EIGEN-GRACE01S, UCPH2002_0.5, EGM96

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Summary. Spherical wavelets have been developed by the Geomathematics Group Kaiserslautern for several years and have been successfully applied to georelevant problems. Wavelets can be considered as consecutive band-pass filters and allow local approximations. The wavelet transform can also be applied to spherical harmonic models of the Earth's gravitational field like the most up-to-date EIGEN-1S, EIGEN-2, EIGEN-GRACE01S, UCPH2002_0.5, and the well-known EGM96. Thereby, wavelet coefficients arise and these shall be made available to other interested groups. These wavelet coefficients allow the reconstruction of the wavelet approximations. Different types of wavelets are considered: bandlimited wavelets (here: Shannon and Cubic Polynomial (CuP)) as well as non-bandlimited ones (in our case: Abel-Poisson). For these types wavelet coefficients are computed and wavelet variances are given. The data format of the wavelet coefficients is also included.

Key words: Multiscale Modeling, Wavelets, Wavelet Variances, Wavelet coefficients, Gravitational Field Model Convertion

1 Introduction

During the last years spherical wavelets have been brought into existence. (cf. e.g. [3], [2], [4] and the references therein). It is time to apply them to well-known models in order to offer easy access to the multiscale methods. Therefore, the spherical harmonics models EIGEN-1S, EIGEN-2, UCPH2002_0.5, EGM96 and also EIGEN-GRACE01S are transformed into bilinear wavelet models (see [3] or [2]) and the coefficients of these models are available via the worldwide web.

2 Wavelet Models

Due to the structure of the gravitational field we leave the first degrees and orders (up to $n = 2$) as an approximation by spherical harmonics which are denoted by $Y_{n,k}$. For the remaining parts of the models, a scaling function and its corresponding wavelets are applied. Thus, we can write the J-level representation of the geopotential V on the sphere Ω_R in terms of a spherical harmonics part $V_{0..2}$ (which we neglect from now on), a low-frequent band V_{j_0} and wavelet bands W_j for $x = R\xi \in \Omega_R$:

$$V_J(x) = V_{j_0}(x) + \sum_{j=j_0}^{J-1} W_j(x)$$
$$= (\Phi_{j_0} * (\Phi_{j_0} * V))(x) + \sum_{j=j_0}^{J-1} (\tilde{\Psi}_j * (\Psi_j * V))(x), \quad (1)$$

where $*$ denotes the $\mathcal{L}^2(\Omega_R)$-convolution and $\lim_{J\to\infty} V_J = V$ in the sense of $\mathcal{L}^2(\Omega_R)$.
The kernel can be expressed by a sum over Legendre polynomials (cf. [3] for $x, y \in \Omega_R$:

$$\Phi_j(x,y) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi R^2} \varphi_j(n) P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right).$$

where $\varphi_j(n)$ denotes its symbol. The kernels $\Psi_j, \tilde{\Psi}_j$ of the corresponding primal and dual wavelets are constructed analogously with their symbols $\psi_j(n)$, $\tilde{\psi}_j(n)$ given by the refinement equation:

$$\psi_j(n)\tilde{\psi}_j(n) = (\varphi_{j+1}(n))^2 - (\varphi_j(n))^2. \quad (2)$$

Since the considered gravitational field models are provided in terms of spherical harmonics, we may regard them to be bandlimited of degree M. For including the maximal information of the models, we choose the highest scale J such that that $2^J > M$.
By using an equiangular grid we integrate exactly spherical harmonics up to the degree of the integrand (if it is bandlimited, otherwise we obtain an approximation) in (1). Thus, we write:

$$V_J(x) \approx \sum_{i=1}^{(N_{j_0}+1)^2} w_i^{j_0} \underbrace{\int_{\Omega_R} \Phi_{j_0}(z, y_i^{j_0}) V(z) d\omega(z)}_{a_i^{j_0}} \Phi_{j_0}(x, y_i^{j_0})$$
$$+ \sum_{j=j_0}^{J-1} \sum_{i=1}^{(N_j+1)^2} w_i^j \underbrace{\int_{\Omega_R} \Psi_j(z, y_i^j) V(z) d\omega(z)}_{c_i^j} \tilde{\Psi}_j(x, y_i^j).$$

In the latter formulae the $(y_i^{j_0}, w_i^{j_0})$ and (y_i^j, w_i^j) denote the locations on Ω_R and corresponding weights of the Driscoll-Healy integration scheme (cf. [1]).
The scaling function and wavelet coefficients $a_i^{j_0}$ and c_i^j are also obtained by numerical integration. Therefore, the coefficients are

$$a_i^{j_0} \approx \sum_{k=1}^{(\tilde{N}_{j_0}+1)^2} \tilde{w}_k^{j_0} \Phi_{j_0}(\tilde{z}_k^{j_0}, y_i^{j_0}) V(\tilde{z}_k^{j_0}), \quad c_i^j \approx \sum_{k=1}^{(\tilde{N}_j+1)^2} \tilde{w}_k^j \Psi_j(\tilde{z}_k^j, y_i^j) V(\tilde{z}_k^j).$$

3 Selected Examples

We have chosen three different types of wavelets: the bandlimited Shannon and Cubic Polynomial and the non-bandlimited Abel-Poisson kernel.

Shannon Wavelets: In the case of Shannon scaling functions the symbol $\varphi_j(n)$ reads as follows

$$\varphi_j^{SH}(n) = \begin{cases} 1 & \text{for} \quad n \in [0, 2^j) \\ 0 & \text{for} \quad n \in [2^j, \infty), \end{cases}$$

and for the corresponding wavelets we choose the P-scale version to resolve the refinement equation (2), i.e.

$$\tilde{\psi}_j^{SH}(n) = \psi_j^{SH}(n) = \sqrt{\left(\varphi_{j+1}^{SH}(n)\right)^2 - \left(\varphi_j^{SH}(n)\right)^2}.$$

Cubic Polynomial (CuP) Wavelets: In the CuP case the symbol takes the following form:

$$\varphi_j^{CP}(n) = \begin{cases} (1 - 2^{-j}n)^2(1 + 2^{-j+1}n) & \text{for} \quad n \in [0, 2^j) \\ 0 & \text{for} \quad n \in [2^j, \infty) \end{cases}$$

and for the corresponding wavelets we apply again the P-scale version. The parameters for the discretization of the integrals in (1) are also taken as above.

Abel-Poisson Wavelets: For the Abel-Poisson scaling function the symbol takes the following form:

$$\varphi_j^{AP}(n) = e^{-2^{-j}\alpha n}, \quad n \in [0, \infty), \quad \text{with some constant } \alpha > 0.$$

We choose $\alpha = 1$. Since $\varphi_j^{AP}(n) \neq 0$ for all $n \in \mathbb{N}$ this symbol leads to a non-bandlimited kernel.

It should be noted that the Abel-Poisson scaling function has a closed form representation which allows the omission of a series evaluation and truncation, i.e. for $x, y \in \Omega_R$, i.e. $|x| = |y| = R$ we have

$$\begin{aligned} \Phi_j^{AP}(x, y) &= \sum_{n=0}^{\infty} \frac{2n+1}{4\pi R^2} \varphi_j^{AP}(n) P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right) \\ &= \frac{1}{4\pi R^2} \frac{1 - e^{-2^{-j+1}}}{\left(1 + e^{-2^{-j+1}} - 2e^{-2^{-j}}t\right)^{\frac{3}{2}}}. \end{aligned}$$

When constructing bilinear Abel-Poisson wavelets we want to keep such a representation as an elementary function. Thus, we decide to use M-scale wavelets whose symbols are deduced from the refinement equation (2) by the third binomial formula:

$$\begin{aligned} \psi_j^{AP}(n) &= \left(\varphi_{j+1}^{AP}(n) - \varphi_j^{AP}(n)\right) \\ \tilde{\psi}_j^{AP}(n) &= \left(\varphi_{j+1}^{AP}(n) + \varphi_j^{AP}(n)\right). \end{aligned}$$

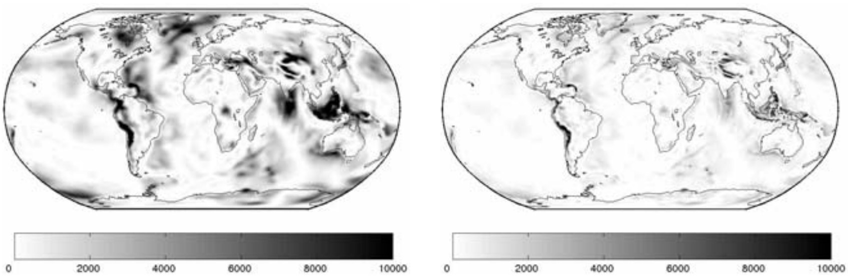
Since the Abel-Poisson scaling function and its corresponding wavelets are non-bandlimited we obtain just a good approximation by the method based on an equiangular grid (we choose the parameter of polynomial exactness sufficiently large enough).

4 Wavelet Variances

The wavelet coefficients c_i^j can also be used to compute the scale and space variance of V at the positions y_i^j and scale j. These variances at the positions y_i^j are given by

$$\mathrm{Var}_{j\,;\,y_i^j}(V) = \int_{\Omega_R}\int_{\Omega_R} V(x)V(z)\Psi_j(x,y_i^j)\Psi_j(z,y_i^j)d\omega(x)d\omega(z) = \left(c_i^j\right)^2 .$$

The scale variance of V at scale j, $\mathrm{Var}_j(V)$, is then defined as the integral over the scale and space variances which can be evaluated using the coefficients c_i^j. For more about wavelet variances, see [4].



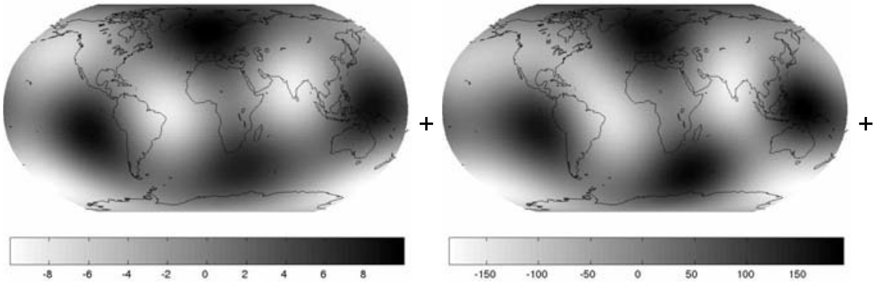
CP-wavelet variances for scales 6 (left) and 7 (right) of EGM96, in $[m^4/s^4]$

5 Reconstruction

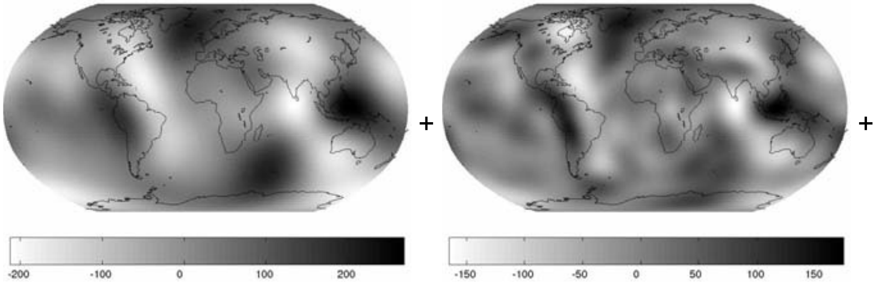
We supply to the end-user the scaling function or wavelet coefficients, $a_i^{j_0}$ or c_i^j corresponding to some locations on Ω_R. One reconstructs the J-level representation of the potential V by

$$V_J(x) \approx \sum_{i=1}^{(N_{j_0}+1)^2} w_i^{j_0} a_i^{j_0} \Phi_{j_0}(x, y_i^{j_0}) + \sum_{j=j_0}^{J-1} \sum_{i=1}^{(N_j+1)^2} w_i^j c_i^j \tilde{\Psi}_j(x, y_i^j). \qquad (3)$$

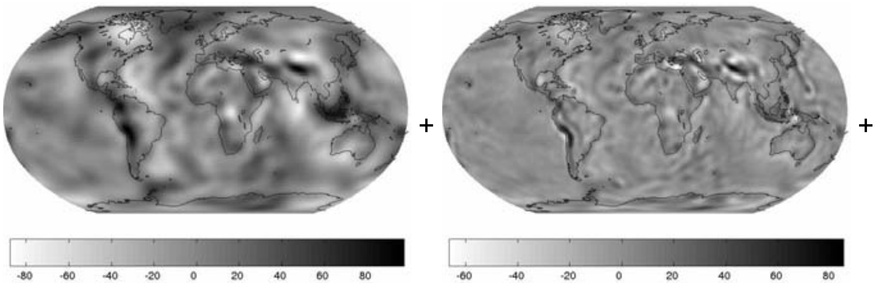
A full multiscale analysis of the EIGEN2 with CP-wavelets is exemplarily given below. (Note that all figures are in $[m^2/s^2]$.)



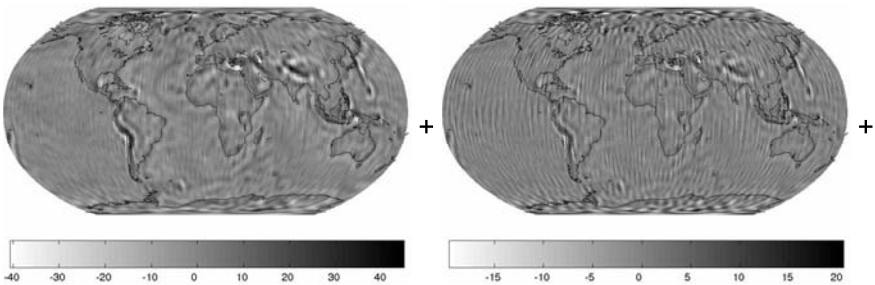
The CP-scaling function with $j_0 = 2$ (left) and the CP-wavelet with $j = 2$ (right)



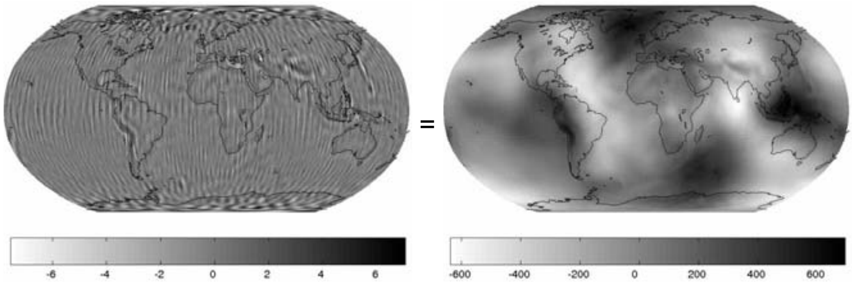
The CP-wavelet with $j = 3$ (left) and $j = 4$ (right)



The CP-wavelet with $j = 5$ (left) and $j = 6$ (right)

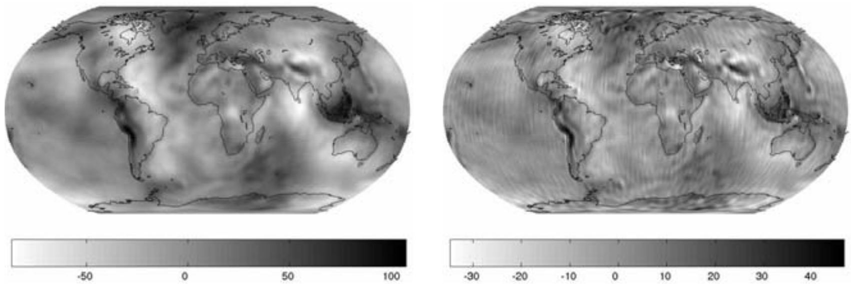


The CP-wavelet with $j = 7$ (left) and $j = 8$ (right)



The CP-wavelet with $j = 9$ *(left) and the whole sum (3) (right)*

Moreover, we present some details of the Abel-Poisson multiresolution of EIGEN2. The sectorial parts of the model are resolved more and more by the higher scales.



The AP-wavelets of scale 6 (left) and 8 (right) in $[m^2/s^2]$.

Location in the Web

The coefficients, a detailed model description as well as further figures can be found and downloaded at the following web page:

http://www.mathematik.uni-kl.de/˜wwwgeo/waveletmodels.html

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