Application of Eigenvalue Decomposition in the Parallel Computation of a CHAMP 100x100 Gravity Field

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Summary. To obtain an alternative gravity solution to that of EIGEN1S, the author's Singular Value Decomposition(SVD) tool, **P**arallel **LA**rge **S**vd **S**olver (PLASS), was applied to the CHAMP normal matrix **ngl-eigen-1s** [2] to perform an Eigenvalue Decomposition (EVD) analysis. The EIGEN1S solution is based on the Tikhonov regularization method of approximating the ill-conditioned system of equations in a subspace of lower rank. In the EVD solution, poorly determined linear combinations of parameter corrections are removed in the culpable eigenspace of the unconstrained least-squares normal equation. The selection of eigenvalues to be removed, is based upon a new method and four different common optimization (truncation) criteria. The new method, the Kaula Eigenvalue (KEV) relation, optimizes the removal of eigenvalues to best satisfy Kaula's Rule. The four other techniques are: inspection, relative error, norm-norm minimization, and finding the minimum trace of the mean square error (MSE) matrix. Analysis of the five different EVD gravity fields was performed. Two of them were shown to be comparable to the EIGEN1S CHAMP solution obtained by the GeoForschungsZentrum Potsdam (GFZ) [2]. The best of the five optimal solutions, that of the KEV, is presented. The number of estimated parameters is 11216.

Key words: eigenvalue disposal, Kaula's rule

1 Eigenvalue Disposal

To illustrate the effect of eigenvalue truncation on the inversion for solution, the inspection analysis, which is based on the graph of the eigenvalues versus number (where "number" is the ith eigenvalue), is discussed first. Figure 1 shows the spread of eigenvalues. The largest and smallest eigenvalues are $7.17x10^{25}$ and $8.08x10^{9}$, respectively, which yields a condition number of $8.87x10^{15}$; indicating an ill-conditioned system of equations. The eigenvalues are displayed from largest to smallest.

Because the smallest eigenvalue is much greater than zero and there is a smooth transition throughout most of the graph, it is difficult to determine which of the eigenvalues are responsible for the ill-conditioned nature of the normal matrix. To illustrate the stabilization effect of eigenvalue disposal, solutions were calculated in which the smallest 4000, 6000, and 10000 eigenvalues were set to zero. These are compared to a solution retaining all

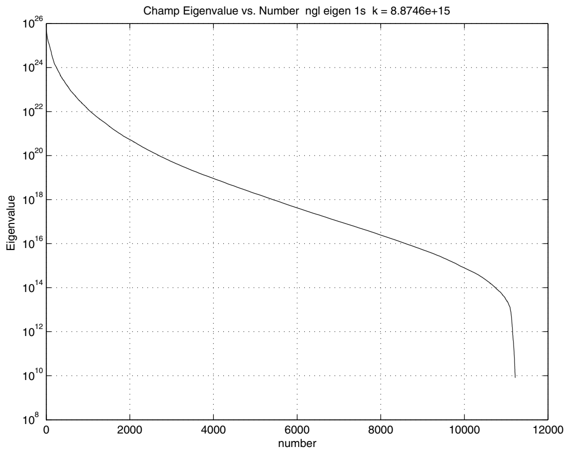

Fig. 1. CHAMP Eigenvalue vs. Number.

eigenvalues. Figure 2 is a graph of degree amplitude versus harmonic degree overlayed with Kaula's rule. This shows that much of the excess power above harmonic degree of about 35 is removed, when the culpable eigenvectors contributing to this inflation (through their linear combination in the eigenspace), are eliminated by setting their eigenvalues to zero. Because the inspection method is somewhat subjective, it is useful only for illustrative purposes.

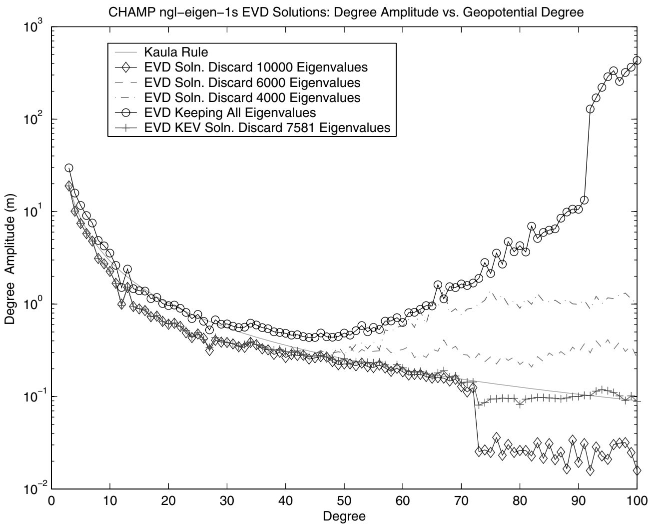

Fig. 2. CHAMP: All Eigenvalues Soln., Inspection Soln.'s, KEV EVD Soln.

2 Kaula Eigenvalue (KEV) Relation

The KEV method relates the disposal of eigenvalues, in the EVD stabilization of a gravity field solution, to Kaula's power rule of thumb. Since an EVD solution is affected by eigenvalue inclusion/exclusion, there must exist a relation between eigenvalue truncation and the equations of gravity field estimation. A gravity field solution is an estimated parameter vector, whose elements are the scaled dimensionless coefficients, $C_{l,m}$ and $S_{l,m}$. These parameters are the constants that are multiplied against the basis functions appearing in the spherical harmonic expansion, which is used in the equation to describe a three dimensional gravitational potential in the free (zero density) space above the Earth. The connection between eigenvalues and these spherical harmonic coefficients is revealed through the use of the degree variance equation, $\sigma_l^2 = \sum_{m=0}^{l}(C_{l,m}^2 + S_{l,m}^2)$. Since the power of these coefficients at a particular harmonic degree l are closely approximated by Kaula's rule, eigenvalue truncation/disposal can be manipulated to best satisfy this criterion. Thus, a series of these scalar power values can be monitored as eigenvalues are truncated (one at a time) for each inversion case and a minimum difference between the generated power curves of Kaula's rule and that of the truncated EVD solutions, can be found. The following illustrates this concept.

For ease of description, let us label the one dimensional storage array containing the Kaula power at all harmonic degrees and another which contains the degree variance as determined by an EVD solution, as the Kaula and EVD "vector", respectively. Thus the "Kaula vector" $\overline{v}_K$ can be constructed using Kaula's rule, and the estimated coefficients of the EVD solution defines the elements of the "EVD-vector", $\overline{v}_{EVD}$. The *ordering* of the elements for both vectors are identical and is based upon the sequence of the estimated coefficients in the EVD solution. A relation between Kaula's power rule and eigenvalue truncation/disposal is discovered by taking the two-norm of the difference of these two vectors, yielding the scalar, $\alpha = \|\overline{v}_{EVD} - \overline{v}_K\|_2$. This is equivalent to taking the square root of the sum of the squares of the differences between the vectors for every "jth" solution for a particular number of used eigenvalues "u". The following equations illustrate this.

$$\alpha(j;u) = \left\{ \sum_{l=1}^{lmax} [v_{EVDl} - v_{Kl}]^2 \right\}^{\frac{1}{2}}, \tag{1}$$

where,

$$v_{EVDl} = \left[\sum_{m=0}^{l} (C_{l,m}^2 + S_{l,m}^2) \right] \text{ and } v_{Kl} = \left\{ \frac{10^{-10}(2l+1)}{l^4} \right\}. \tag{2}$$

By constructing this "vector" pair for each new EVD solution, according to each new combination of eigenvalues, the behavior of the dimensionless scalar

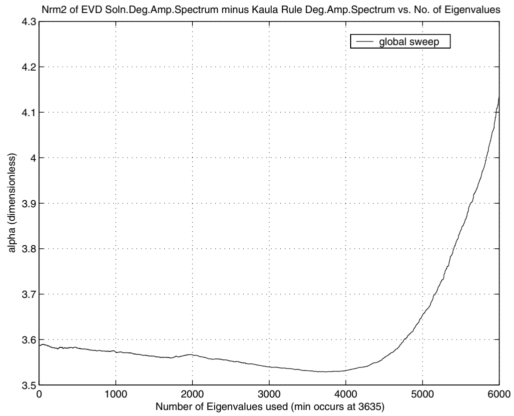

Fig. 3. CHAMP: Global Minimum of Used Eigenvalue Cases.

$\alpha(j;u)$ may be plotted against u, the number of used eigenvalues. Thus a function relating Kaula's rule to eigenvalue truncation may be plotted. It is the minimum of this function that corresponds to the optimal choice of eigenvalues, for the gravity solution that best satisfies Kaula's rule. By sweeping through many solutions, the global optimum (minimum) is very quickly found. Figure 3 displays the magnified view of the area where the global minimum occurs for "ngl-eigen-1s", which is at 7581 discarded eigenvalues (3635 included). Figure 2 also includes this KEV EVD gravity solution expressed as degree amplitude versus harmonic degree overlayed with Kaula's rule. It is the EVD solution which closely follows the entire length of Kaula's power curve. About 68 percent of all eigenvalues were discarded for this optimum solution.

3 Evaluation of Gravity Field

Orbital Arc Fit Computations. The satellites selected to fly through the estimated EVD gravity fields are shown in Table 1. All arc fits were computed using UTOPIA [1] and compared with the actual observation data for a chosen satellite. Table 1 shows the SLR orbit fits in centimeters RMS of all five candidate gravity fields on all selected satellites. Notice, that for the Inspection case (1216 eigenvalues used), all satellites fall out of orbit, leading to its rejection. The case EIGEN1S is the gravity field produced by the GFZ from the same CHAMP normal matrix "ngl-eigen-1s" used in this investigation. It is this EIGEN1S gravity field to which all EVD fields of this investigation were compared. Other than the GFZ1 satellite, the RMS fits are fairly similar for all cases of the EVD gravity fields and the EIGEN1S. However, the EVD deflation effects are best seen in the orbit fit residual of the low altitude satellite GFZ1.

Case	GFZ1	Lageos 1	Lageos 2	Starlette	Stella	Topex
Inspection	crash	crash	crash	crash	crash	crash
KEV	11.52	8.12	10.77	3.08	3.64	2.32
MSE	15.65	10.77	10.77	2.95	3.64	2.32
Norm-Norm	11.34	8.13	10.77	3.08	3.64	2.39
Relative Error	11.22	8.12	10.77	2.97	3.64	2.34
EIGEN1S	74.03	8.11	10.76	3.07	3.31	2.37

Table 1. Orbital Arc Fits of Candidate Gravity Fields (cm. radial RMS).

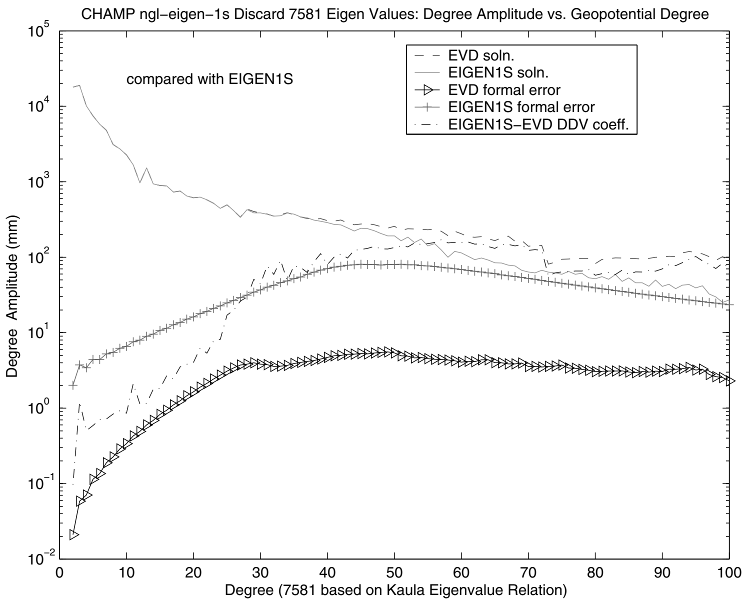

Fig. 4. CHAMP: KEV Degree Error Var. and Var. Geopotential Difference to EIGEN1S.

EVD Degree Error Variance and Geopotential Variance Difference vs. EIGEN1S. The covariance matrix corresponding to a truncated Eigenvalue Decomposition (EVD) estimated solution, is not an adequate measure of error for an estimate. Because not all eigenvectors were included into the estimation process, the estimate is biased. The calculated gravity field is "shifted" by some amount away from the true gravity field and the confidence in the estimated coefficients may be too optimistic, i.e. their variances are not an accurate indication of the difference between the estimated gravity field and the true gravity field. However, if the bias introduced by an EVD estimate is "small", its covariance may be considered unbiased in an approximate sense. Although the KEV EVD estimate is biased, its variances and those of an unbiased gravity solution (reference field) may be compared to evaluate the difference between the two fields with respect to the error variance of the reference field. Figure 4 shows the degree error variance of the KEV EVD solution in comparison to that of the EIGEN1S reference field. The differences between their geopotential coefficient variance spectra along with the EIGEN1S solution, is shown (Degree Difference Variance (DDV)). The formal error of the biased KEV solution are all within the error variances of EIGEN1S, implying that this candidate EVD field is within the uncertainty (in a random sense) of the EIGEN1S gravity field. However the differences in the coefficients between the two solutions, as seen in the DDV curve, become larger than the EIGEN1S formal error above a harmonic degree of about 30. This indicates that the KEV EVD solution may be too optimistic in this region.

Conclusion. **PLASS** demonstrates a new feasibility in the application of the EVD in the solution for large gravity fields. Employing the KEV technique, the removal of 7581 eigenvalues, was deemed optimum. The bias in this solution caused no deleterious effects detected by the analyzes of this investigation.

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