2 A Little Dynamics

2.1 Single-Degree-of-Freedom Systems

Precisely as for the treatment of fluid-borne sound, the kinematic and dynamic field variables constitute the primary ingredients in the treatment of structure-borne sound. An important difference is, however, that in most measurements of fluid-borne sound, the primary quantity is dynamic i.e., the sound pressure whereas it is a kinematic in measurements of structure-borne sound i.e., the displacement, velocity or acceleration. The dynamic quantities such as forces or stresses can be obtained via the constitutive relations for the material in the latter case. The reason for this difference is that the pressure is a scalar quantity and hence most manageable whilst the vectorial motion quantities, despite their direction dependence, are simpler to measure for structures.

As in classical dynamics, a corner stone for structure-borne sound is the single-degree-of-freedom system, often called the mass-spring-damper system. For such a system, the equation of motion can be written as

$$m\frac{d^2\xi}{dt^2} + r\frac{d\xi}{dt} + s\xi = F \tag{2.1}$$

where ξ is the displacement from equilibrium.

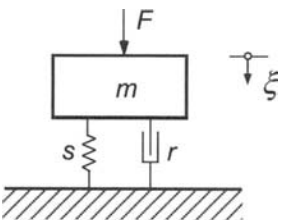


Fig. 2.1. Single-degree-of-freedom system with excitation F and displacement ξ

In this system, depicted in Fig. 2.1, the mass is denoted *m,* the viscous damping *r* and the stiffness *s*. If there is no excitation such that $F = 0$, the general solution is

$$\xi(t) = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t}.$$

Here the two constants C_1 and C_2 are to be determined from the initial conditions at time $t = 0$. α_1 and α_2 are the eigen-values i.e., the two roots of the characteristic equation

$$m\alpha^2 + c\alpha + s = 0.$$

With an external excitation ($F \neq 0$) the particular solution must be added such that

$$\xi(t) = \left[C_1 + \frac{1}{\alpha_1 - \alpha_2} g_1(t)\right] e^{\alpha_1 t} + \left[C_2 + \frac{1}{\alpha_2 - \alpha_1} g_2(t)\right] e^{\alpha_2 t},$$

where g_1 and g_2 represent the (indefinite) integrals

$$g_1 = \int \frac{F(t)}{m} e^{-\alpha_1 t} dt$$

$$g_2 = \int \frac{F(t)}{m} e^{-\alpha_2 t} dt,$$

and C_1 and C_2, as before, are the integration constants which satisfy the initial conditions.

Assuming now that the system is subject to a steady-state harmonic excitation at the angular frequency ω such that the excitation is given in the form

$$F(t) = \hat{F} e^{j\omega t},$$

where $\hat{F}$ is the force amplitude and

$$e^{j\omega t} = \cos\omega t + j\sin\omega t.$$

The use of the complex exponential $e^{j\omega t}$ to describe the time dependence of the physical force means that implicitly either is understood

$$F(t) = \mathrm{Re}\left[\hat{F} e^{j\omega t}\right] = \hat{F}\cos\omega t,$$

or, alternatively,

$$F(t) = \mathrm{Im}\left[\hat{F} e^{j\omega t}\right] = \hat{F}\sin\omega t.$$

The selection of the real or imaginary part is arbitrary as long as this choice is retained throughout the analysis and the physical interpretation of the results is made consistently. Herein, the first alternative will be employed.

Now, once the initial transients have faded, the steady-state response must also be harmonic of the same frequency as that of the excitation and one can assume that it can be written in the form

$$\xi(t) = D(\omega)\hat{F}e^{j\omega t},$$

where $D(\omega)$ is the complex frequency response function of the system. Upon substituting this into the equation of motion (2.1), it is clear that

$$\left(-\omega^2 m + jr\omega + s\right)D(\omega) = 1,$$

which means that

$$D(\omega) = \frac{1}{-m\omega^2 + jr\omega + s}.$$

If the undamped natural frequency squared is defined by

$$\omega_0^2 = \frac{s}{m},$$

together with the damping ratio as

$$\zeta = \frac{r}{2m\omega_0},$$

the frequency response function can be re-written as

$$D(\omega) = \frac{1}{m\left[-\omega^2 + 2\zeta\, j\omega\omega_0 + \omega_0^2\right]}.$$

From this complex-valued function, the amplitude ratio between the response and excitation can be developed to

$$|D| = \frac{1}{m\omega_0^2\left[\left(2\zeta\dfrac{\omega}{\omega_0}\right)^2 + \left(1 - \dfrac{\omega^2}{\omega_0^2}\right)^2\right]^{1/2}},$$

and the associated phase relation

$$\varphi_D = \arctan\left[\frac{\zeta\dfrac{\omega}{\omega_0}}{\left(1-\dfrac{\omega^2}{\omega_0}\right)}\right].$$

So far, all what has been discussed follows the classical presentation of mechanical vibrations. For different reasons, however, use is made of velocity instead of displacement in acoustics and structure-borne sound. One such reason is that the scalar product of force and velocity yields the power, which is a convinient measure for stationary processes. Therefore, upon rewriting the equation of motion for the mass-spring-damper system in terms of the velocity,

$$\left(j\omega m + r + \frac{s}{j\omega}\right)v = F.$$

In this case, the employment of the frequency response function leads to the response

$$v(t) = Y(\omega)\hat{F}e^{j\omega t},$$

such that by means of a substitution into the equation of motion,

$$Y(\omega) = \frac{1}{j\omega m + r + \dfrac{s}{j\omega}}. \tag{2.2}$$

In Fig. 2.2, the magnitude and phase of this version of the frequency response function – conventionally termed the mobility – is plotted versus non-dimensional frequency ω/ω_0 for some different damping ratios. As can be seen, the low frequency region is featured by a positive slope, proportional to frequency. This corresponds to a stiffness controlled behaviour of the system i.e., the term $s/j\omega$ in the denominator is the governing for small ω. Close to the undamped natural frequency ω_0, the response grows markedly, only constrained by the damping r. In this range, ωm and s/ω are numerically quite close but of opposite signs, precisely to cancel at the undamped natural frequency. The fact that for the damped system, the maximum of the mobility is slightly shifted from ω_0 is related to the minimum of the denominator. The maximum is shifted downwards from the frequency where the imaginary part vanishes and the phase therefore has a zero crossing. For the high frequency region, the mass term in Eq. (2.2) dominates the denominator, which means that the mobility becomes inversely proportional to frequency, resulting in a negative slope.

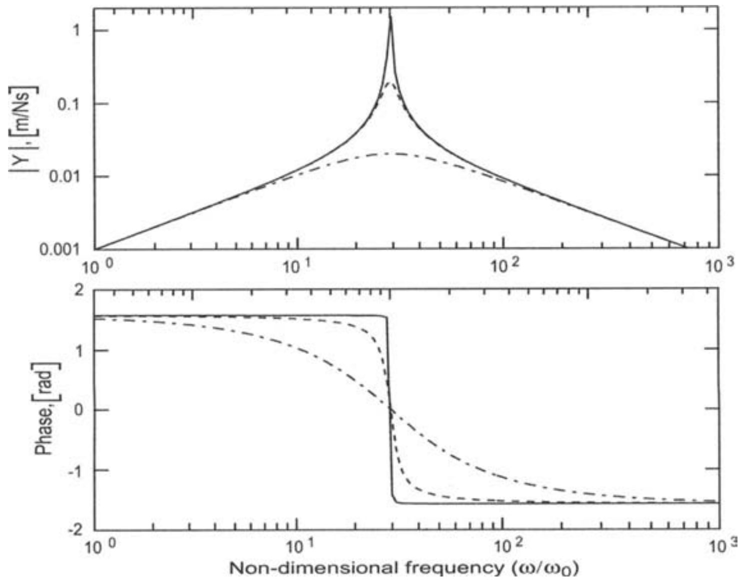


Fig. 2.2. Mobility of mass-spring-damper system for three values of damping versus non-dimensional frequency

In the phase, the stiffness controlled region is associated with a positive sign – the velocity leads the force – whereas for the mass controlled range the phase is negative – the velocity lags the force. As can be expected, there in between is a region in which the phase approaches zero and the velocity is in phase with the force. This is the resonantly or resistively controlled region where the damper essentially dictates the behavior of the system.

For the upside-down system, see Fig. 2.3, where the supporting structure realizes a constant harmonic motion, the free body diagram gives

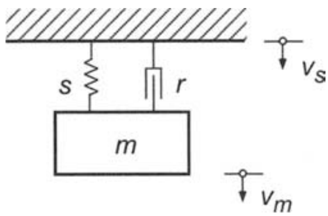


Fig. 2.3. Resiliently suspended mass with a moving base

$$\left(v_s - v_m\right) r + \left(v_s - v_m\right) \frac{k}{j\omega} = F, \tag{2.3}$$

$$j\omega m v_m = F. \tag{2.4}$$

Upon equating these linear relations the velocity ratio

$$\frac{v_m}{v_s} = \frac{1 + \dfrac{jr\omega}{k}}{1 - \dfrac{\omega^2 m}{k} + \dfrac{j\omega r}{k}} = \frac{1}{1 - \dfrac{\omega^2}{\omega_0^2} + \dfrac{j\omega^2}{\omega_0^2} 2\zeta}$$

results, demonstrating that for frequencies below the undamped natural frequency, the velocity of the mass is essentially that of the supporting structure. For high frequencies on the other hand, the mass velocity is substantially smaller than that of the supporting structure. Only in the vicinity of the undamped natural frequency ω_0, the mass velocity exceeds that of the supporting structure. Again the maximum is slightly displaced from the undamped natural frequency, see Fig. 2.4

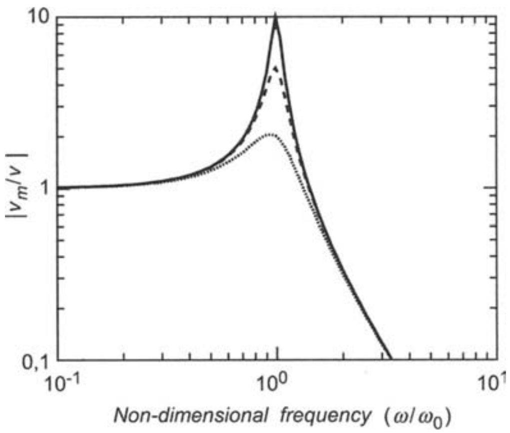


Fig. 2.4. Transfer function $H(\omega) = v_m / v_s$ for a resiliently suspended mass for three values of damping

2.2 Lagrange's Equations

Naturally, the approach of raising the equations of motion via force equilibrium can be used also for more complicated configurations than the two previously discussed. Another possibility, which can be rather advantageous, is that termed Lagrange's equations of motion. Thereby, the explicit need of the force is removed and since the terms involved are primarily quadratic, the problem of sign errors markedly reduced. The Lagrange's equations are an alternative, energy based statement of Newton's law of motion and a derivation can start either from Newton's laws or, as will be seen in subsequently, from Hamilton's principle [2.1]. The method for deriving the equations of motion using Lagrange's equations consists of two steps:

i) Determination of the total kinetic and potential energies of the system and

ii) differentiation of the energies with respect to velocities and displacements.

To demonstrate the method, the system depicted in Fig. 2.5 will be considered. The total kinetic energy for the system can be written as

$$E_{kin} = \frac{1}{2}\sum_{\nu}^{6} m_{\nu} v_{\nu}^{2}. \tag{2.5}$$

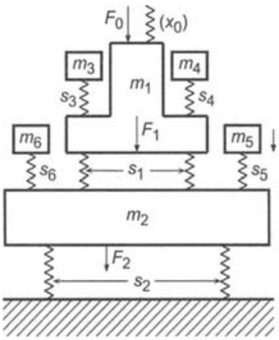
The potential energy is contained only in the springs in this case and it can be obtained via the assumption that each spring is compressed in several small steps of length Δl. After the q^{th} compression step, the total compression is $l_q = q \cdot \Delta l$ and at the end of the compression i.e., $q = Q$, the total compression is $Q\Delta l$. For each step, the compression force can be found from Hooke's law as $F_q = \Delta l \cdot s$, where s is the spring stiffness and adding all these steps the potential energy is given by

$$\begin{aligned} E_{pot} &= \sum_{q=0}^{Q} F_q \cdot l_q = s\sum_{q=0}^{Q} \Delta l \cdot l_q = s\sum_{q=0}^{Q} q\left(\Delta l\right)^2 = s\left(\Delta l\right)^2 \sum_{q=0}^{Q} q \\ &= s\left(\Delta l\right)^2 \frac{Q\left(Q+1\right)}{2} \approx \frac{1}{2} s\left(\Delta l \cdot Q\right)^2 = \frac{1}{2} s\left(l_Q\right)^2 . \end{aligned}$$

The potential energy stored in a spring, therefore, is proportional to the total compression squared. When this relation is applied to the system in Fig. 2.5, the potential energy is found to be given by

$$\begin{aligned} E_{pot} = \frac{1}{2}\Big\{ & s_1\left(\xi_1 - \xi_2\right)^2 + s_2\xi_2^2 + s_3\left(\xi_1 - \xi_3\right)^2 + s_4\left(\xi_1 - \xi_4\right)^2 \\ & s_5\left(\xi_2 - \xi_5\right)^2 + s_6\left(\xi_2 - \xi_6\right)^2 + s_0\left(\xi_0 - \xi_1\right)^2 \Big\}. \end{aligned} \tag{2.6}$$

It should be noted that the resulting energy is independent of whether, for example, $(\xi_1 - \xi_2)$ or $(\xi_2 - \xi_1)$ is used and the sign does not cause a problem in this case. In the expression above are included the dummy spring s_0 and its (zero) compression. This establishes a way to account for a possible external force.



$$E_0 - E_1 + E_3 + E_4 = -\omega^2 m_1 \xi_1 ;\; E_1 = s_1(\xi_1 - \xi_2)$$

$$E_1 - E_2 + E_5 + E_6 = -\omega^2 m_2 \xi_2 ;\; E_2 = s_2 \xi_2$$

$$0 - E_3 = -\omega^2 m_3 \xi_3 ;\; E_3 = s_3(\xi_3 - \xi_1)$$

$$0 - E_4 = -\omega^2 m_4 \xi_4 ;\; E_4 = s_4(\xi_4 - \xi_1)$$

$$0 - E_5 = -\omega^2 m_5 \xi_5 ;\; E_5 = s_5(\xi_5 - \xi_2)$$

$$0 - E_6 = -\omega^2 m_6 \xi_6 ;\; E_6 = s_6(\xi_6 - \xi_2)$$

Fig. 2.5. Multi-degree-of-freedom system

As is readily found from a textbook on methods of theoretical physics, the procedure with the generalized Lagrange's equation consists in forming the expression

$$\frac{d}{dt}\frac{\partial\left(E_{pot} - E_{kin}\right)}{\partial v_i} - \frac{\partial\left(E_{pot} - E_{kin}\right)}{\partial \xi_i} = 0, \qquad (2.7)$$

for i = 1, 2, 3, 4,… Here, v_i and ξ_i are the unknown velocities and displacements respectively but not the prescribed and thus known displacement ξ_0. It must be pointed out that in these calculations, v_i and ξ_i are real, time-dependent variables and no phasors.

In all cases treated herein as for many other problems in dynamics, it is possible to choose the co-ordinates in such a way that the kinetic energy depends only on the velocities whilst the potential energy only on the displacements. Equation (2.7) can accordingly be rewritten as

$$\frac{d}{dt}\frac{\partial E_{kin}}{\partial v_i} + \frac{\partial E_{pot}}{\partial \xi_i} = 0 \;\; ; \;\; i = 1, \; 2, \; 3, \; 4, ... \qquad (2.8)$$

Upon substituting Eqs. (2.5) and (2.6) into (2.7), the differentiations furnish six linear equations, establishing the equations of motion

$$\begin{bmatrix} a_{11} & -s_1 & -s_3 & -s_4 & 0 & 0 \\ -s_1 & a_{22} & 0 & 0 & -s_5 & -s_6 \\ -s_3 & 0 & a_{33} & 0 & 0 & 0 \\ -s_4 & 0 & 0 & a_{44} & 0 & 0 \\ 0 & -s_5 & 0 & 0 & a_{55} & 0 \\ 0 & -s_6 & 0 & 0 & 0 & a_{66} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \end{bmatrix} = \begin{bmatrix} s_0\xi_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

in which the abbreviations

$$a_{11} = m_1 \frac{d^2}{dt^2} + s_1 + s_3 + s_4, \quad a_{22} = m_2 \frac{d^2}{dt^2} + s_1 + s_2 + s_5 + s_6,$$

$$a_{33} = m_3 \frac{d^2}{dt^2} + s_3, \quad a_{44} = m_{14} \frac{d^2}{dt^2} + s_4, \; a_{55} = m_5 \frac{d^2}{dt^2} + s_5$$

$$\text{and} \;\; a_{66} = m_6 \frac{d^2}{dt^2} + s_6$$

have been used. An external excitation of the system can readily be accounted for by selecting the spring stiffness s_0 small so that $s_0\xi_1 \rightarrow 0$ and ξ_0 such that the product $s_0\xi_0$ equals the external force. By means of an inversion of the matrix, the unknown displacements can be found.

As an example, the mobility $Y = j\omega\xi_1/F_0$ at the excitation point is shown in Fig. 2.6 for a set of given parameters.

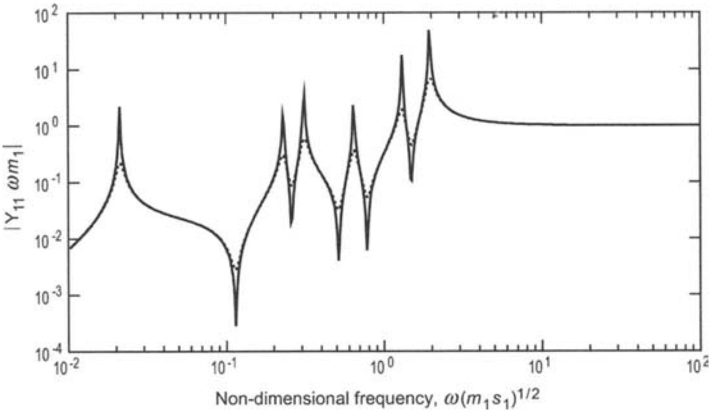


Fig. 2.6. Point mobility of the system in Fig. 2.4 at the position of F_0. Parameters: $m_2/m_1 = 1$, $m_3/m_1 = 3$, $m_4/m_1 = 1.8$, $m_5/m_1 = 5$, $m_6/m_1 = 32$, $s_2/s_1 = 0.02$, $s_3/s_1 = 0.8$, $s_4/s_1 = 1.1$, $s_5/s_1 = 0.4$, $s_6/s_1 = 0.76$, (——) $\eta = 0.01$ and (- - -) $\eta = 0.1$

The advantage of Lagrange's equations is perhaps most easily seen when the vibration of complicated system are considered. The finite element method serves a suitable example whereby every part of the system is modelled by small elements whose motion are obtained from Eq. (2.7) or some of its derivatives. Without further pursuing the features of FEM, the strength of Lagrange's equations will be illuminated by another example where the complexity is enhanced by allowing for different translatory motions as well as rotations. Considered is the system depicted in Fig. 2.7, which can translate in the x- and y-directions as well as rotate around the z-axis. For the kinetic energy, therefore, the translatory velocities v_{cx} and v_{cy} as well as the rotatory velocity $\dot{\varphi}_{cz} = d\varphi_{cz}/dt$ have to be taken into account, which yields

$$E_{kin} = \frac{1}{2}m\left(v_{cx}^2 + v_{cy}^2\right) + \frac{1}{2}\Theta\dot{\varphi}_{cz}^2, \tag{2.9}$$

where m is the mass and Θ the mass moment of inertia of the rigid block.

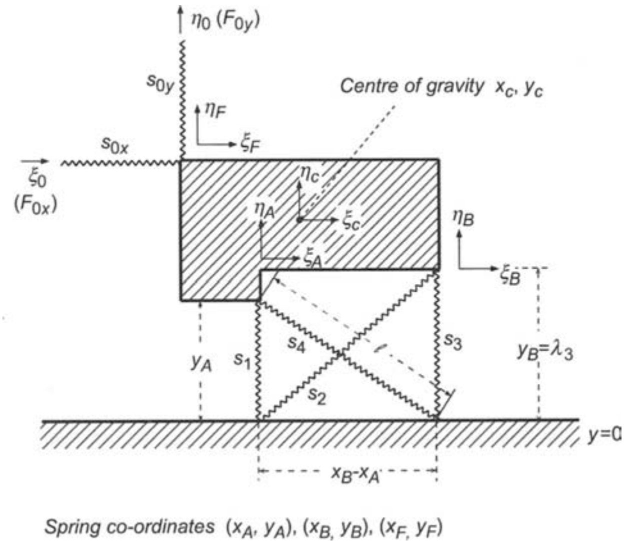


Fig. 2.7. Rigid body supported by several springs

For the potential energy a few extra steps have to be taken since the components of motion are coupled. In view of Eq. (2.6), a spring of stiffness s connecting the coordinates (x_ν, y_ν) with (x_μ, y_μ) gives

$$E_{pot} = \frac{s}{2}(l_c - l_e)^2$$

where l_c means the length of the spring after deformation and l_e is that at equilibrium. Relying upon Pythagoras, a virtual movement as shown in Fig. 2.8 means that

$$l_c = \left\{\left[(x_\nu + \xi_\nu) - (x_\mu + \xi_\mu)\right]^2 + \left[(y_\nu + \eta_\nu) - (y_\mu + \eta_\mu)\right]^2\right\}^{\frac{1}{2}}$$

The length of the spring at equilibrium is given by

$$l_e = \left\{(x_\nu - x_\mu)^2 + (y_\nu - y_\mu)^2\right\}^{\frac{1}{2}}$$

After substitution into the expression for the potential energy, it is found that

$$E_{pot} = \frac{s}{2}\; l_e^2 \left[\left\{\begin{array}{l} 1 + 2\dfrac{x_\nu - x_\mu}{l_e^2}(\xi_\nu - \xi_\mu) + 2\dfrac{y_\nu - y_\mu}{l_e^2}(\eta_\nu - \eta_\mu) \\ + \dfrac{(\xi_\nu - \xi_\mu)^2}{l_e^2} + \dfrac{(\eta_\nu - \eta_\mu)^2}{l_e^2} \end{array}\right\}^{1/2} - 1\right]^2 . \qquad (2.10)$$

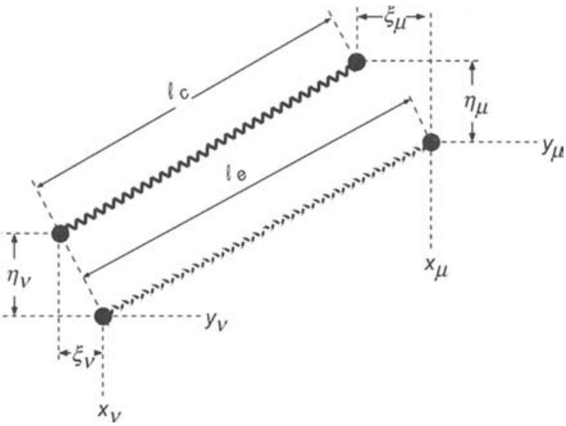


Fig. 2.8. Co-ordinates of a spring at equilibrium and in a tensioned state

Since $\left|x_\nu - x_\mu\right| \le l_e$ and $\left|\xi_\nu - \xi_\mu\right| << l_e$ as well as the corresponding for y and η, all the terms with l_e^2 in the denominator are much smaller than unity and accordingly the asymptote

$$(1+\varepsilon)^{\frac{1}{2}} \approx 1 + \frac{\varepsilon}{2},$$

is valid. Thus, the potential energy simplifies to

$$E_{pot} = \frac{s}{2} l_e^2 \left[\frac{x_\nu - x}{l_e^2}\left(\xi_\nu - \xi_\mu\right) + \frac{y_\nu - y_\mu}{l_e^2}\left(\eta_\nu - \eta_\mu\right) + R_\xi + R_\mu \right]^2$$

wherein the remainders R_ξ and R_μ represent second order terms of the motion which can be viewed as "geometric non-linearities". Such non-linearities appear even when the spring material is completely linear. Obviously, they would play an important role whenever $\left|\xi_\nu - \xi_\mu\right| \ge \left|x_\nu - x_\mu\right|$ or $\left|\eta_\nu - \eta_\mu\right| \ge \left|y_\nu - y_\mu\right|$ i.e., when the movement is of the same order of magnitude as the smallest dimension of system. This means that for very oblique springs and large displacements the geometrical non-linearities do play a role. For instance, the geometrical non-linearities are definitely present when the displacements of a plate movement are of the same order of magnitude as the plate thickness [2.2]. Also, the amplitude dependence of the contact stiffness described by Hertz [2.3] is of this type.

In this textbook, the focus is on linear oscillations, which means that the remainders R_ξ and R_η can be omitted such that the potential energy, stored in a spring, in a two-dimensional configuration can be written as

$$E_{pot} = \frac{s}{2} \left[\frac{x_\nu - x_\mu}{l_e}\left(\xi_\nu - \xi_\mu\right) + \frac{y_\nu - y_\mu}{l_e}\left(\eta_\nu - \eta_\mu\right) \right]^2 .$$

The next step is to express the displacements ξ, η relative to an arbitrary origin for the system co-ordinates. With the assumption that the centre of gravity at equilibrium is situated at (x_c, y_c) and its displacement is (ξ_c, η_c), see Fig. 2.9, then the rotational part of the translation is obtained from

$$\begin{aligned} \xi_R &= (x - x_c)\cos\varphi - (y - y_c)\sin\varphi - (x - x_c) \\ \eta_R &= (x - x_c)\sin\varphi + (y - y_c)\cos\varphi - (y - y_c). \end{aligned} \tag{2.11}$$

The trigonometric functions in Eq. (2.11), indeed, realizes a non-linear relationship between the rotation and the translation. For the oscillations of interest in this context, however, the approximations cos $\varphi \approx 1$ and sin $\varphi \approx \varphi$ are applicable and yield

$$\xi_R = -(y - y_c)\varphi,$$
$$\eta_R = (x - x_c)\varphi.$$

One can note that this kind of non-linearity comes into play by amplitudes, which are certainly larger than those associated with Eq. (2.11).

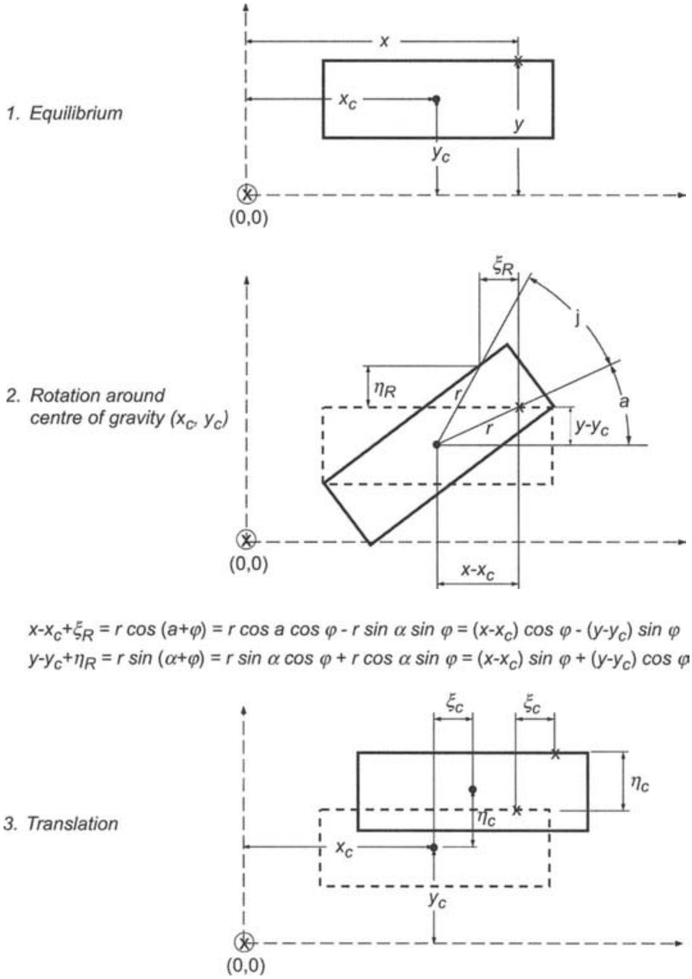


Fig. 2.9. Rotation and translation of a rigid body

By adding the translatory and rotatory parts, the net displacement of any position (x, y) on the rigid body becomes

$$\begin{aligned}\xi &= \xi_c + \xi_R = \xi_c - (y - y_c)\varphi \\ \eta &= \eta_c + \eta_R = \eta_c + (x - x_c)\varphi.\end{aligned} \tag{2.12}$$

With these components introduced in the expression for the potential energy stored in a spring and summing over the individual springs, the total potential energy of the system considered is obtained as

$$\begin{aligned}E_{pot} &= \frac{s_1}{2}\left[\frac{y_A}{l_1}\eta_A\right]^2 + \frac{s_2}{2}\left[\frac{x_B - x_A}{l_2}\xi_B + \frac{y_B}{l_2}\eta_B\right]^2 \\ &+ \frac{s_3}{2}\left[\frac{y_B}{l_3}\eta_B\right]^2 + \frac{s_4}{2}\left[\frac{x_A - x_B}{l_4}\xi_A + \frac{y_A}{l_4}\eta_A\right]^2 \\ &+ \frac{s_{0x}}{2}\left[\xi_F - \xi_0\right]^2 + \frac{s_{0y}}{2}\left[\eta_F - \eta_0\right]^2.\end{aligned} \tag{2.13}$$

In this expression s_1 to s_4 are the stiffnesses of the individual springs and l_1 to l_4 their length at equilibrium. $s_{0\mathrm{x}}$ and $s_{0\mathrm{y}}$ are the auxiliary springs for the external excitation. The explicit calculation is principally quite simple but somewhat tedious. Facilitating is a mathematical computer application with symbolic capabilities. The individual steps are:

- Substitution of (2.12) into (2.13) with $x = x_A$, $y = y_A$, $x = x_B$, $y = y_B$ and $x = x_F$, $y = y_F$ respectively.
- Substitution of the resulting total potential energy as well as the total kinetic energy in (2.9) into (2.8).
- Differentiation with respect to v_{cx}, v_{cy}, $\dot{\varphi}$, ξ_c, η_c and φ.
- Transformation to phasor notation.
- Letting the stiffnesses $s_{0\mathrm{x}}$ and $s_{0\mathrm{y}}$ approach zero with a simultaneous increase of the displacements such that the products $\xi_0 s_{0x}$ and $\eta_0 s_{0y}$ yield the forces F_{0x} and F_{0y} respectively

The resulting set of equations can be written on the form

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{Bmatrix} \xi_c \\ \eta_c \\ \varphi \end{Bmatrix} = \begin{Bmatrix} F_{0x} \\ F_{0y} \\ F_{0x} + F_{0y} \end{Bmatrix},$$

whereby $A_{\nu\mu}$ are the rather lengthy expressions containing the geometrical information.

In Fig. 2.10 are exemplified the point mobilities for two excitation positions. Observed is the strong dependence on excitation position since rotations are easily excited for this configuration. Observed are also the two 'side-resonances', which appear on either side of the main resonance. Although they are clearly visible for both excitation positions, they become particularly pronounced when the excitation is remote from the centre of gravity. This is a consequence of the fact that the springs are neither symmetrically arranged nor equal. The coupling of the different degrees of freedom and thereof resulting multiple resonance's (maximum six) are often overlooked by measurements on quasi-rigid bodies and can be the reason for errors.

The procedure with the Lagrange's equations is not only a very useful tool to establish the equations of motion but is also most suitable for deriving important general theorems for linear multiple-degree-of-freedom system. Owing to the fact that those theorems are thoroughly treated in textbooks on mechanical vibration, the most important theorems are herein given without proofs.

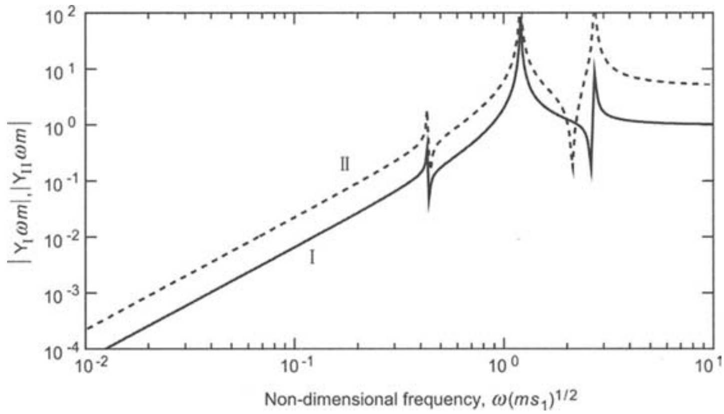


Fig. 2.10. Point mobilities at different positions of a rigid body. (——) above center of gravity and (- - -) at the edge. Parameters: $s_2 = 0.1\ s_1$, $s_3 = s_1$, $s_4 = 0.2\ s_1$, $r_Q = 0.23l$, $Q = mr_Q^2$

2.3 Reciprocity and Mutual Energy

Owing to the quadratic forms of the in this context interesting energy relations, the systems of equations are consistently symmetric. One of the consequences is the reciprocity principle. This principle establishes a relation between the field variables such that the excitation and response positions can be interchanged. The prerequisite is that the product of the variables to be interchanged yields the power or energy. A generalization of the recip-

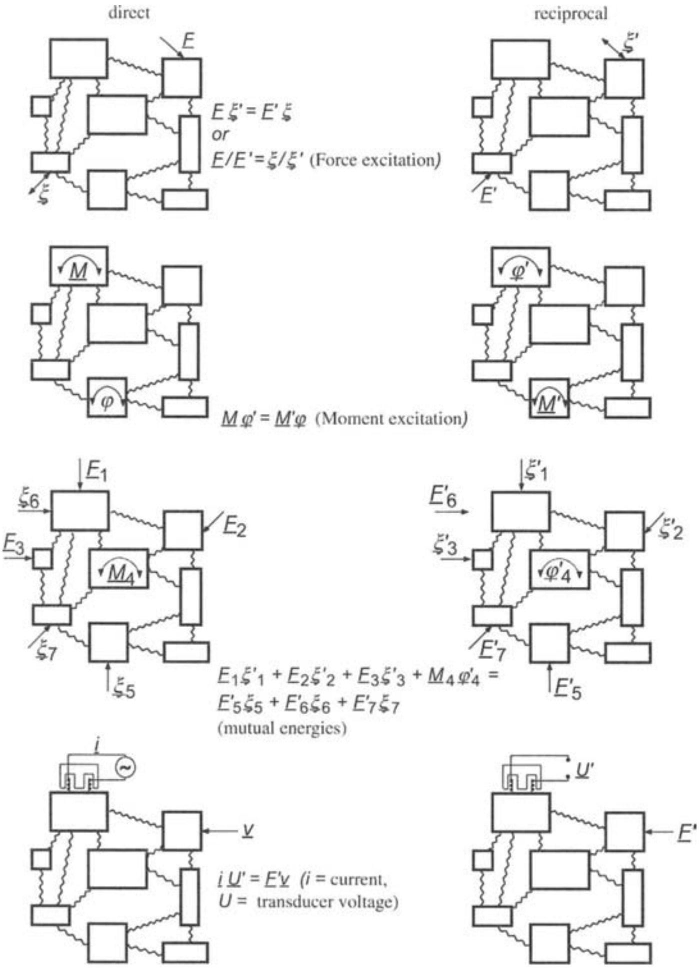


Fig. 2.11. Reciprocity and mutual energies

rocity principle is the theorem of the mutual energies [2.4]. In Fig. 2.11 are exemplified the corresponding relations in phasor notation.

For systems comprising reciprocal transducers, the mutual product of current and voltage can be comprised in the reciprocity relations. The proof can be obtained via Hamilton's principle when this is employed on electrical energy reservoirs, see Eq. (8.53).

2.4 Modal Synthesis

From linear algebra it can be shown that multiple-degree-of-freedom systems exhibit a sequence of eigen-frequencies and that those frequencies are given by the singularities of the coefficient matrix [2.5]. Corresponding to every resonance frequency are one or more characteristic amplitude distributions termed eigen-vectors or modes. Furthermore, it can be shown that any amplitude distribution can be described as a sum of modes. For low frequencies, often only a few modes suffice to give a good approximation of the vibration. The advantage is that with a knowledge of the eigenfrequencies and eigen-vectors, e.g. from an FE-analysis or an experimental modal analysis, the descriptions of the low frequency behaviour is readily established [2.5, 2.6].

2.5 Energy Considerations

In the previous sections, the kinetic and potential energies were used to develop the equations of motion. This section aims at demonstrating that the energy expressions also are highly suitable for the treatment of vibration problems.

The energy flow can be predicted from the equations of motions, which represent the equilibrium of forces, through a multiplication with the velocities of the oscillators. This is readily seen for the damped mass-spring system excited by an external force *F*. Upon multiplying the equation of motion

$$m\ddot{\xi} + r\dot{\xi} + s\xi = F \tag{2.14}$$

with the velocity $\dot{\xi}$, the power balance

$$m\ddot{\xi}\dot{\xi} + r\dot{\xi}^2 + s\xi\dot{\xi} = \frac{1}{2}\frac{d}{dt}\left(m\dot{\xi}^2\right) + \frac{1}{2}\frac{d}{dt}\left(s\xi^2\right) + r\dot{\xi}^2 = F\dot{\xi}$$

is obtained. By re-writing it as

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{\xi}^2 + \frac{1}{2}s\xi^2\right) + r\dot{\xi}^2 = F\dot{\xi},$$

it is seen that this is equivalent with

$$\frac{d}{dt}\left(E_{kin} + E_{pot}\right) + W_d = W. \tag{2.15}$$

This expression reveals that the transmitted power to the system $W = F \cdot v$ is either dissipated by the damper or serves as to change the total energy combined of the kinetic and potential parts. In stationary state, the part preceded by *d/dt* i.e., any change in total energy vanishes per definition and the power transmission equals that dissipated. In Eqs. (2.14) and (2.15) the instantaneous values are used which means that for the employment of phasor notation, it has to be observed that power or energy are products of field variables. Therefore,

$$W = F\dot{\xi} = F \cdot v = \mathrm{Re}\left\{Fe^{j\omega t}\right\}\mathrm{Re}\left\{ve^{j\omega t}\right\},$$

since the operator Re { } and the time base $e^{j\omega t}$ cannot be omitted.

Letting $\underline{F} = |\underline{F}|\ e^{j\varphi}$ and $\underline{v} = |\underline{v}|\ e^{j\psi}$ respectively, the power can be rewritten as

$$W = |\underline{F}|\ |\underline{v}|\ \mathrm{Re}\left\{e^{j(\omega t+\varphi)}\right\}\mathrm{Re}\left\{e^{j(\omega t+\psi)}\right\}$$

which is equivalent to

$$W = |\underline{F}|\ |\underline{v}|\ \cos(\omega t+\varphi)\cos(\omega t+\psi)$$

or

$$W = \frac{1}{2}|\underline{F}|\ |\underline{v}|\ \left[\cos(2\omega t+\varphi+\psi)+\cos(\varphi-\psi)\right].$$

For harmonic vibrations accordingly, the power consists of a part with the double frequency and a part that is time invariant. This means that for the usually most interesting temporal average

$$W = \frac{1}{2}|\underline{F}|\ |\underline{v}|\ \cos(\varphi-\psi) = \frac{1}{2}|\underline{F}|\ |\underline{v}|\mathrm{Re}\left\{e^{j(\varphi-\psi)}\right\} = \frac{1}{2}\mathrm{Re}\left\{|\underline{F}|e^{j\varphi}\ |\underline{v}|e^{-j\psi}\right\},$$

which clearly is equivalent to

$$W = \frac{1}{2}\mathrm{Re}\{\underline{F}\cdot\underline{v}^*\}, \tag{2.16}$$

where * denotes the complex conjugate. With this relation applied to Eq. (2.15)

$$\frac{1}{2}r\,\mathrm{Re}\left\{\dot{\underline{\xi}}\ \dot{\underline{\xi}}^*\right\} = \frac{r}{2}\left|\dot{\underline{\xi}}\right|^2 = \frac{r}{2}|\underline{v}|^2 = \frac{1}{2}\mathrm{Re}\{\underline{F}\cdot\underline{v}^*\}.$$

An extension to multiple-degree-of-freedom systems is rather simple starting from Eq. (2.7) and performing the same steps after introduction of the damping terms and external forces.

2.5.1 Minimization of the Average Energy Difference (Hamilton's Principle)

One of the most important laws in physics is Hamilton's principle. When applied in classical dynamics it states that the motion of the different components of a system always adjust themselves so that an extreme value of the temporally averaged difference between kinetic and potential energy is obtained. For the cases studied in this context the extremum is always a minimum. It is astonishing that so simply stated a principle can represent such a multitude of physical processes. Furthermore, it is noteworthy that by employing Hamilton's principle, all physical processes can be described without the concept of force i.e., "without pressure or enforcement/constraint" albeit transcriptions might be required in some cases. Some implicit problems associated with the force concept are discussed in [2.7].

The mathematical representation of Hamilton's principle can be written as

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2}\left(E_{kin} - E_{pot}\right)dt = \text{Extremum (Minimum)}\,. \tag{2.17}$$

Since a minimum or an extremum of a smooth, well-behaved function is featured by no or smaller than linear variations of the ordinate in its vicinity, the statement in Eq. (2.17) is equivalent to a vanishing variation of the energy difference i.e.,

$$\delta\int_{t_1}^{t_2}\left(E_{kin} - E_{pot}\right)dt = 0\,. \tag{2.17a}$$

The constant factor $t_2 - t_1$ thus disappears. δ denotes the variation and in the next example will be shown how it can be obtained through differentiations. Again, a single-degree-of-freedom system is considered, excited by a prescribed deformation ξ_0 of a spring s_0 as shown in Fig. 2.12. Clearly, the expression

$$\delta\int_{t_1}^{t_2}\left(\frac{1}{2}m\dot{\xi}^2 - \frac{1}{2}s\xi^2 - \frac{1}{2}s_0(\xi_0 - \xi)^2\right)dt = 0 \tag{2.18}$$

is to be manipulated. For the variation, the fact that small changes (variations) of an arbitrary function $g(x_1, x_2, \ldots)$ can be obtained from the changes of the arguments as

$$\delta\left\{g\left(x_1, x_2, \ldots.\right)\right\} = \frac{\partial g}{\partial x_1}\delta x_1 + \frac{\partial g}{\partial x_2}\delta x_2 + \ldots.$$

Herein, $\delta x_1$and δx_2 etc. are the variations of the arguments. Application on Eq. (2.18) yields

$$\int_{t_1}^{t_2} \left(m\dot{\xi}\delta\dot{\xi} - s\xi\delta\xi + s_0(\xi_0 - \xi)\delta\xi \right) dt = 0 \,.$$

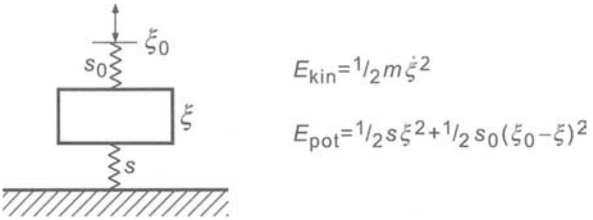


Fig. 2.12. Mass-spring system excited by means of a prescribed displacement via a spring

By means of integration by parts, the first term can be brought onto the, in this case, more useful form of

$$m\int_{t_1}^{t_2} \dot{\xi}\delta\dot{\xi}\, dt = m\int_{t_1}^{t_2} \frac{d\xi}{dt}\delta\left(\frac{d\xi}{dt}\right) dt = m\left[\frac{d\xi}{dt}\delta\xi\right]_{t_1}^{t_2} - m\int_{t_1}^{t_2} \frac{d^2\xi}{dt^2}\delta\xi\, dt \,.$$

If the integration limits are chosen such that the velocity $d\xi/dt$ is zero at those points, the term in the bracket vanishes and hence Eq. (2.18) becomes

$$\int_{t_1}^{t_2} \left(-m\frac{d^2\xi}{dt^2} - s\xi - s_0\xi + s_0\xi_0 \right) \delta\xi\, dt = 0 \,.$$

The expression above must be equal to zero for all, possibly time dependent, variations $\delta\xi$. This is only possible if the expression with the parenthesis is identically zero and accordingly

$$m\frac{d^2\xi}{dt^2} + s\xi + s_0\xi = s_0\xi_0$$

which is the equation of motion sought for the system in Fig. 2.12. With several masses and springs involved, the procedure is principally the same but since the corresponding operations are identical to those described previously in conjunction with the Lagrange's equations (Eq. (2.7)) it is here refrained from such developments. Subsequently, Hamilton's principle will be employed on other interesting cases.

2.5.2 The Rayleigh Quotient

So far, the important relations discussed have been derived from sums and differences of energies. In this section is described a procedure based on equality in energies [2.8, 2.9].

A prerequisite for the application of the Rayleigh quotient is the assumption that at all resonances (but not outside resonances) and free vibrations

$$\overline{E_{pot}} = \overline{E_{kin}} \tag{2.19}$$

is valid whereby the over bar denotes averaging over one or more periods of vibration.

If, for example, a multi-degree-of-freedom system vibrates at one of its eigen-frequencies ω_n then the displacement and the velocity of the ν-th mass have the time dependencies

$$\xi_\nu(t) = \xi_{\nu A}\cos(\omega_n t + \varphi_\nu)\,; \quad v_\nu(t) = -\omega_n \xi_{\nu A}\sin(\omega_n t + \varphi_\nu)\,,$$

respectively. Upon averaging over one or more periods is obtained

$$\overline{\xi_\nu(t)^2} = \frac{1}{2}\xi_{\nu A}^2\,; \quad \overline{v_\nu(t)^2} = \frac{1}{2}\omega_n^2 \xi_{\nu A}^2\,.$$

Accordingly, the eigen-frequency squared (eigen-value) of a multi-degree-of-freedom system, where the ν -th mass has a displacement amplitude $\xi_{\nu A}$, is given by

$$\omega_n^2 = \frac{\overline{E_{pot}}}{\frac{1}{2}\sum \frac{1}{2} m_\nu \xi_{\nu A}^2} \tag{2.20}$$

Owing to the temporal average, no phase information is required and the potential energy is also a function of the amplitude $\xi_{\nu A}$ only.

The advantages of the expression in Eq. (2.20) for the estimation of eigen-frequencies (resonance frequencies) are that

- surprisingly good approximations are obtained for the first few eigen-frequencies when realistic assumptions for the amplitudes are introduced and
- expression (2.20) has the character of a minimum such that when guessed amplitudes are employed, an upper limit is found for the eigen-frequency. Indeed, this minimum character of the Rayleigh quotient can be used to find the true amplitude iteratively by means of variation until the eigen-value ω_n^2 becomes a minimum [2.1].

To highlight this, a simple mass-spring-mass system can be considered where the two masses have the amplitudes ξ_{1A} and ξ_{2A} respectively. The temporally averaged kinetic and potential energies are thus given by

$$\overline{E_{kin}} = \frac{1}{4}\omega^2 (m_1 \xi_{1A}^2 + m_2 \xi_{2A}^2)\,; \quad \overline{E_{pot}} = \frac{1}{4} s(\xi_{1A} - \xi_{2A})^2\,,$$

which substituted into Eq. (2.20) yields

$$\omega_n^2 = s\frac{(\xi_{1A} - \xi_{2A})^2}{(m_1 \xi_{1A}^2 + m_2 \xi_{2A}^2)} = s\frac{(1-\alpha)^2}{m_1 + m_2 \alpha^2}\,,$$

in which $\alpha = \xi_{2A}/\xi_{1A}$ is introduced in the last equality. To find the eigenvalue ω_n^2, α must be chosen such that a minimum is established. As can be readily demonstrated from a vanishing first derivative, there are two extremes. Those are obtained for

$$\alpha = 1 \Leftrightarrow \xi_{2A} = \xi_{1A}$$

and

$$\alpha = -m_1/m_2 \Leftrightarrow m_2 \xi_{2A} = m_1 \xi_{1A}\ .$$

The associated eigen-frequencies are thus found to be given by

$$\omega_n^2 = 0\,; \quad \omega_\nu^2 = s\frac{m_1 + m_2}{m_1 m_2}$$

respectively, both of which are in agreement with classical results.

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