Green's Function Method Extended by Successive Approximations and Applied to Earth's Gravity Field Recovery

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Abstract

The aim of the paper is to implement the Green's function method for the solution of the Linear Gravimetric Boundary Value Problem. The approach is iterative by nature. A transformation of spatial (ellipsoidal) coordinates is used that offers a possibility for an alternative between the boundary complexity and the complexity of the coefficients of Laplace's partial differential equation governing the solution. The solution domain is carried onto the exterior of an oblate ellipsoid of revolution. Obviously, the structure of Laplace's operator is more complex after the transformation. It was deduced by means of tensor calculus and in a sense it reflects the geometrical nature of the Earth's surface. Nevertheless, the construction of the respective Green's function is simpler for the solution domain transformed. It gives Neumann's function (Green's function of the second kind) for the exterior of an oblate ellipsoid of revolution. In combination with successive approximations it enables to meet also Laplace's partial differential equation expressed in the system of new (i.e. transformed) coordinates.

Keywords

Boundary value problems · Integral kernels · Laplace's operator · Method of successive approximations · Transformation of spatial coordinates

1 Introduction

Green's functions are an important tool in solving problems of mathematical physics. Equally this holds for applications in gravity field studies. The mathematical apparatus of classical physical geodesy is a typical example. Green's function is an integral kernel, which, convolved with input values, gives the solution of the particular problem considered. Regarding its construction, there exist elegant and powerful methods for one or two dimensional problems. However, only very few of these methods carried over to higher dimensions, indeed the higher the dimension of the Euclidean space the simpler the boundary of the region of interest had to be, see Roach (1982). In order to preserve the benefit of the Green's function method a suitable approximation procedure is discussed. The aim of the paper is to implement the procedure with the particular focus on the solution of the linear gravimetric boundary value problem. Two approaches immediately suggest themselves; either to approximate the boundary of the region of interest or approximate the domain functional (partial differential operator). We follow still another alternative that merges both of these approaches.

In this paper x_i, $i = 1, 2, 3$, mean rectangular Cartesian coordinates with the origin at the center of gravity of the Earth. We identify W and U with the gravity and a standard (or normal) potential of the Earth, respectively. Under this notation $\boldsymbol{g} = \boldsymbol{grad}\, W$ is the gravity vector and its

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length $g = |\,\boldsymbol{grad}\, W\,|$ is the measured gravity. By analogy we put $\boldsymbol{\gamma} = \boldsymbol{grad}\, U$ and $\gamma = |\,\boldsymbol{grad}\, U\,|$ for the normal gravity. Finally, in the general point $\boldsymbol{x} = (x_1, x_2, x_3)$ we have $T(\boldsymbol{x}) = W(\boldsymbol{x}) - U(\boldsymbol{x})$ for the disturbing potential and $\delta g(\boldsymbol{x}) = g(\boldsymbol{x}) - \gamma(\boldsymbol{x})$ for the gravity disturbance.

We will discuss the Linear Gravimetric Boundary Value Problem (LGBVP). It is an oblique derivative problem. Its solution domain is the exterior of the Earth. We will denote it by Ω. The problem may be formulated as follows

$$\Delta T = div\, \boldsymbol{grad}\, T = 0 \quad \text{in} \quad \Omega, \tag{1}$$

$$\frac{\partial T}{\partial \boldsymbol{s}} = \langle \boldsymbol{s}, \boldsymbol{grad}\, T \rangle = -\delta g \quad \text{on} \quad \partial\Omega, \tag{2}$$

where

$$\boldsymbol{s} = -\frac{1}{\gamma}\, \boldsymbol{grad}\, U, \tag{3}$$

$\langle\, ,\rangle$ is the inner product, Δ means Laplace's operator and $\partial\Omega$ represents the boundary of Ω, see Koch and Pope (1972), Bjerhammar and Svensson (1983), Grafarend (1989) and Holota (1997). Let us add in this connection that the vector $\boldsymbol{s}$ is assumed to be nowhere tangential to $\partial\Omega$.

Now we introduce ellipsoidal coordinates u, β, λ (β is the reduced latitude and λ is the geocentric longitude in the usual sense) related to Cartesian coordinates x_1, x_2, x_3 by the equations

$$x_1 = \sqrt{u^2 + E^2} \cos\beta \cos\lambda, \tag{4}$$

$$x_2 = \sqrt{u^2 + E^2} \cos\beta \sin\lambda, \tag{5}$$

$$x_3 = u \sin\beta, \tag{6}$$

where $E = \sqrt{a^2 - b^2}$ is the linear eccentricity of an ellipsoid of revolution with semiaxes a and b, $a \geq b$, whose center is in the origin of our Cartesian system and whose axis of rotation coincides with the x_3-axis.

In our considerations we will suppose that $h(\beta, \lambda)$ is a function that describes the boundary $\partial\Omega$ of our solution domain Ω with respect to the level ellipsoid $u = b$, i.e. $\partial\Omega$ is represented by

$$x_1 = \sqrt{[b + h(\beta, \lambda)]^2 + E^2} \cos\beta \cos\lambda, \tag{7}$$

$$x_2 = \sqrt{[b + h(\beta, \lambda)]^2 + E^2} \cos\beta \sin\lambda, \tag{8}$$

$$x_3 = [b + h(\beta, \lambda)] \sin\beta. \tag{9}$$

In addition, referring to Heiskanen and Moritz (1967), we can reproduce that $\partial U/\partial\lambda = 0$ for the normal (Somigliana-Pizzeti) potential U and that for $h = 0$ we have $\partial U/\partial\beta = 0$. Moreover, for $\partial\Omega$ close to the level ellipsoid, we can even adopt that with a high (sufficient) accuracy $\partial U/\partial\beta = 0$ is valid for a realistic range of h representing the boundary $\partial\Omega$ (surface of the Earth). In consequence the boundary condition above, Eq. (2), can be interpreted in terms of a derivative of T with respect to u, i.e.,

$$\frac{\partial T}{\partial u} = -w(b + h, \beta)\, \delta g \quad \text{on} \quad \partial\Omega, \tag{10}$$

where

$$w(u, \beta) = \sqrt{\frac{u^2 + E^2 \sin^2\beta}{u^2 + E^2}}. \tag{11}$$

In the following approach to the solution the LGBVP a transformation (small modification) of ellipsoidal coordinates will be applied together with an attenuation function. This will open a way for an alternative between the boundary complexity and the complexity of the coefficients of the partial differential equation governing the solution. The approach represents a generalization of the concept discussed in Holota (1985, 1986, 1989, 1992a, b, 2016) and Holota and Nesvadba (2016).

2 Transformation of Coordinates and an Attenuation Function

Our starting point will be the mapping given by Eqs. (4)–(6), but with

$$u = z + \omega(z) h(\beta, \lambda), \tag{12}$$

where z is a new coordinate and $\omega(z)$ is a twice continuously differentiable attenuation function defined for $z \in [b, \infty)$, such that

$$\omega(z) h(\beta, \lambda) > -b, \tag{13}$$

$$\omega(b) = 1, \quad \frac{d\omega}{dz}(b) = 0 \tag{14}$$

and

$$\omega(z) = 0 \quad \text{for} \quad z \in [z_{ext}, \infty)\,, \text{where } b < z_{ext}. \tag{15}$$

Stress that the assumption concerning the continuity of ω and its first and the second derivatives implies

$$\lim \omega(z) = 0, \quad \lim \frac{d\omega(z)}{dz} = 0, \quad \lim \frac{d^2\omega(z)}{dz^2} = 0 \tag{16}$$

for $z \to z_{ext}^-$, i.e. for z approaching z_{ext} from the left. Obviously, z, β, λ form a system of new curvilinear coordinates and in case that

$$\frac{du}{dz} = 1 + \frac{d\omega}{dz} h > 0 \tag{17}$$

the transformation given by Eqs. (4)–(6) with u as in Eq. (12) represents a one-to-one mapping between the original solution domain Ω and the outer space Ω_{ell} of our oblate ellipsoid of revolution.

The construction of the attenuation function $\omega(z)$ in the interval $[b, z_{ext})$, i.e. for $b \le z < z_{ext}$, deserves some attention. Here we give an example, which is also applied in this work. We put

$$\omega(z) = \exp\left[2 - \frac{2\,(\Delta z)^2}{(\Delta z)^2 - (z-b)^2}\right], \tag{18}$$

where $\Delta z = z_{ext} - b$. By direct computation we can verify that $\omega(b) = 1$ and $\lim\limits_{z \to z_{ext}^-} \omega(z) = 0$. For the first derivative of $\omega(z)$ we obtain

$$\frac{d\omega(z)}{dz} = -\frac{4\,(\Delta z)^2\,(z-b)}{\left[(\Delta z)^2 - (z-b)^2\right]^2}\;\omega(z), \tag{19}$$

$$\frac{d\omega(b)}{dz} = 0 \quad \text{and} \quad \lim_{z \to z_{ext}^-} \frac{d\omega(z)}{dz} = 0. \tag{20}$$

Similarly for the second derivative of $\omega(z)$ we can verify that

$$\begin{aligned} \frac{d^2\omega(z)}{dz^2} &= -\frac{4\,(\Delta z)^2 (z-b)}{\left[(\Delta z)^2-(z-b)^2\right]^2}\,\frac{d\omega(z)}{dz} - \\ &- \left\{ \frac{4\,(\Delta z)^2}{\left[(\Delta z)^2-(z-b)^2\right]^2} + \frac{16\,(\Delta z)^2 (z-b)^2}{\left[(\Delta z)^2-(z-b)^2\right]^3} \right\} \omega(z) \end{aligned} \tag{21}$$

and

$$\lim_{z \to z_{ext}^-} \frac{d^2\omega(z)}{d^2 z} = 0. \tag{22}$$

3 Transformation of the Boundary Condition

In the coordinates z, β, λ the boundary $\partial\Omega$ is defined by $z = b$ and its image $\partial\Omega_{ell}$ coincides with our oblate ellipsoid of revolution. In addition the transformation changes the formal representation of the LGBVP. Indeed, the boundary condition turns into

$$\frac{\partial T}{\partial z} = -w\left[z + \omega(z) h(\beta,\lambda), \beta\right] \delta g \quad \text{for} \quad z = b. \tag{23}$$

Hence, denoting by $\partial/\partial n$ the derivative in the direction of the unit (outer) normal $\boldsymbol{n}$ of $\partial\Omega_{ell}$ and recalling $\partial T/\partial n = (\partial T/\partial z)\,(dz/dn)$, where $dz/dn = 1/w(z,\beta)$, which follows from differential geometry considerations, we obtain

$$\frac{\partial T}{\partial n} = -\sqrt{1+\varepsilon}\;\delta g \quad \text{on} \quad \partial\Omega_{ell}, \tag{24}$$

where

$$\varepsilon = \frac{E^2\left(2bh + h^2\right)\cos^2\beta}{\left(a^2\sin^2\beta + b^2\cos^2\beta\right)\left[(b+h)^2 + E^2\right]} \tag{25}$$

may practically be neglected (in our case). Using the values of the parameters a and b as, e.g., in the Geodetic Reference System 1980, see Moritz (1992), together with $h_{\max} = 8848\,\text{m}$, we can deduce that $\varepsilon < 1.9 \times 10^{-5}\cos^2\beta$.

4 Metric Tensor

Expressing Laplace's operator of T in terms of the curvilinear coordinates z, β, λ, which do not form an orthogonal system, is somewhat more complicated. In the first step we approach the construction of the metric tensor. Putting

$$y_1 = z, \quad y_2 = \beta, \quad y_3 = \lambda, \tag{26}$$

we easily deduce that the Jacobian (Jacobian determinant)

$$J = \left|\frac{\partial x_i}{\partial y_j}\right| = -\left(1 + \frac{d\omega}{dz}h\right)\left[(z+\omega h)^2 + E^2\sin^2\beta\right]\cos\beta \tag{27}$$

of the transformation in Sect. 2 is negative (apart from its zero values for $\beta = -\pi/2$ and $\pi/2$). Thus, the transformation is a one-to-one mapping. Now we use the tensor calculus and by means of some algebra we obtain the components of the metric tensor

$$g_{ij}(\boldsymbol{y}) = \frac{\partial x_k}{\partial y_i} \cdot \frac{\partial x_k}{\partial y_j} \tag{28}$$

in the coordinates y_i. In the original notation this means that

$$g_{11} = \left(1 + \frac{d\omega}{dz}h\right)^2 \alpha, \; g_{12} = \left(1 + \frac{d\omega}{dz}h\right)\alpha\,\omega\,\frac{\partial h}{\partial \beta}, \tag{29}$$

$$g_{13} = \left(1 + \frac{d\omega}{dz}h\right)\alpha\,\omega\,\frac{\partial h}{\partial \lambda}, \tag{30}$$

$$g_{22} = (z+\omega h)^2 + E^2\sin^2\beta + \alpha\,\omega^2\left(\frac{\partial h}{\partial \beta}\right)^2, \tag{31}$$

$$g_{23} = \alpha\,\omega^2 \frac{\partial h}{\partial \beta}\,\frac{\partial h}{\partial \lambda}, \tag{32}$$

$$g_{33} = \left[(z+\omega h)^2 + E^2\right]\cos^2\beta + \alpha\,\omega^2\left(\frac{\partial h}{\partial \lambda}\right)^2, \tag{33}$$

where $\alpha = w^2(z+\omega\, h, \beta)$.

5 Associated (Conjugate) Metric Tensor

Of similar importance is the associate (conjugate) metric tensor. For the determinant $g = |g_{ij}|$ we have $g = J^2$. Denoting the cofactor of g_{ij} in the determinant g by G^{ij} and putting $g^{ij} = G^{ij}/g$ for the components of the associated metric tensor, we get

$$\begin{aligned} g^{11} &= \tfrac{1}{\alpha}\left(1+\tfrac{d\omega}{dz}h\right)^{-2} + \left(1+\tfrac{d\omega}{dz}h\right)^{-2}\times \\ &\times\left\{\tfrac{\omega^2}{(z+\omega h)^2+E^2\sin^2\beta}\left(\tfrac{\partial h}{\partial\beta}\right)^2 + \tfrac{\omega^2}{\left[(z+\omega h)^2+E^2\right]\cos^2\beta}\left(\tfrac{\partial h}{\partial\lambda}\right)^2\right\}, \end{aligned} \tag{34}$$

$$g^{12} = -\left(1+\frac{d\omega}{dz}h\right)^{-1}\frac{\omega}{(z+\omega h)^2 + E^2\sin^2\beta}\,\frac{\partial h}{\partial\beta}, \tag{35}$$

$$g^{13} = -\left(1+\frac{d\omega}{dz}h\right)^{-1}\frac{\omega}{\left[(z+\omega h)^2 + E^2\right]\cos^2\beta}\,\frac{\partial h}{\partial\lambda}, \tag{36}$$

$$g^{22} = \frac{1}{(z+\omega h)^2 + E^2\sin^2\beta}, \tag{37}$$

$$g^{23} = 0 \quad \text{and} \quad g^{33} = \frac{1}{\left[(z+\omega h)^2 + E^2\right]\cos^2\beta}. \tag{38}$$

6 Laplacian and Topography-Dependent Coefficients

Now we are ready to approach Laplace's operator applied on T. In terms of the curvilinear coordinates y_i (i.e. in z, β, λ) it has the following general form

$$\Delta T = \tfrac{1}{\sqrt{g}}\tfrac{\partial}{\partial y_i}\left(\sqrt{g}\, g^{ij}\,\tfrac{\partial T}{\partial y_j}\right) = g^{ij}\,\tfrac{\partial^2 T}{\partial y_i \partial y_j} + \tfrac{1}{\sqrt{g}}\tfrac{\partial\sqrt{g}\, g^{ij}}{\partial y_i}\,\tfrac{\partial T}{\partial y_j}, \tag{39}$$

see Sokolnikoff (1971). After some algebra and neglecting the difference

$$w^2(z+\omega h,\beta) - w^2(z,\beta) \le \tfrac{E^2}{z^2}\left[2\omega\,\tfrac{h}{z} + \left(\omega\,\tfrac{h}{z}\right)^2\right]\cos^2\beta, \tag{40}$$

which for $h_{\max} = 8848$ m and the values of E^2 and $z = b$ taken from the Geodetic Reference System 1980, see Moritz (1992), can be estimated from above by $1.9\times10^{-5}\cos^2\beta$, we can deduce that

$$\Delta T = \frac{z^2 + E^2\sin^2\beta}{(z+\omega h)^2 + E^2\sin^2\beta}\left[\Delta_{ell}T - \delta(T,h)\right], \tag{41}$$

where

$$\begin{aligned} \Delta_{ell}T &= \tfrac{1}{z^2+E^2\sin^2\beta}\Big[\left(z^2+E^2\right)\tfrac{\partial^2 T}{\partial z^2} + 2z\tfrac{\partial T}{\partial z} + \\ &+ \tfrac{\partial^2 T}{\partial\beta^2} - \tfrac{\sin\beta}{\cos\beta}\tfrac{\partial T}{\partial\beta} + \tfrac{z^2+E^2\sin^2\beta}{\left(z^2+E^2\right)\cos^2\beta}\tfrac{\partial^2 T}{\partial\lambda^2}\Big], \end{aligned} \tag{42}$$

$$\begin{aligned} \delta(T,h) &= A_1\tfrac{\partial T}{\partial z} + A_2\tfrac{\partial^2 T}{\partial z^2} + \\ &+ A_3\tfrac{1}{\sqrt{z^2+E^2\sin^2\beta}}\tfrac{\partial^2 T}{\partial z\partial\beta} + A_4\tfrac{\sqrt{\alpha}}{\sqrt{z^2+E^2}\cos\beta}\tfrac{\partial^2 T}{\partial z\partial\lambda} \end{aligned} \tag{43}$$

and A_i are topography dependent coefficients given by

$$\begin{aligned} A_1 &= \left(1+\tfrac{d\omega}{dz}h\right)^{-1}\left[2\left(\tfrac{d\omega}{dz}-\tfrac{\omega}{z}\right)\tfrac{zh}{z^2+E^2\sin^2\beta} + \omega\,\Delta_E h\right] - \\ &- 2\left(1+\tfrac{d\omega}{dz}h\right)^{-2}\omega\,\tfrac{d\omega}{dz}|\boldsymbol{grad}_E h|^2 + \\ &+ \left(1+\tfrac{d\omega}{dz}h\right)^{-3}\left[\tfrac{(z+\omega h)^2+E^2}{z^2+E^2\sin^2\beta} + \omega^2|\boldsymbol{grad}_E h|^2\right]\tfrac{d^2\omega}{dz^2}h, \end{aligned} \tag{44}$$

$$\begin{aligned} A_2 &= \left(1+\tfrac{d\omega}{dz}h\right)^{-2}\Big\{2\left(\tfrac{d\omega}{dz}-\tfrac{\omega z}{z^2+E^2}\right)h + \\ &+ \left[\left(\tfrac{d\omega}{dz}\right)^2 - \tfrac{\omega^2}{z^2+E^2}\right]h^2\Big\}\tfrac{z^2+E^2}{z^2+E^2\sin^2\beta} - \\ &- \left(1+\tfrac{d\omega}{dz}h\right)^{-2}\omega^2|\boldsymbol{grad}_E h|^2, \end{aligned} \tag{45}$$

$$A_3 = \left(1+\frac{d\omega}{dz}h\right)^{-1}\frac{2\omega}{\sqrt{z^2+E^2\sin^2\beta}}\,\frac{\partial h}{\partial\beta}, \tag{46}$$

$$A_4 = \left(1+\frac{d\omega}{dz}h\right)^{-1}\frac{2\,\omega\,\sqrt{\alpha}}{\sqrt{z^2+E^2}\cos\beta}\,\frac{\partial h}{\partial\lambda} \tag{47}$$

with

$$|\boldsymbol{grad}_E h|^2 = \frac{1}{z^2+E^2\sin^2\beta}\left[\left(\frac{\partial h}{\partial \beta}\right)^2 + \frac{z^2+E^2\sin^2\beta}{(z^2+E^2)\cos^2\beta}\left(\frac{\partial h}{\partial \lambda}\right)^2\right] \tag{48}$$

and

$$\Delta_E h = \frac{1}{z^2+E^2\sin^2\beta}\left[\frac{\partial^2 h}{\partial \beta^2} - \frac{\sin\beta}{\cos\beta}\frac{\partial h}{\partial \beta} + \frac{z^2+E^2\sin^2\beta}{(z^2+E^2)\cos^2\beta}\frac{\partial^2 h}{\partial \lambda^2}\right] \tag{49}$$

being the first and the second Beltrami differential operators.

7 Linear GBVP and Neumann's Function

The disturbing potential T is a harmonic function in the original solution domain Ω. In the space of the curvilinear coordinates z, β, λ, therefore, T satisfies Laplace's equation $\Delta T = 0$ for $z > b$, which in view of Eq. (41) yields

$$\Delta_{ell} T = \delta(T, h) \quad \text{for} \quad z > b, \tag{50}$$

where $\delta(T, h)$ is given by Eq. (43). Hence in combination with Eq. (24) the linear gravimetric boundary value problem in terms of the curvilinear coordinates z, β, λ attains the form

$$\Delta_{ell} T = f \quad \text{in} \quad \Omega_{ell}, \tag{51}$$

$$\frac{\partial T}{\partial n} = -\sqrt{1+\varepsilon}\ \delta g \quad \text{on} \quad \partial\Omega_{ell}, \tag{52}$$

where $f = \delta(T, h)$ and ε given by Eq. (25) is as small that it may be omitted.

Neglecting the fact that $f = \delta(T, h)$ depends on T, we can represent the solution of the problem formally by means of a classical apparatus of mathematical physics. The natural point of departure is Green's third identity (Green's representation formula)

$$\begin{aligned} T_P = {} & \frac{1}{4\pi}\int_{\partial\Omega_{ell}}\left[T\frac{\partial}{\partial n}\left(\frac{1}{l}\right) - \frac{1}{l}\frac{\partial T}{\partial n}\right]dS \\ & - \frac{1}{4\pi}\int_{\Omega_{ell}}\frac{1}{l}\,\Delta_{ell} T\, dV \end{aligned} \tag{53}$$

with l being the distance between the computation and the variable point of integration and dS and dV denoting the surface and the volume element, respectively. Similarly, the quantities with and without the subscript P are referred to the computation and the variable point of integration. We will generalize the formula a little. To do that, we take into consideration a function H harmonic in Ω_{ell}. Hence $\Delta H = 0$ in Ω_{ell} and by Green's second identity we have

$$\int_{\partial\Omega_{ell}}\left(T\frac{\partial H}{\partial n} - H\frac{\partial T}{\partial n}\right)dS = \int_{\Omega_{ell}} H\Delta_{ell} T\, dV. \tag{54}$$

Writing now

$$G = \frac{1}{l} - H \tag{55}$$

and combining Eqs. (53) and (54), we obtain the generalized Green representation formula

$$T_P = \frac{1}{4\pi}\int_{\partial\Omega_{ell}}\left(T\frac{\partial G}{\partial n} - G\frac{\partial T}{\partial n}\right)dS - \frac{1}{4\pi}\int_{\Omega_{ell}} G\Delta_{ell} T\, dV. \tag{56}$$

In the following we will use the function G constructed under Neumann's boundary condition, i.e.

$$\frac{\partial G}{\partial n} = 0 \quad \text{on} \quad \partial\Omega_{ell}, \tag{57}$$

which means that we have to look for a function $H = H(z, \beta, \lambda)$ such that

$$\frac{\partial H}{\partial n} = \frac{\partial}{\partial n}\left(\frac{1}{l}\right) \quad \text{for} \quad z = b. \tag{58}$$

In this case G represents Green's function of the second kind, usually called Neumann's function. We will denote the function G by N and from Eq. (56) we obtain that

$$T_P = \frac{1}{4\pi}\int_{z=b} N\,\delta g\ dS - \frac{1}{4\pi}\int_{b<z<z_{ext}} N\,\delta(T, h)\ dV, \tag{59}$$

where in addition we took into consideration Eq. (50) and the properties of the attenuation function $\omega(z)$, see Sect. 2. On the other hand the construction of Neumann's function itself for the exterior of an oblate ellipsoid of revolution is not routine as yet in contrast to problems formulated for a spherical boundary, as e.g. in Holota (2003). For an oblate ellipsoid of revolution the construction is discussed in Holota (2004, 2011), Holota and Nesvadba (2014, 2018b) and in particular in Holota and Nesvadba (2018a), equally as its relation to Green's function of the first kind and to the so-called reproducing kernel.

8 Iteration Process

The integral formula (59) represents an integro-differential equation for T. For clarity we put

$$F_P = \frac{1}{4\pi}\int\limits_{z=b} N\ \delta g\ \ dS, \tag{60}$$

$$(K\,T)_P = -\frac{1}{4\pi}\int\limits_{b<z<z_{ext}} N\ \delta\,(T,h)\ \ dV, \tag{61}$$

where F is a harmonic function and KT is an integro-differential operator applied on T, such that

$$\Delta_{ell}\,K\,T = \delta\,(T,h) \quad \text{in} \quad \Omega_{ell} \tag{62}$$

and

$$\frac{\partial K\,T}{\partial n} = 0 \quad \text{on} \quad \partial\Omega_{ell}, \tag{63}$$

which follows from general principles applied in constructing Neumann's function. Under this notation the problem is to find T from

$$T = F + K\ T. \tag{64}$$

Our aim is to apply the method of successive approximations, i.e.

$$T = \lim_{n} T_n, \qquad T_n = F + K\ T_{n-1}, \tag{65}$$

where $n = 1, 2, \ldots\ \infty$ and T_0 is the starting approximation, e.g. $T_0 = F$.

9 Operator with Reduced Degree of Derivatives

For practical use it is convenient to modify the operator K in order to reduce the degree of derivatives involved in $\delta(T,h)$ and to display the mutual interplay of individual terms in $\delta(T,h)$ more explicitly. Integrating by parts and neglecting terms multiplied by E^2/z^3, we get

$$\begin{aligned}(K\,T)_P = &-\tfrac{1}{4\pi}\int\limits_{z=b} NA_2\ \delta g dS - \tfrac{1}{4\pi}\int\limits_{b<z<z_{ext}} NA_5\tfrac{\partial T}{\partial z}dV + \\ &+\tfrac{1}{4\pi}\int\limits_{b<z<z_{ext}}\left(A_2\tfrac{\partial N}{\partial z} + \tfrac{A_3}{\sqrt{z^2+E^2\sin^2\beta}}\tfrac{\partial N}{\partial \beta} + \tfrac{A_4\sqrt{\alpha}}{\sqrt{z^2+E^2}\cos\beta}\tfrac{\partial N}{\partial \lambda}\right)\tfrac{\partial T}{\partial z}dV,\end{aligned} \tag{66}$$

where

$$\begin{aligned}A_5 = A_1 - \tfrac{\partial A_2}{\partial z} - \tfrac{2}{\sqrt{z^2+E^2\sin^2\beta}}A_2 - \\ -\tfrac{1}{\sqrt{z^2+E^2\sin^2\beta}}\left(\tfrac{\partial A_3}{\partial \beta} - \tfrac{\sin\beta}{\cos\beta}A_3\right) - \tfrac{\sqrt{\alpha}}{\sqrt{z^2+E^2}\cos\beta}\tfrac{\partial A_4}{\partial \lambda}.\end{aligned} \tag{67}$$

Note. It may be interesting that for $\omega(z) = 1$, i.e. $z_{ext} = \infty$, we get $A_5 = -\ \Delta_E h$ directly from Eq. (49).

10 Conclusions

Loosely speaking, the operator K "consumes" derivatives. The question is how the operator transforms the differentiability of the function T or what is the range of the operator for an initially chosen function space, i.e. an initially chosen domain of the operator? This feature is of considerable importance. Its impact will take effect immediately in case that we try to proof the convergence of the iteration procedure as in Eq. (65) by means of tools of functional analysis. The key step is to show that K is a contraction mapping which (if proved) guarantees the convergence of the iteration procedure on the basis of Banach's fixed point theorem, see e.g. Lyusternik and Sobolev (1965). This approach was already discussed in Holota (1985, 1986, 1989, 1992a, b) for $E = 0$ and functions from Sobolev's space $W_2^{(2)}$ produced (roughly speaking) by functions which together with their (generalized) derivatives of the 1st and the 2nd order are square integrable on a spherical layer. In this case it was shown that K is as mapping from $W_2^{(2)}$ onto $W_2^{(2)}$ and its contractivity depends on essential supreme values of the topography dependent coefficients $A_i, i = 1, 2, 3, 4$. The most intricate step to estimate the second order derivatives of KT has been done by means of the Calderon-Zygmund inequality (which belongs to L_p estimates for Poisson's equation), see Gilbarg and Trudinger (1983). As a result the convergence of the iteration procedure was proved for a realistic range of heights and relatively gentle slopes and curvatures of the topography, see Holota (1992b).

Nevertheless, by nature these are a priori estimates and the results concerning the solvability of the LGBVP may differ a bit. Indeed, studies on the existence, uniqueness and stability of the LGBVP, as e.g. in Holota (1997) and by Sansò in Sansò and Sideris (2013), show that the requirements on the topography may be considerably milder. In particular, in his proof Sansò shows that the inclination should be smaller than about 89°. In addition also the use of the ellipsoidal apparatus for the construction of the iteration procedure has its impact on the behavior and the speed of the convergence of the successive approximations.

For all these reasons it may be very instructive to use a numerical approach. The idea is given attention in the ongoing research. First step in this direction was the application of the integration by parts in Sect. 9 that decreases the order of derivatives in the operator K and keeps Lebesgue integrability at the same time. Considerable attention is also given to the investigation on how the successive approximations of the solution behave close to the boundary and how they attain the boundary values. Preference is given to the classical (pointwise) definition of these properties. These goals are challenging, but we believe they will enrich the solution of the problem.

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