# Height System Unification Based on the Fixed GBVP Approach

Thomas Grombein, Kurt Seitz, and Bernhard Heck

**Abstract**

In general, any national or regional height reference system is related to an individual vertical datum, defined by one or several tide gauges. The discrepancies of these local vertical datums cause height datum offsets in a range of about $\pm 1$–2 m at a global scale. For the purpose of height system unification, global geopotential models derived from homogeneous satellite data provide an important contribution. However, to achieve a unification of high precision, the use of local terrestrial gravity data in the framework of a Geodetic Boundary Value Problem (GBVP) is required. By solving the GBVP at GNSS/leveling benchmarks, the unknown height datum offsets can be estimated in a least squares adjustment. In contrast to previous studies, related to the scalar free GBVP based on gravity anomalies, this paper discusses the alternative use and benefit of the fixed GBVP. This modern formulation of the GBVP is related to gravity disturbances, using the surface of the Earth as boundary surface. In contrast to gravity anomalies, gravity disturbances are not affected by the discrepancies of the local height datum. Therefore, in comparison to a scalar free GBVP approach, the proposed method is not affected by indirect bias terms, which will simplify a height system unification. In this paper, the theory of the fixed GBVP approach is developed and formulas in spherical approximation are derived. Moreover, the method is validated using a closed loop simulation based on the global geopotential model EGM2008, showing mm-accuracy of the estimated height datum offsets.

**Keywords**

Height system unification • Geodetic boundary value problem (GBVP) • Hotine's integral formula

## 1 Introduction

In geodesy, there are two different types of height systems: geometrical and physical. In the former, geometrically defined ellipsoidal heights are used, related to the orthogonal distance to a reference ellipsoid. In the latter, physical heights are utilized that refer to a physically defined reference surface linked to the Earth's gravity potential $W$.

The ellipsoidal height $h(P)$ of a point $P$ on the Earth's surface can directly be measured using methods of GNSS positioning (Global Navigation Satellite Systems). By combining GNSS observations with other space techniques, global three-dimensional terrestrial reference frames have been established that provide sub-cm consistency in the vertical component, e.g., ITRF2008 (Altamimi et al. 2011).

For physical (or national) height systems the situation is quite different. Physical heights are determined by a combination of spirit leveling and gravimetry with respect to a fixed datum point $P_0$. These observations are then used

T. Grombein (✉) • K. Seitz • B. Heck
Geodetic Institute, Karlsruhe Institute of Technology (KIT), Englerstr. 7, Karlsruhe, 76128 Germany
e-mail: grombein@kit.edu


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to derive geopotential numbers

$$C(P) := W_0 - W(P), \quad (1)$$

representing the difference of the gravity potential value between a leveling point $P$ and the datum point $P_0$, i.e., $W_0 := W(P_0)$. In that way, the equipotential surface passing through $P_0$ is chosen as the reference level of the height system. Dividing Eq. (1) by the mean normal gravity value $\overline{\gamma}$ along the normal plumb line, the geopotential number $C(P)$ is transformed to the (metric) normal height $H(P)$ (Heiskanen and Moritz 1967, p. 170 f.).

For the practical realization of a physical height system, the height reference level is conventionally linked to the mean sea level (MSL), observed at one or several tide gauges, i.e., the datum point $P_0$ is selected such that the zero level is fixed to the local MSL. As the leveling networks of different national surveys mostly refer to individual tide gauges, hundreds of different national height systems exist worldwide that are realized by their own local vertical datum. Due to the sea surface topography, different tide gauges do not refer to the same equipotential surface. Therefore, the reference levels of different physical height systems are inconsistent by about $\pm 1$–2 m at a global scale (Heck 1990; Gerlach and Rummel 2013).

On the other hand, many global and regional applications such as monitoring of sea level change, ice sheet melting, or post-glacial rebound require a high-precision and consistent global physical height system. Moreover, this is also relevant for establishing the Global Geodetic Observing System (Ihde and Sánchez 2005). In order to overcome the problem of height datum inconsistencies, different strategies and approaches for height system unification have been discussed and proposed in various publications (e.g., Colombo 1980; Rapp 1988; Heck and Rummel 1990; Sansò and Venuti 2002; Sánchez 2009).

Considering a local height datum zone $\sigma^i$ that is linked to the gravity potential value $W_0^i$, the geopotential number in Eq. (1) analogously reads

$$C^i(P) = W_0^i - W(P). \quad (2)$$

Combining Eqs. (1) and (2), the relation between the local datum zone $\sigma^i$ and a global datum specified by the gravity potential value $W_0$ is described by the height datum offset

$$\delta H^i := \frac{C(P) - C^i(P)}{\overline{\gamma}} = \frac{W_0 - W_0^i}{\overline{\gamma}}. \quad (3)$$

For the determination of $\delta H^i$, observation points that combine physical and geometrical height information are of particular interest, i.e., GNSS/leveling benchmarks. For these points, global geopotential models (GGM) can be used to determine approximated values $C(P) = W_0 - W_{\mathrm{GGM}}(P)$, which can be inserted in Eq. (3). In this context, GGM derived from recent gravity field satellite missions like GRACE and GOCE provide an important contribution, as they provide a homogeneous reference surface that is not affected by a height datum offset (Rummel 2002; Gatti et al. 2013). Due to the limited resolution of the used GGM, such an approach suffers from an omission error. Although this error can be reduced, representing shorter wavelengths by the high-resolution EGM2008 (Pavlis et al. 2012) or regional geoid models, the expected accuracy for $\delta H^i$ is limited to cm–dm level (Gruber et al. 2012; Rülke et al. 2012).

To achieve a unification at sub-cm level, the use of terrestrial gravity data in a Geodetic Boundary Value Problem is indispensable (GBVP, Heiskanen and Moritz 1967, p. 36 f.). For this purpose, the solution of the GBVP is used to estimate height datum offsets in a least squares approach (e.g., Heck and Rummel 1990). In contrast to previous publications, mostly related to the scalar free GBVP approach (Rummel and Teunissen 1988; Xu 1992; Gerlach and Rummel 2013), this paper discusses perspectives and benefits of the alternative use of a fixed GBVP approach for height system unification. In order to reduce systematic errors, a combination with a GGM and topographic information in a remove-compute-restore approach is advisable, as frequently used in gravimetric (quasi-)geoid determination (Forsberg and Tscherning 1997). However, such a combination is beyond the scope of this article. Therefore, the presented formulas will be restricted to the use of terrestrial gravity data.

The paper is organized as follows: in Sect. 2 the proposed fixed GBVP approach is presented and formulas in spherical approximation are derived. In order to validate the method and analyze its accuracy, a closed loop simulation based on EGM2008 is presented in Sect. 3. Finally, in Sect. 4, a summary and an outlook to ongoing research are provided.

## 2 Fixed GBVP Approach

Let the Earth's surface $S$ be partitioned into $n$ disjoint local height datum zones $\sigma^i$, $i = 1, \ldots, n$, i.e., $S = \bigcup_{i=1}^n \sigma^i$ with $\sigma^i \bigcap \sigma^k = \emptyset$ for $i \neq k$. Each datum zone is assumed to be linked to an individual equipotential surface defined by the gravity potential value $W_0^i$. Furthermore, let each datum zone $\sigma^i$ contain $m_i$ GNSS/leveling benchmarks $P_j^i$, $j = 1, \ldots, m_i$, where the (unbiased) ellipsoidal height $h$ and the (biased) normal height $H^i$ are known. For these benchmarks, the (biased) height anomaly $\zeta^i = h - H^i$ can be calculated, which is linked to the disturbing potential $T$ by the generalized Bruns' formula (Heiskanen and Moritz 1967, p. 100):

$$\zeta^i(P_j^i) = \frac{T(P_j^i) - (W_0^i - U_0)}{\gamma} = \frac{T(P_j^i) - \Delta W_0}{\gamma} + \delta H^i, \quad (4)$$

where $U_0$ denotes the constant normal gravity potential value of the used reference ellipsoid, $\gamma$ is the normal gravity value at the Earth's surface, and

$$\Delta W_0 := W_0 - U_0. \tag{5}$$

To determine the disturbing potential $T$, the fixed GBVP will be used that is based on gravity disturbances

$$\delta g := g(P) - \gamma(P) \approx -\left.\frac{\partial T}{\partial r}\right|_S \tag{6}$$

resulting from the difference between the measured gravity $g(P)$ and the normal gravity $\gamma(P)$, both defined at the Earth's surface point $P \in S$. Here, $\partial/\partial r$ denotes the partial derivative with respect to the geocentric radius $r$. Considering the normal gravity formula (Heiskanen and Moritz 1967, p. 79), the ellipsoidal height $h(P)$ of the gravity measurement benchmark is required to obtain $\gamma(P)$. Thus, in the case of the fixed GBVP, the geometry of the Earth's surface $S$ is assumed to be known, e.g, by GNSS positioning.

Utilizing the analytical solution of the fixed GBVP, the disturbing potential $T$ can be obtained in constant radius approximation by Hotine's spherical integral formula (Hotine 1969, p. 311 ff.; Heck 2011):

$$T(\varphi, \lambda) = \frac{R}{4\pi} \iint_\sigma \delta g(\varphi', \lambda') \cdot \mathrm{H}(\psi)\, \mathrm{d}\sigma, \tag{7}$$

where

$$\mathrm{H}(\psi) = \frac{1}{\sin(\psi/2)} - \ln\left(1 + \frac{1}{\sin(\psi/2)}\right) \tag{8}$$

and $\psi$ is the spherical distance between the position vectors of the computation point $P$ $(r = R, \varphi, \lambda)$ and the running integration point $P'$ $(\varphi', \lambda')$, both located on the sphere with radius $R$. The surface of the unit sphere is denoted by $\sigma$ with the corresponding surface element $\mathrm{d}\sigma = \cos\varphi' \mathrm{d}\varphi' \mathrm{d}\lambda'$.

Applying Eq. (7) to Eq. (4) leads to

$$\zeta^i(P^i_j) = \frac{R}{4\pi\gamma} \iint_\sigma \delta g \cdot \mathrm{H}(\psi)\, \mathrm{d}\sigma - \frac{\Delta W_0}{\gamma} + \delta H^i, \tag{9}$$

which is the basic equation of the fixed GBVP approach that can already be used for the estimation of the unknown height datum offsets $\delta H^i$ at GNSS/leveling benchmarks $P^i_j$.

However, the lacking availability of globally distributed gravity disturbances $\delta g$ complicates the practical evaluation of Eq. (9). Since for most (historical) gravity measurement benchmarks of the pre-GNSS era the ellipsoidal height $h$ has not been determined, gravity disturbances $\delta g$ according to Eq. (6) could not be compiled. Instead, gravity measurements $g$ have frequently been used to derive gravity anomalies $\Delta g$ that serve as boundary values for the traditional scalar free GBVP. Taking into account the present situation, Eq. (9) will be extended by considering the transformation of gravity anomalies $\Delta g$ to gravity disturbances $\delta g$.

## 2.1 Extension to Gravity Anomalies

Following the theory of Molodensky (Heiskanen and Moritz 1967, p. 291 ff.), gravity anomalies

$$\Delta g := g(P) - \gamma(Q) \approx \left.\left(-\frac{\partial T}{\partial r} - \frac{2\gamma}{r}\zeta\right)\right|_\Sigma \tag{10}$$

differ from gravity disturbances $\delta g$ in the normal gravity $\gamma(Q)$, evaluated at the telluroid $\Sigma \ni Q$ instead of the Earth's surface $S$. Considering that $h(Q) = H^i(P)$ (Heiskanen and Moritz 1967, p. 293), the normal gravity value $\gamma(Q)$ depends on the (biased) normal height. Thus, in contrast to gravity disturbances, gravity anomalies are affected by the height datum offset $\delta H^i$ of the local datum zone $\sigma^i$ (Heck 1990). This becomes clear when inserting Eq. (4) into Eq. (10):

$$\Delta g^i = \left.\left(-\frac{\partial T}{\partial r} - \frac{2}{r}T + \frac{2}{r}\Delta W_0 - \frac{2\gamma}{r}\delta H^i\right)\right|_\Sigma. \tag{11}$$

Combining the boundary conditions of Eqs. (6) and (11), the (unbiased) gravity disturbance $\delta g$ can be expressed as a function of the (biased) gravity anomaly $\Delta g^i$ and the height datum offset $\delta H^i$ using the linear approximation

$$\delta g = \Delta g^i + \left.\left(\frac{2}{r}T - \frac{2}{r}\Delta W_0 + \frac{2\gamma}{r}\delta H^i\right)\right|_S + \delta_{\mathrm{BS}}, \tag{12}$$

where $\delta_{\mathrm{BS}}$ denotes the error induced by the different boundary surfaces ($S$ and $\Sigma$), which is neglected in the following.

Splitting Eq. (12) into three components

$$\delta g_0 := \Delta g^i + \frac{2}{r}T,\quad \delta g_1 := -\frac{2}{r}\Delta W_0,\quad \delta g_2 := \frac{2\gamma}{r}\delta H^i, \tag{13}$$

and inserting them separately into Eq. (9) results in

$$\zeta^i(P^i_j) = \zeta_0 + \zeta_1 + \zeta_2 - \frac{\Delta W_0}{\gamma} + \delta H^i, \tag{14}$$

where

$$\zeta_m := \frac{R}{4\pi\gamma} \iint_\sigma \delta g_m \cdot \mathrm{H}(\psi)\, \mathrm{d}\sigma, \quad m = 0, 1, 2. \tag{15}$$

Applying constant radius approximation, i.e., $r = R$, the evaluation of Eq. (15) leads to

$$\zeta_0 = \frac{R}{4\pi\gamma} \iint\limits_{\sigma} \left( \Delta g^i + \frac{2}{R} T \right) \cdot \mathrm{H}(\psi)\, \mathrm{d}\sigma, \tag{16}$$

$$\zeta_1 = -\frac{\Delta W_0}{2\pi\gamma} \iint\limits_{\sigma} \mathrm{H}(\psi)\, \mathrm{d}\sigma = -\frac{\Delta W_0}{2\pi\gamma} \cdot 4\pi = -\frac{2\Delta W_0}{\gamma}, \tag{17}$$

$$\zeta_2 = \sum_{i=1}^{n} \frac{\delta H^i}{2\pi} \iint\limits_{\sigma^i} \mathrm{H}(\psi)\, \mathrm{d}\sigma, \tag{18}$$

where in the case of $\zeta_2$, the (global) integral domain $\sigma$ is decomposed into the disjoint height datum zones $\sigma^i$.

Finally, inserting Eqs. (16) – (18) into Eq. (14) results in

$$\zeta^i(P^i_j) = \zeta_0(\Delta g^i, T) + \delta H^0 + \delta H^i + \sum_{k=1}^{n} \delta H^k \cdot \mathrm{G}^{i,k}_j, \tag{19}$$

where

$$\delta H^0 := -\frac{3\Delta W_0}{\gamma}, \quad \text{and} \quad \mathrm{G}^{i,k}_j := \frac{1}{2\pi} \iint\limits_{\sigma^k} \mathrm{H}(\psi)\, \mathrm{d}\sigma \Big|_{P^i_j}. \tag{20}$$

In Eq. (19) different kinds of height datum offsets occur. The height datum offset $\delta H^i$ represents the direct influence of the datum zone $\sigma^i$ containing $P^i_j$. This offset, also occurring in the basic Eq. (9), is frequently called direct bias term.

Moreover, Eq. (19) also comprises the height datum offsets $\delta H^k$ $(k = 1, \ldots, n)$ of all datum zones, i.e., $\delta H^1, \ldots, \delta H^n$. These offsets are a consequence of the global integration of biased gravity anomalies $\Delta g^i$ and are named indirect bias terms (Gerlach and Rummel 2013). Particularly, the evaluation of the corresponding factors $\mathrm{G}^{i,k}_j$ in Eq. (20) is complicated, as the separate integration requires the coordinates of the bounding polygon for each datum zone. While the indirect bias terms amount to about $\pm$1–2 m, simulation studies for the scalar free GBVP approach presented by Gerlach and Rummel (2013) demonstrate that their influence can be reduced to a level below 1 cm, when a satellite-derived GGM is employed for representing the long-wavelength parts of $\zeta_0$. However, it is worthwhile mentioning that the basic approach in Eq. (9) is not affected by the indirect bias terms. Therefore, if gravity disturbances $\delta g$ become globally available, the indirect bias terms can be avoided, demonstrating the advantage of the fixed GBVP approach in future applications.

The parameter $\delta H^0$ in Eq. (19) comprises $\Delta W_0$, defining the reference level of the global datum. As this global offset cannot be uniquely estimated within this approach, $W_0$ is assumed to be equal to $U_0$, i.e., $\Delta W_0$ in Eq. (5) and $\delta H^0$ in Eq. (20) are set to zero. By this procedure, an "absolute" vertical datum is defined by convention (e.g., Heck 2004).

## 2.2 Least Squares Adjustment

Using Eq. (19) with $\delta H^0 = 0$, the observation equation for least squares adjustment (LSA) is provided by

$$L^i_j = \zeta^i - \zeta_0(\Delta g^i, T)\Big|_{P^i_j} = \delta H^i + \sum_{k=1}^{n} \delta H^k \cdot \mathrm{G}^{i,k}_j, \tag{21}$$

where $\Delta g^i$ are the observed (biased) gravity anomalies and $T$ the (unbiased) disturbing potential values, derived from an a priori model (e.g., EGM2008). The quantities on the left hand side of Eq. (21) are the known observations and those on the right hand side contain the unknowns to be estimated. The functional model according to Eq. (21) is specified by

$$\underbrace{\begin{pmatrix} l^1_1 + v^1_1 \\ l^1_2 + v^1_2 \\ \vdots \\ l^2_1 + v^2_1 \\ \vdots \\ l^n_{m_n} + v^n_{m_n} \end{pmatrix}}_{\mathbf{l+v}} = \underbrace{\begin{pmatrix} 1 + \mathrm{G}^{1,1}_1 & \mathrm{G}^{1,2}_1 & \cdots & \mathrm{G}^{1,n}_1 \\ 1 + \mathrm{G}^{1,1}_2 & \mathrm{G}^{1,2}_2 & \cdots & \mathrm{G}^{1,n}_2 \\ \vdots & & & \\ \mathrm{G}^{2,1}_1 & 1 + \mathrm{G}^{2,2}_1 & \cdots & \mathrm{G}^{2,n}_1 \\ \vdots & & & \\ \mathrm{G}^{n,1}_{m_n} & \mathrm{G}^{n,2}_{m_n} & \cdots & 1 + \mathrm{G}^{n,n}_{m_n} \end{pmatrix}}_{\mathbf{A}} \cdot \underbrace{\begin{pmatrix} \delta H^1 \\ \delta H^2 \\ \delta H^3 \\ \vdots \\ \delta H^n \end{pmatrix}}_{\mathbf{x}},$$

where $\mathbf{l}$ is the observation vector, $\mathbf{v}$ the inconsistency vector, and $\mathbf{x}$ the vector of unknowns. The design matrix $\mathbf{A}$ contains the partial derivatives of the observations with respect to the unknowns. Using a standard LSA, the unknown height datum offsets are estimated by

$$\hat{\mathbf{x}} = \mathbf{N}^{-1} \cdot \mathbf{A}^{\mathrm{T}}\mathbf{P} \cdot \mathbf{l}, \tag{22}$$

where $\mathbf{N} = \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}$ is the normal matrix and $\mathbf{P}$ is the weight matrix of the observations, which can be specified by an additional stochastic model.

# 3 Closed Loop Simulation

Using the presented fixed GBVP approach, a closed loop simulation is performed following a four-step sequence:

1. Definition of eight height datum zones $\sigma^i$ with individual height datum offsets $\delta H^i$ $(i = 1, \ldots, 8)$.
2. Addition of $\delta H^i$ to EGM2008-derived observations.

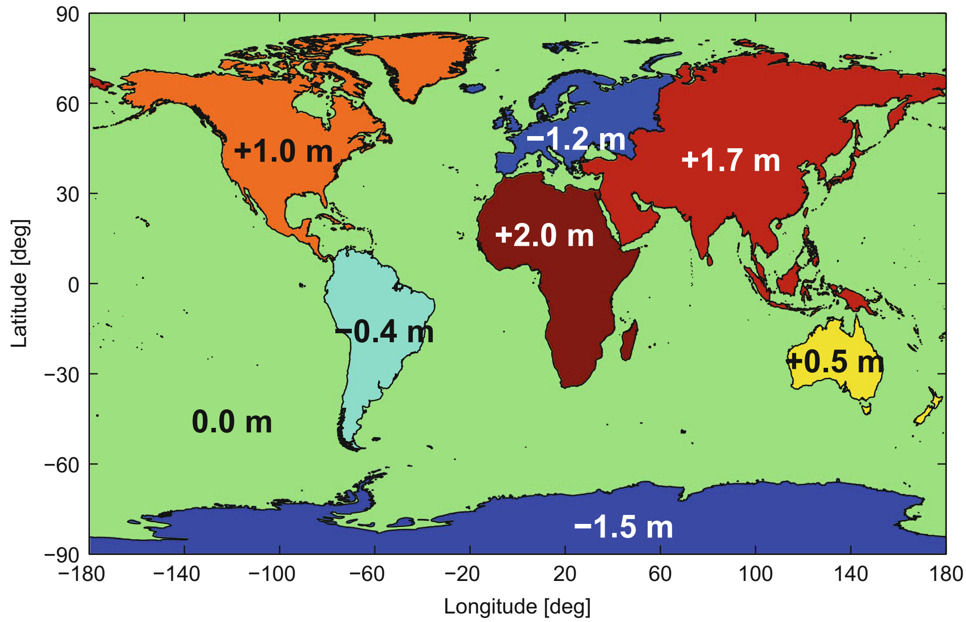


**Fig. 1** Visualization of the height datum zones $\sigma^i$ and their assumed height datum offsets $\delta H^i$ used for the closed loop simulation

**Table 1** Specification of the height datum zones $\sigma^i$, their assumed height datum offsets $\delta H^i$, and the error values $\varepsilon^i$ according to Eq. (24) for the scenarios (a) – (d)

| | | | Error values $\varepsilon^i$ [mm] | | | |
|---|---|---|---|---|---|---|
| $i$ | Datum zone $\sigma^i$ | $\delta H^i$ [m] | (a) | (b) | (c) | (d) |
| 1 | Asia | 1.7 | −0.2 | 1.3 | −19.9 | −0.6 |
| 2 | North America | 1.0 | −0.4 | 1.5 | −177.8 | −4.1 |
| 3 | Europe | −1.2 | 0.3 | 1.7 | 22.0 | 2.1 |
| 4 | Africa | 2.0 | 0.3 | 1.7 | −24.4 | 2.2 |
| 5 | South America | −0.4 | −2.7 | −0.9 | 10.6 | −0.7 |
| 6 | Australia | 0.5 | −0.5 | 2.2 | 10.1 | 2.2 |
| 7 | Antarctica | −1.5 | 2.4 | 4.5 | −25.7 | 3.8 |
| 8 | Ocean | 0.0 | 0.0 | −2.7 | −2.1 | −2.5 |

3. Estimation of $\hat{\mathbf{x}}$ by Eq. (22) with $\mathbf{P} = \mathbf{I}$ (identity matrix).
4. Comparison of estimated and reference values.

As illustrated in Fig. 1 and specified by Table 1, the Earth's continents and oceans are utilized as height datum zones $\sigma^i$, where height datum offsets $\delta H^i$ are assumed that cover the range of $\pm 1$–$2$ m. Using EGM2008 to degree and order 2190, global grids of consistent height anomalies $\zeta_{\mathrm{EGM}}$ ($5° \times 5°$), gravity anomalies $\Delta g_{\mathrm{EGM}}$ ($5' \times 5'$) and disturbing potential values $T_{\mathrm{EGM}}$ ($5' \times 5'$) are generated on a sphere with radius $R = 6{,}371$ km and normal gravity $\overline{\gamma} = \gamma = 9.81$ m s$^{-2}$. Applying the height datum offsets $\delta H^i$, simulated observations according to Eq. (21) are calculated by

$$L^i_j = \underbrace{\zeta_{\mathrm{EGM}} + \delta H^i}_{\zeta^i} - \zeta_0(\underbrace{\Delta g_{\mathrm{EGM}} - \frac{2\gamma}{R}\delta H^i}_{\Delta g^i}, \underbrace{T_{\mathrm{EGM}}}_{T})\Big|_{P^i_j}, \qquad (23)$$

where the integration is performed by Gauss–Legendre quadrature (e.g., Schwarz 1989, p. 361 ff.).

To analyze the impact of the global distribution of the used benchmarks $P^i_j$, four different scenarios (a) – (d) are considered as displayed in Fig. 2. In scenario (a), all 2,592 observations $L^i_j$ of the $5° \times 5°$ global grid are used in the LSA. In scenario (b), observations are restricted to continental areas (879 benchmarks), while in scenario (c) only observations in Europe, South America, and Australia are included (161 benchmarks). Scenario (d) is similar to (c), but additionally at least one benchmark is included in each datum zone (166 benchmarks). In each scenario, the height datum offsets of all datum zones are estimated.

In Table 1 the numerical results for the scenarios (a) – (d) are presented in terms of error values

$$\varepsilon^i = \delta H^i - \hat{x}^i, \qquad (24)$$

where the estimated height datum offsets are denoted by $\hat{x}^i$, i.e., the components of $\hat{\mathbf{x}}$. Moreover, to quantify the stability of the LSA, Table 2 specifies the spectral condition number $\kappa_2$ of the normal matrix $\mathbf{N}$, i.e., the ratio of the largest to the smallest eigenvalue of the matrix (Schwarz 1989, p. 24f.).

In the ideal scenario (a), the error values attain a sub-mm level, only in South America and Antarctica slightly larger values occur. Excluding the observations of the oceans, scenario (b) produces error values at lower mm level. Going a step further towards a realistic scenario, case (c) demonstrates that the error values are increased to cm level or even dm level in North America. In contrast to the other scenarios, the large condition number of (c) indicates the instability of

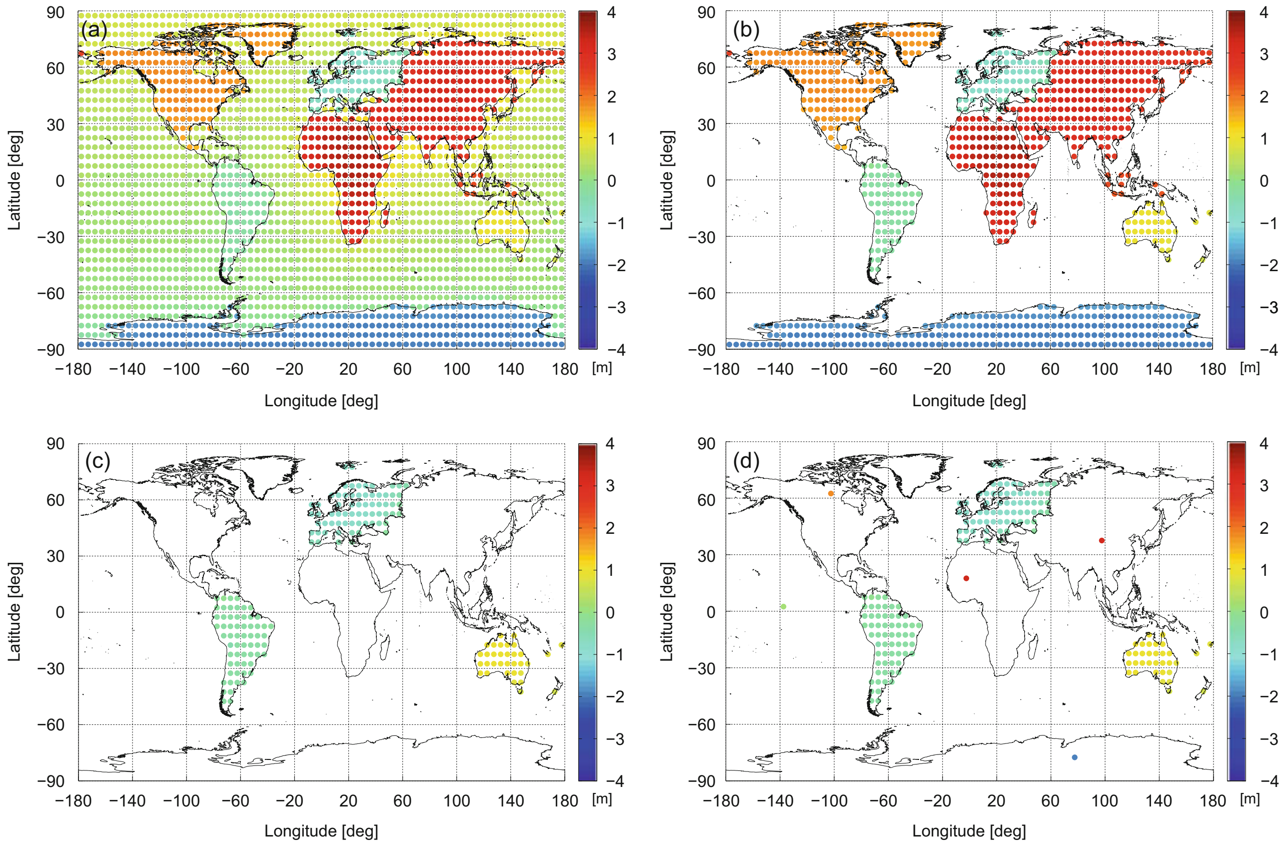


**Fig. 2** Visualization of the global distribution of benchmarks $P_j^i$ used in the scenarios (**a**) – (**d**). *Each dot* represents the value of an observation equation $L_j^i$ according to Eq. (23)

**Table 2** Spectral condition number $\kappa_2$ of the normal matrix **N**, quantifying the stability of the LSA for the scenarios (a) – (d)

| Scenario | (a) | (b) | (c) | (d) |
|---|---|---|---|---|
| Condition number $\kappa_2(\mathbf{N})$ | 221 | 404 | 118480 | 246 |

the LSA. Concerning scenario (d), it is demonstrated that if at least one observation is added in each datum zone, this instability can be mitigated. Thus, scenario (d) provides an error level comparable to (b), showing that mm-accuracy can be achieved in principle. However, these accuracy values are quite optimistic and must be seen in the context of the assumed error-free observation data of the closed loop simulation. To obtain realistic values for practical applications, a formal error propagation procedure would have to be taken into account.

## 4 Summary and Outlook

In contrast to geometrically defined global terrestrial reference systems, physical height systems suffer from discrepancies of about $\pm 1$–$2$ m due to the individual definition of their local vertical datum. In order to realize a comparison of physical heights, a height system unification is required.

In this paper, a method based on the solution of a fixed GBVP has been presented, where height datum offsets are estimated in a least squares adjustment. In contrast to previous approaches using the traditional scalar free GBVP, the formulation of the proposed method is based on (unbiased) gravity disturbances that do not cause indirect bias terms. Therefore, the fixed GBVP approach simplifies the estimation of height datum offsets, when gravity disturbances become globally available in the future. However, considering the current situation of the global gravity data base, the approach is extended by a transformation of gravity anomalies to gravity disturbances also comprising indirect bias terms. By conducting a closed loop simulation based on eight height datum zones and EGM2008-derived observations, the fixed GBVP approach has been validated, showing a mm-accuracy of the estimated height datum offsets. Furthermore, the stability of the adjustment has been analyzed showing a dependency on the global distribution of the observations; at least one observation should be located in each datum zone.

As future work, the impact of approximation errors on the presented spherical solution will be analyzed and taken into account by suitable reductions. First results concerning the fixed GBVP are presented by Müßle et al. (2014). In addition, the combination of terrestrial gravity data with a GGM and topographic information will be investigated as well as a modification of Hotine's integral kernel to restrict the global integration area (Featherstone 2013).

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