Aggregation of Fuzzy Relations and Preservation of Transitivity

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Abstract. This contribution provides a comprehensive overview on the theoretical framework of aggregating fuzzy relations under the premise of preserving underlying transitivity conditions. As such it discusses the related property of dominance of aggregation operators. After a thorough introduction of all necessary and basic properties of aggregation operators, in particular dominance, the close relationship between aggregating fuzzy relations and dominance is shown. Further, principles of building dominating aggregation operators as well as classes of aggregation operators dominating one of the basic t-norms are addressed. In the paper by Bodenhofer, Küng and Saminger, also in this volume, the interested reader finds an elaborated (real world) example, i.e., an application of the herein contained theoretical framework.

1 Introduction

Flexible (fuzzy) querying systems are designed not just to give results that match a query exactly, but to give a list of possible answers ranked by their closeness to the query—which is particularly beneficial if no record in the database matches the query in an exact way (see [11, 12, 28, 29] for overviews and [7, 8, 9, 10] for particular related examples). The closeness of a single value of a record to the respective value in the query is usually measured by a fuzzy equivalence relation, that is, a reflexive, symmetric, and T-transitive fuzzy relation. Recently, a generalization has been proposed [7, 8, 9] which also allows flexible interpretation of ordinal queries (such as “at least” and “at most”) by using fuzzy orderings [5]. In any case, if a query consists of at least two expressions that are to be interpreted vaguely, it is necessary to combine the degrees of matching with respect

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to the different fields in order to obtain an overall degree of matching — a typical example of an aggregation task. More precisely, assume that we have a query $(q_1, \ldots, q_n)$, where each $q_i \in X_i$ is a value referring to the i-th field of the query. Given a data record $(x_1, \ldots, x_n)$ such that $x_i \in X_i$ for all $i = 1, \ldots, n$, the overall degree of matching is computed as

$$\tilde{R}\big((q_1, \ldots, q_n), (x_1, \ldots, x_n)\big) = \mathbf{A}\big(R_1(q_1, x_1), \ldots, R_n(q_n, x_n)\big),$$

where every R_i is a T-transitive binary fuzzy relation on X_i which measures the degree to which the value x_i matches the query value q_i.

It is natural to require that $\tilde{R}$ is fuzzy relation on the Cartesian product of all X_i and, therefore, that the range of the operation $\mathbf{A}$ should be the unit interval, i.e., $\mathbf{A} : [0,1]^n \to [0,1]$. Furthermore, it is desirable that if a data record matches one of the criteria of the query better than a second one, then the overall degree of matching for the first should be higher or at least the same as the overall degree of matching for second one. Clearly, if some data record matches all criteria, i.e., all $R_i(x_i, q_i) = 1$, then the overall degree of matching should also be 1. On the other hand, if a data record fulfills none of the criteria to any level, i.e., all $R_i(x_i, q_i) = 0$, then the overall degree should vanish to 0. Aggregation operators are exactly such functions which guarantee all these properties [13, 14, 15, 21].

In addition, it would be desirable that, if all relations R_i on X_i are T-transitive, also $\tilde{R}$ is still T-transitive in order to have a clear interpretation of the aggregated fuzzy relation $\tilde{R}$. It is, therefore, necessary to investigate which aggregation operators are particularly able to guarantee that $\tilde{R}$ maintains T-transitivity.

This contribution provides an overview on results on the aggregation of fuzzy relations and the related property of dominance of aggregation operators which have been achieved by collaboration among different research groups within the EU COST Action TARSKI. The present part focusses on the theoretical background, as such provides a comprehensive overview of the theory of aggregation operators dominating triangular norms as well as depends on results already published in [27, 30, 32]. In addition, in [10], the interested reader finds an elaborated (real world) example, i.e., an application of the herein contained theoretical framework. Next, we provide a thorough introduction of all necessary and basic properties of aggregation operators, in particular dominance. Then we turn to the close relationship between the aggregation of fuzzy relations and dominance. In Section IV, we discuss principles of building dominating aggregation operators and focus in Section V on the class of aggregation operators dominating one of the basic t-norms.

2 Basic Definitions and Preliminaries

In order to be self-contained and to provide a compact overview we provide basic definitions and results about aggregation operators and dominance. For more details on aggregation operators as well as t-norms we refer the interested reader to [2, 14, 21].

2.1 Aggregation Operators

Definition 1. [14] An *aggregation operator* is a function $\mathbf{A} : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ which fulfills the following properties:

(AO1) $\mathbf{A}(x_1,\ldots,x_n) \le \mathbf{A}(y_1,\ldots,y_n)$ whenever $x_i \le y_i$ for all $i \in \{1,\ldots,n\}$,
(AO2) $\mathbf{A}(x) = x$ for all $x \in [0,1]$,
(AO3) $\mathbf{A}(0,\ldots,0) = 0$ and $\mathbf{A}(1,\ldots,1) = 1$.

Each aggregation operator $\mathbf{A}$ can be represented by a family $(\mathbf{A}_{(n)})_{n\in\mathbb{N}}$ of n-ary operations, i.e., functions $\mathbf{A}_{(n)} : [0,1]^n \to [0,1]$ given by

$$\mathbf{A}_{(n)}(x_1,\ldots,x_n) = \mathbf{A}(x_1,\ldots,x_n)$$

being non-decreasing and fulfilling $\mathbf{A}_{(n)}(0,\ldots,0) = 0$ and $\mathbf{A}_{(n)}(1,\ldots,1) = 1$. Such operations $\mathbf{A}_{(n)}$ are referred to as *n-ary aggregation operators.* Note also that in such a case $\mathbf{A}_{(1)} = \mathrm{id}_{[0,1]}$. Usually, the aggregation operator $\mathbf{A}$ and the corresponding family $(\mathbf{A}_{(n)})_{n\in\mathbb{N}}$ of n-ary operations are identified with each other.

Unless explicitly mentioned otherwise, we will restrict to aggregation operators acting on the unit interval (according to Definition 1). With only simple and obvious modifications, aggregation operators can be defined to act on any closed interval $I = [a,b] \subseteq [-\infty,\infty]$. Consequently, we will speak of an *aggregation operator acting on I.*

Particularly, such operators can be constructed by rescaling the input and output data, and as such creating isomorphic aggregation operators.

Consider an aggregation operator $\mathbf{A} : \bigcup_{n\in\mathbb{N}}[a,b]^n \to [a,b]$ on $[a,b]$ and a monotone bijection $\varphi : [c,d] \to [a,b]$. The operator $\mathbf{A}_\varphi : \bigcup_{n\in\mathbb{N}}[c,d]^n \to [c,d]$ defined by

$$\mathbf{A}_\varphi(x_1,\ldots,x_n) = \varphi^{-1}\big(\mathbf{A}(\varphi(x_1),\ldots,\varphi(x_n))\big)$$

is an aggregation operator on $[c,d]$, which is isomorphic to $\mathbf{A}$.

A particularly important transformation is duality induced by $\varphi_d : [0,1] \to [0,1]$, $\varphi_d(x) = 1 - x$. Applying this transformation to an aggregation operator $\mathbf{A}$ on the unit interval leads to the so-called *dual aggregation operator* $\mathbf{A}^d$.

Couples of dual aggregation operators are, e.g., the minimum and the maximum. The arithmetic mean is dual to itself. Such aggregation operators, i.e., $\mathbf{A} = \mathbf{A}^d$, are called *self-dual* (compare also [38] where these operators are called symmetric sums).

Let us now briefly summarize further properties of aggregation operators.

Definition 2. Consider some aggregation operator $\mathbf{A} : \bigcup\limits_{n\in\mathbb{N}}[0,1]^n \to [0,1]$.

(i) $\mathbf{A}$ is called *symmetric*, if for all $n \in \mathbb{N}$ and for all $x_1,\ldots,x_n \in [0,1]$:

$$\mathbf{A}(x_1,\ldots,x_n) = \mathbf{A}(x_{\alpha(1)},\ldots,x_{\alpha(n)})$$

for all permutations $\alpha = (\alpha(1),\ldots,\alpha(n))$ of $\{1,\ldots,n\}$.

(ii) $\mathbf{A}$ is called *associative* if for all $n, m \in \mathbb{N}$ and for all $x_1, \ldots, x_n, y_1, \ldots, y_m \in [0,1]$:

$$\mathbf{A}(x_1, \ldots, x_n, y_1, \ldots, y_m) = \mathbf{A}(\mathbf{A}(x_1, \ldots, x_n), \mathbf{A}(y_1, \ldots, y_m)).$$

(iii) An element $e \in [0,1]$ is called a *neutral element* of $\mathbf{A}$ if for all $n \in \mathbb{N}$ and for all $x_1, \ldots, x_n \in [0,1]$:

$$\mathbf{A}(x_1, \ldots, x_n) = \mathbf{A}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

whenever $x_i = e$ for some $i \in \{1, \ldots, n\}$.

(iv) $\mathbf{A}$ is *subadditive* on $[0,1]$, if the following inequality holds for all $x_i, y_i \in [0,1]$ with $x_i + y_i \in [0,1]$:

$$\mathbf{A}(x_1 + y_1, \ldots, x_n + y_n) \leq \mathbf{A}(x_1, \ldots, x_n) + \mathbf{A}(y_1, \ldots, y_n).$$

Observe that, for a given aggregation operator $\mathbf{A}$, the operators $\mathbf{A}_{(n)}$ and $\mathbf{A}_{(m)}$ need not be related in general, if $n \neq m$. However, if $\mathbf{A}$ is an associative aggregation operator, all n-ary operators $\mathbf{A}_{(n)}$, $n \geq 3$, can be identified with recursive extensions of the binary operator $\mathbf{A}_{(2)}$. Therefore, in case of associative aggregation operators, the distinction between $\mathbf{A}_{(2)}$ and $\mathbf{A}$ itself is often omitted.

Example 1. A typical example of a symmetric, but non-associative aggregation operator without neutral element is the *arithmetic mean* $\mathbf{M}\colon \bigcup_{n\in\mathbb{N}} [a,b]^n \to [a,b]$ defined for any interval $[a,b] \subseteq [-\infty, \infty]$ by

$$\mathbf{M}(x_1, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

If for some practical purposes some of the properties of the arithmetic mean do not fit the demands of the aggregation process the arithmetic mean is usually modified with respect to the violated property but by preserving as many as possible other properties of the original aggregation operator. Three different approaches can be mentioned — introduction of weights, ordering of the inputs and transformation of the aggregation operator.

We briefly summarize the formal definitions of weighted means, (weighted) quasi-arithmetic means and OWA operators (see also, e.g., [14, 40]). Recall that for a fixed $n \in \mathbb{N}$, weighting vectors $\overrightarrow{w} = (w_1, \ldots, w_n)$ are characterized by fulfilling $\overrightarrow{w} \in [0,1]^n$ and $\sum_{i=1}^{n} w_i = 1$.

Definition 3. For a continuous strictly monotone function $f : [a,b] \to [-\infty, \infty]$, the *quasi-arithmetic mean* $\mathbf{M}_f\colon \bigcup_{n\in\mathbb{N}} [a,b]^n \to [a,b]$ is given by

$$\mathbf{M}_f(x_1, \ldots, x_n) = f^{-1}(\tfrac{1}{n}\sum_{i=1}^{n} f(x_i)).$$

Consider for arbitrary $n \in \mathbb{N}$, a weighting vector $\overrightarrow{w}$. Then the *weighted mean* $\mathbf{W}\colon [a,b]^n \to [a,b]$ is given by

$$\mathbf{W}(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i x_i$$

and the *weighted quasi-arithmetic mean* $\mathbf{W}_f \colon \bigcup_{n\in\mathbb{N}} [a,b]^n \to [a,b]$ by

$$\mathbf{W}_f(x_1,\ldots,x_n) = f^{-1}(\sum_{i=1}^{n} w_i f(x_i)).$$

with $f\colon\ [a,b] \to [-\infty,\infty]$ again some continuous strictly monotone function. An *OWA operator* $\mathbf{W}'\colon \bigcup_{n\in\mathbb{N}} [a,b]^n \to [a,b]$ is characterized by

$$\mathbf{W}'(x_1,\ldots,x_n) = \sum_{i=1}^{n} w_i x_i'$$

where x_i' denotes the i-th order statistics from the sample $(x_1,\ldots,x_n)$ and w_i the corresponding weights.

2.2 Triangular Norms

Triangular norms can be interpreted as a particular class of aggregation operators which were originally introduced in the context of probabilistic metric spaces [25, 35, 36]. We just briefly state the formal definitions and introduce the four basic t-norms. For further details and properties about t-norms we refer to [22, 23, 24] or to the monographs [2, 21].

Definition 4. A *triangular norm* (t-norm for short) is a binary operation T on the unit interval which is commutative, associative, non-decreasing in each component, and has 1 as a neutral element.

Example 2. The following are the four basic t-norms:

Minimum: $T_{\mathbf{M}}(x,y) = \min(x,y)$,

Product: $T_{\mathbf{P}}(x,y) = x \cdot y$,

Łukasiewicz t-norm: $T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$,

Drastic product: $T_{\mathbf{D}}(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,1[^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$

Several construction principles are known for t-norms. Here we just mention the concept of ordinal sums which allow to define t-norms by a particular behaviour on subdomains and, moreover, gave rise for a construction principle for aggregation operators.

Definition 5. Let $(T_i)_{i\in I}$ be a family of t-norms and let $(]a_i,e_i[)_{i\in I}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. Then the following function $T : [0,1]^2 \to [0,1]$ is a t-norm [21]:

$$T(x,y) = \begin{cases} T_i^*(x,y) = a_i + (e_i - a_i)\cdot T(\frac{x-a_i}{e_i-a_i}, \frac{y-a_i}{e_i-a_i}), & \text{if } (x,y) \in [a_i,e_i]^2, \\ \min(x,y), & \text{otherwise.} \end{cases}$$

The t-norm T is called the *ordinal sum* of the *summands* $\langle a_i, e_i, T_i\rangle, i \in I$, and we shall write $T = (\langle a_i, e_i, T_i\rangle)_{i\in I}$.

Corresponding to t-norms, aggregation operators can also be constructed from several aggregation operators acting on non-overlapping domains. We will use the *lower ordinal sum* of aggregation operators [14,26]. Observe that this ordinal sum was originally proposed only for finitely many summands, however, we generalize this concept to an arbitrary (countable) number of summands.

Definition 6. Consider a family of aggregation operators

$$\big(\mathbf{A}_i : \bigcup_{n\in\mathbb{N}} [a_i, e_i]^n \to [a_i, e_i]\big)_{i\in\{1,\ldots,k\}}$$

acting on non-overlapping domains $[a_i, e_i]$ with $i \in \{1, \ldots, k\}$ and

$$0 \le a_1 < e_1 \le a_2 < e_2 \le \ldots \le e_k \le 1.$$

The aggregation operator $\mathbf{A}^{(w)}$ defined by [14]

$$\mathbf{A}^{(w)}(x_1, \ldots, x_n) = \begin{cases} 0, & \text{if } u < a_1, \\ \mathbf{A}_i\big(\min(x_1, e_i), \ldots, \min(x_n, e_i)\big), & \text{if } a_i \le u < a_{i+1}, \\ 1, & \text{if } u = 1. \end{cases}$$

with $u = \min(x_1, \ldots, x_n)$ is called the *lower ordinal sum* (of aggregation operators $\mathbf{A}_i$) and it is the weakest aggregation operator (with respect to the standard ordering of n-ary functions) that coincides with $\mathbf{A}_i$ at inputs from $[a_i, e_i]$.

If $(\mathbf{A}_i)_{i\in I}$ is a family of aggregation operators on $[0, 1]$ and $(]a_i, e_i[)_{i\in I}$ a (countable) family of non-empty, pairwise disjoint open subintervals of $[0, 1]$, then the lower ordinal sum of this family $\mathbf{A}^{(w)} = (\langle a_i, e_i, \mathbf{A}_i\rangle)_{i\in I}$ can be constructed in the following way:

$$\mathbf{A}^{(w)}(x_1, \ldots, x_n) = \begin{cases} \sup_{i\in I}\{\mathbf{A}_i^*\big(\min(x_1, e_i), \ldots, \min(x_n, e_i)\big) \mid a_i \le u\}, & \\ & \text{if } u < 1, \\ 1, & \text{otherwise,} \end{cases}$$

with $\sup \emptyset = 0$ and $u = \min(x_1, \ldots, x_n)$. $\mathbf{A}_i^*$ denotes the aggregation operator $\mathbf{A}_i$, scaled for acting on $[a_i, e_i]$ by

$$\mathbf{A}_i^*(x_1, \ldots, x_n) = a_i + (e_i - a_i) \cdot \mathbf{A}_i\big(\tfrac{x_1 - a_i}{e_i - a_i}, \ldots, \tfrac{x_n - a_i}{e_i - a_i}\big).$$

2.3 Transitivity and Preservation of Transitivity

We have already mentioned that binary fuzzy relations R_i on the subspaces X_i can be used for the comparison of two objects on the subspaces' level. For details on fuzzy relations, especially fuzzy equivalence relations we recommend [3, 16, 17, 19, 42] and for fuzzy orderings [4, 5, 6, 20, 42]. We only recall the definition of T-transitivity, since we are interested in its preservation during the aggregation process.

Definition 7. Consider a binary fuzzy relation R on some universe X and an arbitrary t-norm T. R is called *T-transitive* if and only if, for all $x, y, z \in X$ the following property holds

$$T\big(R(x,y), R(y,z)\big) \leq R(x,z).$$

Definition 8. An aggregation operator $\mathbf{A}$ *preserves T-transitivity* if, for all $n \in \mathbb{N}$ and for all binary T-transitive fuzzy relations R_i on X_i with $i \in \{1,\ldots,n\}$, the aggregated relation $\tilde{R} = \mathbf{A}(R_1,\ldots,R_n)$ on the Cartesian product of all X_i, i.e.,

$$\tilde{R}(A,B) = \tilde{R}((a_1,\ldots,a_n),(b_1,\ldots,b_n)) = \mathbf{A}\big(R_1(a_1,b_1),\ldots,R_n(a_n,b_n)\big),$$

is also T-transitive, that means, for all $A, B, C \in \prod_{i=1}^{n} X_i$,

$$T\big(\tilde{R}(A,B), \tilde{R}(B,C)\big) \leq \tilde{R}(A,C).$$

Without loss of generality, we will restrict our considerations to fuzzy relations on the same universe $X_i = X$.

2.4 Dominance — Basic Notions and Properties

Similar to t-norms, the concept of dominance has been introduced in the framework of probabilistic metric spaces [37,39] when constructing the Cartesian products of such spaces. In the framework of t-norms, dominance is also needed when constructing T-equivalence relations and fuzzy orderings [4, 6, 16, 17] on some Cartesian product.

Definition 9. Consider two t-norms T_1 and T_2. We say that T_1 *dominates* T_2 if for all $x, y, u, v \in [0,1]$ the following inequality holds

$$T_2(T_1(x,y), T_1(u,v)) \leq T_1(T_2(x,u), T_2(y,v)).$$

It can be easily verified (see also, e.g., [21]) that for any t-norm T, it holds that T itself and $T_{\mathbf{M}}$ dominate T. Furthermore, for any two t-norms T_1, T_2, $T_1 \gg T_2$ implies $T_1 \geq T_2$ and, therefore, we know that $T_{\mathbf{D}} \gg T$ if and only if $T = T_{\mathbf{D}}$ and $T \gg T_{\mathbf{M}}$ if and only if $T = T_{\mathbf{M}}$, since $T_{\mathbf{D}}$ is the weakest and $T_{\mathbf{M}}$ the strongest t-norm.

We have already mentioned before that t-norms can be interpreted as particular aggregation operators. Therefore, we extend the concept of dominance to the framework of aggregation operators [32].

Definition 10. Consider an n-ary aggregation operator $\mathbf{A}_{(n)}$ and an m-ary aggregation operator $\mathbf{B}_{(m)}$. We say that $\mathbf{A}_{(n)}$ *dominates* $\mathbf{B}_{(m)}$, $\mathbf{A}_{(n)} \gg \mathbf{B}_{(m)}$, if, for all $x_{i,j} \in [0,1]$ with $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,n\}$, the following property holds

$$\begin{aligned}\mathbf{B}_{(m)}\big(\mathbf{A}_{(n)}(x_{1,1},\ldots,x_{1,n}),\ldots,\mathbf{A}_{(n)}(x_{m,1},\ldots,x_{m,n})\big)\\ \leq \mathbf{A}_{(n)}\big(\mathbf{B}_{(m)}(x_{1,1},\ldots,x_{m,1}),\ldots,\mathbf{B}_{(m)}(x_{1,n},\ldots,x_{m,n})\big). \quad (1)\end{aligned}$$

Note that if either n or m or both are equal to 1, because of the boundary condition **(AO2)**, $\mathbf{A}_{(n)} \gg \mathbf{B}_{(m)}$ is trivially fulfilled for any two aggregation operators $\mathbf{A}, \mathbf{B}$.

Definition 11. Let $\mathbf{A}$ and $\mathbf{B}$ be aggregation operators. We say that $\mathbf{A}$ *dominates* $\mathbf{B}$, $\mathbf{A} \gg \mathbf{B}$, if $\mathbf{A}_{(n)}$ dominates $\mathbf{B}_{(m)}$ for all $n, m \in \mathbb{N}$.

Note that, if two aggregation operators $\mathbf{A}$ and $\mathbf{B}$ are both acting on some closed interval $I = [a, b] \subseteq [-\infty, \infty]$, then the property of dominance can be easily adapted by requiring that (1) must hold for all arguments $x_{i,j} \in I$ and for all $n, m \in \mathbb{N}$. Further note that the concept of dominance relates to the fact that aggregation operators are operators on posets. Therefore, dominance can and has been introduced for arbitrary operations on posets (see, e.g., [37]).

Due to the monotonicity of aggregation operators, the minimum $T_{\mathbf{M}}$ dominates not only all t-norms, but also any aggregation operator $\mathbf{A}$,

$$\mathbf{A}(\min(x_1, y_1), \ldots, \min(x_n, y_n)) \leq \min(\mathbf{A}(x_1, \ldots, x_n), \mathbf{A}(y_1, \ldots, y_n)).$$

however, as will be shown later, not all aggregation operators dominate $T_{\mathbf{D}}$. Similarly, not all aggregation operators dominate the weakest aggregation operator

$$\mathbf{A}_w(x_1, \ldots, x_n) = \begin{cases} 1, & \text{if } x_1 = \ldots = x_n = 1, \\ 0. & \text{otherwise.} \end{cases}$$

Further on, we will denote the class of all aggregation operators $\mathbf{A}$ which dominate an aggregation operator $\mathbf{B}$ by

$$\mathcal{D}_{\mathbf{B}} = \{\mathbf{A} \mid \mathbf{A} \gg \mathbf{B}\}.$$

Since t-norms are special kinds of associative aggregation operators, the following proposition will be helpful for considering the dominance of an aggregation operator over a t-norm T.

Proposition 1. [32] *Let* $\mathbf{A}, \mathbf{B}$ *be two aggregation operators. Then the following holds:*

(i) If $\mathbf{B}$ *is associative and* $\mathbf{A}_{(n)} \gg \mathbf{B}_{(2)}$ *for all* $n \in \mathbb{N}$*, then* $\mathbf{A} \gg \mathbf{B}$.
(ii) If $\mathbf{A}$ *is associative and* $\mathbf{A}_{(2)} \gg \mathbf{B}_{(m)}$ *for all* $m \in \mathbb{N}$*, then* $\mathbf{A} \gg \mathbf{B}$.

Consequently, if two aggregation operators $\mathbf{A}$ and $\mathbf{B}$ are both associative, as it would be in the case of two t-norms, it is sufficient to show that $\mathbf{A}_{(2)} \gg \mathbf{B}_{(2)}$ for proving that $\mathbf{A} \gg \mathbf{B}$.

In case of a common neutral element, the property of dominance induces the order of the involved aggregation operators.

Lemma 1. [30] *Consider two aggregation operators* $\mathbf{A}$, $\mathbf{B}$ *with a common neutral element* $e \in [0, 1]$. *If* $\mathbf{A}$ *dominates* $\mathbf{B}$, *i.e.,* $\mathbf{A} \gg \mathbf{B}$, *then* $\mathbf{A} \geq \mathbf{B}$.

As a consequence, it is clear that dominance is a reflexive and antisymmetric relation on the set of all t-norms, but it is not transitive as could be shown in [34] (for a counter example see also [33]). Note that transitivity of dominance in the framework of aggregation operators does not hold in general, since, e.g., $\mathbf{A}_w \gg T_{\mathbf{M}}$ and $T_{\mathbf{M}} \gg \mathbf{M}$ but $\mathbf{A}_w$ does not dominate $\mathbf{M}$ (see also [30]).

Further note, that the property of selfdominance of an aggregation operator, i.e., $\mathbf{A} \gg \mathbf{A}$, is nothing else than the property of bisymmetry in the sense of Aczél [1], i.e., for all $n, m \in \mathbb{N}$ and all $x_{i,j} \in [0,1]$ with $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$

$$\begin{aligned}&\mathbf{A}_{(m)}\big(\mathbf{A}_{(n)}(x_{1,1}, \dots, x_{1,n}), \dots, \mathbf{A}_{(n)}(x_{m,1}, \dots, x_{m,n})\big) \\ &\qquad = \mathbf{A}_{(n)}\big(\mathbf{A}_{(m)}(x_{1,1}, \dots, x_{m,1}), \dots, \mathbf{A}_{(m)}(x_{1,n}, \dots, x_{m,n})\big).\end{aligned}$$

Another interesting aspect is the invariance of dominance with respect to transformations.

Proposition 2. [32] *Consider two aggregation operators* $\mathbf{A}$ *and* $\mathbf{B}$ *on* $[a, b]$.

(i) $\mathbf{A} \gg \mathbf{B}$ *if and only if* $\mathbf{A}_\varphi \gg \mathbf{B}_\varphi$ *for all strictly increasing bijections* $\varphi : [c, d] \to [a, b]$.
(ii) $\mathbf{A} \gg \mathbf{B}$ *if and only if* $\mathbf{B}_\varphi \gg \mathbf{A}_\varphi$ *for all strictly decreasing bijections* $\varphi : [c, d] \to [a, b]$.

3 *T*-Transitivity and Dominance

Standard aggregation of fuzzy equivalence relations and fuzzy orderings preserving the T-transitivity has been done either by means of T itself or $T_{\mathbf{M}}$, but in fact, any t-norm $\tilde{T}$ dominating T can be applied, i.e., if R_1, R_2 are two T-transitive binary relations on a universe X and $\tilde{T} \gg T$, then also $\tilde{T}(R_1, R_2)$ is T-transitive [4, 6, 16].

As already mentioned above, in several applications, other types of aggregation processes preserving T-transitivity are required [8, 10] Especially the introduction of different weights (degrees of importance) for input fuzzy equivalences and orderings cannot be properly done by aggregation with t-norms, because of the commutativity. Therefore, we investigated aggregation operators preserving the T-transitivity of the aggregated fuzzy relations. The following theorem generalizes the result known for triangular norms [16].

Theorem 1. [32] *Let* $|X| \geq 3$ *and let* T *be an arbitrary t-norm. An aggregation operator* $\mathbf{A}$ *preserves the* T*-transitivity of fuzzy relations on* X *if and only if* $\mathbf{A} \in \mathcal{D}_T$.

4 Construction of Dominating Aggregation Operators

Since we have shown the close relationship between the preservation of T-transitivity and the dominance of the involved aggregation operator $\mathbf{A}$ over T, we are interested in the characterization of $\mathcal{D}_T$ for some t-norm T. Particularly, we are interested in the introduction of weights, respectively determining operations by its behaviour on subdomains.

4.1 Generated and Weighted T-Norms

Before turning to aggregation operators dominating a continuous, Archimedean t-norm T, recall that they are characterized by having a continuous *additive generator*, i.e., a continuous, strictly decreasing function $t : [0,1] \to [0,\infty]$ which fulfils $t(1) = 0$, and for all $x, y \in [0,1]$:

$$T(x,y) = t^{-1}\big(\min(t(0), t(x) + t(y))\big).$$

Then we also have that $T(x_1, \dots, x_n) = t^{-1}\big(\min(t(0), \sum_{i=1}^{n} t(x_i))\big)$.

Theorem 2. [32] *Consider some continuous, Archimedean t-norm T with an additive generator $t : [0,1] \to [0,c]$, with $t(0) = c$ and $c \in]0,\infty]$. Furthermore, let $\mathbf{A} : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ be an aggregation operator. Then $\mathbf{A} \in \mathcal{D}_T$ if and only if the aggregation operator $\mathbf{H} : \bigcup_{n\in\mathbb{N}} [0,c]^n \to [0,c]$ defined by*

$$\mathbf{H}(z_1, \dots, z_n) = t(\mathbf{A}(t^{-1}(z_1), \dots, t^{-1}(z_n))) \tag{2}$$

for all $n \in \mathbb{N}$ and all $z_i \in [0,c]$ with $i \in \{1, \dots, n\}$ is subadditive on $[0,c]$.

One of the main purposes for investigating aggregation operators dominating t-norms was the request for introducing weights into the aggregation process. Hence, considering continuous Archimedean t-norms, we have to find subadditive aggregation operators, which provide this possibility.

Example 3. Consider some some weights $w_1, \dots, w_n \in [0,\infty]$, $n \geq 2$, and some $c \in]0,\infty]$, then $\mathbf{H}_{(n)} : [0,c]^n \to [0,c]$ given by

$$\mathbf{H}_{(n)}(x_1, \dots, x_n) = \min(c, \sum_{i=1}^{n} w_i x_i)$$

is an n-ary, subadditive aggregation operator on $[0,c]$, fulfilling $\mathbf{H}_{(n)}(c, \dots, c) = c$, whenever $c \leq c \cdot \sum_{i=1}^{n} p_i$. This means, with convention $0 \cdot \infty = 0$, if $c = \infty$, the sum must fulfill $\sum_{i=1}^{n} w_i > 0$ and if $c < \infty$, then also $\sum_{i=1}^{n} w_i \geq 1$.

If we combine such an aggregation operator with an additive generator of a continuous Archimedean t-norm by applying the construction method as proposed in Theorem 2 we can introduce weights into the aggregation process without losing T-transitivity.

Corollary 1. *Consider a continuous Archimedean t-norm T with additive generator t, $t(0) = c$, and a weighting vector $\overrightarrow{w} = (w_1, \dots, w_n)$, $n \geq 2$, with weights $w_i \in [0,\infty]$ fulfilling $c \leq c \cdot \sum_{i=1}^{n} w_i$. Further, let $\mathbf{A}_{(n)} : [0,1]^n \to [0,1]$ be an n-ary aggregation operator defined by Eq. (2) from the aggregation operator $\mathbf{H}_{(n)}$ introduced in Example 3. Then the n-ary aggregation operator can be rewritten by*

$$\mathbf{A}_{(n)}(x_1, \dots, x_n) = t^{-1}\big(\min(t(0), \sum_{i=1}^{n} w_i \cdot t(x_i))\big) \tag{3}$$

and it dominates the t-norm T, i.e., $\mathbf{A}_{(n)} \gg T$.

Remark 1. Note that the n-ary aggregation operator defined by Equation (3) is also called weighted t-norm $T_{\overrightarrow{w}}$ ([15, 21]). Further, for any strict t-norm T, it holds, that not only $T_{\overrightarrow{w}} \gg T$, but also $T \gg T_{\overrightarrow{w}}$. In case of some nilpotent t-norm T it is clear, that $T_{\overrightarrow{w}} \gg T$, but $T \gg T_{\overrightarrow{w}}$ only if all weights $w_i \notin]0, 1[$. In case that $\sum_{i=1}^{n} w_i = 1$ we can apply Corollary 1 independently of $t(0)$. Thus for a continuous Archimedean t-norm T with additive generator t, any weighted quasi-arithmetic mean $\mathbf{W}_t$ dominates T. Especially, any weighted arithmetic mean $\mathbf{W}$ dominates $T_{\mathbf{L}}$ and any weighted geometric mean dominates $T_{\mathbf{P}}$.

Example 4. The strongest subadditive aggregation operator acting on $[0, c]$ is given by $\mathbf{H} : \bigcup_{n \in \mathbb{N}} [0, c]^n \to [0, c]$ with

$$\mathbf{H}(u_1, \ldots, u_n) = \begin{cases} 0, & \text{if } u_1 = \ldots = u_n = 0, \\ c, & \text{otherwise.} \end{cases}$$

Then, for any additive generator $t : [0, 1] \to [0, \infty]$ with $t(0) = c$, we have

$$t(\mathbf{A}(x_1, \ldots, x_n)) = \mathbf{H}\big(t(x_1), \ldots, t(x_n)\big),$$

for all $x_i \in [0, 1]$ with $i \in \{1, \ldots, n\}$ and some $n \in \mathbb{N}$, if and only if

$$\mathbf{A}(x_1, \ldots, x_n) = \begin{cases} 1, & \text{if } x_1 = \ldots = x_n = 1, \\ 0, & \text{otherwise,} \end{cases}$$

i.e., $\mathbf{A} = \mathbf{A}_w$ is the weakest aggregation. Observe that $\mathbf{A}_w$ dominates all t-norms, but not all aggregation operators, e.g., $\mathbf{A}_w$ does not dominate the arithmetic mean.

4.2 Ordinal Sums

Proposition 3. [32] *Let $(T_i)_{i \in I}$ be a family of t-norms, $(\mathbf{A}_i)_{i \in I}$ a family of aggregation operators, and $(]a_i, e_i[)_{i \in I}$ a family of non-empty, pairwise disjoint open subintervals of $[0, 1]$. If for all $i \in I$: $\mathbf{A}_i \in \mathcal{D}_{T_i}$, then the lower ordinal sum $A^{(w)} = (\langle a_i, e_i, \mathbf{A}_i \rangle)_{i \in I}$ dominates the ordinal sum $T = (\langle a_i, e_i, T_i \rangle)_{i \in I}$, i.e., $A^{(w)} \in \mathcal{D}_T$.*

Note that not all dominating aggregation operators are lower ordinal sums of dominating aggregation operators, e.g., the aggregation operator $\mathbf{A}_w$ introduced in Example 4 dominates all t-norms T, but is not a lower ordinal sum constructed by means of some index set I (in fact it is the empty lower ordinal sum). On the other hand, in case of summand t-norms the lower ordinal sum $\mathbf{A}_w = (\langle a_i, e_i, T_i \rangle)_{i \in I}$ coincides with the standard ordinal sum of t-norms $T = (\langle a_i, e_i, T_i \rangle)_{i \in I}$. Moreover, as shown in [31], the condition of Proposition 3 is not only sufficient but also necessary.

The following example also shows that weighted t-norms as proposed by Calvo and Mesiar [15] dominate the original t-norm but are no lower ordinal sums as proposed here. As a consequence we can conclude that $\big(\langle a_i, e_i, \mathcal{D}_{T_i}\rangle\big)_{i\in I} \subset \mathcal{D}_T$, whenever $T = (\langle a_i, e_i, T_i)_{i\in I}$.

Let $(]a_i, e_i[)_{i\in I}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$ and let $t_i : [a_i, e_i] \to [0,\infty]$ be continuous, strictly decreasing mappings fulfilling $t_i(e_i) = 0$. Then (and only then) the following function $T : [0,1]^2 \to [0,1]$ is a continuous t-norm [15]:

$$T(x,y) = \begin{cases} t_i^{-1}\big(\min(t_i(0), t_i(x) + t_i(y)\big), & \text{if } (x,y) \in [a_i, e_i]\,, \\ \min(x,x), & \text{otherwise.} \end{cases}$$

The corresponding weighted t-norm $T_{\overrightarrow{w}}$ in the sense of Calvo and Mesiar [15] is defined by

$$T_{\overrightarrow{w}}(x_1,\ldots,x_n) = \begin{cases} t_i^{-1}(\min(t_i(a_i), \sum_{i=1}^n w_i \cdot t_i(\min(x_i, e_i)))), & \text{if } u \in [a_i, e_i[\,, \\ \min(x_i \mid w_i > 0), & \text{otherwise,} \end{cases}$$

with $u = \min(x_i \mid w_i > 0)$ and some weighting vector $\overrightarrow{w} = (w_1,\ldots,w_n) \neq (0,\ldots,0)$ such that, if $a_i = 0$ for some $i \in I$ and the corresponding $t_i(a_i)$ is finite, then $\sum_{i=1}^n w_i \geq 1$.

Example 5. Consider the t-norm $T = (\langle 0, \frac{1}{2}, T_{\mathbf{P}}\rangle)$, i.e.,

$$T(x,y) = \begin{cases} 2xy, & \text{if } (x,y) \in \left[0, \frac{1}{2}\right]^2, \\ \min(x,y), & \text{otherwise.} \end{cases}$$

We know that the geometric mean $G(x,y) = \sqrt{x \cdot y} = T_{\mathbf{P}(\frac{1}{2},\frac{1}{2})}$ dominates $T_{\mathbf{P}}$. Therefore we can construct

– the lower ordinal sum $\mathbf{A}^{(w)} = (\langle 0, \frac{1}{2}, G\rangle)$ with

$$\mathbf{A}^{(w)}(x,y) = \begin{cases} 1, & \text{if} (x,y) = (1,1), \\ \sqrt{\min(x, \frac{1}{2}) \cdot \min(y, \frac{1}{2})}, & \text{otherwise} \end{cases}$$

– and the weighted t-norm $T_{\overrightarrow{w}} = T_{(\frac{1}{2},\frac{1}{2})}$ by

$$T_{(\frac{1}{2},\frac{1}{2})}(x,y) = \begin{cases} \min(x,y), & \text{if } (x,y) \in \left]\frac{1}{2}, 1\right]^2, \\ \sqrt{\min(x, \frac{1}{2}) \cdot \min(y, \frac{1}{2})}, & \text{otherwise.} \end{cases}$$

Both aggregation operators — $\mathbf{A}^{(w)}$ as well as $T_{\overrightarrow{w}}$ — dominate the t-norm T and they coincide in any values except for arguments $(x,y) \in \left]\frac{1}{2}, 1\right]^2 \setminus \{(1,1)\}$. Observe that this example also demonstrates that not all aggregation operators dominating an ordinal sum t-norm T are necessarily lower ordinal sums of dominating aggregation operators as given in Proposition 3.

5 Dominance of Basic T-Norms

Finally we will discuss the classes of aggregation operators dominating one of the basic t-norms as introduced in Example 2.

5.1 Dominance of the Minimum

As already observed, $T_{\mathbf{M}}$ dominates any t-norm T and any aggregation operator $\mathbf{A}$, but no t-norm T, except $T_{\mathbf{M}}$ itself, dominates $T_{\mathbf{M}}$. The class of all aggregation operators dominating $T_{\mathbf{M}}$ is described in the following proposition.

Proposition 4. [32] *For any $n \in \mathbb{N}$, the class of all n-ary aggregation operators $\mathbf{A}_{(n)}$ dominating the strongest t-norm $T_{\mathbf{M}}$ is given by*

$$\begin{aligned}\mathcal{D}_{\min}^{(n)} = \{\min{}_{\mathcal{F}} \mid \mathcal{F} = (f_1, \ldots, f_n),\ & \\ & f_i : [0,1] \to [0,1],\ \textit{non-decreasing},\ \textit{with} \\ & f_i(1) = 1 \textit{ for all } i \in \{1, \ldots, n\}, \\ & f_i(0) = 0 \textit{ for at least one } i \in \{1, \ldots, n\}\},\end{aligned}$$

where $\min_{\mathcal{F}}(x_1, \ldots, x_n) = \min(f_1(x_1), \ldots, f_n(x_n))$.

Evidently, $\mathbf{A}_{(n)} \in \mathcal{D}_{\min}^{(n)}$ is symmetric if and only if

$$\mathbf{A}_{(n)}(x_1, \ldots, x_n) = f\big(\min(x_1, \ldots, x_n)\big)$$

for some non-decreasing function $f : [0,1] \to [0,1]$ with $f(0) = 0$ and $f(1) = 1$.

Example 6. As already observed in Example 4, the weakest aggregation operator $\mathbf{A}_w$ dominates all t-norms T. Since this aggregation operator is symmetric, it can be described by $\mathbf{A}_w(x_1, \ldots, x_n) = f\big(\min(x_1, \ldots, x_n)\big)$ with $f : [0,1] \to [0,1]$ given by

$$f(x) = \begin{cases} 1, & \text{if } x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 2. Any aggregation operator $\mathbf{A}$ dominating $T_{\mathbf{M}}$ is also dominated by $T_{\mathbf{M}}$, i.e., for arbitrary $n, m \in \mathbb{N}$ and for all $x_{i,j} \in [0,1]$ with $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$ the following equality holds

$$\begin{aligned}&\mathbf{A}\big(\min(x_{1,1}, \ldots, x_{1,n}), \ldots, \min(x_{m,1}, \ldots, x_{m,n})\big) \\ &\quad = \min\big(\mathbf{A}(x_{1,1}, \ldots, x_{m,1}), \ldots, \mathbf{A}(x_{1,n}, \ldots, x_{m,n})\big).\end{aligned}$$

5.2 Dominance of the Drastic Product

Oppositely to the case of $T_{\mathbf{M}}$, the weakest t-norm $T_{\mathbf{D}} : [0,1]^2 \to [0,1]$ is dominated by any t-norm T. This can also be seen from the characterization of all aggregation operators dominating $T_{\mathbf{D}}$ as given in the next proposition.

Proposition 5. [32] *Consider an arbitrary $n \in \mathbb{N}$ and an n-ary aggregation operator $\mathbf{A}_{(n)} : [0,1]^n \to [0,1]$. Then $\mathbf{A}_{(n)} \gg T_{\mathbf{D}}$ if and only if there exists a non-empty subset $I = \{k_1, \ldots, k_m\} \subseteq \{1, \ldots, n\}, k_1 < \ldots < k_m$, and a non-decreasing mapping $B : [0,1]^m \to [0,1]$ satisfying the following conditions*

(i) $B(0, \ldots, 0) = 0$,
(ii) $B(u_1, \ldots, u_m) = 1$ *if and only if* $u_1 = \ldots = u_m = 1$,

such that $\mathbf{A}(x_1, \ldots, x_n) = B(x_{k_1}, \ldots, x_{k_m})$.

Observe that the mapping B in the above proposition is an m-ary aggregation operator whenever $m \geq 2$. However, if $m = 1$, i.e., $I = \{k\}$, then $B : [0,1] \to [0,1]$ is a non-decreasing mapping with strict maximum $B(1) = 1$ and $B(0) = 0$ as well as $A(x_1, \ldots, x_n) = B(x_k)$ and is therefore a distortion of the k-th projection.

Concerning t-norms, for any t-norm T, we have $T(x_1, \ldots, x_n) = 1$ if and only if $x_i = 1$ for all $i \in \{1, \ldots, n\}$ and thus $I = \{1, \ldots, n\}$. Therefore $B = T$ and $T \in \mathcal{D}_{T_{\mathbf{D}}}$.

5.3 Dominance of the Łukasiewicz T-Norm

Summarizing the results from Section 4.1 we can characterize aggregation operators dominating the Łukasiewicz t-norm $T_{\mathbf{L}}$ by means of the subadditivity of the corresponding dual operator.

Theorem 3. [27] *An aggregation operator $\mathbf{A}\colon \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ dominates $T_{\mathbf{L}}$ if and only if its dual aggregation operator $\mathbf{A}^d\colon \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ is subadditive.*

Note that as a consequence of Proposition 2 an aggregation operator is dominated by $T_{\mathbf{L}}$ if and only if its dual aggregation operator $\mathbf{A}^d$ is superadditive.

As already mentioned in Remark 1, any weighted arithmetic mean $\mathbf{W}$ dominates $T_{\mathbf{L}}$. Moreover, due to Corollary 1, for any constant $c \in [1, \infty[$ we have also that $\mathbf{B}\colon \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$, defined by

$$\mathbf{B}(x_1, \ldots, x_n) = \max(0, c \cdot \mathbf{W}(x_1, \ldots, x_n) + 1 - x)$$

dominates $T_{\mathbf{L}}$.

Based on Theorem 3 several other aggregation operators dominating $T_{\mathbf{L}}$ can be introduced. For example, the function $H\colon \bigcup_{n\in\mathbb{N}} [0,\infty]^n \to [0,\infty]$ given by

$$H(x_1, \ldots, x_n) = (\sum_{i=1}^{n} x_i^{\lambda})^{\frac{1}{\lambda}}$$

is subadditive for any $\lambda \geq 1$. Therefore, also the Yager t-conorm $S_\lambda^{\mathbf{Y}} = \min(H, 1)$ is subadditive such that the Yager t-norm $T_\lambda^{\mathbf{Y}}$ dominates $T_{\mathbf{L}}$ for all $\lambda \in [1, \infty[$.

Similarly any root-power operator [18] $\mathbf{A}_\lambda\colon \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ given by

$$\mathbf{A}_\lambda(x_1, \ldots, x_n) = (\tfrac{1}{n} \sum_{i=1}^{n} x_i^{\lambda})^{\frac{1}{\lambda}}$$

is subadditive for any $\lambda \geq 1$. As a consequence its dual aggregation operator $\mathbf{A}_\lambda^d \colon \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$

$$\mathbf{A}_\lambda^d(x_1,\ldots,x_n) = 1 - (\tfrac{1}{n}\sum_{i=1}^{n}(1-x_i)^\lambda)^{\frac{1}{\lambda}}$$

dominates $T_\mathbf{L}$.

For the aggregation of fuzzy relations, the introduction of weights in the aggregation process has been of importance. Therefore, the dominance of OWA operators over $T_\mathbf{L}$ is an interesting problem.

Proposition 6. [27] *Consider an n-ary OWA operator $\mathbf{W}'_{(n)}$, $n \in \mathbb{N}$, with weights $w_1,\ldots,w_n$. Then $\mathbf{W}'_{(n)}$ dominates $T_\mathbf{L}$ if and only if $w_1 \geq w_2 \geq \ldots \geq w_n$.*

If we consider an OWA operator $\mathbf{W}' : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$, it is clear that $\mathbf{W}' \gg T_\mathbf{L}$ if and only if $\mathbf{W}'_{(n)} \gg T_\mathbf{L}$ for all $n \in \mathbb{N}$.

It has been proposed in [41] to derive the weights for an OWA operator from some quantifier function $q : [0,1] \to [0,1]$, which is a monotone real function such that $\{0,1\} \subseteq \operatorname{Ran} q$. As a consequence, q can either be non-decreasing with $q(0) = 0$ and $q(1) = 1$ or can be non-increasing with $q(0) = 1$ and $q(1) = 0$.

Since we are looking for aggregation operators dominating $T_\mathbf{L}$, the corresponding weights for each n-ary operator must be non-increasing. Therefore we are looking for additional properties for the quantifier function, such that the non-increasingness of the weights is guaranteed. It will turn out, that non-increasingness of the weights is closely related to the concavity, resp. the convexity of the involved quantifier.

Definition 12. A function f on some convex domain A is *convex*, if the following inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

holds for all $\lambda \in [0,1]$ and $x, y \in A$. The function is said to be *concave*, if the inequality

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$$

holds for all $\lambda \in [0,1]$ and $x, y \in A$.

First, we will restrict our considerations to non-decreasing quantifiers. Some examples for such functions are shown in Fig. 1. The weights derived from such a quantifier can be computed by

$$w_{in} = q(\tfrac{i}{n}) - q(\tfrac{i-1}{n}).$$

Lemma 2. *If $q : [0,1] \to [0,1]$ is a non-decreasing quantifier for some OWA operator and the generated weights fulfill $w_{1,n} \geq \ldots \geq w_{n,n}$ for all $n \in \mathbb{N}$ and $i \in \{1,\ldots,n\}$, then q is continuous on $]0,1]$.*

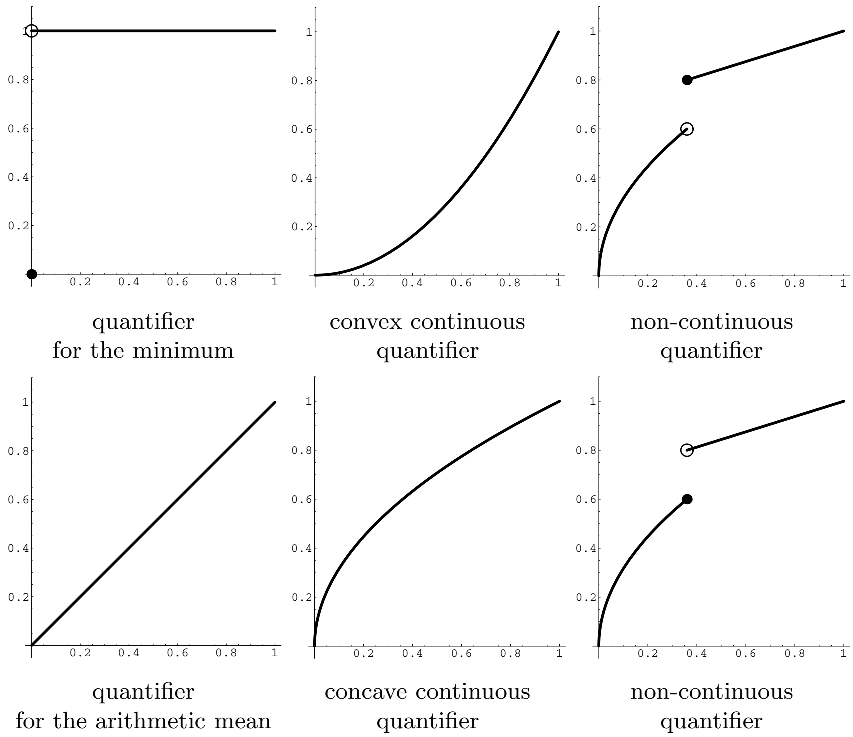

Fig. 1. Some examples of non-decreasing quantifier functions

Proposition 7. [27] *Consider some OWA operator with non-decreasing quantifier* $q : [0,1] \to [0,1]$ *and generated weights* $w_{1,n}, \dots, w_{n,n}$ *for all* $n \in \mathbb{N}$*. Then these weights fulfill* $w_{1,n} \geq \dots \geq w_{n,n}$ *for all* $n \in \mathbb{N}$ *if and only if* q *is concave on* $]0,1]$*, i.e.,* $\forall x, y \in [0,1], \forall \lambda \in [0,1]$

$$q(\lambda x + (1-\lambda)y) \geq \lambda q(x) + (1-\lambda)q(y).$$

Example 7. A typical example of an OWA operator $\mathbf{W}'$ dominating $T_{\mathbf{L}}$ is generated by the quantifier function $q(x) = 2x - x^2$. Observe that for any $n \in \mathbb{N}$ the corresponding weights are given by

$$w_{in} = \tfrac{2(n-i)+1}{n^2}, \quad i \in \{1, \dots, n\}.$$

If a quantifier function is non-increasing then the weights can be computed by

$$w_{in} = q(\tfrac{i-1}{n}) - q(\tfrac{i}{n}).$$

For a few examples of non-increasing quantifiers see Fig. 2. The following properties can be shown analogously to the case of non-decreas-ing quantifiers.

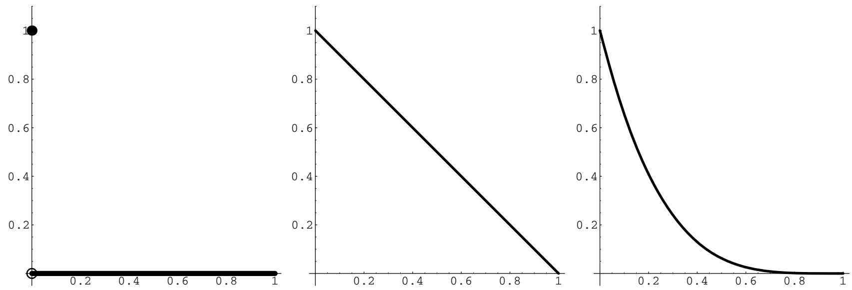

Fig. 2. Some examples of non-increasing quantifier functions

Corollary 2. *If* $q : [0,1] \to [0,1]$ *is a non-increasing quantifier for some OWA operator and the generated weights fulfill* $w_{1n} \geq \ldots \geq w_{nn}$ *for all* $n \in \mathbb{N}$ *and* $i \in \{1, \ldots, n\}$*, then q is continuous on* $]0,1]$.

Corollary 3. *Consider some OWA operator with non-increasing quantifier* $q : [0,1] \to [0,1]$. *Then the generated weights fulfill* $w_{1n} \geq \ldots \geq w_{nn}$ *for all* $n \in \mathbb{N}$ *if and only if q is convex on* $]0,1]$*, i.e.,* $\forall x, y \in [0,1], \forall \lambda \in [0,1]$

$$q(\lambda x + (1-\lambda)y) \leq \lambda q(x) + (1-\lambda)q(y).$$

Remark 3. Any nilpotent t-norm T is isomorphic to the Łukasiewicz t-norm $T_{\mathbf{L}}$, i.e., $T{=}(T_{\mathbf{L}})_\varphi$ with $\varphi : [0,1] \to [0,1]$ a strictly increasing bijection. According to Proposition 2, we know that if $T_{\mathbf{L}}$ is dominated by an OWA operator $\mathbf{W}'$ then an isomorphic t-norm $T = (T_{\mathbf{L}})_\varphi$ is dominated by the aggregation operator $\mathbf{W}'_\varphi$. In fact $\mathbf{W}'_\varphi$ is nothing else than an ordered weighted quasi-arithmetic mean (OWQA) with respect to the strictly increasing bijection $\varphi : [0,1] \to [0,1]$ with corresponding weights $w_{1n} \geq w_{2n} \geq \ldots \geq w_{nn}$ for all $n \in \mathbb{N}$, i.e.,

$$\begin{aligned} \mathbf{W}'_\varphi(x_1, \ldots, x_n) &= \varphi^{-1}(\mathbf{W}'(\varphi(x_1), \ldots, \varphi(x_n))) \\ &= \varphi^{-1}(\tfrac{1}{n}\sum_{i=1}^{n} w_{in}\varphi(x_i)') = \varphi^{-1}(\tfrac{1}{n}\sum_{i=1}^{n} w_{in}\varphi(x_i')). \end{aligned}$$

5.4 Dominance of the Product

Concerning dominance over the product $T_{\mathbf{P}}$, Theorem 2 transforms as follows.

Theorem 4. [27] *An aggregation operator* $\mathbf{A}\colon \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ *dominates* $T_{\mathbf{P}}$ *if and only if the function* $f_n\colon [0,\infty]^n \to [0,\infty]$ *given by*

$$f_n(x_1, \ldots, x_n) = -\log(\mathbf{A}(e^{-x_1}, \ldots, e^{-x_n}))$$

is subadditive for each $n \in \mathbb{N}$.

Again an aggregation operator $\mathbf{A}$ is dominated by $T_{\mathbf{P}}$ if and only if each f_n as given by Theorem 4 is superadditive.

As already mentioned any weighted geometric mean dominates $T_{\mathbf{P}}$. Moreover, for any $n \geq 2$ and any $\overrightarrow{w} = (w_1, \ldots, w_n)$ with $\sum_{i=1}^n w_i > 0$ and $w_i \in [0, \infty]$, the function $\mathbf{H} : [0, \infty]^n \to [0, \infty]$ defined by

$$\mathbf{H}(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i \cdot x_i$$

is an n-ary, subadditive aggregation operator acting on $[0, \infty]$. Therefore, any n-ary aggregation operator

$$\mathbf{A}_{\overrightarrow{w}}(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i^{w_i}$$

dominates the product $T_{\mathbf{P}}$.

However, observing that for all $\lambda \geq 1$, the function

$$\mathbf{H}_\lambda : [0, \infty]^2 \to [0, \infty] \,, \mathbf{H}_\lambda(x, y) = (x^\lambda + y^\lambda)^{\frac{1}{\lambda}},$$

is also a binary, subadditive aggregation operator acting on $[0, \infty]$, also any member of the Aczél-Alsina family of t-norms $(T_\lambda^{\mathbf{AA}})_{\lambda \in [1, \infty]}$, is contained in $\mathcal{D}_{T_{\mathbf{P}}}$ because of Theorem 2.

Similar as in the case of the Łukasiewicz t-norm $T_{\mathbf{L}}$, we can show the next result.

Proposition 8. [27] *For a fixed $n \in \mathbb{N}$ and some weighting vector $\overrightarrow{w} = (w_1, \ldots, w_n)$, let $\mathbf{A}\colon [0,1]^n \to [0,1]$ be an ordered weighted geometric mean, i.e., $\mathbf{A}(x_1, \ldots, x_n) = \prod_{i=1}^n (x_i')^{w_i}$ where x_i' is again the i-th order statistic of $(x_1, \ldots, x_n)$. Then $\mathbf{A} \gg T_{\mathbf{P}}$ if and only if $w_1 \geq w_2 \geq \ldots \geq w_n$.*

Due to the isomorphism of any strict t-norm to the product $T_{\mathbf{P}}$, similar considerations are valid for any strict t-norm.

5.5 Final Remarks Related to Continuous Archimedean T-Norms

In Section 5.3 we have shown how the dominance of $T_{\mathbf{L}}$ by an OWA operator $\mathbf{W}'$ restricts the possible choices for weights. When considering some similar constraints reflecting $\mathbf{W}' \gg T$ for some other continuous Archimedean t-norm T, we cannot exploit the isomorphism of $T_{\mathbf{L}}$ and nilpotent t-norms (then also $\mathbf{W}'$ should be isomorphically transformed). Thus as a separate problem let us consider a continuous Archimedean t-norm T with additive generator t and an OWA operator $\mathbf{W}' : \bigcup_{n \in \mathbb{N}} [0,1]^n \to [0,1]$ which is supposed to dominate T, i.e., for all $n \in \mathbb{N}$ and for all $x_i, y_i \in [0,1], i \in \{1, \ldots, n\}$

$$\mathbf{W}'(T(x_1, y_1), \ldots, T(x_n, y_n)) \geq T(\mathbf{W}'(x_1, \ldots, x_n), \mathbf{W}'(y_1, \ldots, y_n)).$$

If we concentrate on the binary case and choose $x_1 = 0$, $y_1 = 1$, $x_2 = 1$, $y_2 > 0$ then we see that necessarily

$$\begin{aligned}\mathbf{W}'(0, y_2) = w_2 y_2 &\geq T(w_2, w_1 y_2 + w_2)\\ &= t^{-1}(\min(t(0), t(w_2) + t(w_1 y_2 + w_2))),\end{aligned}$$

i.e., for all $y_2 \in [0, 1[$

$$t(w_2 y_2) \leq t(w_2) + t(w_1 y_2 + w_2).$$

Evidently if $t(0) = +\infty$ then we get that $w_2 = 0$ because of the continuity of t. Similarly we can show in the general case with $n \in \mathbb{N}$ that $w_i = 0$ for $i > 1$. It follows that for any strict t-norm T only one OWA dominates T, namely the minimum.

In the case of nilpotent t-norms, equation (5.5) gives a necessary condition for $\mathbf{W}' \gg T$.

For $y_2 \to 0^+$ we get that for normed additive generators $1 \leq 2t(w_2)$, i.e., $w_2 \leq t^{-1}(\frac{1}{2})$ holds. This fact can be exploited in determination of OWA operators dominating a specific t-norm. For example, it can be conjectured that an OWA operator with weights $(w_1, \ldots, w_n)$ dominates

– Yager's t-norm T_p^Y [21] with parameter $p \in]0, \infty[$ and normed additive generator $t_p(x) = (1 - x)^p$ if and only if

$$w_i \geq \tfrac{1}{2^{1/p} - 1} w_{i+1}, i = 1, \ldots, n-1,$$

– Schweizer-Sklar's t-norm T_λ^{SS} [21] with parameter $\lambda \in]0, \infty[$ and normed additive generator $t_\lambda(x) = 1 - x^\lambda$ if and only if

$$w_i \geq (2^{1/\lambda} - 1) w_{i+1}.$$

Observe that the arithmetic mean $\mathbf{M} \gg T_p^Y$ if and only if $p \leq 1$ and $\mathbf{M} \gg T_\lambda^{SS}$ if and only if $\lambda \geq 1$. Recall that $T_\mathbf{L} = T_1^Y = T_1^{SS}$.

6 Conclusion

We have discussed the aggregation of fuzzy relations and the preservation of their transitivity. In particular, the aggregation operator $\mathbf{A}$ preserves the T-transitivity of fuzzy relations if and only if it dominates the corresponding t-norm T ($\mathbf{A} \in \mathcal{D}_T$). Several methods for constructing aggregation operators within a certain class $\mathcal{D}_T$ have been mentioned with a particular emphasis on the introduction of weights. Further, a characterization of $\mathcal{D}_T$ for the four basic t-norms has been provided.

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