# Fuzzy Information Relations and Operators: An Algebraic Approach Based on Residuated Lattices*

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**Abstract.** We discuss fuzzy generalisations of information relations taking two classes of residuated lattices as basic algebraic structures. More precisely, we consider commutative and integral residuated lattices and *extended residuated lattices* defined by enriching the signature of residuated lattices by an antitone involution corresponding to the De Morgan negation. We show that some inadequacies in representation occur when residuated lattices are taken as a basis. These inadequacies, in turn, are avoided when an extended residuated lattice constitutes the basic structure. We also define several fuzzy information operators and show characterizations of some binary fuzzy relations using these operators.

**Keywords:** Information relations, Information operators, Residuated lattices, Fuzzy sets, Fuzzy logical connectives.

## 1 Introduction

In real–life problems we usually deal with incomplete information. Generally speaking, there are two reasons for incompleteness of information. Firstly, we often have only partial data about a domain under considerations. Secondly, the acquired information, if available, is often imprecise (e.g. when expressed by means of linguistic terms like "quite good" or "rather tall"). Formal methods for representing and analyzing incomplete information have been extensively developed within the theory of rough sets ([26]). In these approaches an information relation is any relation defined on a set of objects of an information system and determined by the properties of these objects. Since properties of an object can be represented by a set of values of its attributes (properties), any information relation is formally a binary relation between two subsets of a domain in discourse. Examples of some information relations (in information systems) and

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their theories can be found, for example, in [6], [10], [25], and [26]. A comprehensive exposition of logical and algebraic theories of information relations and their applications can be found in [7].

When imprecise information is admitted, it is clear that it cannot be adequately represented by means of standard methods based on classical two–valued structures. A natural solution seems to be fuzzy generalisations of the respective methods. Multi–valued generalisations of information relations based on residuated lattices were developed in [27].

In the present paper we continue our studies of fuzzy information relations and information operators. In [30], [31], and [33] we discussed fuzzy generalisations of information relations taking the unit interval $[0, 1]$ and traditional fuzzy logical connectives as the basis. In this framework the relationships between objects are real numbers from $[0, 1]$, so they are always comparable. In real–life problems, however, such relationships need not have this property. For instance, a child is usually similar to both parents, but it is often hard to say to which of his/her parents the child is more (or less) similar. Therefore, some lattice–based approaches seems to be more adequate.

We present some fuzzy generalisations of information relations and information operators taking two classes of residuated lattices ([4],[8],[15],[16],[22],[42]) as basic algebraic structures. Our approach is motivated by the role these algebras play in fuzzy set theory ([18],[19],[20],[23],[43]) and by the rough set–style data analysis ([26]). In a residuated lattice a product operator and its residuum are abstract counterparts of a triangular norm ([41]) and a fuzzy residual implication ([23]), respectively. However, traditional residuated lattices do not provide sufficiently general counterparts of other fuzzy logical connectives, in particular triangular conorms, fuzzy negations, and fuzzy S–implications. Consequently, in generalisations of information relations some inadequacies occur. From this reason, *double residuated lattices* were introduced ([28],[29]) and some fuzzy information relations and operators were investigated. In the signature of these algebras there are two independent operations corresponding to a triangular norm and a triangular conorm. Yet these structures do not give us the algebraic counterpart of the De Morgan negation. Therefore, while some inadequacies are avoided, other drawbacks in representation still remain. To cope with these problems, we propose yet another class of residuated lattices, called *extended residuated lattices* ([12]), which are an extension of residuated lattices by an antitone involution. This operation, together with the operations of residuated lattices, allows us to define algebraic counterparts of the main classes of fuzzy logical connectives. Basing on these algebras, we extend the results obtained in [36] and discuss another generalisation of information relations. We show how these representations allow us to avoid inadequacies occurring when residuated lattices are taken as a basis.

It is well–known that binary relations determine modal–like operators which, in turn, are the abstract counterparts of the information operators derived from information systems ([7]). Generally speaking, an information operator is any mapping defined on binary relations on a non–empty universe and subsets of this universe. A general theory of the classical abstract information operators was developed in [7], [10], and [11]. A fuzzy generalisation of some information operators, based on the interval $[0,1]$, were presented in [32], [33], and [38]. In [34], [35], [36], and [37] fuzzy approximation operators based on residuated lattices were discussed.

In this paper we propose a generalisation of information operators determined by information relations based on extended residuated lattices. This approach might be a basis for developing multi–valued logics and algebras. On the other hand, this is a generalisation of approximation operators, which are the main tools in rough set–style data analysis. We show that properties of main classes of fuzzy information relations can be expressed by means of these operators.

The paper is organized as follows. In Section 2 we provide some algebraic foundations to our discussion. In particular, the notions of residuated lattices and extended residuated lattices will be presented. Also, the notion of fuzzy sets and fuzzy relations will be recalled. In Section 3 we define several fuzzy information relations taking a commutative and integral residuated lattice as a basic structure. Main properties of these relations will be presented. We will point out some drawbacks of this representation and propose another generalisation of some information relations, where extended residuated lattices are taken as basic structures. Next, in Section 4, we discuss some fuzzy information operators. It will be shown that these operators are useful for characterisations of fuzzy binary relations. Some concluding remarks will complete the paper.

## 2 Algebraic Foundations

### 2.1 Residuated Lattices

A *monoid* is a system $(M, \circ, \varepsilon)$, where $M$ is a non–empty set, $\circ$ is an associative operation in $M$, and $\varepsilon \in M$ is such that $\varepsilon \circ a = a \circ \varepsilon = a$ for every $a \in M$. A monoid $(M, \circ, \varepsilon)$ is called *commutative* iff $\circ$ is commutative.

Typical examples of monoid operations are *triangular norms* (t–norms) and *triangular conorms* (t–conorms). Recall ([41]) that a triangular norm $t$ (resp. a triangular conorm $s$) is a $[0,1]^2 - [0,1]$ mapping, non-decreasing in both arguments, associative, commutative, and satisfying for every $a \in [0,1]$ the boundary condition $t(a,1) = a$ (resp. $s(0,a) = a$). The well–known t–norms and t–conorms, $t_Z$ and $s_Z$ (the Zadeh's t–norm and the t–conorm), $t_P$ and $s_P$ (the algebraic product and the bounded sum), and $t_L$ and $s_L$ (the Łukasiewicz t–norm and the Łukasiewicz t–conorm), are given in Table 1.

**Table 1.** Well–known t–norms and t–conorms

| | |
|---|---|
| $t_Z(a,b) = \min(a,b)$ | $s_Z(a,b) = \max(a,b)$ |
| $t_P(a,b) = a \cdot b$ | $s_P(a,b) = a + b - a \cdot b$ |
| $t_L(a,b) = \max(0, a+b-1)$ | $s_L(a,b) = \min(1, a+b)$ |

Let $(L, \leqslant)$ be a poset and let $\circ$ be a binary operation in $L$. Define two binary operations in $L$, $\rightarrow_r$, $\rightarrow_l$, satisfying the *residuation conditions* for all $a, b, c \in L$,

$$a \circ b \leqslant c \quad \textit{iff} \quad a \leqslant b \rightarrow_l c \tag{1}$$
$$a \circ b \leqslant c \quad \textit{iff} \quad b \leqslant a \rightarrow_r c \tag{2}$$

The operations (1) and (2) are called the *left residuum of* $\circ$ and the *right residuum of* $\circ$, respectively. It can be easily shown that if the respective residua exist, then

$$a \rightarrow_l b = \sup\{c \in L : c \circ a \leqslant b\}$$
$$a \rightarrow_r b = \sup\{c \in L : a \circ c \leqslant b\}.$$

Clearly, if $\circ$ is commutative, then $\rightarrow_l = \rightarrow_r$.

Residua of left–continuous[1] t–norms are called *fuzzy residual implications* ([23]). Three well–known residual implications, $\rightarrow_Z$, $\rightarrow_P$, and $\rightarrow_L$, determined by $t_Z$, $t_P$ and $t_L$, respectively, are given in Table 2.

**Table 2.** Well–known residual implications

| | |
|---|---|
| Gödel implication | $i_Z(a,b) = 1$ iff $a \leqslant b$ and $i_Z(a,b) = b$ otherwise |
| Gaines implication | $i_P(a,b) = 1$ iff $a \leqslant b$ and $i_P(a,b) = \frac{b}{a}$ otherwise |
| Łukasiewicz implication | $i_L(a,b) = \min(1, 1-a+b)$ |

**Definition 1.** *A* ***residuated lattice*** *is an algebra* $(L, \wedge, \vee, \otimes, \rightarrow_l, \rightarrow_r, 0, 1, 1')$ *such that*

**(i)** $(L, \wedge, \vee, 0, 1)$ *is a bounded lattice with the least element* $0$ *and the greatest element* $1$,
**(ii)** $(L, \otimes, 1')$ *is a monoid, and*
**(iii)** $\rightarrow_l$ *and* $\rightarrow_r$ *are the left and the right residuum of* $\otimes$*, respectively.*

*The operation* $\otimes$ *of a residuated lattice* $L$ *is called its product.* □

We say that a residuated lattice $(L, \wedge, \vee, \otimes, \rightarrow_l, \rightarrow_r, 0, 1, 1')$ is

- *integral* iff $1' = 1$,
- *commutative* iff $\otimes$ is commutative,
- *complete* iff the underlying lattice $(L, \wedge, \vee, 0, 1)$ is complete.

*Remark 1.* Some researchers (in particular, fuzzy logicians) assume that residuated lattices are commutative by definition (e.g. [3],[19]). Others, however, consider these structures in a more general framework and assume that the product operation of residuated lattices need not be commutative (see, for example, [4] and [22]). □

Throughout this paper we consider only integral and commutative residuated lattices, which will be referred to as *R–lattices* and written simply $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$.

[1] A t–norm is *called left–continuous* iff it has left–continuous partial mappings.

Given an R–lattice $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$, we define the following *precomplement* operation for every $a \in L$:

$$\neg a = a \rightarrow 0. \tag{3}$$

Note that this operation is a generalisation of the pseudo–complement in a lattice ([39]). If $\wedge = \otimes$, then $\rightarrow$ is the relative pseudo–complement, $\neg$ is the pseudo–complement and $(L, \wedge, \vee, \rightarrow, \neg, 0, 1)$ is a Heyting algebra.

*Example 1.* Let $t$ be a left–continuous t–norm and let $i_t$ be the fuzzy residual implication based on $t$. Put $\mathcal{L} = [0, 1]$. Then the algebra $(\mathcal{L}, \min, \max, t, i_t, 0, 1)$ is an R–lattice. □

The following lemma will be useful later.

**Lemma 1.** *Let $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be an R–lattice such that its product $\otimes$ satisfies the following condition: for all $a, b \in L$,*

$$a \neq 0 \ \& \ b \neq 0 \Longrightarrow a \otimes b \neq 0. \tag{4}$$

*Then for every $a \in L$, $\neg a = 0$ iff $a \neq 0$ and $\neg a = 1$ iff $a = 0$.*

*Proof.* Analogous to the proof presented in [5]. ■

Following the terminology from fuzzy set theory, we say that the product $\otimes$ satisfying (4) *has no zero divisors*. Notice that among t–norms given in Table 1, the Zadeh's t–norm $t_Z$ and the algebraic product $t_P$ have this property, while the Łukasiewicz t–norm $t_L$ does not. The family of all R–lattices, which product satisfy (4), will be denoted by $RL^+$.

For the recent results on residuated lattices we refer to [2], [4], [21], and [22].

Given an R–lattice $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$, its product $\otimes$ is an algebraic counterpart of a left–continuous t–norm, the residuum $\rightarrow$ of $\otimes$ corresponds to a fuzzy residual implication determined by $\otimes$, and the precomplement $\neg$ corresponds to a fuzzy negation.[2] However, in general $\neg$ is not involutive. Moreover, the signature of R–lattices do not give algebraic counterparts of t–conorms. From this reason *double residuated lattices* were proposed (see [28],[29]).

First, let us recall the following notions. Given a poset $(L, \leqslant)$, and a binary operation $\circ$ in $L$, the following binary operations in $L$, $\leftarrow_l$ and $\leftarrow_r$, respectively called the *dual left residuum of* $\circ$ and the *dual right residuum of* $\circ$, are defined as follows: for all $a, b \in L$,

$$c \leqslant a \circ b \quad \textit{iff} \quad c \leftarrow_l b \leqslant a \tag{5}$$

$$c \leqslant a \circ b \quad \text{iff} \quad c \leftarrow_r a \leqslant b. \tag{6}$$

If the respective dual residua of $\circ$ exist, then

$$a \leftarrow_l b = \inf\{c \in L : a \leqslant c \circ b\}$$
$$a \leftarrow_r b = \inf\{c \in L : a \leqslant b \circ c\}.$$

[2] A fuzzy negation ([23]) is a non–increasing mapping $n : [0, 1] \rightarrow [0, 1]$ satisfying $n(0) = 1$ and $n(1) = 0$.

Dual residua of a lattice join were studied by Rauszer ([40]) in the context of Heyting–Brouwer logic. In [1] dual residua of a monoid operator are discussed. The dual residua of the most famous t–conorms are presented in Table 3.

**Table 3.** The dual residua of well–known t–conorms

| |
|---|
| $a \leftarrow_Z b = 0$ iff $b \leqslant a$ and $a \leftarrow_Z b = b$ otherwise |
| $a \leftarrow_P b = 0$ iff $b \leqslant a$ and $a \leftarrow_P b = \frac{b-a}{1-a}$ otherwise |
| $a \leftarrow_L b = \max(0, b-a)$ |

**Definition 2.** ([28],[29]) *A* ***double residuated lattice*** *is an algebra* $(L, \wedge, \vee, \otimes, \oplus, \rightarrow_l, \rightarrow_r, \leftarrow_l, \leftarrow_r, 0, 1, 0', 1')$ *such that* $(L, \wedge, \vee, \otimes, \rightarrow_l, \rightarrow_r, 0, 1, 1')$ *is a residuated lattice,* $(L, \oplus, 0')$ *is a monoid, and* $\leftarrow_l$ *and* $\leftarrow_r$ *are respectively the dual left and the dual right residuum of* $\oplus$. □

A double residuated lattice is called *commutative* (resp. *integral*) iff $\otimes$ and $\oplus$ are commutative (resp. $1' = 1$ and $0' = 0$). Commutative and integral double residuated lattices will be written $(L. \wedge, \vee, \otimes, \oplus, \rightarrow, \leftarrow, 0, 1)$.

Given a commutative and integral double residuated lattice, define the *dual precomplement* operation as

$$\ulcorner a = 1 \leftarrow a \quad \text{for every } a \in L. \tag{7}$$

This operation is a generalisation of a dual pseudo–complement ([39]). The dual pseudo–complement is one of the operations in double Stone algebras. However, it is a primitive operation there, residuation operations are not in the signature of Stone algebras.

Let $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ be a bounded lattice with its ordering $\leqslant$. We write $\mathcal{L}^{-1}$ to denote the lattice obtained from $\mathcal{L}$ by reversing its ordering, i.e. the lattice with the ordering $\leqslant^{-1} = \geqslant$. Then the join $\vee^{-1}$ (resp. the meet $\wedge^{-1}$) of $\mathcal{L}^{-1}$ is the meet $\wedge$ (resp. the join $\vee$) of $\mathcal{L}$ and the greatest (resp. the least) element of $\mathcal{L}^{-1}$ is the least (resp. the greatest) element of $\mathcal{L}$. In other words, $\mathcal{L}^{-1} = (L, \vee, \wedge, 1, 0)$.

**Proposition 1.** [29] *Let* $(L, \wedge, \vee, \otimes, \oplus, \rightarrow_l, \rightarrow_r, \leftarrow_l, \leftarrow_r, 0, 1, 0', 1')$ *be a double residuated lattice. Then the algebras* $(L, \wedge, \vee, \otimes, \rightarrow_l, \rightarrow_r, 0, 1, 1')$ *and* $(L, \vee, \wedge, \oplus, \leftarrow_l, \leftarrow_r, 1, 0, 0')$ *are residuated lattices.* ■

In view of the above proposition it is easily observed that in double residuated lattices the analogon of Lemma 1 holds. Namely, if $\oplus$ satisfies the condition

$$a \neq 1 \ \&\ b \neq 1 \Longrightarrow a \oplus b \neq 1 \quad \text{for all } a, b \in L,$$

then $\ulcorner a = 1$ iff $a \neq 1$ and $\ulcorner a = 0$ iff $a = 1$. This means that $\ulcorner$ can be reduced to the binary case.

Observe that the signature of a commutative and integral double residuated lattice $(L, \wedge, \vee, \otimes, \oplus, \rightarrow, \leftarrow, 0, 1)$ gives two independent algebraic counterparts of a t–norm ($\otimes$) and a t–conorm ($\oplus$). Also, the residuum $\rightarrow$ of $\otimes$ corresponds to a residual implication (determined by $\otimes$). However, we do not have algebraic counterpart of the second main class of fuzzy implications called *S–implications*. Recall ([23]) that an S–implication, determined by a t–conorm $s$ and a fuzzy negation $n$, is defined by: $i_{s,n}(a,b) = s(n(a),b)$ for all $a,b \in [0,1]$ (the most famous S–implications, based respectively on $s_Z$ and $n$, $s_P$ and $n$, and $s_L$ and $n$, where $n$ is the standard fuzzy negation $n(a)=1-a$ for $a \in [0,1]$, are given in Table 4). Moreover, neither the precomplement $\neg$ nor the dual precomplement $\ulcorner$ are sufficiently general counterparts of fuzzy negations, since (under some conditions) can be reduced to the binary case. Therefore, in general case we cannot obtain the counterpart of the De Morgan negation. Having this on mind, the *extended residuated lattices* were defined ([12],[34],[36]).

**Table 4.** Well–known S–implications

| | |
|---|---|
| Kleene–Dienes implication | $i_{s_Z,n}(a,b) = \max(1-a,n)$ |
| Reichenbach implication | $i_{s_P,n}(a,b) = 1-a+a \cdot b$ |
| Łukasiewicz implication | $i_{s_L,n}(a,b) = \min(1,1-a+b)$ |

**Definition 3.** *By an* ***extended residuated lattice*** *we mean a system* $(L,\wedge,\vee,\otimes,\rightarrow_l,\rightarrow_r,\sim,0,1,1')$ *such that*

**(i)** $(L,\wedge,\vee,\otimes,\rightarrow_l,\rightarrow_r,0,1,1')$ *is a residuated lattice*

**(ii)** $\sim$ *is an antitone involution satisfying* $\sim 0=1$ *and* $\sim 1=0$. □

Analogously, an extended residuated lattice is integral (resp. commutative) iff the underlying residuated lattice is integral (resp. commutative). Any integral and commutative extended residuated lattice will be referred to as an *ER–lattice* and written $(L,\wedge,\vee,\otimes,\rightarrow,\sim,0,1)$.

Let $(L,\wedge,\vee,\otimes,\rightarrow,\sim,0,1)$ be an ER–lattice. Let us define the following operations in $L$: for all $a,b \in L$,

$$a \oplus b = \sim(\sim a \otimes \sim b) \tag{8}$$

$$a \Rightarrow b = \sim a \oplus b \tag{9}$$

$$a \leftarrow b = \sim(\sim a \rightarrow \sim b) \tag{10}$$

$$a \Leftarrow b = \sim(\sim a \Rightarrow \sim b). \tag{11}$$

*Remark 2.* Assume that $\sim$ and $\rightarrow$ are respectively the classical negation and implication. From the definition (10) it follows that $a \leftarrow b = \sim(b \rightarrow a)$, so $a \leftarrow b$ is a generalisation of the classical conjunction $b \wedge \sim a$. The operation (11) has the similar interpretation. □

By straightforward verification one can easily check the following

**Proposition 2.** *Let* $(L, \wedge, \vee, \otimes, \rightarrow, \sim, 0, 1)$ *be an ER–lattice. Then*

**(i)** $(L, \oplus, 0)$ *is a commutative monoid*
**(ii)** $\leftarrow$ *is the dual residuum of* $\oplus$
**(iii)** $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ *and* $(L, \vee, \wedge, \oplus, \leftarrow, 1, 0)$ *are R–lattices*
**(iv)** $(L, \wedge, \vee, \otimes, \oplus, \rightarrow, \leftarrow, 0, 1)$ *is a commutative and integral double residuated lattice.* ■

Given an ER-lattice $(L, \wedge, \vee, \otimes, \rightarrow, \sim, 0, 1)$, its product $\otimes$ and its sum $\oplus$ are algebraic counterparts of a triangular norm and a triangular conorm. Also, $\rightarrow$ and $\Rightarrow$ correspond to a fuzzy residual implication and an S–implication, respectively. Finally, $\sim$ corresponds to the De Morgan negation. Therefore, ER–lattices allow us to get algebraic counterparts of all main classes of fuzzy logical connectives.

Main properties of ER–lattices are given in the following two lemmas.

**Lemma 2.** *For every ER–lattice* $(L, \wedge, \vee, \otimes, \rightarrow, \sim, 0, 1)$ *and for all* $a, b, c \in L$, *the following properties hold:*

| | |
|---|---|
| **(i)** $a \leqslant b$ *implies* $a \otimes c \leqslant b \otimes c$, $b \rightarrow c \leqslant a \rightarrow c$, $c \rightarrow a \leqslant c \rightarrow b$, $b \Rightarrow c \leqslant a \Rightarrow c$, $c \Rightarrow a \leqslant c \Rightarrow b$, $\neg b \leqslant \neg a$ | **(i')** $a \leqslant b$ *implies* $a \oplus c \leqslant b \oplus c$, $b \leftarrow c \leqslant a \rightarrow c$, $c \leftarrow a \leqslant c \rightarrow b$, $b \Leftarrow c \leqslant a \Rightarrow c$, $c \Leftarrow a \leqslant c \Rightarrow b$, $\ulcorner b \leqslant \ulcorner a$ |
| **(ii)** $a \otimes b \leqslant a$ | **(ii')** $a \leqslant a \oplus b$ |
| **(iii)** $a \otimes b \leqslant a \wedge b$ | **(iii')** $a \vee b \leqslant a \oplus b$ |
| **(iv)** $a \otimes 0 = 0$ | **(iv')** $a \oplus 1 = 1$ |
| **(v)** $a \leqslant b$ *iff* $a \rightarrow b = 1$ | **(v')** $b \leftarrow a = 0$ *iff* $a \leqslant b$ |
| **(vi)** $1 \rightarrow a = 1 \Rightarrow a = a$ | **(vi')** $0 \leftarrow a = 0 \Leftarrow a = a$ |
| **(vii)** $a \otimes (a \rightarrow b) \leqslant b$ | **(vii')** $b \leqslant a \oplus (a \leftarrow b)$ |
| **(viii)** $a \otimes (b \rightarrow c) \leqslant b \rightarrow (a \otimes c)$ | **(viii')** $b \leftarrow (a \oplus c) \leqslant a \oplus (b \leftarrow c)$ |
| **(ix)** $(a \rightarrow b) \otimes (b \rightarrow c) \leqslant (a \rightarrow c)$ | **(ix')** $(a \leftarrow c) \leqslant (a \leftarrow b) \oplus (b \leftarrow c)$ |
| **(x)** $(a \Rightarrow c) \leqslant (a \Rightarrow b) \oplus (b \Rightarrow c)$ | **(x')** $(a \Leftarrow b) \otimes (b \Leftarrow c) \leqslant (a \Leftarrow c)$ |
| **(xi)** $(a \rightarrow b) \leqslant (c \rightarrow a) \rightarrow (c \rightarrow b)$ | **(xi')** $(c \leftarrow a) \leftarrow (c \leftarrow b) \leqslant (a \leftarrow b)$ |
| **(xii)** $(a \rightarrow b) \leqslant (a \otimes c) \rightarrow (b \otimes c)$ | **(xii')** $(a \oplus c) \leftarrow (b \oplus c) \leqslant (a \leftarrow b)$ |
| **(xiii)** $b \leqslant a \rightarrow (a \otimes b)$ | **(xiii')** $a \leftarrow (a \oplus b) \leqslant b$ |
| **(xiv)** $a \rightarrow (b \rightarrow c) = (a \otimes b) \rightarrow c$ | **(xiv')** $a \leftarrow (b \leftarrow c) = (a \oplus b) \leftarrow c$ |
| **(xv)** $a \Rightarrow (b \Rightarrow c) = (a \otimes b) \Rightarrow c$ | **(xv')** $a \Leftarrow (b \Leftarrow c) = (a \oplus b) \Leftarrow c$ |
| **(xvi)** $a \rightarrow \neg b = \neg(a \otimes b)$ | **(xvi')** $a \leftarrow \ulcorner b = \ulcorner (a \oplus b)$ |
| **(xvii)** $a \Rightarrow \sim b = \sim(a \otimes b)$ | **(xvii')** $a \Leftarrow \sim b = \sim (a \oplus b)$ |
| **(xviii)** $a \rightarrow b \leqslant \neg b \rightarrow \neg a$ | **(xviii')** $\ulcorner b \leftarrow \ulcorner a \leqslant a \leftarrow b$ |
| **(xix)** $a \Rightarrow b = \sim b \Rightarrow \sim a$ | **(xix')** $a \Leftarrow b = \sim b \Leftarrow \sim a$ |
| **(xx)** $a \otimes \neg b \leqslant \neg(a \rightarrow b)$ | **(xx')** $a \otimes \sim b = \sim(a \Rightarrow b)$ |
| **(xxi)** $a \leqslant \neg\neg a$ | **(xxi')** $\ulcorner\ulcorner a \leqslant a.$ |

*Proof.* Note that the properties in the right column can be easily obtained from the properties in the left column by the definitions (7)–(11). Moreover, all properties, where the operations $\otimes$, $\rightarrow$, and $\neg$ occur, are well–known properties of residuated lattices (see, e.g., [19],[22],[42]). Then it remains to show **(x)**, **(xv)**, **(xvii)**, and **(xix)**. By way of example we show **(x)** and **(xv)**.

**(x)** By **(ii')**, for all $a, b, c \in L$, $\sim a \oplus b \geqslant \sim a$ and $\sim b \oplus c \geqslant c$. Then, by the definition (9) and the property **(i')**, $(a \Rightarrow b) \oplus (b \Rightarrow c) \geqslant (\sim a \oplus c) = (a \Rightarrow c)$.

**(xv)** For all $a, b, c \in L$, it holds:

$$\begin{aligned} & a \Rightarrow (b \Rightarrow c) \\ & \quad = \sim a \oplus (\sim b \oplus c) && \text{by the definition (9)} \\ & \quad = (\sim a \oplus \sim b) \oplus c && \text{by associativity of } \oplus \\ & \quad = \sim(a \otimes b) \oplus c && \text{by the definition (8)} \\ & \quad = (a \otimes b) \Rightarrow c. \end{aligned}$$

■

**Lemma 3.** *For every ER–lattice $(L, \wedge, \vee, \otimes, \rightarrow, \sim, 0, 1)$, for every $a \in L$, and for all families $(b_i)_{i \in I}$ and $(c_i)_{i \in I}$ of elements of $L$, if the respective infima and suprema exist, then the following properties hold:*

**(i)** $a \otimes \sup_{i \in I} c_i = sup_{i \in I}(a \otimes c_i)$

**(ii)** $a \rightarrow \inf_{i \in I} c_i = \inf_{i \in I}(a \rightarrow c_i)$

**(iii)** $a \Rightarrow (\inf_{i \in I} c_i) = \inf_{i \in I}(a \Rightarrow c_i)$

**(iv)** $(\sup_{i \in I} c_i) \rightarrow a = \inf_{i \in I}(c_i \rightarrow a)$

**(v)** $(\sup_{i \in I} c_i) \Rightarrow a = \inf_{i \in I}(c_i \Rightarrow a)$

**(vi)** $\sup_{i \in I} c_i \leqslant \neg \inf_{i \in I} \neg c_i$

**(vii)** $(\inf_{i \in I} b_i) \otimes (\inf_{i \in I} c_i) \leqslant \inf_{i \in I}(b_i \otimes c_i)$

**(viii)** $\inf_{i \in I} \neg c_i = \neg \sup_{i \in I} c_i$

**(ix)** $\sup_{i \in I} \neg b_i \leqslant \neg(\inf_{i \in I} b_i)$.

**(i')** $a \oplus (\inf_{i \in I} c_i) = \inf_{i \in I}(a \oplus c_i)$

**(ii')** $a \leftarrow (\sup_{i \in I} c_i) = \sup_{i \in I}(a \leftarrow c_i)$

**(iii')** $a \Leftarrow (\sup_{i \in I} c_i) = \sup_{i \in I}(a \Leftarrow c_i)$

**(iv')** $(\inf_{i \in I} c_i) \leftarrow a = \sup_{i \in I}(c_i \leftarrow a)$

**(v')** $(\inf_{i \in I} c_i) \Leftarrow a = \sup_{i \in I}(c_i \Leftarrow a)$

**(vi')** $\sup_{i \in I} c_i = \sim \inf_{i \in I} \sim c_i$

**(vii')** $\sup_{i \in I}(b_i \oplus c_i) \leqslant (\sup_{i \in I} b_i) \oplus (\sup_{i \in I} c_i)$.

*Proof.* As in Lemma 2, the properties in the right column are easily obtained from the respective properties in the left column using the definitions (7)–(11). Notice that all properties except from **(iii)** and **(v)** are known properties of residuated lattices.

**(iii)** By the definition of $\Rightarrow$ and **(i')**, we easily get $a \Rightarrow (\inf_{i \in I} c_i) = \sim a \oplus \inf_{i \in I} c_i = \inf_{i \in I}(\sim a \oplus c_i) = \inf_{i \in I}(a \Rightarrow c_i)$.

**(v)** can be proved in the analogous way. ■

*Example 2.* Let $\mathcal{L} = [0, 1]$ and let $(\mathcal{L}, \min, \max, t, i_t, 0, 1)$ be the R–lattice as in Example 1. Also, let $n$ be the standard fuzzy negation $n(a) = 1 - a$ for every $a \in [0, 1]$. Then $(\mathcal{L}, \min, \max, t, i_t, n, 0, 1)$ is an ER–lattice. □

*Remark 3.* Note that properties **(xviii)** and **(xix)** of Lemma 2 correspond to the contraposition law. In general, however, we do not have analogous links between $a \rightarrow b$ and $\sim a \rightarrow \sim b$. For example, consider the ER–lattice as in Example 2 and let $\rightarrow$ and $\sim$ be the Gödel implication (see Table 2) and the standard fuzzy negation.

Then for $a = 0.8$ and $b = 0.4$ we have: $a \to b = 0.4$ and $\sim b \to \sim a = 0.2$. Hence $a \to b > \sim b \to \sim a$. Taking $c = 0.1$, we easily get: $b \to c = 0.1$ and $\sim c \to \sim b = 0.6$, so $b \to c < \sim c \to \sim b$. □

## 2.2 *L*–fuzzy Sets and *L*–fuzzy Relations

**Fuzzy sets.** Let $L$ be a residuated lattice (in particular, R–lattice or ER–lattice) and let $X$ be a non–empty domain. By an *$L$–fuzzy set in $X$* we mean any mapping $F : X \to L$. For every $x \in X$, $F(x)$ is the degree of membership of $x$ to $F$. Two specific $L$–fuzzy sets in $X$, $\emptyset$ and $X$, are respectively defined by: $\emptyset(x) = 0$ and $X(x) = 1$ for every $x \in X$. The family of all $L$–fuzzy sets in $X$ will be denoted by $\mathcal{F}_L(X)$.

Recall the basic operations on $L$–fuzzy sets. First, let $L$ be an R–lattice. For all $A, B \in \mathcal{F}_L(X)$ and for every $x \in X$,

$$\begin{aligned}(A \sqcup_L B)(x) &= A(x) \vee B(x)\\ (A \sqcap_L B)(x) &= A(x) \wedge B(x)\\ (A \cap_L B)(x) &= A(x) \otimes B(x)\\ (\neg_L A)(x) &= \neg A(x).\end{aligned}$$

If $L$ is an ER–lattice, we additionally define:

$$\begin{aligned}(A \cup_L B)(x) &= A(x) \oplus B(x)\\ (\sim_L A)(x) &= \sim A(x)\\ (\ulcorner_L A)(x) &= \ulcorner A(x).\end{aligned}$$

For $A \in \mathcal{F}_L(X)$, we will write $A \gg \emptyset$ iff $A(x) \neq 0$ for every $x \in X$. Also, for two $L$–fuzzy sets $A, B \in \mathcal{F}_L(X)$, we will write $A \subseteq_L B$ iff $A(x) \leqslant B(x)$ for every $x \in X$ (Zadeh's inclusion). If $L$ is complete, then for any indexed family $(A_i)_{i \in I}$ of $L$–fuzzy sets in $X$, $\bigcup_{i \in I} A_i$ and $\bigcap_{i \in I} A_i$ are $L$–fuzzy sets in $X$ defined as: for every $x \in X$, $(\bigcup_{i \in I} A_i)(x) = \sup_{i \in I} A_i(x)$ and $(\bigcap_{i \in I} A_i)(x) = \inf_{i \in I} A_i(x)$.

**Fuzzy relations.** An *$L$–fuzzy relation on $X$* is a mapping $R : X \times X \to L$. The family of all $L$–fuzzy relations on $X$ will be denoted by $\mathcal{R}_L(X)$.

An $L$–fuzzy relation $R \in \mathcal{R}_L(X)$ is called

- *reflexive* iff $R(x, x) = 1$ for every $x \in X$
- *irreflexive* iff $R(x, x) = 0$ for every $x \in X$
- *symmetric* iff $R(x, y) = R(y, x)$ for all $x, y \in X$
- $\otimes$*–transitive* iff $R(x, y) \otimes R(y, z) \leqslant R(x, z)$ for all $x, y, z \in X$
- $\oplus$*–cotransitive* iff $R(x, y) \oplus R(y, z) \geqslant R(x, z)$ for all $x, y, z \in X$
- $\otimes$*–quasi ordering* iff it is reflexive and $\otimes$–transitive
- $\otimes$*–equivalence* iff it is reflexive, symmetric, and $\otimes$–transitive
- *crisp* iff $R(x, y) \in \{0, 1\}$ for all $x, y \in X$.

Note that if $R$ is crisp and $\oplus = \vee$, then cotransitivity of $R$ means that the complement of $R$ is transitive. Hence, $\oplus$–cotransitivity is a fuzzy generalisation of this property.

# 3 Fuzzy Information Relations

In this section we define several fuzzy information relations measuring degrees of relationship between two fuzzy sets. We take two classes of residuated lattices as an algebraic basis: complete R–lattices and complete ER–lattices.

Let $L$ be a complete R–lattice or an ER–lattice and let $X \neq \emptyset$. By an *$L$–fuzzy information relation* we mean any $L$–fuzzy relation on $\mathcal{F}_L(X)$.

## 3.1 Fuzzy Information Relations Based on R–Lattices

Let us define several $L$–information relations.

**Definition 4.** *Let $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be a complete R–lattice and let $X \neq \emptyset$. Define the following $L$–fuzzy information relations: for all $A, B \in \mathcal{F}_L(X)$,*

**(i)** *$L$–fuzzy inclusion:*
$$inc_L(A,B) = \inf_{x \in X}(A(x) \rightarrow B(x))$$
**(ii)** *$L$–fuzzy noninclusion:*
$$ninc_L(A,B) = \sup_{x \in X}(A(x) \otimes \neg B(x))$$
**(iii)** *$L$–fuzzy compatibility:*
$$com_L(A,B) = \sup_{x \in X}(A(x) \otimes B(x))$$
**(iv)** *$L$–fuzzy orthogonality:*
$$ort_L(A,B) = inc_L(A, \neg_L B)$$
**(v)** *$L$–fuzzy exhaustiveness:*
$$exh_L(A,B) = \inf_{x \in X}(A(x) \vee B(x))$$
**(vi)** *$L$–fuzzy nonexhaustiveness:*
$$nexh_L(A,B) = com_L(\neg_L A, \neg_L B)$$
**(vii)** *$L$–fuzzy indiscernibility:*
$$ind_L(A,B) = inc_L(A,B) \otimes inc_L(B,A)$$
**(viii)** *$L$–fuzzy diversity:*
$$div_L(A,B) = ninc_L(A,B) \vee ninc_L(\neg_L A, B).$$ □

For two $L$–fuzzy sets $A, B \in \mathcal{F}_L(X)$, $inc_L(A,B)$ (resp. $ninc_L(A,B)$) is the degree, to which $A$ is included (resp. not included) in $B$. Note that the formula for $ninc_L$ is the straightforward generalisation of the classical equivalence: $A \not\subseteq B \Leftrightarrow (\exists x \in X)\,(x \in A \;\&\; x \notin B)$. Next, $com_L(A,B)$ (resp. $ort_L(A,B)$) represents the degree, to which $A$ and $B$ overlap (resp. are disjoint). The formulation for $ort_L$ results from the generalisation of the classical equivalence: $A \cap B = \emptyset \Leftrightarrow A \subseteq -B$, where $-B = X \setminus B$. Furthermore, $exh_L(A,B)$ (resp. $nexh_L(A,B)$) is the degree, to which $A$ and $B$ cover (resp. do not cover) the whole domain $X$. Note that in the classical case, $A \cup B \neq X \Leftrightarrow (-A \cap -B \neq \emptyset)$. This equivalence underlies the formulation for $nexh_L$. Finally, $ind_L(A,B)$ (resp. $div_L(A,B)$) is the degree, to which $A$ is equal to $B$ (resp. $A$ differs from $B$). The formulation for $div_L$ is again a generalisation of the classical equivalence: $A \neq B \Leftrightarrow (A \cap -B \neq \emptyset) \vee (-A \cap B \neq \emptyset)$.

The following proposition provides main properties of these relations.

**Proposition 3.** *Let* $(L, \wedge.\vee, \otimes, \rightarrow 0, 1)$ *be a complete R–lattice. Then*

- **(i)** $inc_L$ *is an* $L$*–quasi ordering*
- **(ii.1)** $ninc_L$ *is irreflexive*
- **(ii.2)** *if* $L \in RL^+$*, then for any* $A \in \mathcal{F}_L(X)$ *and for any* $B \in \mathcal{F}_L(X)$ *satisfying* $B \gg \emptyset$, $ninc_L(A, B) = 0$
- **(iii)** $com_L$ *and* $exh_L$ *are symmetric*
- **(iv.1)** $ort_L$ *is symmetric*
- **(iv.2)** *if* $L \in RL^+$*, then* $ort_L$ *is crisp*
- **(v.1)** $nexh_L$ *is symmetric;*
- **(v.2)** *if* $L \in RL^+$*, then for all* $A, B \in \mathcal{F}_L(X)$ *such that* $A(x) \neq 0$ *or* $B(x) \neq 0$ *for any* $x \in X$, $nexh_L(A, B) = 0$
- **(vi)** $ind_L$ *is an* $L$*–fuzzy equivalence*
- **(vii.1)** $div_L$ *is irreflexive and symmetric*
- **(vii.2)** *if* $L \in RL^+$*, then for all* $A, B \in \mathcal{F}_L(X)$ *such that* $A \gg \emptyset$ *and* $B \gg \emptyset$*, it holds* $div_L(A, B) = 0$.

*Proof.*

**(i)** See [3].
**(ii.1)** For every $A \in \mathcal{F}_L(X)$, we have: $ninc_L(A, A) = \sup_{x \in X}(A(x) \otimes \neg A(x)) = \sup_{x \in X}(A(x) \rightarrow (A(x) \rightarrow 0)) = 0$ by Lemma 2**(vii)**.
**(ii.2)** Assume that $L \in RL^+$ (i.e. $\otimes$ has no zero divisors) and take an arbitrary $A \in \mathcal{F}_L(X)$ and $B \in \mathcal{F}_L(X)$ such that $B \gg \emptyset$, i.e. $B(x) \neq 0$ for every $x \in X$. Then by Lemma 1, $\neg_L B = \emptyset$, so we have: $ninc_L(A, B) = \sup_{x \in X}(A(x) \otimes \neg B(x)) = \sup_{x \in X}(A(x) \otimes 0) = 0$ by Lemma 2**(iv)**.
**(iii)** Symmetry of $com_L$ (resp. $exh_L$) directly follows from commutativity of $\otimes$ (resp. $\vee$).
**(iv.1)** By Lemma 2**(xvi)**, for all $a, b \in L$, $a \rightarrow \neg b = \neg(a \otimes b) = \neg(b \otimes a) = b \rightarrow \neg a$. Then for every $A, B \in \mathcal{F}_L(X)$, $ort_L(A, B) = \inf_{x \in X}(A(x) \rightarrow \neg B(x)) = \inf_{x \in X}(B(x) \rightarrow \neg A(x)) = ort_L(B, A)$.
**(iv.2)** Assume that $L \in RL^+$. Let $A, B \in \mathcal{F}_L(X)$ and take an arbitrary $x \in X$. If $B(x) \neq 0$, then by Lemma 1, $\neg B(x) = 0$, so $A(x) \rightarrow \neg B(x) = \neg A(x) \in \{0, 1\}$. If $B(x) = 0$, then $A(x) \rightarrow \neg B(x) = A(x) \rightarrow 1 = 1$ by Lemma 2**(v)**. Then $ort_L(A, B) = \inf_{x \in X}(A(x) \rightarrow \neg B(x)) \in \{0, 1\}$.
**(v.1)** Follows directly from symmetry of $com_L$.
**(v.2)** Assume that $L \in RL^+$ and consider $A, B \in \mathcal{F}_L(X)$ such that for every $x \in X$, $A(x) \neq 0$ or $B(x) \neq 0$. By Lemma 1, it implies that for every $x \in X$, $\neg A(x) = 0$ or $\neg B(x) = 0$, so using Lemma 2**(iv)**, $\neg A(x) \otimes \neg B(x) = 0$ for every $x \in X$. Hence $nexh_L(A, B) = 0$.
**(vi)** Reflexivity and $\otimes$–transitivity of $ind_L$ follows directly from **(i)**, symmetry of $ind_L$ results from commutativity of $\otimes$.
**(vii.1)** Let $A \in \mathcal{F}_L(X)$. For every $x \in X$, $A(x) \otimes \neg A(x) = A(x) \otimes (A(x) \rightarrow 0) = 0$ by Lemma 2**(vii)**, so $com_L(A, \neg_L A) = 0$. By symmetry of $com_L$, $com_L(\neg_L A, A) = 0$. Hence $div_L(A, A) = 0$. Symmetry of $div_L$ follows from symmetry of $com_L$.

**(vii.2)** Assume that $L \in RL^+$ and consider $A, B \in \mathcal{F}_L(X)$ such that $A(x) \neq 0$ and $B(x) \neq 0$ for every $x \in X$. By Lemma 1, $\neg A(x) = \neg B(x) = 0$ for every $x \in X$. Then we have $A(x) \otimes \neg B(x) = \neg A(x) \otimes B(x) = 0$ for every $x \in X$, which implies $\sup_{x\in X}(A(x) \otimes \neg B(x)) = \sup_{x\in X}(\neg A(x) \otimes B(x)) = 0$, so $div_L(A, B) = 0$. ∎

In the crisp case the relation of set inclusion (resp. compatibility, exhaustiveness, indiscernibility) is complementary to noninclusion (resp. orthogonality, nonexhaustiveness, diversity). While generalising these relations on the basis of R–lattices only the weaker form of complementarity holds, as the following proposition states.

**Proposition 4.** *For every complete R–lattice* $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$,

- **(i)** $ninc_L \subseteq_L \neg_L inc_L$
- **(ii)** $ort_L = \neg_L com_L$ *and* $com_L \subseteq_L \neg_L ort_L$
- **(iii)** $exh_L \subseteq_L \neg_L nexh_L$
- **(iv)** $div_L \subseteq_L \neg_L ind_L$.

*Proof.*

**(i)** For every $A, B \in \mathcal{F}_L(X)$,

$$\begin{aligned}
&\neg inc_L(A, B)\\
&\quad = \neg(\inf_{x\in X}(A(x) \rightarrow B(x)))\\
&\quad \geqslant \sup_{x\in X} \neg(A(x) \rightarrow B(x)) && \text{by Lemma 3}\textbf{(ix)}\\
&\quad \geqslant \sup_{x\in X} \neg(\neg B(x) \rightarrow \neg A(x)) && \text{by Lemma 2}\textbf{(xviii)}\\
&\quad = \sup_{x\in X} \neg\neg(\neg B(x) \otimes A(x)) && \text{by Lemma 2}\textbf{(xvi)}\\
&\quad \geqslant \sup_{x\in X}(\neg B(x) \otimes A(x)) && \text{by Lemma 2}\textbf{(xxi)}\\
&\quad = ninc_L(A, B).
\end{aligned}$$

**(ii)** For every $A, B \in \mathcal{F}_L(X)$,

$$\begin{aligned}
&\neg com_L(A, B)\\
&\quad = \neg \sup_{x\in X}(A(x) \otimes B(x))\\
&\quad = \inf_{x\in X} \neg(A(x) \otimes B(x)) && \text{by Lemma 3}\textbf{(viii)}\\
&\quad = \inf_{x\in X}(A(x) \rightarrow \neg B(x)) && \text{by Lemma 2}\textbf{(xvi)}\\
&\quad = ort_L(A, B).
\end{aligned}$$

Since $ort_L(A, B) = \neg com_L(A, B)$, from Lemma 2**(xxi)** we immediately obtain $\neg ort_L(A, B) = \neg\neg com_L(A, B) \geqslant com_L(A, B)$.

**(iii)** For all $A, B \in \mathcal{F}_L(X)$,

$$\begin{aligned}
&\neg nexh_L(A, B)\\
&\quad = \neg \sup_{x\in X}(\neg A(x) \otimes \neg B(x))\\
&\quad = \inf_{x\in X} \neg(\neg A(x) \otimes \neg B(x)) && \text{by Lemma 3}\textbf{(viii)}\\
&\quad \geqslant \inf_{x\in X} \neg(\neg A(x) \wedge \neg B(x)) && \text{by Lemma 2}\textbf{(iii)}\\
&\quad \geqslant \inf_{x\in X}(\neg\neg A(x) \vee \neg\neg B(x)) && \text{by Lemma 3}\textbf{(ix)}\\
&\quad \geqslant \inf_{x\in X}(A(x) \vee B(x)) && \text{by Lemma 2}\textbf{(xxi)}.
\end{aligned}$$

**(iv)** For every $A, B \in \mathcal{F}_L(X)$,

$\neg ind_L(A, B)$
$= \neg((\inf_{x\in X}(A(x) \to B(x))) \otimes (\inf_{x\in X}(B(x) \to A(x))))$
$\geqslant \neg(\inf_{x\in X}(A(x) \to B(x)) \wedge (\inf_{x\in X}(B(x) \to A(x))))$ by Lemma 2**(iii)**
$\geqslant \neg(\inf_{x\in X}(A(x) \to B(x))) \vee \neg(\inf_{x\in X}(B(x) \to A(x)))$ by Lemma 3**(ix)**
$\geqslant \sup_{x\in X} \neg(A(x) \to B(x)) \vee \sup_{x\in X} \neg(B(x) \to A(x))$ by Lemma 3**(ix)**
$\geqslant \sup_{x\in X}(A(x) \otimes \neg B(x)) \vee \sup_{x\in X}(B(x) \otimes \neg A(x))$ by Lemma 2**(xx)**
$= ninc_L(A, B) \vee ninc_L(B, A)$
$= div_L(A, B)$. ∎

Proposition 3 shows that most properties of the $L$–fuzzy information relations discussed here coincide with their properties in the crisp case. Unfortunately, some properties are counterintuitive. First, fuzzy orthogonality should not reduce to the binary case. Moreover, the properties **(iv.2)**, **(v.2)**, and **(vii.2)** also do not coincide with what is expected, as the following example shows.

*Example 3.* Let $(\mathcal{L}, \min, \max, t, i_t, 0, 1)$ be the R–lattice as in Example 1, where $\mathcal{L} = [0, 1]$ and $t$ is a left–continuous t–norm without zero divisors (e.g., $t_Z$ or $t_P$). Consider an $\mathcal{L}$–fuzzy set $A$ in $X \neq \emptyset$ given by: $A(x) = 0.001$ for every $x \in X$. The intuition dictates that $X$ is not included in $A$ to a very high degree. However, by Proposition 3**(ii.2)**, $ninc_{\mathcal{L}}(X, A) = 0$, which means that in fact $X$ is totally included in $A$. Also, it is clear that $\emptyset$ and $A$ do not cover the universe $X$ to a high degree, but $nexh_{\mathcal{L}}(A, \emptyset) = 0$. Finally, $A$ and $X$ are totally different, yet $div_{\mathcal{L}}(A, X) = 0$. □

In order to overcome these inadequacies, we take another class of residuated lattices, namely ER–lattices.

## 3.2 Fuzzy Information Relations Based on ER–Lattices

In this part we discuss another fuzzy generalisation of some information relations taking any complete ER–lattice $(L, \wedge, \vee, \otimes, \to, \sim, 0, 1)$ as a basic algebraic structure.

**Definition 5.** *For a complete ER–lattice* $(L, \wedge, \vee, \otimes, \to, \sim, 0, 1)$, *define the following L–fuzzy information relations: for all* $A, B \in \mathcal{F}_L(X)$,

**(i)** *L–fuzzy noninclusion:*
$$Ninc_L(A, B) = \sup_{x\in X}(B(x) \leftarrow A(x))$$
**(ii)** *L–fuzzy orthogonality:*
$$Ort_L(A, B) = \inf_{x\in X}(A(x) \Rightarrow \sim B(x))$$
**(iii)** *L–fuzzy exhaustiveness:*
$$Exh_L(A, B) = \inf_{x\in X}(A(x) \oplus B(x))$$
**(iv)** *L–fuzzy nonexhaustiveness:*
$$Nexh_L(A, B) = com_L(\sim_L A, \sim_L B)$$
**(v)** *L–fuzzy diversity:*
$$Div_L(A, B) = Ninc_L(A, B) \otimes Ninc_L(B, A).$$
□

The definition of $Ninc_L$, $Exh_L$, and $Div_L$ were presented in [29], where $L$ was any complete double residuated lattice.

In view of Remark 2, $B(x) \leftarrow A(x)$ is a generalisation of the classical implication $A(x) \wedge \neg B(x)$, so $Ninc_L(A, B)$ is the fuzzy counterpart of the classical formula $(\exists x \in X)\,(x \in A \;\&\; x \notin B)$ and indeed represents the degree, to which $A$ is not included in $B$. In the definition of $Ort_L$, $Exh_L$, $Nexh_L$, and $Div_L$ we substitute the operations $\rightarrow$, $\neg$, and $\vee$ by $\Rightarrow$, $\sim$, and $\oplus$, respectively.

**Proposition 5.** *For every complete ER–lattice* $(L, \wedge, \vee, \otimes, \rightarrow, \sim, 0, 1)$*,*

**(i)** $Ninc_L$ *is irreflexive and* $\oplus$*–cotransitive.*
**(ii)** $Ort_L$*,* $Exh_L$*, and* $Nexh_L$ *are symmetric*
**(iii)** $Div_L$ *is irreflexive and symmetric.*

*Proof.*

**(i)** Irreflexivity of $Ninc_L$ results from Lemma 2**(v')**. To show that it is also $\oplus$–cotransitive, let us take $A, B, C \in \mathcal{F}_L(X)$. Then

$$\begin{aligned}
&Ninc_L(A,B) \oplus Ninc_L(B,C) \\
&\quad = \sup_{x\in X}(B(x) \leftarrow A(x)) \oplus \sup_{x\in X}(C(x) \leftarrow B(x)) \\
&\quad \geqslant \sup_{x\in X}((B(x) \leftarrow A(x)) \oplus (C(x) \leftarrow B(x))) && \text{by Lemma 3}\textbf{(vii')} \\
&\quad = \sup_{x\in X}((C(x) \leftarrow B(x)) \oplus (B(x) \leftarrow A(x))) && \text{by commutativity of } \oplus \\
&\quad \geqslant \sup_{x\in X}(C(x) \leftarrow A(x)) && \text{by Lemma 2}\textbf{(ix')} \\
&\quad = Ninc_L(A,C).
\end{aligned}$$

**(ii)** Symmetry of $Ort_L$ follows from Lemma 2**(xvii)** and commutativity of $\otimes$, symmetry of $Exh_L$ immediately follows from commutativity of $\oplus$, and symmetry of $Nexh_L$ results from symmetry of $com_L$.
**(iii)** Irreflexivity of $Div_L$ follows from irreflexivity of $Ninc_L$, while symmetry of $Div_L$ results from commutativity of $\oplus$. ■

*Example 4.* Put $\mathcal{L} = [0,1]$ and consider the lattice $(\mathcal{L}, \min, \max, t, i_t, n, 0, 1)$ as in Example 2 (recall that $t$ is a left–continuous t–norm, $i_t$ is the residual implication determined by $t$, and $n$ is the standard fuzzy negation). Let $X \neq \emptyset$ be an arbitrary domain and let $A \in \mathcal{F}_{\mathcal{L}}(X)$ be defined as in Example 3, i.e. $A(x) = 0.001$ for every $x \in X$. By simple calculations we get $Ninc_{\mathcal{L}}(X, A) = 1$ for $\leftarrow \in \{\leftarrow_Z, \leftarrow_P\}$. Of course, this result coincides with our intuition.

Let $B \in \mathcal{F}_{\mathcal{L}}(X)$ be such that $B(x) = 0.999$ for every $x \in X$. Then $Ort_{\mathcal{L}}(A, B) = t(n(0.001), n(0.999)) = t(0.999, 0.001) \notin \{0, 1\}$ for any $t$ without zero divisors. Hence $Ort_{\mathcal{L}}$ does not reduce to a crisp relation. Moreover, for any t-norm $t$, $Nexh_{\mathcal{L}}(A, \emptyset) = 0.999$. Clearly, this is again the expected result: $A$ and $\emptyset$ do not cover the universe $X$ up to the very high degree. Finally, $Ninc_{\mathcal{L}}(A, X) = 0$ for the dual residuum of any (right–continuous) t–conorm $s$. Also, note that $Ninc_{\mathcal{L}}(X, A) = (0.001 \leftarrow 1) = 1$ for $\leftarrow \in \{\leftarrow_Z, \leftarrow_P\}$. Then $Div_{\mathcal{L}}(X, A) = 0 \oplus 1 = 1$. So, as expected, $A$ differs from $X$ to the very high degree. □

In view of the above example, it is now clear that the inadequacies in representation, which occur when R–lattices of the class $RL^+$ were taken as basic structures, are avoided.

Note also:

**Proposition 6.** *For every complete ER–lattice* $(L, \wedge, \vee, \otimes, \rightarrow, \sim, 0, 1)$,

**(i)** $Ninc_L(A,B) = \sim inc_L(\sim_L B, \sim_L A)$ *and* $Div_L(A,B) = \sim ind_L(\sim_L A, \sim_L B)$ *for every* $A, B \in \mathcal{F}_L(X)$,

**(ii)** $Ort_L = \sim_L com_L$ *and* $Exh_L = \sim_L Nexh_L$.

*Proof.*

**(i)** For every $A, B \in \mathcal{F}_L(X)$,

$$\begin{aligned} &Ninc_L(A,B)\\ &\quad = \sup_{x\in X}(B(x) \leftarrow A(x))\\ &\quad = \sup_{x\in X} \sim(\sim B(x) \rightarrow \sim A(x)) && \text{by (10)}\\ &\quad = \sim \inf_{x\in X}(\sim B(x) \rightarrow \sim A(x)) && \text{by Lemma 3}\textbf{(vi')}\\ &\quad = \sim inc_L(\sim_L B, \sim_L A). \end{aligned}$$

Similarly, using (10), Lemma 3**(vi')**, and (8), we get for every $A, B \in \mathcal{F}_L(X)$,

$$\begin{aligned} &Div_L(A,B)\\ &\quad = Ninc_L(A,B) \oplus Ninc_L(B,A)\\ &\quad = \sup_{x\in X}(B(x) \leftarrow A(x)) \oplus \sup_{x\in X}(A(x) \leftarrow B(x))\\ &\quad = \sup_{x\in X} \sim(\sim B(x) \rightarrow \sim A(x)) \oplus \sup_{x\in X} \sim(\sim A(x) \rightarrow \sim B(x))\\ &\quad = \sim \inf_{x\in X}(\sim B(x) \rightarrow \sim A(x)) \oplus \sim \inf_{x\in X}(\sim A(x) \rightarrow \sim B(x))\\ &\quad = \sim(\inf_{x\in X}(\sim B(x) \rightarrow \sim A(x)) \otimes \inf_{x\in X}(\sim A(x) \rightarrow \sim B(x)))\\ &\quad = \sim ind_L(\sim_L A, \sim_L B). \end{aligned}$$

The proof of **(ii)** is similar. ■

Note that the properties stated in the above proposition coincide with the respective properties of these relations in the crisp case. Clearly, for every crisp subsets $A, B \subseteq X$, $A = B \Leftrightarrow -A = -B$. Yet in general $ind_L(A,B) \neq ind_L(\sim_L A, \sim_L B)$. Similarly, $inc_L(A,B) \neq inc_L(\sim_L B, \sim_L A)$. It follows from the fact that in an arbitrary ER–lattice $L$, we do not have any relationship between $a \rightarrow b$ and $\sim b \rightarrow \sim a$, as observed in Remark 2.

## 4 Fuzzy Information Operators

Let $(L, \wedge, \vee, \otimes, \rightarrow, \sim, 0, 1)$ be a complete ER–lattice. By an *L–fuzzy information operator* we mean any mapping $\Omega_L : \mathcal{R}_L(X) \times \mathcal{F}_L(X) \rightarrow \mathcal{F}_L(X)$. Below we define several $L$–information operators.

**Definition 6.** *For every complete ER–lattice* $(L, \wedge, \vee, \otimes, \rightarrow, \sim, 0, 1)$, *for every* $R \in \mathcal{R}_L(X)$, *for every* $A \in \mathcal{F}_L(X)$, *and for every* $x \in X$,

**(O.1)** $[R]_{\rightarrow}A(x) = \inf_{y\in X}(R(x,y) \rightarrow A(y))$
**(O.2)** $[R]_{\Rightarrow}A(x) = \inf_{y\in X}(R(x,y) \Rightarrow A(y))$
**(O.3)** $[R]_{\leftarrow}A(x) = \sup_{y\in X}(R(x,y) \leftarrow A(y))$
**(O.4)** $[R]_{\Leftarrow}A(x) = \sup_{y\in X}(R(x,y) \Leftarrow A(y))$
**(O.5)** $\langle R\rangle_{\otimes}A(x) = \sup_{y\in X}(R(x,y) \otimes A(y))$
**(O.6)** $\langle R\rangle_{\oplus}A(x) = \inf_{y\in X}(R(x,y) \oplus A(y))$. □

It is worth noting that $[\,]_{\rightarrow}$ (resp. $[\,]_{\Rightarrow}$) and $\langle\,\rangle_{\otimes}$ correspond to fuzzy modalities ([13],[14],[17]), i.e. $[R]_{\rightarrow}$ and $[R]_{\Rightarrow}$ are fuzzy generalisations of the necessity operator, while $\langle\,\rangle_{\otimes}$ is the counterpart of the possibility operator. Also, these operators are fuzzy approximation operators well–known in the theory of fuzzy rough sets (see, e.g., [32], [34]), as well as fuzzy morphological operators which are basic tools in mathematical morphology ([24]).

Let $R\in\mathcal{R}_L(X)$. For any $x\in X$ we write $xR$ to denote the $L$–fuzzy set in $X$ defined as: $(xR)(y)=R(x,y)$ for every $y\in X$. Note that for every $A\in\mathcal{F}_L(X)$ and for every $x\in X$,

$$\begin{array}{ll} [R]_{\rightarrow}A(x) = inc_L(xR,A) & [R]_{\leftarrow}A(x) = Ninc_L(xR,A) \\ \langle R\rangle_{\otimes}A(x) = com_L(xR,A) & \langle R\rangle_{\oplus}A(x) = Exh_L(xR,A). \end{array}$$

**Definition 7.** *Let $\Omega_1,\Omega_2:\mathcal{R}_L(X)\times\mathcal{F}_L(X)\rightarrow\mathcal{F}_L(X)$ be two $L$–fuzzy information operators, let $\circ$ be a unary operation in $L$, and let $\circ_L:\mathcal{F}_L(X)\rightarrow\mathcal{F}_L(X)$ be such that $(\circ_L A)(x)=\circ A(x)$ for every $x\in X$. We say that $\Omega_1$ and $\Omega_2$ are*

- *$\circ_L$-dual iff $\Omega_1(R,A)=\circ_L\,\Omega_2(R,\circ_L A)$ for every $R\in\mathcal{R}_L(X)$ and for every $A\in\mathcal{F}_L(X)$*
- *weakly $\circ_L$–dual iff $\Omega_1(R,A)\subseteq_L \circ_L\Omega_2(R,\circ_L A)$ for every $R\in\mathcal{R}_L(X)$ and for every $A\in\mathcal{F}_L(X)$*
- *$\circ_L$–codual iff $\Omega_1(R,A)=\circ_L\Omega_2(\circ_L R,\circ_L A)$ for every $R\in\mathcal{R}_L(X)$ and for every $A\in\mathcal{F}_L(X)$.* □

Basic properties of the operators **(O.1)–(O.6)** are given in the following proposition.

**Proposition 7.** *For every complete ER–lattice $L$ and for every $R\in\mathcal{F}_L(X)$,*

**(i)** $[R]_{\rightarrow}X=[R]_{\Rightarrow}X=\langle R\rangle_{\oplus}X=X,\quad [R]_{\leftarrow}\emptyset=[R]_{\Leftarrow}\emptyset=\langle R\rangle_{\otimes}\emptyset=\emptyset$

**(ii)** *for every $A,B\in\mathcal{F}_L(X)$ and for every $\Omega\in\{[\,]_{\rightarrow},[\,]_{\Rightarrow},[\,]_{\leftarrow},[\,]_{\Leftarrow},\langle\,\rangle_{\otimes},\langle\,\rangle_{\oplus}\}$, $A\subseteq_L B$ implies $\Omega(A)\subseteq_L\Omega(B)$*

**(iii)** *for every $A\in\mathcal{F}_L(X)$,*

$$\begin{array}{ll} [R]_{\rightarrow}A\subseteq_L\neg\langle R\rangle_{\otimes}\neg A & [R]_{\Rightarrow}A=\sim\langle R\rangle_{\otimes}\sim A \\ \langle R\rangle_{\otimes}A\subseteq_L\neg[R]_{\rightarrow}\neg A & \langle R\rangle_{\otimes}A=\sim[R]_{\rightarrow}\sim A \\ \ulcorner[R]_{\leftarrow}\ulcorner A\subseteq_L\langle R\rangle_{\oplus}A & \langle R\rangle_{\oplus}A=\sim[R]_{\Leftarrow}\sim A \\ \ulcorner\langle R\rangle_{\oplus}\ulcorner A\subseteq_L[R]_{\leftarrow}A & [R]_{\Leftarrow}A=\sim\langle R\rangle_{\oplus}\sim A \end{array}$$

**(iv)** *for every $A\in\mathcal{F}_L(X)$,*

$$\begin{array}{ll} [R]_{\rightarrow}A=\sim[\sim R]_{\leftarrow}\sim A & [R]_{\Rightarrow}A=\sim[\sim R]_{\Leftarrow}\sim A \\ [R]_{\leftarrow}A=\sim[\sim R]_{\rightarrow}\sim A & [R]_{\Leftarrow}A=\sim[\sim R]_{\Rightarrow}\sim A \end{array}$$

**(v)** *for every indexed family $(A_i)_{i\in I}$ of $L$–fuzzy sets in $X$,*

$$\begin{array}{ll} [R]_{\rightarrow}(\bigcap_{i\in I}A_i)\ =\ \bigcap_{i\in I}[R]_{\rightarrow}A_i & [R]_{\Rightarrow}(\bigcap_{i\in I}A_i)\ =\ \bigcap_{i\in I}[R]_{\Rightarrow}A_i \\ [R]_{\rightarrow}(\bigcup_{i\in I}A_i)\ {}_L\!\supseteq\ \bigcup_{i\in I}[R]_{\rightarrow}A_i & [R]_{\Rightarrow}(\bigcup_{i\in I}A_i)\ {}_L\!\supseteq\ \bigcup_{i\in I}[R]_{\Rightarrow}A_i \\ [R]_{\leftarrow}(\bigcap_{i\in I}A_i)\subseteq_L\bigcap_{i\in I}[R]_{\leftarrow}A_i & [R]_{\Leftarrow}(\bigcap_{i\in I}A_i)\subseteq_L\bigcap_{i\in I}[R]_{\Leftarrow}A_i \\ [R]_{\leftarrow}(\bigcup_{i\in I}A_i)\ =\ \bigcup_{i\in I}[R]_{\leftarrow}A_i & [R]_{\Leftarrow}(\bigcup_{i\in I}A_i)\ =\ \bigcup_{i\in I}[R]_{\Leftarrow}A_i \\ \langle R\rangle_{\otimes}(\bigcap_{i\in I}A_i)\subseteq_L\bigcap_{i\in I}\langle R\rangle_{\otimes}A_i & \langle R\rangle_{\oplus}(\bigcap_{i\in I}A_i)\ =\ \bigcap_{i\in I}\langle R\rangle_{\oplus}A_i \\ \langle R\rangle_{\otimes}(\bigcup_{i\in I}A_i)\ =\ \bigcup_{i\in I}\langle R\rangle_{\otimes}A_i & \langle R\rangle_{\oplus}(\bigcup_{i\in I}A_i)\ {}_L\!\supseteq\ \bigcup_{i\in I}\langle R\rangle_{\oplus}A_i. \end{array}$$

*Proof.* Straightforward verification. ∎

The property **(ii)** of the above proposition states the monotonicity of $L$–fuzzy information operators w.r.t. Zadeh's inclusion. Also, **(iii)** states the $\sim$–duality and weak $\neg$–duality between these operators, and **(iv)** establishes $\sim$–coduality between $L$–fuzzy information operators.

**Corollary 1.** *For every complete ER–lattice $L$,*

**(i)** $[\,]_{\Rightarrow}$ *and* $\langle\,\rangle_{\otimes}$*, as well as* $[\,]_{\Leftarrow}$ *and* $\langle\,\rangle_{\oplus}$*, are* $\sim$*–dual,*

**(ii)** $[\,]_{\rightarrow}$ *and* $\langle\,\rangle_{\otimes}$ *are weakly* $\neg$*–dual*

**(iii)** $[\,]_{\rightarrow}$ *and* $[\,]_{\leftarrow}$*, as well as* $[\,]_{\Rightarrow}$ *and* $[\,]_{\Leftarrow}$*, are* $\sim$*–codual.* ∎

It is well-known that traditional information operators are useful for characterizing particular classes of (binary) relations. This is also the case for fuzzy information operators. The following theorem presents complete characterizations of some basic classes of fuzzy relations.

**Theorem 1.** *For every complete ER–lattice* $(L,\wedge,\vee,\otimes,\rightarrow,\sim,0,1)$*, for every* $R\in\mathcal{R}_L(X)$*, and for every* $A\in\mathcal{F}_L(X)$ *the following statements hold:*

**(i)** *$R$ is reflexive* *iff* $[R]_{\rightarrow}A \subseteq_L A$ *iff* $[R]_{\Rightarrow}A \subseteq_L A$ *iff* $A \subseteq_L \langle R\rangle_{\otimes}A$

**(ii)** *$R$ is irreflexive* *iff* $A \subseteq_L [R]_{\leftarrow}A$ *iff* $A \subseteq_L [R]_{\Leftarrow}A$ *iff* $\langle R\rangle_{\oplus}A \subseteq_L A$

**(iii)** *$R$ is symmetric* *iff* $\langle R\rangle_{\otimes}[R]_{\rightarrow}A \subseteq_L A$ *iff* $[R]_{\leftarrow}\langle R\rangle_{\oplus}A \subseteq_L A$ *iff* $A \subseteq_L [R]_{\rightarrow}\langle R\rangle_{\otimes}A$ *iff* $A \subseteq_L \langle R\rangle_{\oplus}[R]_{\leftarrow}A$

**(iv)** *$R$ is $\otimes$–transitive* *iff* $[R]_{\rightarrow}A \subseteq_L [R]_{\rightarrow}[R]_{\rightarrow}A$ *iff* $[R]_{\Rightarrow}A \subseteq_L [R]_{\Rightarrow}[R]_{\Rightarrow}A$ *iff* $\langle R\rangle_{\otimes}\langle R\rangle_{\otimes}A \subseteq_L \langle R\rangle_{\otimes}A$

**(v)** *$R$ is $\oplus$–cotransitive* *iff* $[R]_{\leftarrow}[R]_{\leftarrow}A \subseteq_L [R]_{\leftarrow}A$ *iff* $[R]_{\Leftarrow}[R]_{\Leftarrow}A \subseteq_L [R]_{\Leftarrow}A$ *iff* $\langle R\rangle_{\oplus}A \subseteq_L \langle R\rangle_{\oplus}\langle R\rangle_{\oplus}A.$

*Proof.* By way of example we prove **(ii)** and **(iv)**.

**(ii)** First, consider the inclusion $A \subseteq_L [R]_{\leftarrow}A$.

($\subseteq$) Assume that $R$ is irreflexive. Then for every $A\in\mathcal{F}_L(X)$ and for every $x\in X$,

$$[R]_{\leftarrow}A(x) = \sup_{y\in X}(R(x,y) \leftarrow A(y)) \geqslant R(x,x)\leftarrow A(x) = 0\leftarrow A(x) = A(x).$$

by Lemma 2**(vi')**.

($\supseteq$) Assume that $R$ is not irreflexive. Then $R(x_0,x_0)\neq 0$ for some $x_0\in X$. Put $A = x_0R$. Then we have:

$$[R]_{\leftarrow}A(x_0) = \sup_{y\in X}(R(x_0,y)\leftarrow R(x_0,y)) = 0$$

by Lemma 2(**v'**). Hence $A(x_0) \to [R]_{\leftarrow}A(x_0) = R(x_0,x_0) \to 0 \neq 1$ by Lemma 2(**v**), so $A \not\subseteq_L [R]_{\leftarrow}A$.

Consider the second equivalence.
($\subseteq$) For every $A \in \mathcal{F}_L(X)$ and for every $x \in X$,

$$\begin{aligned}[R]_{\Leftarrow}A(x) &= \sup_{y\in X}(R(x,y)\Leftarrow A(y)) \\ &\geqslant R(x,x)\Leftarrow A(x) = 0\Leftarrow A(x) = A(x).\end{aligned}$$

by Lemma 2(**vi'**).

($\supseteq$) As before, assume that $R$ is not irreflexive, i.e. $R(x_0,x_0)\neq 0$ for some $x_0 \in X$. For $A=\{x_0\}$ we have:

$$\begin{aligned}&[R]_{\Leftarrow}A(x_0) \\ &\quad = \sup_{y\in X}(R(x_0,y)\Leftarrow A(y)) \\ &\quad = \sup_{y\in X}\sim(\sim R(x_0,y)\Rightarrow\sim A(y)) && \text{by (11)} \\ &\quad = \sim\inf_{y\in X}(\sim R(x_0,y)\Rightarrow\sim A(y)) && \text{by Lemma 3(vi')} \\ &\quad = \sim\inf_{y\in X}(R(x_0,y)\oplus\sim A(y)) && \text{by (9)} \\ &\quad = \sim R(x_0,x_0).\end{aligned}$$

Since $R(x_0,x_0)\neq 0$, we have $\sim R(x_0,x_0)\neq 1$, so $[R]_{\Leftarrow}A(x_0)\neq 1$. But $A(x_0)=1$. Therefore, $A \not\subseteq_L [R]_{\Leftarrow}A$.

Now, consider the third equivalence.
($\subseteq$) For any $A \in \mathcal{F}_L(X)$ and for any $x \in X$,

$$\langle R\rangle_{\oplus}A(x) = \inf_{y\in X}(R(x,y)\oplus A(y)) \leqslant R(x,x)\oplus A(x) = 0\oplus A(x) = A(x).$$

($\supseteq$) Assume that $R$ is not irreflexive, i.e. $R(x_0,x_0)\neq 0$ for some $x_0 \in X$. For $A = X\setminus\{x_0\}$ we have:

$$\langle R\rangle_{\oplus}A(x_0) = \inf_{y\in X}(R(x_0,y)\oplus A(y)) = R(x_0,x_0)\oplus 0 = R(x_0,x_0).$$

Since $A(x_0)=0$, we get $\langle R\rangle_{\oplus}A(x_0) \not\leqslant A(x_0)$, which implies $\langle R\rangle_{\oplus}A \not\subseteq_L A$.

**(iv)** We show the first equivalence.

($\subseteq$) For every $A \in \mathcal{F}_L(X)$ and for every $x \in X$,

$$\begin{aligned}&[R]_{\to}[R]_{\to}A(x) \\ &\; = \inf_{y\in X}(R(x,y)\to(\inf_{z\in X}(R(y,z)\to A(z)))) \\ &\; = \inf_{z\in X}\inf_{y\in X}(R(x,y)\to(R(y,z)\to A(z))) && \text{by Lemma 3(ii)} \\ &\; = \inf_{z\in X}\inf_{y\in X}(R(x,y)\otimes R(y,z)\to A(z)) && \text{by Lemma 2(xiv)} \\ &\; \geqslant \inf_{z\in X}\inf_{y\in X}(R(x,z)\to A(z)) && \text{by assumption, Lemma 2(i)} \\ &\; = \inf_{z\in X}(R(x,z)\to A(z)) \\ &\; = [R]_{\to}A(x).\end{aligned}$$

($\supseteq$) Assume now that $R$ is not $\otimes$–transitive, i.e. $R(x_0,y_0)\otimes R(y_0,z_0)\not\leqslant R(x_0,z_0)$ for some $x_0,y_0,z_0 \in X$. By Lemma 2(**v**), this means that

**(iv.1)** $(R(x_0,y_0)\otimes R(y_0,z_0))\to R(x_0,z_0)\neq 1$.

Consider $A = x_0R$. Using again Lemma 2(**v**) we get

**(iv.2)** $[R]_{\to}A(x_0) = \inf_{y\in X}(R(x_0,y)\to R(x_0,y)) = 1$.

Next,

$$\begin{aligned}
&[R]_{\rightarrow}[R]_{\rightarrow}A(x_0)\\
&\quad = \inf_{y\in X}(R(x_0,y)\rightarrow(\inf_{z\in X}(R(y,z)\rightarrow R(x_0,z)))\\
&\quad = \inf_{z\in X}\inf_{y\in X}(R(x_0,y)\rightarrow(R(y,z)\rightarrow R(x_0,z))) && \text{by Lemma 3}\mathbf{(ii)}\\
&\quad = \inf_{z\in X}\inf_{y\in X}((R(x_0,y)\otimes R(y,z))\rightarrow R(x_0,z)) && \text{by Lemma 2}\mathbf{(xiv)}\\
&\quad \leqslant (R(x_0,y_0)\otimes R(y_0,z_0))\rightarrow R(x_0,z_0)\\
&\quad \neq 1 && \text{by } \mathbf{(iv.1)}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&[R]_{\rightarrow}A(x_0)\rightarrow[R]_{\rightarrow}[R]_{\rightarrow}A(x_0)\\
&\quad = 1\rightarrow[R]_{\rightarrow}[R]_{\rightarrow}A(x_0) && \text{by } \mathbf{(iv.2)}\\
&\quad = [R]_{\rightarrow}[R]_{\rightarrow}A(x_0) && \text{by Lemma 2}\mathbf{(vi)}\\
&\quad \neq 1.
\end{aligned}$$

Then, by Lemma 2**(v)**, $[R]_{\rightarrow}A(x_0)\nleqslant[R]_{\rightarrow}[R]_{\rightarrow}A(x_0)$, so $[R]_{\rightarrow}A \not\subseteq_L [R]_{\rightarrow}[R]_{\rightarrow}A$.

Now, we show the second equivalence.

($\subseteq$) For every $A\in\mathcal{F}_L(X)$ and for every $x\in X$,

$$\begin{aligned}
&[R]_{\Rightarrow}[R]_{\Rightarrow}A(x)\\
&\quad = \inf_{y\in X}(R(x,y)\Rightarrow(\inf_{z\in X}(R(y,z)\Rightarrow A(z))))\\
&\quad = \inf_{z\in X}\inf_{y\in Z}(R(x,y)\Rightarrow(R(y,z)\Rightarrow A(z))) && \text{by Lemma 3}\mathbf{(iii)}\\
&\quad = \inf_{z\in X}\inf_{y\in X}(R(x,y)\otimes R(y,z)\Rightarrow A(z)) && \text{by Lemma 2}\mathbf{(xv)}\\
&\quad \geqslant \inf_{z\in X}\inf_{y\in X}(R(x,z)\Rightarrow A(z)) && \text{by assumption, Lemma 2}\mathbf{(i)}\\
&\quad = [R]_{\Rightarrow}A(x).
\end{aligned}$$

($\supseteq$) Assume that $R$ is not $\otimes$–transitive, i.e. there exist $x_0,y_0,z_0\in X$ such that $R(x_0,y_0)\otimes R(y_0,z_0)\nleqslant R(x_0,z_0)$. Then $\sim R(x_0,z_0)\nleqslant\sim(R(x_0,y_0)\otimes R(y_0,z_0))$, which by Lemma 2**(v)** gives

**(iv.3)** $\quad \sim R(x_0,z_0)\rightarrow\sim(R(x_0,y_0)\otimes R(y_0,z_0))\neq 1.$

Take $A = X\backslash\{z_0\}$. Since for every $a\in L$, $a\oplus 1 = 1$, we easily get for every $y\in X$,

**(iv.4)** $\quad [R]_{\Rightarrow}A(y) = \inf_{z\in X}(\sim R(y,z)\oplus A(z)) = \sim R(y,z_0).$

Furthermore,

$$\begin{aligned}
&[R]_{\Rightarrow}[R]_{\Rightarrow}A(x_0)\\
&\quad = \inf_{y\in X}(R(x_0,y)\Rightarrow[R]_{\Rightarrow}A(y))\\
&\quad = \inf_{y\in X}(R(x_0,y)\Rightarrow\sim R(y,z_0)) && \text{by } \mathbf{(iv.4)}\\
&\quad = \inf_{y\in X}(\sim R(x_0,y)\oplus\sim R(y,z_0)) && \text{by (9)}\\
&\quad = \inf_{y\in X}\sim(R(x_0,y))\otimes R(y,z_0)) && \text{by (8)}\\
&\quad = \sim\sup_{y\in X}(R(x_0,y))\otimes R(y,z_0)) && \text{by Lemma 3}\mathbf{(vi')}.
\end{aligned}$$

Then we get

$$\begin{aligned}
&[R]_{\Rightarrow}A(x_0)\rightarrow[R]_{\Rightarrow}[R]_{\Rightarrow}A(x_0)\\
&\quad = \sim R(x_0,z_0)\rightarrow\sim\sup_{y\in X}(R(x_0,y)\otimes R(y,z_0)) && \text{by } \mathbf{(iv.4)}\\
&\quad \leqslant\sim R(x_0,z_0)\rightarrow\sim(R(x_0,y_0)\otimes R(y_0,z_0)) && \text{by Lemma 2}\mathbf{(i)}\\
&\quad \neq 1 && \text{by } \mathbf{(iv.3)}.
\end{aligned}$$

By Lemma 2(**v**), this implies $[R]_{\Rightarrow}A(x_0) \not\leqslant [R]_{\Rightarrow}[R]_{\Rightarrow}A(x_0)$. Therefore, we get $[R]_{\Rightarrow}A \not\subseteq_L [R]_{\Rightarrow}[R]_{\Rightarrow}A$.

In the similar way the third equivalence can be proved.

## 5 Conclusions

In this paper we have presented fuzzy generalisations of several information relations and operators. Two classes of residuated lattices have been taken as basic algebraic structures: traditional residuated lattices (commutative and integral) and so–called extended residuated lattices (ER–lattices). It has been shown that ER–lattices allow us to define abstract counterparts of the main classes of fuzzy logical connectives. We have indicated that some inadequacies in representation occur when residuated lattices constitute the basic structures and that these drawbacks can be avoided on the basis of ER–lattices. Some fuzzy information operators have been presented. We have shown that these operators are useful for characterizations of main classes of fuzzy relations.

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