Relational Logics and Their Applications

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Abstract. Logics of binary relations corresponding, among others, to the class RRA of representable relation algebras and the class FRA of full relation algebras are presented together with the proof systems in the style of dual tableaux. Next, the logics are extended with relational constants interpreted as point relations. Applications of these logics to reasoning in non-classical logics are recalled. An example is given of a dual tableau proof of an equation which is RRA-valid, while not RA-valid.

1 Introduction

We present a survey of relational logics which provide a general framework for specification and reasoning (verification of validity, model checking and entailment) in non-classical logics. They also provide a common background for a broad class of relational structures used in computer science. We present the logics step by step, starting with a logic of binary relations with basic relational operations of relation algebras (RL-logic), then expanding the language with the constant 1 (RL(1)-logic), next with the constant $1'$ (RL($1'$)-logic), then with the constants 1 and $1'$ put together (RL($1, 1'$)-logic), and finally adding relational constants interpreted as point relations ($\mathsf{RL}_{ax}(C)$-logic and $\mathsf{RL}_{df}(C)$-logic). The logics are based on various classes of models which differ in the interpretation of relational constants, for example, 1 may be interpreted as a universal relation or as an equivalence relation, $1'$ may be interpreted as an equivalence relation or an identity. We present completeness theorems with respect to all those classes of models. We also show which classes of models of RL($1, 1'$)-language enable us to simulate the RRA-validity and FRA-validity. Logic RL($1, 1'$) with the class of models corresponding to full relation algebras plays the role of a generic logic within which many non-classical logics can be expressed. Its applications to modal logics originated in [15]. Then, after few more examples of logics treated in a relational framework (see e.g., [16], [17]), a paradigm 'formulas are relations' has been formulated in [18]. Since then relational proof systems have been developed for several theories, see e.g., [3], [4], [8], [11], [12], [13], [19], [20], [21], [10] and [9]. Any particular relational proof system consists of the deduction system for RL($1, 1'$) augmented with the specific rules which reflect properties of accessibility relations from the models of a non-classical logic in question. An important feature of RL($1, 1'$)-logic is that it is expressive enough for performing the major

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logical tasks, namely verification of validity, entailment, model checking and satisfiability, as it is shown in Sections 10, 11, 12, and 13. A correspondence theory for relational proof systems is considered in [14]. A general method of defining deduction rules reflecting various constraints imposed on relations in the models of $\mathsf{RL}(1,1')$-logic is presented in that paper.

Recent implementations of the proof system for $\mathsf{RL}(1,1')$-logic are described in [2] and [6]. The first one is available at `http://logic.stfx.ca/reldt`. In [5] an implementation of translation procedures from the languages of non-classical logics to relational languages is presented. The system can be downloaded from `http://www.di.univaq.it/TARSKI/transIt/`. For the algebraic background of the relational logics see [24], [25] and [23].

2 A General Scheme of Relational Logics

Each relational logic L is determined by its language and its class of models. In this paper we consider logics of binary relations. There are two kinds of expressions of relational languages: terms and formulas. Terms represent relations and formulas express the facts that a pair of objects stands in a relation.

The vocabulary $\mathcal{V}_\mathsf{L}$ of L-language consists of the symbols from the following pairwise disjoint sets:

- a countable infinite set of object variables $\mathbb{OV}_\mathsf{L}$;
- a countable (possibly empty) set of object constants $\mathbb{OC}_\mathsf{L}$;
- a countable (possibly empty) set of relational variables $\mathbb{RV}_\mathsf{L}$;
- a countable (possibly empty) set of relational constants $\mathbb{RC}_\mathsf{L}$;
- a set of relational operation symbols $\mathbb{OP}_\mathsf{L} = \{-, \cup, \cap, ;, ^{-1}\}$, where $-$, $\cup$, $\cap$ are Boolean operations, ; is a relative product, and $^{-1}$ is the operation of converse;
- a set of parentheses $\{(,)\}$.

The set $\mathbb{RA}_\mathsf{L} = \mathbb{RV}_\mathsf{L} \cup \mathbb{RC}_\mathsf{L}$ is called the *set of atomic relational terms*. The set $\mathbb{OS}_\mathsf{L} = \mathbb{OV}_\mathsf{L} \cup \mathbb{OC}_\mathsf{L}$ is called the *set of objects symbols*. The set $\mathbb{RT}_\mathsf{L}$ of *relational terms* is the smallest (wrt inclusion) set of expressions that includes all atomic relational terms and is closed with respect to all relational operation symbols. L-*formulas* are of the form xRy, where $x, y \in \mathbb{OS}_\mathsf{L}$ and $R \in \mathbb{RT}_\mathsf{L}$. An L-formula xRy is said to be *atomic* whenever $R \in \mathbb{RA}_\mathsf{L}$.

With an L-language a class of L-models is associated. An L-*model* is a structure $\mathcal{M} = (U, m)$, where U is a non-empty set and m is a meaning function which assigns:

- elements of U to object constants, that is $m(c) \in U$, for every $c \in \mathbb{OC}_\mathsf{L}$;
- binary relations on U to atomic relational terms, that is $m(R) \subseteq U \times U$, for every $R \in \mathbb{RA}_\mathsf{L}$;

and extends to compound relational terms as follows:

- some condition about $m(-R)$ is assumed (see Sections 4 and 5 for the examples of the definitions of the complement operations);

- $m(R \cup S) = m(R) \cup m(S)$;
- $m(R \cap S) = m(R) \cap m(S)$;
- $m(R^{-1}) = (m(R))^{-1} = \{(x, y) \in U \times U : (y, x) \in m(R)\}$;
- $m(R; S) = m(R); m(S) = \{(x, y) \in U \times U : \exists z((x, z) \in m(R) \wedge (z, y) \in m(S))\}$;
- some additional conditions about m may be assumed (see Sections 5 and 6).

Let $\mathcal{M} = (U, m)$ be an L-model. An L-*valuation* in $\mathcal{M}$ is any function $v : \mathbb{OS}_{\mathsf{L}} \rightarrow U$ such that $v(c) = m(c)$, for every $c \in \mathbb{OC}_{\mathsf{L}}$. Let $\mathcal{M}$ be an L-model, let v be an L-valuation in $\mathcal{M}$ and let xRy be an L-formula. *Satisfiability* of xRy by v in $\mathcal{M}$ is defined as follows:

- If $1 \notin \mathbb{RC}_{\mathsf{L}}$, then $\mathcal{M}, v \models xRy$ iff $(v(x), v(y)) \in m(R)$.
- If $1 \in \mathbb{RC}_{\mathsf{L}}$, then $\mathcal{M}, v \models xRy$ iff $(v(x), v(y)) \notin m(1)$ or $(v(x), v(y)) \in m(1) \cap m(R)$.

Note that in the latter case, satisfiability is defined in a non-standard way. This is because we want to relativize satisfiability to the interpretation of the relational constant 1. In the general case, this interpretation need not be the universal relation. In the case it is, clearly the two definitions are equivalent.

An L-formula xRy is *true* in $\mathcal{M}$ whenever it is satisfied in $\mathcal{M}$ by all L-valuations. An L-formula xRy is L-*valid* whenever it is true in all L-models.

Fact 1
Let L and L' be relational logics such that every L-model is an L'-model. Then for any relational formula xRy, if xRy is L'-valid, then it is L-valid.

3 A General Scheme of Relational Proof Systems

Relational proof systems in the style of dual tableaux are founded on the Rasiowa-Sikorski system for the first order logic [22]. They are powerful tools for performing the major reasoning tasks: verification of validity, verification of entailment, model checking, and verification of satisfiability. Every relational proof system is determined by its axiomatic sets of formulas and rules which most often apply to finite sets of relational formulas. Some relational proof systems with infinitary rules are known in the literature, but in the present paper we confine ourselves to finitary rules only. The axiomatic sets take the place of axioms. The rules are intended to reflect properties of relational operations and constants. There are two groups of rules: decomposition rules and specific rules. Given a formula, the decomposition rules of the system enable us to transform it into simpler formulas, or the specific rules enable us to replace a formula by some other formulas. The rules have the following general form:

$$(*) \qquad \frac{\Phi}{\Phi_1 \mid \ldots \mid \Phi_n}$$

where $\Phi_1, \ldots, \Phi_n$ are finite non-empty sets of formulas, $n \geq 1$, and Φ is a finite (possibly empty) set of formulas. A rule of the form $(*)$ is said to be *applicable* to a set X of formulas whenever $\Phi \subseteq X$. As a result of an application of a rule of the form $(*)$ to a set X, we obtain the sets $(X \setminus \Phi) \cup \Phi_i$, $i = 1, \ldots, n$. A set to which a rule has been applied is called the *premise* of the rule, and the sets obtained by an application of the rule are called its *conclusions*. As usual, any concrete rule will always be presented in a short form, that is we will indicate only the formulas which are essential for a transformation to be performed by the rule and also we will omit set brackets. Given a formula, successive applications of the rules result in a tree whose nodes consist of finite sets of formulas. Each node includes all the formulas of its predecessor node, possibly except for those which have been transformed. A node of the tree does not have successors whenever its set of formulas includes an axiomatic subset or none of the rules is applicable to it. We say that a variable in a rule is *new* whenever it appears in a conclusion of the rule and does not appear in its premise.

Let L be a relational logic. A relational proof system for L (L-system for short) contains a set $\mathcal{DR}_{\mathsf{L}}$ of L-*decomposition rules* and a set $\mathcal{SR}_{\mathsf{L}}$ of L-*specific rules*, where in each particular logic L the terms and the object symbols range over the corresponding sets of L.

The set of decomposition rules $\mathcal{DR}_{\mathsf{L}}$ includes the set $\mathcal{DR}_0$ of rules of the following forms:

Let $x, y, \in \mathbb{OS}_{\mathsf{L}}$ and $R, S \in \mathbb{RT}_{\mathsf{L}}$.

$(\cup)$ $\quad \dfrac{x(R \cup S)y}{xRy, xSy}$ $\qquad (-\cup)$ $\quad \dfrac{x-(R \cup S)y}{x-Ry \mid x-Sy}$

$(\cap)$ $\quad \dfrac{x(R \cap S)y}{xRy \mid xSy}$ $\qquad (-\cap)$ $\quad \dfrac{x-(R \cap S)y}{x-Ry, x-Sy}$

$(--)$ $\quad \dfrac{x--Ry}{xRy}$

$(^{-1})$ $\quad \dfrac{xR^{-1}y}{yRx}$ $\qquad (-^{-1})$ $\quad \dfrac{x-R^{-1}y}{y-Rx}$

$(;)$ $\quad \dfrac{x(R;S)y}{xRz, x(R;S)y \mid zSy, x(R;S)y}$ $\quad z \in \mathbb{OS}_{\mathsf{L}}$

$(-;)$ $\quad \dfrac{x-(R;S)y}{x-Rz, z-Sy}$ $\quad z \in \mathbb{OV}_{\mathsf{L}}$ and z is new

The set of specific rules includes the rules that reflect the properties of constants assumed in an L-language in question.

In all the systems considered in this paper the sets containing a subset $\{xRy, x-Ry\}$, for $x, y \in \mathbb{OS}_{\mathsf{L}}, R \in \mathbb{RT}_{\mathsf{L}}$, are assumed to be L-*axiomatic sets*. A finite set of formulas $\{\varphi_1, \ldots, \varphi_n\}$ is said to be an L-*set* whenever for every L-model $\mathcal{M}$ and for every L-valuation v in $\mathcal{M}$ there exists $i \in \{1, \ldots, n\}$ such that φ_i is satisfied by v in $\mathcal{M}$. Let Φ be a non-empty set of L-formulas. A rule $\dfrac{\Phi}{\Phi_1 | \ldots | \Phi_n}$ is L-*correct* whenever it holds: Φ is an L-set if and only if Φ_i is an L-set, for every $i \in \{1, \ldots, n\}$. In the case when Φ is empty, L-correctness can be expressed as follows: a rule $\frac{}{\Phi_1 | \ldots | \Phi_n}$ is L-*correct* whenever there exists $i \in \{1, \ldots, n\}$ such that Φ_i is not an L-set. It follows that the rules are semantically invertible. It is a characteristic feature of all Rasiowa-Sikorski style deduction systems (see [22] and [9]). A transfer of validity from the bottom sets of a rule to the upper set is needed for soundness of the system. The other direction is used in a proof of completeness. Observe that the classical tableau system for first-order logic has in fact the analogous property of preserving and reflecting unsatisfiability. Although this fact is not provable directly from the definition of tableau rules, it can be proved under the additional assumptions on repetition of some formulas in the process of application of the rules. In tableau system this assumption is hidden, it is shifted to a strategy of building the proof trees. In our systems the required repetitions are explicitly indicated in the rules.

Let xRy be an L-formula. An L-*proof tree for* xRy is a tree with the following properties:

– the formula xRy is at the root of this tree;
– each node except the root is obtained by an application of an L-rule to its predecessor node;
– a node does not have successors whenever it is an L-axiomatic set.

A branch of an L-proof tree is said to be L-*closed* whenever it contains a node with an L-axiomatic set of formulas. A tree is L-*closed* iff all of its branches are L-closed.

Due to the forms of decomposition rules of $\mathcal{DR}_0$ we obtain the following:

Fact 2
Let L*-system consists of decomposition rules from* $\mathcal{DR}_0$*. If a node of an* L*-proof tree does not contain an* L*-axiomatic subset and contains an* L*-formula* xRy *or* $x{-}Ry$*, for atomic* R*, then all of its successors contain this formula as well.*

An L-formula xRy is L-*provable* whenever there is a closed L-proof tree for it.

Fact 3
For every relational logic L*, if we show that:*

1. *All* L*-rules are* L*-correct.*
2. *All* L*-axiomatic sets are* L*-sets.*

then we obtain the soundness theorem for L*-logic: if an* L*-formula* xRy *is* L*-provable, then it is* L*-valid.*

As usual in proof theory a concept of completeness of a non-closed proof tree is needed. Intuitively, completeness of a non-closed tree means that all the rules that can be applied have been applied. By abusing the notation, for any branch b and a formula xRy, we write $xRy \in b$, if xRy belongs to a set of formulas of a node of branch b.

A non-closed branch b of an L-proof tree is said to be L-*complete* whenever it satisfies L-completion conditions. L-completion conditions determined by the rules of $\mathcal{DR}_0$ are the following:

For all $x, y \in \mathbb{OS}_{\mathsf{L}}$ and for all $R, S \in \mathbb{RT}_{\mathsf{L}}$:

Cpl($\cup$) (resp. Cpl($-\cap$)) If $x(R \cup S)y \in b$ (resp. $x-(R \cap S)y \in b$), then both $xRy \in b$ (resp. $x-Ry \in b$) and $xSy \in b$ (resp. $x-Sy \in b$).

Cpl($\cap$) (resp. Cpl($-\cup$)) If $x(R \cap S)y \in b$ (resp. $x-(R \cup S)y \in b$), then either $xRy \in b$ (resp. $x-Ry \in b$) or $xSy \in b$ (resp. $x-Sy \in b$).

Cpl($-$) If $x(--R)y \in b$, then $xRy \in b$.

Cpl($^{-1}$) If $xR^{-1}y \in b$, then $yRx \in b$.

Cpl($-^{-1}$) If $x-R^{-1}y \in b$, then $y-Rx \in b$.

Cpl($;$) If $x(R;S)y \in b$, then for every $z \in \mathbb{OS}_{\mathsf{L}}$, either $xRz \in b$ or $zSy \in b$.

Cpl($-;$) If $x-(R;S)y \in b$, then for some $z \in \mathbb{OV}_{\mathsf{L}}$, both $x-Rz \in b$ and $z-Sy \in b$.

An L-proof tree is said to be L-*complete* iff all of its non-closed branches are L-complete. An L-complete non-closed branch is said to be L-*open*.

By Fact 2 and since the set containing a subset $\{xRy, x-Ry\}$ is L-axiomatic, in every L-system containing only decomposition rules of $\mathcal{DR}_0$ the following holds:

Fact 4
Let L-system be a system with decomposition rules of $\mathcal{DR}_0$ as the only rules and let b be an L-open branch of an L-proof tree. Then there is no atomic L-formula xRy such that $xRy \in b$ and $x-Ry \in b$.

Due to Facts 2 and 4 it is easy to prove the following proposition:

Proposition 1
Let L-system be a system with decomposition rules of $\mathcal{DR}_0$ as the only rules and let b be a branch of an L-proof tree. If there are $x, y \in \mathbb{OS}_{\mathsf{L}}$ and $R \in \mathbb{RT}_{\mathsf{L}}$ such that $xRy \in b$ and $x-Ry \in b$, then b is closed.

Sometimes if the logic L is clear from the context we will omit the index L.

4 Basic Relational Logic **RL**

The logic presented in this section is a common core of all the logics relevant for binary relations. The vocabulary of the language of RL-logic is defined as in Section 2 where:

– $\mathbb{RC}_{\mathsf{RL}} = \emptyset$.

An RL-*model* is a structure $\mathcal{M} = (U, m)$, where U is a non-empty set and $m: \mathbb{RV}_{\mathsf{RL}} \cup \mathbb{OC}_{\mathsf{RL}} \to \mathcal{P}(U \times U) \cup U$ is a meaning function such that m extends to all compound relational terms as defined in Section 2 with the condition:

$$m(-R) = (U \times U) \setminus m(R)$$

where on the right hand side '\' denotes the set difference.

The decomposition rules $\mathcal{DR}_{\mathsf{RL}}$ of the RL-system are the rules of $\mathcal{DR}_0$ presented in Section 3 adjusted to the RL-language. There are no specific rules in this system. RL-*axiomatic* set is any set containing $\{xRy, x{-}Ry\}$, as defined in Section 3, where $x, y \in \mathbb{OS}_{\mathsf{RL}}$ and R is a relational term of $\mathbb{RT}_{\mathsf{RL}}$.

For each rule $(\#) \in \mathcal{DR}_{\mathsf{RL}}$ its correctness follows directly from semantics of relational terms built with the operator #.

Proposition 2

1. *All* RL-*rules are* RL-*correct.*
2. *All* RL-*axiomatic sets are* RL-*sets.*

Due to the above proposition and Fact 3 we obtain:

Theorem 1 (Soundness of RL)
Let xRy be an RL-*formula. If xRy is* RL-*provable, then it is* RL-*valid.*

A non-closed branch b of a proof tree is said to be RL-*complete* whenever it satisfies RL-completion conditions of Section 3 determined by the rules from $\mathcal{DR}_{\mathsf{RL}}$.

Let b be an RL-open branch of an RL-proof tree. We define a branch structure $\mathcal{M}^b = (U^b, m^b)$ as follows:

- $U^b = \mathbb{OS}_{\mathsf{RL}}$;
- $m^b(c) = c$, for every $c \in \mathbb{OC}_{\mathsf{RL}}$
- $m^b(R) = \{(x, y) \in U^b \times U^b : xRy \notin b\}$, for every relational variable R;
- m^b extends homomorphically to all compound relational terms as in the RL-models.

Fact 5
For every RL-*open branch b, $\mathcal{M}^b$ is an* RL-*model.*

Any structure $\mathcal{M}^b$ is referred to as an RL-*branch model.* Let $v^b : \mathbb{OS}_{\mathsf{RL}} \to U^b$ be an RL-valuation in $\mathcal{M}^b$ such that $v^b(x) = x$ for every $x \in \mathbb{OS}_{\mathsf{RL}}$.

Proposition 3
For every open branch b of an RL-*proof tree, and for every* RL-*formula xRy:*

$$(*) \quad \text{if} \quad \mathcal{M}^b, v^b \models xRy, \quad \text{then} \quad xRy \notin b.$$

Proof. The proof is by induction on the complexity of formulas.

Let xRy be an atomic RL-formula. Assume $\mathcal{M}^b, v^b \models xRy$, that is $(x, y) \in m^b(R)$. By the definition of a branch model $xRy \notin b$. Let $R \in \mathbb{RV}$ and $\mathcal{M}^b, v^b \models x{-}Ry$, that is $(x, y) \notin m^b(R)$. Therefore $xRy \in b$. By Fact 4, $x{-}Ry \notin b$.

By way of example we prove $(*)$ for $R = S;T$ and $R = -(S;T)$.

Let $\mathcal{M}^b, v^b \models xRy$, for $R = S;T$. Then $(x,y) \in m^b(S;T)$, that is there exists $z \in \mathbb{OS}_{\mathsf{RL}}$ such that $xSz \notin b$ and $zTy \notin b$. Suppose $x(S;T)y \in b$. By the completion condition Cpl(;), for every $z \in \mathbb{OS}_{\mathsf{RL}}$ either $xSz \in b$ or $zTy \in b$, a contradiction.

Let $\mathcal{M}^b, v^b \models xRy$, for $R = -(S;T)$. Then $(x,y) \notin m^b(S;T)$, that is for every $z \in \mathbb{OS}_{\mathsf{RL}}$ either $xSz \in b$ or $zTy \in b$. Suppose $x-(S;T)y \in b$. By the completion condition Cpl(−;), for some $z \in \mathbb{OV}_{\mathsf{RL}}$ both $x-Sz \in b$ and $z-Ty \in b$. By Proposition 1, b is closed, a contradiction. □

The above proposition enables us to prove the following completeness theorem:

Theorem 2 (Completeness of RL)
Let xRy be an RL-formula. If xRy is RL-valid, then xRy is RL-provable.

Proof. Assume xRy is RL-valid. Suppose there is no any closed RL-proof tree for xRy. Consider a non-closed RL-proof tree for xRy. We may assume that this tree is complete. Let b be an open branch of the complete RL-proof tree for xRy. Since $xRy \in b$, by Proposition 3 in the branch model $\mathcal{M}^b$ valuation v^b does not satisfy xRy. Hence xRy is not RL-valid, a contradiction. □

5 Relational Logics with the Constant 1

In this section we present a relational logic RL(1) obtained from RL by expanding its language with a relational constant 1. There are two classes of models associated with the logic RL(1): in the first one the relational constant 1 is interpreted as an equivalence relation on a non-empty set U, while in the second 1 is interpreted as a universal relation. The vocabulary of the language of RL(1)-logic is defined as in Section 2 with

– $\mathbb{RC}_{\mathsf{RL(1)}} = \{1\}$.

An RLN(1)-*model* is a structure $\mathcal{M} = (U, m)$, where U is a non-empty set and $m\colon \mathbb{RA}_{\mathsf{RL(1)}} \cup \mathbb{OC}_{\mathsf{RL(1)}} \to \mathcal{P}(U \times U) \cup U$ is a meaning function such that:

– $m(1)$ is an equivalence relation on U;
– m extends to all compound relational terms as defined in Section 2 with the following additional condition: $m(-R) = m(1) \cap (U \times U \setminus m(R))$.

An RLN(1)-model is said to be RL(1)-model whenever 1 is interpreted as an universal relation, that is $m(1) = U \times U$. It follows that if $\mathcal{M} = (U, m)$ is RLN(1)-model or RL(1)-model, then truth of a formula xRy in $\mathcal{M}$ is equivalent to $m(1) \subseteq m(R)$.

Due to the definitions of RLN(1)-models and RL(1)-models we obtain the following:

Fact 6
For every $\mathsf{RL}(1)$*-formula* xRy*, if* xRy *is* $\mathsf{RLN}(1)$*-valid, then it is* $\mathsf{RL}(1)$*-valid.*

$\mathsf{RL}(1)$-decomposition rules are precisely the rules of $\mathcal{DR}_{\mathsf{RL}}$, that is $\mathcal{DR}_{\mathsf{RL}(1)} = \mathcal{DR}_{\mathsf{RL}}$. Moreover, the relational proof system for $\mathsf{RL}(1)$-logic ($\mathsf{RL}(1)$-system for short) contains $\mathsf{RL}(1)$-axiomatic sets defined below. A set is an $\mathsf{RL}(1)$-axiomatic whenever it includes any of the subsets (Ax1) or (Ax2), where:

(Ax1) $\{x1y\}$, where $x, y \in \mathbb{OS}_{\mathsf{RL}(1)}$;
(Ax2) $\{xRy, x{-}Ry\}$, where $x, y \in \mathbb{OS}_{\mathsf{RL}(1)}$ and $R \in \mathbb{RT}_{\mathsf{RL}(1)}$.

As in the case of RL-logic, it is easy to prove the following:

Proposition 4

1. *All* $\mathsf{RL}(1)$*-rules are* $\mathsf{RLN}(1)$*-correct.*
2. *All* $\mathsf{RL}(1)$*-axiomatic sets are* $\mathsf{RLN}(1)$*-sets.*

Due to the above proposition and Fact 3 we have the following:

Proposition 5
Let xRy *be an* $\mathsf{RL}(1)$*-formula. If* xRy *is* $\mathsf{RL}(1)$*-provable, then it is* $\mathsf{RLN}(1)$*-valid.*

Due to Fact 6 the following holds:

Corollary 1
Let xRy *be an* $\mathsf{RL}(1)$*-formula. If* xRy *is* $\mathsf{RL}(1)$*-provable, then it is* $\mathsf{RL}(1)$*-valid.*

$\mathsf{RL}(1)$-completion conditions are the same as the completion conditions defined in Section 3 determined by the rules from $\mathcal{DR}_{\mathsf{RL}(1)}$ and adapted to the $\mathsf{RL}(1)$-language.

Let b be an open branch of an $\mathsf{RL}(1)$-proof tree. A branch structure $\mathcal{M}^b = (U^b, m^b)$ is defined as for RL-logic, taking the object symbols of $\mathsf{RL}(1)$ as the elements of U^b, defining m^b for atomic $\mathsf{RL}(1)$-terms and for object constants as in RL-branch model and defining m^b for all $\mathsf{RL}(1)$-terms as in $\mathsf{RL}(1)$-models.

Proposition 6
For every $\mathsf{RL}(1)$*-open branch* b*, a branch structure* $\mathcal{M}^b$ *is an* $\mathsf{RL}(1)$*-model.*

Proof. For all $x, y \in \mathbb{OS}_{\mathsf{RL}(1)}$ $x1y \notin b$, since otherwise b would be closed. So $m^b(1) = U^b \times U^b$. Therefore by the definition, $\mathcal{M}^b$ is an $\mathsf{RL}(1)$-model. □

Let $v^b\colon \mathbb{OS}_{\mathsf{RL}(1)} \to U^b$ be an $\mathsf{RL}(1)$-valuation in $\mathcal{M}^b$ such that $v^b(x) = x$ for every $x \in \mathbb{OS}_{\mathsf{RL}(1)}$.

Proposition 7
For every open branch b *of an* $\mathsf{RL}(1)$*-proof tree, and for every* $\mathsf{RL}(1)$*-formula* xRy*:*

$$(*) \quad \textit{if} \quad \mathcal{M}^b, v^b \models xRy, \quad \text{then} \quad xRy \notin b.$$

Since $m^b(1)$ is the universal relation, the proof is similar to the proof of Proposition 3. Due to Proposition 7 we obtain the following:

Proposition 8
Let xRy be an $\mathsf{RL}(1)$*-formula. If xRy is* $\mathsf{RL}(1)$*-valid, then xRy is* $\mathsf{RL}(1)$*-provable.*

Finally, due to Corollary 1 and Propositions 5 and 8 we obtain the following theorem:

Theorem 3 (Soundness and Completeness of $\mathsf{RL}(1)$)
Let xRy be an $\mathsf{RL}(1)$*-formula. The the following conditions are equivalent:*

– *xRy is* $\mathsf{RL}(1)$*-provable;*
– *xRy is* $\mathsf{RL}(1)$*-valid;*
– *xRy is* $\mathsf{RLN}(1)$*-valid.*

The above theorem confirms the known fact that the classes of equations provable in algebras of relations with 1 being the universal relation and with 1 being an equivalence relation are the same. It will be discussed in more details in Section 14.

6 Relational Logics with Constant $1'$

A logic considered in this section is obtained from RL-logic by expanding its language with a constant $1'$. The vocabulary of the language of $\mathsf{RL}(1')$-logic is defined as in Section 2 with

– $\mathbb{RC}_{\mathsf{RL}(1')} = \{1'\}$.

An $\mathsf{RL}(1')$-*model* is a structure $\mathcal{M} = (U, m)$, where U is a non-empty set and $m: \mathbb{RA}_{\mathsf{RL}(1')} \cup \mathbb{OC}_{\mathsf{RL}(1')} \to \mathcal{P}(U \times U) \cup U$ is a meaning function such that the following conditions are satisfied:

– $m(1')$ is an equivalence relation on U;
– $m(1'); m(R) = m(R); m(1') = m(R)$ for every $R \in \mathbb{RA}_{\mathsf{RL}(1')}$ (extensionality);
– m extends to all compound relational terms as in the RL-models.

By an easy induction the following can be proved :

Proposition 9
Let $\mathcal{M} = (U, m)$ be an $\mathsf{RL}(1')$*-model. Then for every relational term R of* $\mathsf{RL}(1')$*-language, the following extensionality property holds:*

$$m(1'); m(R) = m(R); m(1') = m(R).$$

Proof
By way of example we show that the extensionality property holds for $R = -S$ and $R = (S; T)$.

Proof of $m(-S) = m(1'); m(-S)$

Assume $(x, y) \in m(-S)$. Since $m(1')$ is reflexive, $(x, x) \in m(1')$ and $(x, y) \in m(-S)$. Hence there exists $z \in U$ such that $(x, z) \in m(1')$ and $(z, y) \in m(-S)$. Therefore $(x, y) \in m(1'); m(-S)$.

Assume $(x, y) \in m(1'); m(-S)$, that is there exists $z \in U$ such that $(x, z) \in m(1')$ and $(z, y) \notin m(S)$. By the induction hypothesis, for all $u \in U$ $((z, u) \notin m(1')$ or $(u, y) \notin m(S))$. Let $u := x$. It follows that $(z, x) \notin m(1')$ or $(x, y) \notin m(S)$. Since $m(1')$ is symmetric, it must be $(x, y) \notin m(S)$. Therefore $(x, y) \in m(-S)$.

Proof of $m(S;T) = m(1'); m(S;T)$

Since $m(1')$ is reflexive, $m(S;T) \subseteq m(1'); m(S;T)$.

Assume $(x, y) \in m(1'); m(S;T)$, that is there exist $z, u \in U$ such that $(x, z) \in m(1')$, $(z, u) \in m(S)$ and $(u, y) \in m(T)$. By the induction hypothesis we get $(x, u) \in m(S)$. Therefore $(x, y) \in m(S;T)$. □

Proposition 10
Let $\mathcal{M} = (U, m)$ be a structure such that U is a non-empty set and $m\colon \mathbb{RA}_{\mathsf{RL}(1')} \cup \mathbb{OC}_{\mathsf{RL}(1')} \to \mathcal{P}(U \times U) \cup U$ is a meaning function satisfying the following conditions:

- *$m(1')$ is reflexive;*
- *m extends to all compound relational terms as in the* RL*-models;*
- *$m(1'); m(R) = m(R); m(1') = m(R)$ for every $R \in \mathbb{RT}_{\mathsf{RL}(1')}$.*

Then $\mathcal{M}$ is an RL$(1')$*-model.*

Proof
It suffices to show that $m(1')$ is symmetric and transitive. Let $R = (1')^{-1}$. Then $m(1')^{-1}; m(1') = m(1')^{-1} = m(1'); m(1')^{-1}$, thus $(m(1'); m(1')^{-1}); m(1') = m(1')^{-1}$. It implies that: (*) $(y, x) \in m(1')$ iff there exist $z, u \in U$ such that $(x, z) \in m(1')$, $(u, z) \in m(1')$ and $(u, y) \in m(1')$. Assume $(x, y) \in m(1')$, for some $x, y \in U$. Then $z := y$ and $u := x$ satisfy the right side of condition (*), so $(y, x) \in m(1')$. Therefore $m(1')$ is symmetric. Assume $(x, y) \in m(1')$ and $(y, z) \in m(1')$. Since $m(1'); m(1') \subseteq m(1')$, $(x, z) \in m(1')$. Therefore $m(1')$ is transitive, hence it is an equivalence relation on U. □

It follows that the equivalent set of conditions on the RL$(1')$-models could be reflexivity of $m(1')$ and the extensionality property for all the relational terms.

An RL$(1')$-model $\mathcal{M} = (U, m)$ is said to be *standard* whenever $m(1')$ is the identity on U, that is $m(1') = \{(x, x) : x \in U\}$. Any standard RL$(1')$-model will be referred to as RL$^*(1')$-model. A formula xRy is said to be RL$^*(1')$-valid iff it is true in all standard RL$(1')$-models.

Fact 7
If xRy is $\mathsf{RL}(1')$-valid, then it is $\mathsf{RL}^*(1')$-valid.

The decomposition rules of the $\mathsf{RL}(1')$-system are the rules obtained from the rules in $\mathcal{DR}_0$ presented in Section 3 by adjusting them to the $\mathsf{RL}(1')$-language. The specific rules of $\mathsf{RL}(1')$-system have the following forms:

Let $x, y \in \mathbb{OS}_{\mathsf{RL}(1')}$ and $R \in \mathbb{RA}_{\mathsf{RL}(1')}$.

$$(1'1) \quad \frac{xRy}{xRz, xRy \mid y1'z, xRy} \quad z \in \mathbb{OS}_{\mathsf{RL}(1')}$$

$$(1'2) \quad \frac{xRy}{x1'z, xRy \mid zRy, xRy} \quad z \in \mathbb{OS}_{\mathsf{RL}(1')}$$

A finite set of formulas is $\mathsf{RL}(1')$-*axiomatic* whenever it includes (Ax1) or (Ax2), where:

(Ax1) $\{x1'x\}$, where $x \in \mathbb{OS}_{\mathsf{RL}(1')}$
(Ax2) $\{xRy, x{-}Ry\}$, where $x, y \in \mathbb{OS}_{\mathsf{RL}(1')}$ and $R \in \mathbb{RT}_{\mathsf{RL}(1')}$

It is easy to see that the properties of Facts 2, 4 and Proposition 1 are satisfied in $\mathsf{RL}(1')$, that is in the $\mathsf{RL}(1')$-system the following holds:

Proposition 11
Let b be a branch of an $\mathsf{RL}(1')$-proof tree. If $xRy \in b$ and $x{-}Ry \in b$, for some relational term R and for some $x, y \in \mathbb{OS}_{\mathsf{RL}(1')}$, then b is closed.

Proposition 12

1. *All $\mathsf{RL}(1')$-rules are $\mathsf{RL}(1')$-correct.*
2. *All $\mathsf{RL}(1')$-axiomatic sets are $\mathsf{RL}(1')$-sets.*

Proof
Since $m(1')$ is reflexive, $\{x1'x\}$ is an $\mathsf{RL}(1')$-set. To prove 1. it suffices to show correctness of the specific rules, correctness of the decomposition rules follows from the definitions of the relational operations. Let us prove that the rule $(1'1)_{\mathsf{RL}(1')}$ is correct, for any atomic relational term R. It is easy to see that if $\{xRy\}$ is an $\mathsf{RL}(1')$-set, then $\{xRy, xRz\}$ and $\{y1'z, xRy\}$ are $\mathsf{RL}(1')$-sets. Assume $\{xRy, xRz\}$ and $\{y1'z, xRy\}$ are $\mathsf{RL}(1')$-sets, that is, by symmetry of $m(1')$, for every $\mathsf{RL}(1')$-model $\mathcal{M}$ and for every $\mathsf{RL}(1')$-valuation v:

$$\mathcal{M}, v \models xRz \text{ or } \mathcal{M}, v \models xRy \qquad \text{and} \qquad \mathcal{M}, v \models z1'y \text{ or } \mathcal{M}, v \models xRy$$

Let $\mathcal{M}$ be an $\mathsf{RL}(1')$-model and v be an $\mathsf{RL}(1')$-valuation in $\mathcal{M}$. Suppose $\mathcal{M}, v \models xRz$ and $\mathcal{M}, v \models z1'y$. Then $(v(x), v(z)) \in m(R)$ and $(v(z), v(y)) \in m(1')$. Since $m(R); m(1') \subseteq m(R)$, $(v(x), v(y)) \in m(R)$. Hence $\mathcal{M}, v \models xRy$. In the remaining cases the proofs are obvious. The proof for the rule $(1'2)$ is similar. □

Due to the above proposition and Fact 3 we obtain the following:

Proposition 13
Let xRy be an $\mathsf{RL}(1')$-formula. If xRy is $\mathsf{RL}(1')$-provable, then it is $\mathsf{RL}(1')$-valid.

Corollary 2
Let xRy be an $\mathsf{RL}(1')$*-formula. If xRy is* $\mathsf{RL}(1')$*-provable, then it is* $\mathsf{RL}^*(1')$*-valid.*

A non-closed branch b of an $\mathsf{RL}(1')$-proof tree is said to be $\mathsf{RL}(1')$-*complete* whenever it satisfies $\mathsf{RL}(1')$-completion conditions which consist of the completion conditions determined by decomposition rules of $\mathcal{DR}_{\mathsf{RL}(1')}$ and the following:

For every $R \in \mathbb{RA}_{\mathsf{RL}(1')}$ and for all $x, y \in \mathbb{OS}_{\mathsf{RL}(1')}$:

Cpl($1'1$) If $xRy \in b$, then for every $z \in \mathbb{OS}_{\mathsf{RL}(1')}$, either $xRz \in b$ or $y1'z \in b$.

Cpl($1'2$) If $xRy \in b$, then for every $z \in \mathbb{OS}_{\mathsf{RL}(1')}$, either $x1'z \in b$ or $zRy \in b$.

Let b be an open branch of an $\mathsf{RL}(1')$-proof tree. We define a branch structure $\mathcal{M}^b = (U^b, m^b)$ similarly as for RL-logic adapted to the $\mathsf{RL}(1')$-language. In particular, $m^b(1') = \{(x, y) \in U^b \times U^b : x1'y \notin b\}$.

Proposition 14
For every $\mathsf{RL}(1')$*-open branch b, a branch structure $\mathcal{M}^b$ is an* $\mathsf{RL}(1')$*-model.*

Proof
We need to prove that (1) $m^b(1')$ is an equivalence relation on U^b and (2) $m^b(1'); m^b(R) = m^b(R); m^b(1') = m^b(R)$ for every $R \in \mathbb{RA}_{\mathsf{RL}(1')}$.

Proof of (1)
For every $x \in U^b$, $x1'x \notin b$, since otherwise b would be closed. Therefore $(x, x) \in m^b(1')$, hence $m^b(1')$ is reflexive. Assume $(x, y) \in m^b(1')$, that is $x1'y \notin b$. Suppose $(y, x) \notin m^b(1')$. Then $y1'x \in b$. By the completion condition Cpl($1'1$), either $y1'y \in b$ or $x1'y \in b$, a contradiction. Therefore $m^b(1')$ is symmetric. To prove transitivity, assume $(x, y) \in m^b(1')$ and $(y, z) \in m^b(1')$, that is $x1'y \notin b$ and $y1'z \notin b$. Suppose $(x, z) \notin m^b(1')$. Then $x1'z \in b$. By the completion condition Cpl($1'1$), either $x1'y \in b$ or $z1'y \in b$. In the first case we get a contradiction, so $z1'y \in b$. By the completion condition Cpl($1'1$) applied to $z1'y$, either $z1'z \in b$ or $y1'z \in b$, a contradiction. Therefore $m^b(1')$ is transitive.

Proof of (2)
Since $m^b(1')$ is reflexive, $m^b(R) \subseteq m^b(1'); m^b(R)$ and $m^b(R) \subseteq m^b(R); m^b(1')$.

Now assume $(x, y) \in m^b(1'); m^b(R)$, that is there exists $z \in U^b$ such that $x1'z \notin b$ and $zRy \notin b$. Suppose $(x, y) \notin m^b(R)$. Then $xRy \in b$. By the completion condition Cpl(($1'2$), for every $z \in U^b$, either $x1'z \in b$ or $zRy \in b$, a contradiction.

Assume $(x, y) \in m^b(R); m^b(1')$, that is, by symmetry of $m^b(1')$, there exists $z \in U^b$ such that $xRz \notin b$ and $y1'z \notin b$. Suppose $(x, y) \notin m^b(R)$. Then $xRy \in b$. By the completion condition Cpl($1'1$), for every $z \in U^b$, either $xRz \in b$ or $y1'z \in b$, a contradiction. □

Any structure $\mathcal{M}^b$ is referred to as an $\mathsf{RL}(1')$-*branch model*. Let $v^b \colon \mathbb{OS}_{\mathsf{RL}(1')} \to U^b$ be an $\mathsf{RL}(1')$-valuation in $\mathcal{M}^b$ such that $v^b(x) = x$ for every $x \in \mathbb{OS}_{\mathsf{RL}(1')}$.

Proposition 15
For every open branch b of an $\mathsf{RL}(1')$*-proof tree, and for every* $\mathsf{RL}(1')$*-formula xRy:*

$$(*) \quad \textit{if} \quad \mathcal{M}^b, v^b \models xRy, \quad \text{then} \quad xRy \notin b.$$

The proof is similar to the proof of Proposition 3.

Since $m^b(1')$ is an equivalence relation on U^b, given an $\mathsf{RL}(1')$-branch model $\mathcal{M}^b$, we may define the quotient model $\mathcal{M}^b_q = (U^b_q, m^b_q)$ as follows:

- $U^b_q = \{\|x\| : x \in U^b\}$, where $\|x\|$ is the equivalence class of $m^b(1')$ generated by x;
- $m^q_b(c) = \|c\|$, for every $c \in \mathbb{OC}_{\mathsf{RL}(1')}$;
- $m^b_q(R) = \{(\|x\|, \|y\|)) \in U^b_q \times U^b_q : (x, y) \in m^b(R)\}$, for every $R \in \mathbb{RA}_{\mathsf{RL}(1')}$;
- m^b_q extends for all compound relational terms as in the $\mathsf{RL}(1')$-models.

Since a branch model satisfies the extensionality property, the definition of $m^b_q(R)$ is correct, that is the following condition is satisfied:

$$\text{if } (x, y) \in m^b(R) \text{ and } (x, z), (y, t) \in m^b(1'), \text{ then } (z, t) \in m^b(R).$$

Let v^b_q be an $\mathsf{RL}(1')$-valuation in $\mathcal{M}^b_q$ such that $v^b_q(x) = \|x\|$, for every $x \in \mathbb{OS}_{\mathsf{RL}(1')}$.

Proposition 16

1. *The model $\mathcal{M}^b_q$ is a standard $\mathsf{RL}(1')$-model,*
2. *For every $\mathsf{RL}(1')$-formula xRy:*

$$(*) \quad \mathcal{M}^b, v^b \models xRy \quad \text{iff} \quad \mathcal{M}^b_q, v^b_q \models xRy$$

Proof

1. We have to show that $m^b_q(1')$ is the identity on U^b_q . Indeed, we have:

$$(\|x\|, \|y\|) \in m^b_q(1') \text{ iff } (x, y) \in m^b(1') \text{ iff } \|x\| = \|y\|$$

2. The proof is by an easy induction on the complexity of formulas. □

Proposition 17
Let xRy be an $\mathsf{RL}(1')$-formula. If xRy is $\mathsf{RL}^*(1')$-valid, then xRy is $\mathsf{RL}(1')$-provable.

Proof
Assume xRy is $\mathsf{RL}^*(1')$-valid. Suppose there is no closed $\mathsf{RL}(1')$-proof tree for xRy. Consider a non-closed $\mathsf{RL}(1')$-proof tree for xRy. We may assume that this tree is complete. Let b be an open branch of the complete $\mathsf{RL}(1')$-proof tree for xRy. Since $xRy \in b$, so by Proposition 15, the branch model $\mathcal{M}^b$ does not satisfy xRy. By Proposition 16 condition 2. also the quotient model $\mathcal{M}^b_q$ does not satisfy xRy. Since $\mathcal{M}^b_q$ is a standard $\mathsf{RL}(1')$-model, so xRy is not $\mathsf{RL}(1')$-valid, a contradiction. □

From Fact 7 and Propositions 13, and 17 we obtain:

Theorem 4 (Soundness and Completeness of $\mathsf{RL}(1')$)
Let xRy be an $\mathsf{RL}(1')$-formula. Then the following conditions are equivalent:

- *xRy is $\mathsf{RL}(1')$-provable;*
- *xRy is $\mathsf{RL}(1')$-valid;*
- *xRy is $\mathsf{RL}^*(1')$-valid.*

7 Relational Logics with Constants 1′ and 1

The vocabulary of the language of $\mathsf{RL}(1,1')$ is such that:

- $\mathbb{RC}_{\mathsf{RL}(1,1')} = \{1', 1\}$.

An $\mathsf{RL}(1,1')$-*model* is a structure $\mathcal{M} = (U, m)$, where U is a non-empty set and $m: \mathbb{RA}_{\mathsf{RL}(1,1')} \cup \mathbb{OC}_{\mathsf{RL}(1,1')} \to \mathcal{P}(U \times U) \cup U$ is a meaning function such that $\mathcal{M}$ is an $\mathsf{RL}(1')$-model and $\mathcal{M}$ is an $\mathsf{RL}(1)$-model.

An $\mathsf{RLN}(1,1')$-*model* is a structure $\mathcal{M} = (U, m)$, where U is a non-empty set and $m: \mathbb{RA}_{\mathsf{RL}(1,1')} \cup \mathbb{OC}_{\mathsf{RL}(1,1')} \to \mathcal{P}(U \times U) \cup U$ is a meaning function such that

- $\mathcal{M}$ is an $\mathsf{RLN}(1)$-model;
- $m(1')$ is an equivalence relation on U;
- $m(1'); m(R) = m(R); m(1') = m(R)$ for every atomic R.

An $\mathsf{RL}(1,1')$-model (resp. $\mathsf{RLN}(1,1')$-model) $\mathcal{M} = (U, m)$ is said to be *standard* whenever $m(1')$ is the identity on U. Standard $\mathsf{RL}(1,1')$-models (resp. $\mathsf{RLN}(1,1')$-models) are referred to as $\mathsf{RL}^*(1,1')$-models (resp. $\mathsf{RLN}^*(1,1')$-models).

$\mathsf{RL}(1,1')$-system consists of $\mathsf{RL}(1')$-rules, $\mathsf{RL}(1')$-axiomatic sets, and $\mathsf{RL}(1)$-axiomatic sets adjusted to the language of $\mathsf{RL}(1,1')$-logic.

Note that in order to prove completeness we construct, as usual, the branch model. $m^b(1)$ is the universal relation in a branch model. It follows that completeness and soundness can be proved in a similar way as in $\mathsf{RL}(1')$-logic and then by using Theorems 3 and 4 we obtain the following:

Theorem 5 (Soundness and Completeness of $\mathsf{RL}(1,1')$)
For any $\mathsf{RL}(1,1')$-formula xRy the following conditions are equivalent:

- *xRy is $\mathsf{RL}(1,1')$-provable;*
- *xRy is $\mathsf{RL}(1,1')$-valid;*
- *xRy is $\mathsf{RL}^*(1,1')$-valid;*
- *xRy is $\mathsf{RLN}(1,1')$-valid;*
- *xRy is $\mathsf{RLN}^*(1,1')$-valid.*

The class of $\mathsf{RLN}(1,1')$-models is closely related to the class RRA of representable relation algebras, while the class of $\mathsf{RL}(1,1')$-models corresponds to the class FRA of full relation algebras, as it will be proved in Section 14.

8 Relational Logics with Point Relations Introduced with Axioms

In the present section and in the subsequent Section 9 we consider the logics intended for providing a means of relational reasoning in the theories which refer to objects of their domains. There are two relational formalisms for coping with individual objects. A logic $\mathsf{RL}_{ax}(C)$ presented in this section is a purely relational formalism where objects are introduced through point relations which, in turn are presented axiomatically with a well known set of axioms. The axioms say that a binary relation is a point relation whenever it is non-empty, right ideal relation with one-element domain. A binary relation R on a set U is right ideal whenever $R;1 = R$, where $1 = U \times U$. In other words such an R is of the form $X \times U$, for some $X \subseteq U$. We may think of right ideal relations as representing sets, they are sometimes referred to as vectors (see [23]). Therefore if the domain of a right ideal relation is a singleton set, the relation may be seen as a representation of an individual object. A logic $\mathsf{RL}_{df}(C)$ presented in Section 9 includes object constants in its language interpreted as singletons, and moreover, with each object constant c there is associated a relation R_c such that its meaning in every model is defined as a right ideal relation with the domain consisting of the single element being a meaning of c.

The language of the logics considered in this section includes, apart from the relational constants 1 and $1'$, a family of relational constants interpreted as point relations determined axiomatically by the conditions 1, 2, and 3 below. The vocabulary of the language of $\mathsf{RL}_{ax}(C)$-logic is such that:

– $\mathbb{RC}_{\mathsf{RL}_{ax}(C)} = \{1', 1\} \cup C$, where $C = \{R_i : i \in I\}$ for some fixed set I.

An $\mathsf{RL}_{ax}(C)$-*model* is a structure $\mathcal{M} = (U, m)$, where U is a non-empty set and $m\colon \mathbb{RA}_{\mathsf{RL}_{ax}(C)} \cup \mathbb{OC}_{\mathsf{RL}_{ax}(C)} \rightarrow \mathcal{P}(U \times U) \cup U$ is a meaning function such that $\mathcal{M}$ is an $\mathsf{RL}(1, 1')$-model and the following hold:

– for every $R_i \in C$
  1. $m(R_i) \neq \emptyset$;
  2. $m(R_i) = m(R_i); m(1)$;
  3. $m(R_i); m(R_i)^{-1} \subseteq m(1')$;
– m extends to all compound relational terms as in RL-logic.

An $\mathsf{RL}_{ax}(C)$-model $\mathcal{M} = (U, m)$ is said to be *standard* ($\mathsf{RL}^*_{ax}(C)$-model for short) whenever $m(1')$ is the identity on U.

The above conditions 1., 2., and 3. say that relations R_i are point relations. Condition 2. guarantees that R_i is a right ideal relation, and condition 3. says that in the standard models the domains of relations R_i are singleton sets.

$\mathsf{RL}_{ax}(C)$-system consists of decomposition rules and specific rules of $\mathsf{RL}(1')$-system adjusted to the $\mathsf{RL}_{ax}(C)$-language and additional specific rules of the following forms that characterize relational constants R_i:

Let $x, y \in \mathbb{OS}_{\mathsf{RL}_{ax}(C)}$ and $R_i \in C$.

$(C1)$ $\quad \dfrac{}{z - R_i t} \qquad z, t \in \mathbb{OV}_{\mathsf{RL}_{ax}(C)}$ are new

$(C2)$ $\quad \dfrac{x R_i y}{x R_i y, x R_i z} \qquad z \in \mathbb{OS}_{\mathsf{RL}_{ax}(C)}$

$(C3)$ $\quad \dfrac{x 1' y}{x R_i z, x 1' y \mid y R_i z, x 1' y} \qquad z \in \mathbb{OS}_{\mathsf{RL}_{ax}(C)}$

$\mathsf{RL}_{ax}(C)$-*axiomatic* sets are those of $\mathsf{RL}(1, 1')$ adapted to the $\mathsf{RL}_{ax}(C)$-language. As in the previous cases, the conditions of Facts 2, 4, and Proposition 1 are satisfied in $\mathsf{RL}_{ax}(C)$, that is the $\mathsf{RL}_{ax}(C)$-system satisfies the property of Proposition 11. Therefore the following can be proved easily:

Proposition 18

1. *All* $\mathsf{RL}_{ax}(C)$*-rules are* $\mathsf{RL}_{ax}(C)$*-correct.*
2. *All* $\mathsf{RL}_{ax}(C)$*-axiomatic sets are* $\mathsf{RL}_{ax}(C)$*-sets.*

It is easy to see that correctness of the rules $(C1)$, $(C2)$, and $(C3)$ follows directly from the semantic conditions 1., 2., and 3., respectively.

Due to the above proposition and Fact 3 we obtain:

Proposition 19
Let xRy *be an* $\mathsf{RL}_{ax}(C)$*-formula. If* xRy *is* $\mathsf{RL}_{ax}(C)$*-provable, then it is* $\mathsf{RL}_{ax}(C)$*-valid.*

Corollary 3
Let xRy *be an* $\mathsf{RL}_{ax}(C)$*-formula. If* xRy *is* $\mathsf{RL}_{ax}(C)$*-provable, then it is* $\mathsf{RL}^*_{ax}(C)$*-valid.*

To prove completeness of $\mathsf{RL}_{ax}(C)$-system it suffices to define the branch structure so that it will be an $\mathsf{RL}_{ax}(C)$-model and the usual property will hold: if a formula is satisfied in a branch model determined by an open branch b, then it does not belong to b.

A non-closed branch b of an $\mathsf{RL}_{ax}(C)$-proof tree is said to be $\mathsf{RL}_{ax}(C)$-*complete* whenever it satisfies $\mathsf{RL}_{ax}(C)$-completion conditions which consist of the completion conditions determined by the decomposition rules of $\mathcal{DR}_{\mathsf{RL}_{ax}(C)}$, the specific rules for $1'$, and additionally the following:

For every $R_i \in C$ and for all $x, y \in \mathbb{OS}_{\mathsf{RL}_{ax}(C)}$:

Cpl$(C1)$ There exist $z, t \in \mathbb{OV}_{\mathsf{RL}_{ax}(C)}$ such that $z{-}R_i t \in b$.

Cpl$(C2)$ If $xR_i y \in b$, then for every $z \in \mathbb{OS}_{\mathsf{RL}_{ax}(C)}$ $xR_i z \in b$.

Cpl$(C3)$ If $x1'y \in b$, then for every $z \in \mathbb{OS}_{\mathsf{RL}_{ax}(C)}$ either $xR_i z \in b$ or $yR_i z \in b$.

Let b be an open branch of an $\mathsf{RL}_{ax}(C)$-proof tree. We define a branch structure $\mathcal{M}^b = (U^b, m^b)$ with $U^b = \mathbb{OS}_{\mathsf{RL}_{ax}(C)}$ similarly as in RL-logic by adjusting it to the $\mathsf{RL}_{ax}(C)$-language.

Proposition 20
For every open branch b*, the branch structure* $\mathcal{M}^b$ *is an* $\mathsf{RL}_{ax}(C)$*-model.*

Proof
It suffices to prove that for every $R_i \in C$, (1) $m^b(R_i) \neq \emptyset$, (2) $m^b(R_i) = m^b(R_i); m^b(1)$, and (3) $m^b(R_i); m^b(R_i)^{-1} \subseteq m^b(1')$.

Proof of (1)
By the completion condition Cpl($C1$) there exist $z, t \in U^b$ such that $z{-}R_i t \in b$. Hence $zR_it \notin b$, since otherwise b would be closed. Therefore there exist $z, t \in U^b$ such that $(z, t) \in m^b(R_i)$.

Proof of (2)
Since $m^b(1) = U^b \times U^b$, so $m^b(R_i) \subseteq m^b(R_i); m^b(1)$. Assume there exists $z \in U^b$ such that $(x, z) \in m^b(R_i)$ and $(z, y) \in m^b(1)$, that is $xR_iz \notin b$ and $z1y \notin b$. Suppose $(x, y) \notin m^b(R_i)$. Then $xR_iy \in b$. By the completion condition Cpl($C2$) for every $z \in U^b$, $xR_iz \in b$, a contradiction.

The proof of (3) is similar. □

Note that $m^b(R)$ is defined for all atomic relational terms R. Therefore due to the above proposition, the proof of completeness is similar to that of $\mathsf{RL}(1')$-logic.

Proposition 21
Let xRy be an $\mathsf{RL}_{ax}(C)$-formula. If xRy is $\mathsf{RL}^*_{ax}(C)$-valid, then it is $\mathsf{RL}_{ax}(C)$-provable.

Corollary 4
Let xRy be an $\mathsf{RL}_{ax}(C)$-formula. If xRy is $\mathsf{RL}_{ax}(C)$-valid, then it is $\mathsf{RL}_{ax}(C)$-provable.

Due to Fact 1 and Propositions 19, and 21 we obtain the following:

Theorem 6 (Soundness and Completeness of $\mathsf{RL}_{ax}(C)$)
Let xRy be an $\mathsf{RL}_{ax}(C)$-formula. Then the following conditions are equivalent:

- *xRy is $\mathsf{RL}_{ax}(C)$-provable;*
- *xRy is $\mathsf{RL}_{ax}(C)$-valid;*
- *xRy is $\mathsf{RL}^*_{ax}(C)$-valid.*

9 Relational Logics with Point Relations Introduced with Definitions

The vocabulary of the language of $\mathsf{RL}_{df}(C)$-logic is such that:

- $\mathbb{OC}^0_{\mathsf{RL}_{df}(C)} \subseteq \mathbb{OC}_{\mathsf{RL}_{df}(C)}$, where $\mathbb{OC}^0_{\mathsf{RL}_{df}(C)} = \{c_i : i \in I\}$ for a fixed set I;
- $\mathbb{RC}_{\mathsf{RL}_{df}(C)} = \{1', 1\} \cup C$, where $C = \{R_i : i \in I\}$.

An $\mathsf{RL}_{df}(C)$-*model* is a structure $\mathcal{M} = (U, m)$, where U is a non-empty set and $m: \mathbb{RA}_{\mathsf{RL}_{df}(C)} \cup \mathbb{OC}_{\mathsf{RL}_{df}(C)} \to \mathcal{P}(U \times U) \cup U$ is a meaning function such that $\mathcal{M}$ is an $\mathsf{RL}(1, 1')$-model and the following holds:

- $m(R_i) = \{(x, y) \in U \times U : (x, m(c_i)) \in m(1')\}$, for every $R_i \in C$;
- m extends to all compound relational terms as in RL-models.

An $\mathsf{RL}_{df}(C)$-model $\mathcal{M} = (U, m)$ is said to be *standard* ($\mathsf{RL}^*_{df}(C)$-model for short) whenever $m(1')$ is the identity on U. In the standard models relations R_i are right ideal relations with singleton domains.

$\mathsf{RL}_{df}(C)$-system consists of decomposition rules $\mathcal{DR}_{\mathsf{RL}_{df}(C)}$ obtained from $\mathcal{DR}_{\mathsf{L}}$ by adjusting them to the $\mathsf{RL}_{df}(C)$-language, the specific rules for $1'$ of $\mathsf{RL}(1')$-system adapted to $\mathsf{RL}_{df}(C)$-language, and the specific rules that characterize relational constants R_i:

Let $x, y \in \mathbb{OS}_{\mathsf{RL}_{df}(C)}$, $c_i \in \mathbb{OC}^0_{\mathsf{RL}_{df}(C)}$ and $R_i \in C$.

$$(CD1) \quad \frac{xR_iy}{xR_iy, x1'c_i}$$

$$(CD2) \quad \frac{x{-}R_iy}{x{-}R_iy, x{-}1'c_i}$$

$\mathsf{RL}_{df}(C)$-*axiomatic* sets are those of $\mathsf{RL}(1, 1')$ adjusted to the $\mathsf{RL}_{df}(C)$-language. As in the previous cases, the $\mathsf{RL}_{df}(C)$-system satisfies the property of Proposition 11. Therefore the following holds:

Proposition 22

1. *All* $\mathsf{RL}_{df}(C)$*-rules are* $\mathsf{RL}_{df}(C)$*-correct.*
2. *All* $\mathsf{RL}_{df}(C)$*-axiomatic sets are* $\mathsf{RL}_{df}(C)$*-sets.*

Proof
It suffices to show correctness of the new specific rules. It is easy to see that correctness of the rule $(CD1)$ follows from the property: if $(x, m(c_i)) \in m(1')$, then for every $y \in U$, $(x, y) \in m(R_i)$. The correctness of the rule $(CD2)$ follows from the property: if $(x, m(c_i)) \notin m(1')$, then for every $y \in U$, $(x, y) \notin m(R_i)$.
□

Due to the above proposition and Fact 3 we obtain the following:

Proposition 23
Let xRy *be an* $\mathsf{RL}_{df}(C)$*-formula. If* xRy *is* $\mathsf{RL}_{df}(C)$*-provable, then it is* $\mathsf{RL}_{df}(C)$*-valid.*

Corollary 5
Let xRy *be an* $\mathsf{RL}_{df}(C)$*-formula. If* xRy *is* $\mathsf{RL}_{df}(C)$*-provable, then it is* $\mathsf{RL}^*_{df}(C)$*-valid.*

To prove completeness of $\mathsf{RL}_{df}(C)$-system we define as usual the branch structure satisfying the appropriate conditions.

A non-closed branch b of an $\mathsf{RL}_{df}(C)$-proof tree is said to be $\mathsf{RL}_{df}(C)$-*complete* whenever it satisfies $\mathsf{RL}_{df}(C)$-completion conditions which consist of the completion conditions determined by the decomposition rules, the completion conditions determined by the specific rules for $1'$, and additionally the following completion conditions determined by the specific rules for relational constants R_i:

For every $R_i \in C$ and for all $x, y \in \mathbb{OS}_{\mathsf{RL}_{df}(C)}$:

Cpl($CD1$) If $xR_iy \in b$, then $x1'c_i \in b$.

Cpl($CD2$) If $x{-}R_iy \in b$, then $x{-}1'c_i \in b$.

Let b be an open branch of an $\mathsf{RL}_{df}(C)$-proof tree. We define a branch structure $\mathcal{M}^b = (U^b, m^b)$ as follows:

- $U^b = \mathbb{OS}_{\mathsf{RL}_{df}(C)}$;
- $m^b(c) = c$, for every $c \in \mathbb{OC}_{\mathsf{RL}_{df}(C)}$;
- $m^b(R) = \{(x, y) \in U^b \times U^b : xRy \notin b\}$, for every $R \in \mathbb{RV}_{\mathsf{RL}_{df}(C)} \cup \{1, 1'\}$;
- $m^b(R_i) = \{x \in U^b : (x, c_i) \in m^b(1')\} \times U^b$, for every $R_i \in C$;
- m extends to all compound relational terms as in $\mathsf{RL}_{df}(C)$-models.

Fact 8
For every open branch b, $\mathcal{M}^b$ defined above is an $\mathsf{RL}_{df}(C)$-model.

Proposition 24
Let b be an open branch of an $\mathsf{RL}_{df}(C)$-proof tree and xRy be an $\mathsf{RL}_{df}(C)$-formula. Then

$$(*) \text{ if } \quad \mathcal{M}^b, v^b \models xRy, \text{ then } xRy \notin b.$$

Proof
It suffices to prove that $(*)$ holds for R being R_i or $-R_i$, where $R_i \in C$.

Let $R = R_i$ for some $R_i \in C$. Assume $(x, y) \in m^b(R_i)$, that is $(x, c_i) \in m^b(1')$. Then $x1'c_i \notin b$. Suppose $xR_iy \in b$. By the completion condition determined by the rule $(CD1)$, $x1'c_i \in b$, a contradiction.

Let $R = -R_i$ for some $R_i \in C$. Assume $(x, y) \in m^b(-R_i)$, that is $(x, c_i) \notin m^b(1')$. Then $x1'c_i \in b$. Suppose $x{-}R_iy \in b$. By the completion condition Cpl($CD2$), $x{-}1'c_i \in b$, hence b is closed a contradiction. □

Due to the above proposition, the proof of completeness is similar to that of $\mathsf{RL}(1')$-logic.

Proposition 25
Let xRy be an $\mathsf{RL}_{df}(C)$-formula. If xRy is $\mathsf{RL}^*_{df}(C)$-valid, then it is $\mathsf{RL}_{df}(C)$-provable.

Corollary 6
Let xRy be an $\mathsf{RL}_{df}(C)$-formula. If xRy is $\mathsf{RL}_{df}(C)$-valid, then it is $\mathsf{RL}_{df}(C)$-provable.

Due to Fact 1 and propositions 23, and 25 we obtain the following:

Theorem 7 (Soundness and Completeness of $\mathsf{RL}_{df}(C)$)
Let xRy be an $\mathsf{RL}_{df}(C)$-formula. Then the following conditions are equivalent:

- *xRy is $\mathsf{RL}_{df}(C)$-provable;*
- *xRy is $\mathsf{RL}_{df}(C)$-valid;*
- *xRy is $\mathsf{RL}^*_{df}(C)$-valid.*

10 Applications to Verification of Validity in Non-classical Logics

The logic $\mathsf{RL}(1,1')$ serves as a basis for the relational formalisms for non-classical logics whose Kripke-style semantics is determined by frames with binary accessibility relations. Let L be a modal logic with classical modal operators of possibility ($\langle R\rangle$) and necessity ($[R]$). The relational logic appropriate for expressing L-formulas is $\mathsf{RL_L}(1,1')$ obtained from $\mathsf{RL}(1,1')$ by expanding its language with a relational constant R representing the accessibility relation from the models of L-language and by assuming all the properties of R from these models in the $\mathsf{RL_L}(1,1')$-models. For example, if a relation R in a modal frame of a logic L is assumed to satisfy some conditions, e.g., reflexivity (logic T), symmetry (logic B), transitivity (logic $\mathsf{S4}$) etc., then in the models of the corresponding logic $\mathsf{RL_L}(1,1')$ we add the respective conditions as the axioms of its models. The translation of a modal formula into a relational term starts with an assignment of relational variables to the propositional variables of the formula. Let τ' be such an assignment. Then the translation τ of the modal formulas is defined inductively as follows:

- $\tau(p) := \tau'(p); 1$ for propositional variable p;
- $\tau(\neg\alpha) := -\tau(\alpha)$;
- $\tau(\alpha \vee \beta) := \tau(\alpha) \cup \tau(\beta)$;
- $\tau(\alpha \wedge \beta) := \tau(\alpha) \cap \tau(\beta)$;
- $\tau(\langle R\rangle\alpha) := R; \tau(\alpha)$;
- $\tau([R]\alpha) := -(R; -\tau(\alpha))$.

The translation is defined so that it preserves validity of formulas.

Proposition 26
For every L-formula φ and for every L-model $\mathcal{M}$ there exists $\mathsf{RL}^*_\mathsf{L}(1,1')$-model $\mathcal{M}'$ such that

$$\mathcal{M} \models \varphi \quad \textit{iff} \quad \mathcal{M}' \models x\tau(\varphi)y$$

where x and y are object variables such that $x \neq y$.

Proof
Let φ be an L-formula and let $\mathcal{M} = (U, m)$ be an L-model. We define the corresponding $\mathsf{RL}^*_\mathsf{L}(1,1')$-model $\mathcal{M}' = (U', m')$ as follows:

- $U' = U$;
- $m'(1) = U' \times U'$;
- $m'(1')$ is an identity on U';
- $m'(\tau'(p)) = \{(x,y) \in U' \times U' : x \in m(p)\}$, for any propositional variable p;
- $m'(R) = m(R)$;
- m' extends to all compound relational terms as in $\mathsf{RL}(1,1')$-models.

Given a valuation $v\colon \mathbb{OS}_{\mathsf{RL_L}(1,1')} \to U$ we show by induction on the complexity of φ that the following property holds:

$$\mathcal{M}, v(x) \models \varphi \quad \text{iff} \quad \mathcal{M}', v \models x\tau(\varphi)y.$$

From that, we can conclude that φ is true in $\mathcal{M}$ iff $x\tau(\varphi)y$ is true in $\mathcal{M}'$. By way of example we prove the required condition for the formulas of the form: $\psi_1 \vee \psi_2$ and $\langle R\rangle\psi$.

- If $\varphi = \psi_1 \vee \psi_2$ then $\mathcal{M}, v(x) \models \psi_1 \vee \psi_2$ iff $\mathcal{M}, v(x) \models \psi_1$ or $\mathcal{M}, v(x) \models \psi_2$, iff, by inductive hypothesis, $\mathcal{M}', v \models x\tau(\psi_1)y$ or $\mathcal{M}', v \models x\tau(\psi_2)y$, iff $\mathcal{M}', v \models x(\tau(\psi_1) \cup \tau(\psi_2))y$ iff $\mathcal{M}', v \models x\tau(\psi_1 \cup \psi_2)y$.
- If $\varphi = \langle R\rangle\psi$ then $\mathcal{M}, v(x) \models \langle R\rangle\psi$ iff there exists $s \in U$ such that $(v(x), s) \in m(R)$ and $\mathcal{M}, s \models \psi$ iff, by inductive hypothesis, there exists $s \in U'$ such that $(v(x), s) \in m'(R)$ and $(s, v(y)) \in m'(\tau(\psi))$, iff $(v(x), v(y)) \in m'(R; \tau(\psi))$ iff $\mathcal{M}', v(x) \models x\tau(\langle R\rangle\psi)y$. □

Proposition 27
For every L*-formula* φ *and for every* $\mathsf{RL}^*_{\mathsf{L}}(1,1')$*-model* $\mathcal{M}'$ *there exists* L*-model* $\mathcal{M}$ *such that*

$$\mathcal{M} \models \varphi \quad \textit{iff} \quad \mathcal{M}' \models x\tau(\varphi)y$$

where x *and* y *are object variables such that* $x \neq y$.

Proof
Let φ be an L-formula and let $\mathcal{M}' = (U', m')$ be an $\mathsf{RL}^*_{\mathsf{L}}(1,1')$-model. We define the corresponding L-model $\mathcal{M} = (U, m)$ as follows:

- $U = U'$;
- for every propositional variable p, $s \in m(p)$ iff $(s, s') \in m'(\tau'(p))$ for some $s' \in U'$;
- $m(R) = m'(R)$.

The rest of the proof is similar to the proof of Proposition 26. □

From Theorem 5 and Propositions 26, and 27 we obtain the following:

Theorem 8
For every formula φ *of a logic* L, φ *is valid in* L *iff* $xt(\varphi)y$ *is valid in* $\mathsf{RL}_{\mathsf{L}}(1,1')$, *where* x *and* y *are object variables such that* $x \neq y$.

Once a translation from a non-classical logic L into an appropriate relational logic $\mathsf{RL}_{\mathsf{L}}(1,1')$ is defined, we develop a dual tableau proof system for $\mathsf{RL}_{\mathsf{L}}(1,1')$ which by the above theorem is a validity checker for L. The core of such a system is the $\mathsf{RL}(1,1')$-system. For each particular logic L the rules and/or axiomatic sets must be added reflecting the properties of the constant R. Defining these rules we follow the general principles presented in [14].

For example, a relational formalism for the modal logic K is the logic $\mathsf{RL}_{\mathsf{K}}(1,1')$ obtained from $\mathsf{RL}(1,1')$ by assuming that the set of relational constants includes additionally a relational constant, say R, representing the accessibility relation from the frames of K. Since in the K-models there is no any specific assumption about R, $\mathsf{RL}_{\mathsf{K}}(1,1')$-proof system can be obtained from that of $\mathsf{RL}(1,1')$ by adjusting it to the $\mathsf{RL}_{\mathsf{K}}(1,1')$-language, in particular by postulating $\mathbb{RC}_{\mathsf{RL}_{\mathsf{K}}(1,1')} = \{1', 1, R\}$.

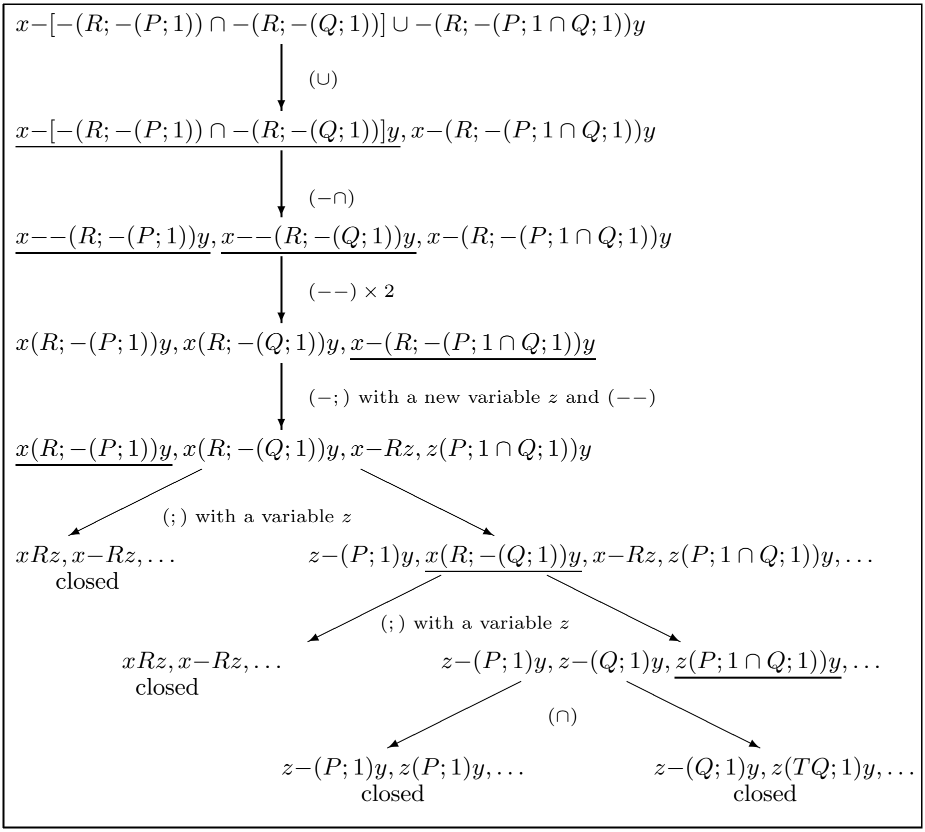

Fig. 1.

Let us consider the following formula φ of modal logic K:

$$\neg([R]p \wedge [R]q) \vee [R](p \wedge q)$$

Let $\tau'(p) = P$ and let $\tau'(q) = Q$. The translation $\tau(\varphi)$ of the above formula into a relational term of $\mathsf{RL}_{\mathsf{K}}(1, 1')$ is:

$$-[-(R; -(P; 1)) \cap -(R; -(Q; 1))] \cup -(R; -(P; 1 \cap Q; 1))$$

We show that the formula φ is K-valid, that is $x\tau(\varphi)y$ is $\mathsf{RL}_{\mathsf{K}}(1, 1')$-valid. In each node of the proof tree we underline a formula which determines an applicable rule. Figure 1 presents a closed $\mathsf{RL}_{\mathsf{K}}(1, 1')$-proof tree for the formula $x\tau(\varphi)y$.

The method of relational formalization of non-classical logics is applicable to a great variety of logics, see e.g., [1], [10], [11], [15], [16] and [7].

11 Applications to Verification of Entailment in Non-classical Logics

The logic $\mathsf{RL}(1,1')$ can be also used to verify the entailment of formulas of non-classical logics, provided that they can be translated into a relational logic. The method is based on the following fact. Let $R_1,\ldots,R_n,R$ be binary relations on a set U and let $1=U\times U$. It is known that $R_1=1,\ldots,R_n=1$ imply $R=1$ iff $(1;-(R_1\cap\ldots\cap R_n);1)\cup R=1$. It follows that for every $\mathsf{RL}(1,1')$-model $\mathcal{M}$, $\mathcal{M}\models xR_1y$, ..., $\mathcal{M}\models xR_ny$ imply $\mathcal{M}\models xRy$ iff $\mathcal{M}\models x(1;-(R_1\cap\ldots\cap R_n);1)\cup R)y$ which means that entailment in $\mathsf{RL}(1,1')$ can be expressed in its language.

For example, in K-logic the formulas $[R]p$ and $[R](p\to q)$ imply $[R]q$. That is in $\mathsf{RL_K}(1,1')$-logic, relations $-(R;-(P;1))$ and $-(R;-(-(P;1)\cup(Q;1)))$ imply $-(R;-(Q;1))$. To prove this we need to show that the formula

$$x[(1;-(-(R;-(P;1))\cap-(R;-(-(P;1)\cup(Q;1))));1)\cup-(R;-(Q;1))]y$$

is $\mathsf{RL_K}(1,1')$-provable. Figure 2 presents a closed $\mathsf{RL_K}(1,1')$-proof tree for this formula.

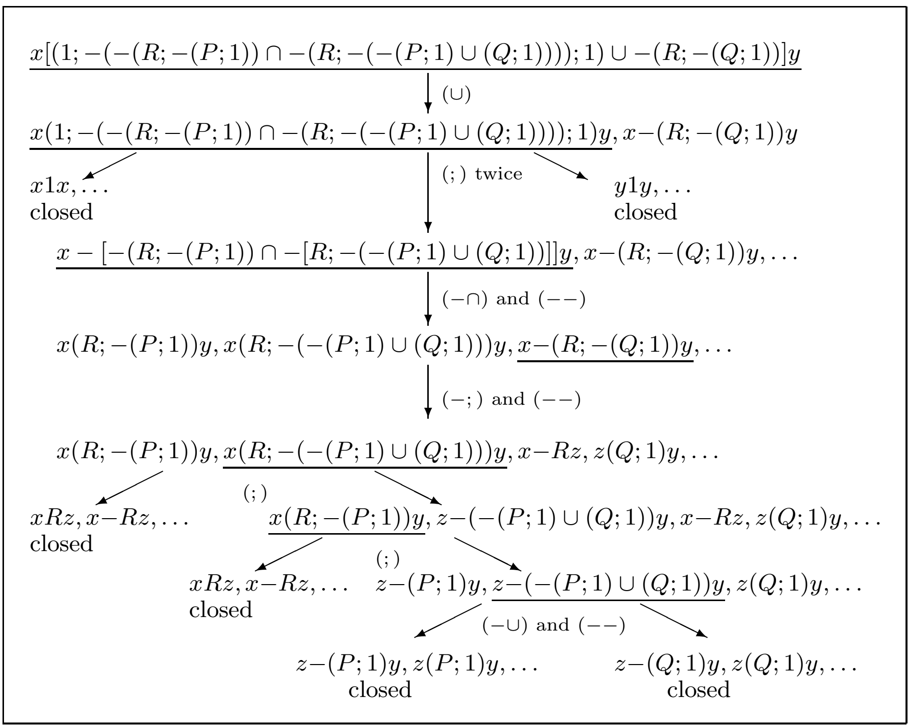

Fig. 2.

12 Applications to Model Checking in Non-classical Logics

The logic $\mathsf{RL}(1,1')$ is used in the formalisms of relational logics whose model checking problem is in question. Let $\mathcal{M} = (U, m)$ be a fixed $\mathsf{RL}^*(1,1')$-model and let $\varphi = xRy$ be an $\mathsf{RL}(1,1')$-formula, where R is a relational term and x, y are any object symbols. In order to obtain the relational formalism for the problem '$\mathcal{M} \models \varphi$?', we consider an instance $\mathsf{RL}_{\mathcal{M},\varphi}$ of the logic $\mathsf{RL}(1,1')$. Its language provides a code of model $\mathcal{M}$ and formula φ, and in its models the syntactic elements of φ are interpreted as in the model $\mathcal{M}$. The vocabulary of the logic $\mathsf{RL}_{\mathcal{M},\varphi}$ consists of the following pairwise disjoint sets:

- $\mathbb{OV}_{\mathsf{RL}_{\mathcal{M},\varphi}}$ a countable infinite set of object variables;
- $\mathbb{OC}_{\mathsf{RL}_{\mathcal{M},\varphi}} = \mathbb{OC}^0_{\mathsf{RL}_{\mathcal{M},\varphi}} \cup \mathbb{OC}^1_{\mathsf{RL}_{\mathcal{M},\varphi}}$, where $\mathbb{OC}^0_{\mathsf{RL}_{\mathcal{M},\varphi}} = \{c_{\mathsf{a}} : \mathsf{a} \in U\}$ and $\mathbb{OC}^1_{\mathsf{RL}_{\mathcal{M},\varphi}} = \{c \in \mathbb{OC}_{\mathsf{RL}(1,1')} : c \text{ occurs in } \varphi\}$;
- $\mathbb{RC}_{\mathsf{RL}_{\mathcal{M},\varphi}} = \{S : S \text{ is an atomic subterm of } R\} \cup \{1, 1'\}$;
- $\mathbb{OP}_{\mathsf{RL}_{\mathcal{M},\varphi}} = \{-, \cup, \cap, ;, ^{-1}\}$;
- a set of parentheses $\{(,)\}$.

Note that the language of $\mathsf{RL}_{\mathcal{M},\varphi}$ does not contain relational variables.

An $\mathsf{RL}_{\mathcal{M},\varphi}$-model is a pair $\mathcal{N} = (W, n)$ where

- $W = U$;
- $n(c) = m(c)$, for every $c \in \mathbb{OC}^1_{\mathsf{RL}_{\mathcal{M},\varphi}}$;
- $n(c_{\mathsf{a}}) = \mathsf{a}$, for any $c_{\mathsf{a}} \in \mathbb{OC}^0_{\mathsf{RL}_{\mathcal{M},\varphi}}$;
- $n(S) = m(S)$, for any atomic subterm S of R;
- $n(1)$, $n(1')$ are defined as in $\mathsf{RL}^*(1,1')$-models;
- n extends to compound terms as in $\mathsf{RL}^*(1,1')$-models.

Observe that the above definition implies: for every atomic subterm S of R, $\mathcal{N}, v \models xSy$ iff there exist $a, b \in U$ such that $(a, b) \in m(S)$ and $v(x) = a$ and $v(y) = b$. Moreover, it is easy to prove that $n(R) = m(R)$. Note also that the class of $\mathsf{RL}_{\mathcal{M},\varphi}$-models has exactly one element up to isomorphism. Therefore, $\mathsf{RL}_{\mathcal{M},\varphi}$-validity is equivalent to the truth in a single $\mathsf{RL}_{\mathcal{M},\varphi}$-model $\mathcal{N}$, that is the following holds:

Proposition 28
The following statements are equivalent:

- $\mathcal{M} \models xRy$
- xRy *is* $\mathsf{RL}_{\mathcal{M},\varphi}$*-valid*

The relational proof system for $\mathsf{RL}_{\mathcal{M},\varphi}$ consists of the rules and axiomatic sets of $\mathsf{RL}(1,1')$-system adapted to the language of $\mathsf{RL}_{\mathcal{M},\varphi}$, and additionally:

– for every atomic subterm S of R and for any $x, y \in \mathbb{OS}_{\mathsf{RL}_{\mathcal{M},\varphi}}$ we add the rules of the following form:

$$(-S) \quad \frac{x{-}Sy}{x{-}1'c_{\mathsf{a}}, y{-}1'c_{\mathsf{b}}, c_{\mathsf{a}}{-}Sc_{\mathsf{b}}, x{-}Sy} \qquad c_{\mathsf{a}}, c_{\mathsf{b}} \in \mathbb{OC}^0_{\mathsf{RL}_{\mathcal{M},\varphi}} \text{ are new}$$

$$(1') \quad \frac{}{x{-}1'c_{\mathsf{a}}} \qquad c_{\mathsf{a}} \text{ is new}$$

$$(\mathsf{a} \neq \mathsf{b}) \quad \frac{}{c_{\mathsf{a}}1'c_{\mathsf{b}}} \qquad \text{for all } \mathsf{a} \neq \mathsf{b}$$

where $c_{\mathsf{a}} \in \mathbb{OC}^0_{\mathsf{RL}_{\mathcal{M},\varphi}}$ is *new* whenever it appears in a conclusion of the rule and does not appear in its premise;

– for every $c \in \mathbb{OC}^1_{\mathsf{RL}_{\mathcal{M},\varphi}}$ and for every $\mathsf{a} \in U$ such that $m(c) \neq \mathsf{a}$ we add the rules of the following form:

$$(c\mathsf{a}) \quad \frac{}{c1'c_{\mathsf{a}}}$$

– a set of formulas is assumed to be an axiomatic set whenever it includes either of the following subsets:
  - $\{c1'c_{\mathsf{a}}\}$, for every $c \in \mathbb{OC}^1_{\mathsf{RL}_{\mathcal{M},\varphi}}$ and for every $\mathsf{a} \in U$ such that $m(c) = \mathsf{a}$;
  - $\{c_{\mathsf{a}}Sc_{\mathsf{b}}\}$, for every atomic subterm S of R and for all $\mathsf{a}, \mathsf{b} \in U$ such that $(\mathsf{a}, \mathsf{b}) \in m(S)$;
  - $\{c_{\mathsf{a}}{-}Sc_{\mathsf{b}}\}$, for every atomic subterm S of R and for all $\mathsf{a}, \mathsf{b} \in U$ such that $(\mathsf{a}, \mathsf{b}) \notin m(S)$.

The correctness of all new rules and the validity of all new axiomatic sets follow directly from the definition of $\mathsf{RL}_{\mathcal{M},\varphi}$-semantics. For example, the correctness of the rule $(-S)$ follows from the following property of $n(S)$: $(v(x), v(y)) \in n(S)$ iff for all $\mathsf{a}, \mathsf{b} \in U$, either $(n(c_{\mathsf{a}}), n(c_{\mathsf{b}})) \notin n(S)$ or $v(x) \neq c_{\mathsf{a}}$ or $v(y) \neq c_{\mathsf{b}}$. Note that for every $x \in \mathbb{OS}_{\mathsf{RL}_{\mathcal{M},\varphi}}$ and for every valuation v in $\mathcal{N}$, there exists $c_{\mathsf{a}} \in \mathbb{OC}^0_{\mathsf{RL}_{\mathcal{M},\varphi}}$ such that the model $\mathcal{N}$ satisfies $v(x) = n(c_{\mathsf{a}})$, hence the rule $(1')$ is correct. The correctness of the rule $(\mathsf{a} \neq \mathsf{b})$ follows form the following property of $\mathcal{N}$-models: for all $\mathsf{a}, \mathsf{b} \in U$, if $\mathsf{a} \neq \mathsf{b}$, then $n(c_{\mathsf{a}}) \neq n(c_{\mathsf{b}})$.

The completion conditions are those of $\mathsf{RL}(1, 1')$-system adapted to the language of $\mathsf{RL}_{\mathcal{M},R}$ and additionally for every atomic subterm S of R we add the following conditions:

Cpl$(-S)$ If $x{-}Sy \in b$, then for some $c_{\mathsf{a}}, c_{\mathsf{b}} \in \mathbb{OC}^0_{\mathsf{RL}_{\mathcal{M},\varphi}}$ all of the following conditions are satisfied: $x{-}1'c_{\mathsf{a}} \in b$, $y{-}1'c_{\mathsf{b}} \in b$ and $c_{\mathsf{a}}{-}Sc_{\mathsf{b}} \in b$.

Cpl$(1')$ For every $x \in \mathbb{OV}_{\mathsf{RL}_{\mathcal{M},\varphi}}$ there exists $c_{\mathsf{a}} \in \mathbb{OC}^0_{\mathsf{RL}_{\mathcal{M},\varphi}}$ such that $x{-}1'c_{\mathsf{a}} \in b$.

Cpl$(\mathsf{a} \neq \mathsf{b})$ For all $\mathsf{a}, \mathsf{b} \in U$ such that $\mathsf{a} \neq \mathsf{b}$, $c_{\mathsf{a}}1'c_{\mathsf{b}} \in b$.

Cpl$(c\mathsf{a})$ For every $c \in \mathbb{OC}^1_{\mathsf{RL}_{\mathcal{M},\varphi}}$ and for every $\mathsf{a} \in U$ such that $n(c) \neq \mathsf{a}$, $c1'c_{\mathsf{a}} \in b$.

A branch model is a structure $\mathcal{N}^b = (W^b, n^b)$ satisfying the following conditions:

– $W^b = \mathbb{OS}_{\mathsf{RL}_{\mathcal{M},\varphi}}$;
– $n^b(c) = c$, for every $c \in \mathbb{OC}_{\mathsf{RL}_{\mathcal{M},\varphi}}$;

- $n^b(S) = \{(x, y) \in W^b \times W^b : xSy \notin b\}$, for $S \in \{1, 1'\}$;
- $n^b(S) = \{(x, y) \in W^b \times W^b :$ there exists $\mathsf{a}, \mathsf{b} \in U$ such that $\gamma(\mathsf{a}, \mathsf{b}, x, y)\}$, where $\gamma(\mathsf{a}, \mathsf{b}, x, y)$ is $[(\mathsf{a}, \mathsf{b}) \in m(S) \wedge (x, c_\mathsf{a}) \in n^b(1') \wedge (y, c_\mathsf{b}) \in n^b(1')]$;
- n^b extends to all compound terms as in $\mathsf{RL}(1, 1')$-models.

As in $\mathsf{RL}(1, 1')$-logic it is easy to prove that $n^b(1')$ and $n^b(1)$ are an equivalence relation and a universal relation, respectively.

Let $v^b\colon \mathbb{OS}_{\mathsf{RL}_{\mathcal{M},\varphi}} \to W^b$ be a valuation in $\mathcal{N}^b$ such that $v^b(x) = x$ for every $x \in \mathbb{OS}_{\mathsf{RL}_{\mathcal{M},\varphi}}$. Then the following holds:

Proposition 29
For every open branch b of an $\mathsf{RL}_{\mathcal{M},\varphi}$-proof tree, and for every $\mathsf{RL}_{\mathcal{M},\varphi}$-formula xRy:

$$(*) \quad \text{if} \quad \mathcal{N}^b, v^b \models xRy, \quad \text{then} \quad xRy \notin b.$$

Proof
The proof is similar to the proof of analogous proposition for $\mathsf{RL}(1, 1')$-logic. That is we need to show that $(*)$ holds for every atomic subterm S of R and its complement.

Let $\varphi = xSy$ for some atomic subterm S of R. Assume $\mathcal{N}^b, v^b \models xSy$. By the definition of $n^b(S)$ there exist $\mathsf{a}, \mathsf{b} \in U$ such that $(\mathsf{a}, \mathsf{b}) \in m(S)$, $x1'c_\mathsf{a} \notin b$ and $y1'c_\mathsf{b} \notin b$. Since $(\mathsf{a}, \mathsf{b}) \in m(S)$, $c_\mathsf{a}Sc_{c_\mathsf{b}} \notin b$, otherwise b would be closed. Therefore the following holds: $c_\mathsf{a}Sc_\mathsf{b} \notin b$, $x1'c_\mathsf{a} \notin b$ and $y1'c_\mathsf{b} \notin b$. Suppose $xSy \in b$. By the completion conditions for the rules $(1'1)$ and $(1'2)$, for all $c_\mathsf{a}, c_\mathsf{b} \in \mathbb{OC}^0_{\mathsf{RL}_{\mathcal{M},\varphi}}$, at least one the following holds: $x1'c_\mathsf{a} \in b$ or $y1'c_\mathsf{b} \in b$ or $c_\mathsf{a}Sc_{c_\mathsf{b}} \in b$, a contradiction.

Let $\varphi = x{-}Sy$, for some atomic subterm S of R. Assume $\mathcal{N}^b, v^b \models x{-}Sy$. Then for all $\mathsf{a}, \mathsf{b} \in U$, $(\mathsf{a}, \mathsf{b}) \notin m(S)$ or $x1'c_\mathsf{a} \in b$ or $y1'c_\mathsf{b} \in b$. Since $(\mathsf{a}, \mathsf{b}) \notin m(S)$, $c_\mathsf{a}{-}Sc_{c_\mathsf{b}} \notin b$, otherwise b would be closed. Therefore for all $\mathsf{a}, \mathsf{b} \in U$, the following holds: if $c_\mathsf{a}{-}Sc_\mathsf{b} \in b$, then $x1'c_\mathsf{a} \in b$ or $y1'c_\mathsf{b} \in b$. Suppose $x{-}Sy \in b$. By the completion condition for the rule $({-}S)$, for some $c_\mathsf{a}, c_\mathsf{b} \in \mathbb{OC}^0_{\mathsf{RL}_{\mathcal{M},\varphi}}$, the following holds: $x{-}1'c_\mathsf{a} \in b$ and $y{-}1'c_\mathsf{b} \in b$ and $c_\mathsf{a}{-}Sc_{c_\mathsf{b}} \in b$, a contradiction. □

Since $n^b(1')$ is an equivalence relation on W^b, we may define the quotient model $\mathcal{N}^b_q = (W^b_q, n^b_q)$ as follows:

- $W^b_q = \{\|x\| : x \in W^b\}$, where $\|x\|$ is the equivalence class of $n^b(1')$ generated by x;
- $n^b_q(c) = \|n^b(c)\|$, for every $c \in \mathbb{OC}_{\mathsf{RL}_{\mathcal{M},\varphi}}$;
- $n^b_q(S) = \{(\|x\|, \|y\|) \in W^b_q \times W^b_q : (x, y) \in n^b(S)\}$, for every atomic S;
- n^b_q extends as in $\mathsf{RL}(1, 1')$-models.

Proposition 30
The quotient model $\mathcal{N}^b_q = (W^b_q, n^b_q)$ satisfies the following conditions:

1. $card(W^b_q) = card(W)$;
2. $c \in \|c_\mathsf{a}\|$ *iff* $n(c) = \mathsf{a}$
3. $n^b_q(S) = \{(\|c_\mathsf{a}\|, \|c_\mathsf{b}\| \in W^b_q \times W^b_q : (n(c_\mathsf{a}), n(c_\mathsf{b})) \in n(S)\}$.

Proof

Proof of 1. For all $\mathsf{a}, \mathsf{b} \in U$, if $\mathsf{a} \neq \mathsf{b}$, then $c_{\mathsf{a}} 1' c_{\mathsf{b}} \in b$. Therefore for all $\mathsf{a}, \mathsf{b} \in U$ such that $\mathsf{a} \neq \mathsf{b}$, $(c_{\mathsf{a}}, c_{\mathsf{b}}) \notin n^b(1')$, hence $card(W^b_q) \geq card(W)$. By the completion condition for $(1')$, for every $x \in W^b$ there is $c_{\mathsf{a}} \in W^b$ such that $x{-}1' c_{\mathsf{a}} \in b$. Therefore for every element x of W^b, $x \in \|c_{\mathsf{a}}\|$ for some $a \in U$. Hence $card(W^b_q) \leq card(W)$.

Proof of 2. For $c \in \mathbb{OC}^0_{\mathsf{RL}_{\mathcal{M},\varphi}}$ the proof is obvious. Let $c \in \mathbb{OC}^1_{\mathsf{RL}_{\mathcal{M},\varphi}}$. If $n(c) = \mathsf{a}$, then $c1'c_{\mathsf{a}} \notin b$, since otherwise b would be closed. Therefore $c \in \|c_{\mathsf{a}}\|$. If $n(c) \neq \mathsf{a}$, then by the completion condition for $(c\mathsf{a})$, $c1'c_{\mathsf{a}} \in b$, hence $c \notin \|c_{\mathsf{a}}\|$.

Proof of 3. This follows directly from the definition of $n^b(S)$. □

The above proposition implies that the function $f\colon W^b_q \to W$ defined as $f(\|c_a\|) = a$ is an isomorphism between $\mathcal{N}^b_q$ and $\mathcal{N}$. Therefore $\mathcal{N}^b_q$ and $\mathcal{N}$ satisfy exactly the same formulas. Now the completeness $\mathsf{RL}_{\mathcal{M},\varphi}$ can be proved similarly as in $\mathsf{RL}(1,1')$-logic.

Theorem 9 (Soundness and completeness of $\mathsf{RL}_{\mathcal{M},\varphi}$)
For every $\mathsf{RL}_{\mathcal{M},R}$-formula xRy the following conditions are equivalent:

- *xRy is $\mathsf{RL}_{\mathcal{M},\varphi}$-provable.*
- *xRy is $\mathsf{RL}_{\mathcal{M},\varphi}$-valid.*

Due to the above theorem and Proposition 28 we obtain the following:

Theorem 10
The following statements are equivalent:

- *$\mathcal{M} \models xRy$,*
- *xRy is $\mathsf{RL}_{\mathcal{M},\varphi}$-provable.*

The method presented above can be also used in the case of non-classical logics for which the problem of model checking is in question. By way of example consider the modal logic K. Let $\mathcal{M} = (U, m)$ be a K-model such that $U = \{\mathsf{a}, \mathsf{b}\}$, $m(p) = \{\mathsf{a}\}$ and the accessibility relation is defined as $m(R) = \{(\mathsf{a}, \mathsf{a}), (\mathsf{b}, \mathsf{a})\}$. Let φ be the formula of the form $\langle R \rangle p$. Let us consider the problem: 'is φ true in $\mathcal{M}$?' The translation of the formula φ is $\tau(\varphi) = (R; (P; 1))$, where $\tau'(p) = P$. Using the construction from the proof of Proposition 26 it is easy to prove that there exist an $\mathsf{RL}_{\mathsf{K}}(1,1')$-model $\mathcal{M}'$ such that the following holds:

$$\mathcal{M} \models \varphi \text{ iff } \mathcal{M}' \models x\tau(\varphi)y.$$

The $\mathsf{RL}_{\mathsf{K}}(1,1')$-model $\mathcal{M}' = (U', m')$ is defined as follows:

- $U' = m'(1) = \{\mathsf{a}, \mathsf{b}\}$;
- $m'(P) = \{(\mathsf{a}, \mathsf{a}), (\mathsf{a}, \mathsf{b})\}$;
- $m'(R) = \{(\mathsf{a}, \mathsf{a}), (\mathsf{b}, \mathsf{a})\}$;

– $m'(1') = \{(\mathtt{a}, \mathtt{a}), (\mathtt{b}, \mathtt{b})\}$;
– m' extends to all compound terms as in $\mathsf{RL}(1, 1')$-models.

Therefore the model checking problem 'is φ true in $\mathcal{M}$?' is equivalent to the problem 'is a formula $x\tau(\varphi)y$ true in $\mathcal{M}'$?'. For the latter we apply the method already presented above. The vocabulary of $\mathsf{RL}_{\mathcal{M}',x\tau(\varphi)y}$-language adequate for testing whether $\mathcal{M}' \models x\tau(\varphi)y$ consists of the following sets of symbols:

– $\mathbb{OV}_{\mathsf{RL}_{\mathcal{M}',x\tau(\varphi)y}}$ a countable infinite set of object variables;
– $\mathbb{OC}_{\mathsf{RL}_{\mathcal{M}',x\tau(\varphi)y}} = \{c_\mathtt{a}, c_\mathtt{b}\}$;
– $\mathbb{RC}_{\mathsf{RL}_{\mathcal{M}',x\tau(\varphi)y}} = \{R, P, 1, 1'\}$;
– $\mathbb{OP}_{\mathsf{RL}_{\mathcal{M}',x\tau(\varphi)y}} = \{-, \cup, \cap, ;, ^{-1}\}$;
– a set of parentheses $\{(,)\}$.

An $\mathsf{RL}_{\mathcal{M}',x\tau(\varphi)y}$-model is the structure $\mathcal{N} = (W, n)$ defined as $\mathsf{RL}_\mathsf{K}(1, 1')$-models with the following additional condition $n(c_\mathtt{a}) = \mathtt{a}$, $n(c_\mathtt{b}) = \mathtt{b}$.

The additional rules of $\mathsf{RL}_{\mathcal{M}',x\tau(\varphi)y}$-system are: $(-R)$, $(-P)$, $(\mathtt{a} \neq \mathtt{b})$ and $(1')$. Additional $\mathsf{RL}_{\mathcal{M}',x\tau(\varphi)y}$-axiomatic sets are those which include one of the following sets: $\{c_\mathtt{a}Rc_\mathtt{a}\}$, $\{c_\mathtt{b}Rc_\mathtt{a}\}$, $\{c_\mathtt{b}{-}Rc_\mathtt{b}\}$, $\{c_\mathtt{a}{-}Rc_\mathtt{b}\}$, $\{c_\mathtt{a}Pc_\mathtt{a}\}$, $\{c_\mathtt{a}Pc_\mathtt{b}\}$, $\{c_\mathtt{b}{-}Pc_\mathtt{b}\}$ or $\{c_\mathtt{b}{-}Pc_\mathtt{a}\}$

By Theorem 10, truth of φ in $\mathcal{M}$ is equivalent to $\mathsf{RL}_{\mathcal{M}',x\tau(\varphi)y}$-provability of φ. Figure 3 presents a closed $\mathsf{RL}_{\mathcal{M}',x\tau(\varphi)y}$-proof tree for $x\tau(\varphi)y$.

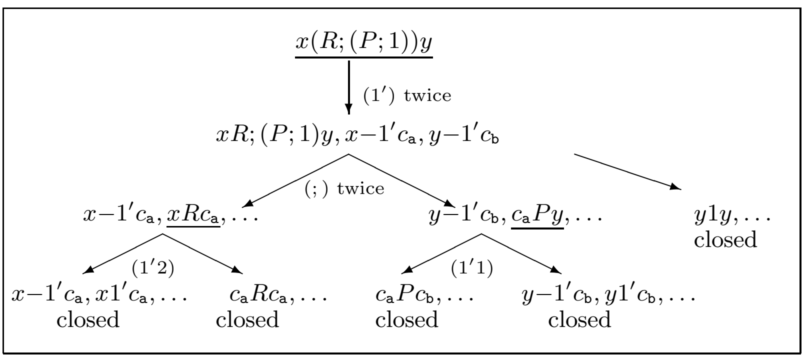

Fig. 3.

13 Applications to Verification of Satisfaction in Non-classical Logics

The logic $\mathsf{RL}_{df}(C)$ is used in the formalisms of relational logics whose problem of verification of satisfaction in a model is in question. Let $\mathcal{M} = (U, m)$ be a fixed $\mathsf{RL}^*(1, 1')$-model, let $\mathtt{a}, \mathtt{b}$ be elements of U and let $\varphi = xRy$ be an $\mathsf{RL}(1, 1')$-formula, where R is a relational term and x, y are any object symbols. In order to obtain the relational formalism for the problem '$(a, b) \in m(R)$?', we consider an instance $\mathsf{RL}_{\mathcal{M},\varphi,\mathsf{a},\mathsf{b}}$ of the logic $\mathsf{RL}_{df}(C)$. The language, the models, and the system of $\mathsf{RL}_{\mathcal{M},\varphi,\mathsf{a},\mathsf{b}}$-logic are constructed in a similar way as in the model checking problem. The vocabulary of the logic $\mathsf{RL}_{\mathcal{M},\varphi,\mathsf{a},\mathsf{b}}$ is defined as in $\mathsf{RL}_{\mathcal{M},\varphi}$-logic with additional set of relational constants:

$C = \{R_c : c \in \mathbb{OC}^0_{\mathsf{RL}_{\mathcal{M},\varphi,\mathsf{a},\mathsf{b}}}\}$.

An $\mathsf{RL}_{\mathcal{M},\varphi,\mathsf{a},\mathsf{b}}$-models are defined as $\mathsf{RL}_{\mathcal{M},\varphi}$-models with the following additional condition:

$n(R_c) = \{n(c)\} \times W$, for every $c \in \mathbb{OC}^0_{\mathsf{RL}_{\mathcal{M},\varphi,\mathsf{a},\mathsf{b}}}$.

As in the case of $\mathsf{RL}_{\mathcal{M},\varphi}$-models, the class of $\mathsf{RL}_{\mathcal{M},\varphi,\mathsf{a},\mathsf{b}}$-models has exactly one element up to isomorphism. Therefore, $\mathsf{RL}_{\mathcal{M},\varphi,\mathsf{a},\mathsf{b}}$-validity is equivalent to the truth in a single $\mathsf{RL}_{\mathcal{M},\varphi,\mathsf{a},\mathsf{b}}$-model $\mathcal{N}$.

Proposition 31
The following statements are equivalent:

- $(\mathsf{a}, \mathsf{b}) \in m(R)$;
- $x[-(R_{c_\mathsf{a}}; R_{c_\mathsf{b}}^{-1}) \cup R]y$ *is* $\mathsf{RL}_{\mathcal{M},\varphi,\mathsf{a},\mathsf{b}}$*-valid.*

Proof
Note that $\mathsf{RL}_{\mathcal{M},\varphi,\mathsf{a},\mathsf{b}}$-validity of $x[-(R_{c_\mathsf{a}}; R_{c_\mathsf{b}}^{-1}) \cup R]y$ is equivalent to the following property: for every $x, y \in W$, if $(x, y) \in n(R_{c_\mathsf{a}}; R_{c_\mathsf{b}}^{-1})$, then $(x, y) \in n(R)$.

($\rightarrow$) Let $(\mathsf{a}, \mathsf{b}) \in m(R)$. Assume $x, y \in W$ and $(x, y) \in n(R_{c_\mathsf{a}}; R_{c_\mathsf{b}}^{-1})$. Then by the semantics of R_{c_a} and R_{c_b}, $x = \mathsf{a}$ and $y = \mathsf{b}$. Since $n(R) = m(R)$, $(x, y) \in n(R)$.

($\leftarrow$) Assume $(\mathsf{a}, \mathsf{b}) \notin m(R)$. We need to show that there exist $x, y \in W$ such that $(x, y) \in n(R_{c_\mathsf{a}}; R_{c_\mathsf{b}}^{-1})$ but $(x, y) \notin n(R)$. Let $x = \mathsf{a}$ and $y = \mathsf{b}$. Then $(x, y) \in n(R_{c_\mathsf{a}}; R_{c_\mathsf{a}}^{-1})$. Since $n(R) = m(R)$, $(x, y) \notin n(R)$. □

$\mathsf{RL}_{\mathcal{M},\varphi,\mathsf{a},\mathsf{b}}$-proof system consists of the rules and axiomatic sets of the systems of $\mathsf{RL}_{\mathcal{M},\varphi}$-logic and $\mathsf{RL}_{df}(C)$-logic adjusted to $\mathsf{RL}_{\mathcal{M},\varphi,\mathsf{a},\mathsf{b}}$-language. The completeness of $\mathsf{RL}_{\mathcal{M},\varphi,\mathsf{a},\mathsf{b}}$-system can be proved in a similar way as in in the case of $\mathsf{RL}_{\mathcal{M},\varphi}$-system.

Theorem 11 (Soundness and completeness of $\mathsf{RL}_{\mathcal{M},\varphi,\mathsf{a},\mathsf{b}}$)
For every $\mathsf{RL}_{\mathcal{M},\varphi,\mathsf{a},\mathsf{b}}$*-formula* φ *the following conditions are equivalent:*

- φ *is* $\mathsf{RL}_{\mathcal{M},\varphi,\mathsf{a},\mathsf{b}}$*-provable.*
- φ *is* $\mathsf{RL}_{\mathcal{M},\varphi,\mathsf{a},\mathsf{b}}$*-valid.*

Due to the above theorem and Proposition 31 we obtain the following:

Theorem 12
The following statements are equivalent:

- $(\mathsf{a}, \mathsf{b}) \in m(R)$;
- $x[-(R_{c_\mathsf{a}}; R_{c_\mathsf{b}}^{-1}) \cup R]y$ *is* $\mathsf{RL}_{\mathcal{M},\varphi,\mathsf{a},\mathsf{b}}$*-provable.*

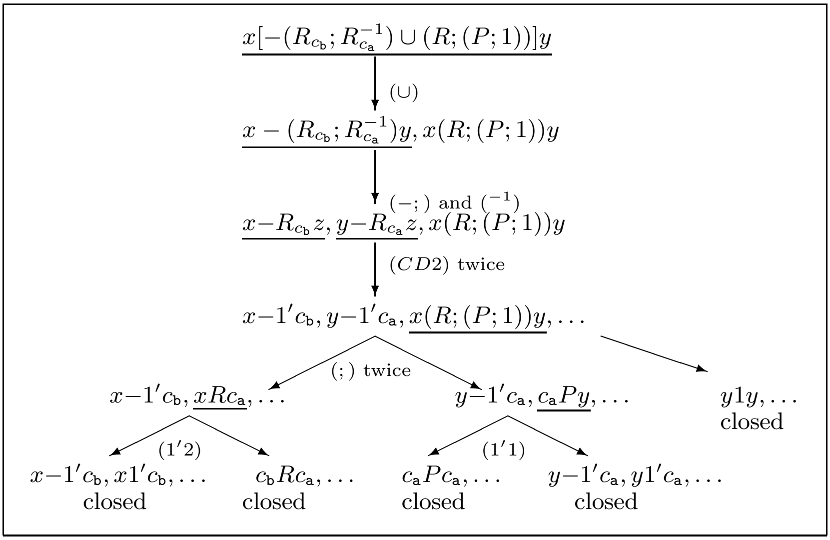

Fig. 4.

As an example of an application of the method presented above, consider the modal logic K. Let $\mathcal{M} = (U, m)$ be a K-model such that $U = \{\mathsf{a}, \mathsf{b}\}$, $m(p) = \{\mathsf{a}\}$ and the accessibility relation is defined as $m(R) = \{(\mathsf{b}, \mathsf{a})\}$. Let φ be the formula of the form $\langle R \rangle p$. Let us consider the problem: 'is φ satisfied in $\mathcal{M}$ by a state b?' The translation of the formula φ is $\tau(\varphi) = (R;(P;1))$, where $\tau'(p) = P$. From the proof of Proposition 26 it follows that there exist an $\mathsf{RL}_{\mathsf{K}}(1,1')$-model $\mathcal{M}'$ and a valuation v_{b} such that the following holds:

$$(\star) \quad \mathcal{M}, \mathsf{b} \models \varphi \text{ iff } \mathcal{M}', v_{\mathsf{b}} \models x\tau(\varphi)y.$$

The $\mathsf{RL}_{\mathsf{K}}(1,1')$-model $\mathcal{M}' = (U', m')$ is defined as follows:

- $U' = m'(1) = \{\mathsf{a}, \mathsf{b}\}$;
- $m'(P) = \{(\mathsf{a}, \mathsf{a}), (\mathsf{a}, \mathsf{b})\}$;
- $m'(R) = \{(\mathsf{b}, \mathsf{a})\}$;
- $m'(1') = \{(\mathsf{a}, \mathsf{a}), (\mathsf{b}, \mathsf{b})\}$;
- m' extends to all compound terms as in $\mathsf{RL}(1,1')$-models.

Let v_{b} be a valuation such that $v_{\mathsf{b}}(x) = b$ and $v_{\mathsf{b}}(y) = a$. Then $\mathcal{M}'$ and v_{b} satisfy the condition $(\star)$.

Therefore the satisfiability problem 'is φ satisfied in $\mathcal{M}$ by a state b?' is equivalent to the problem 'is a formula $x\tau(\varphi)y$ satisfied in $\mathcal{M}'$ by v_{b}?'. By Theorem 12 this is equivalent to $\mathsf{RL}_{\mathcal{M}',x\tau(\varphi)y,\mathsf{b},\mathsf{a}}$-provability of $x[-(R_{c_b};R_{c_a}^{-1}) \cup \tau(\varphi)]y$.

$\mathsf{RL}_{\mathcal{M}',x\tau(\varphi)y,\mathsf{b},\mathsf{a}}$-proof system contains the rules and axiomatic sets of $\mathsf{RL}_{df}(C)$-proof system adjusted to $\mathsf{RL}_{\mathcal{M}',x\tau(\varphi)y,\mathsf{b},\mathsf{a}}$-language and additionally it contains:

– the rules $(-R)$, $(-P)$, $(1')$ and $(\mathsf{a} \neq \mathsf{b})$ of $\mathsf{RL}_{\mathcal{M}',x\tau(\varphi)y}$-system adjusted to $\mathsf{RL}_{\mathcal{M}',x\tau(\varphi)y,\mathsf{b},\mathsf{a}}$-language;
– axiomatic sets that include either of the following subsets: $\{c_{\mathsf{b}}Rc_{\mathsf{a}}\}$, $\{c_{\mathsf{a}}Pc_{\mathsf{a}}\}$, $\{c_{\mathsf{a}}Pc_{\mathsf{b}}\}$, $\{c_{\mathsf{a}}-Rc_{\mathsf{a}}\}$, $\{c_{\mathsf{b}}-Rc_{\mathsf{b}}\}$, $\{c_{\mathsf{a}}-Rc_{\mathsf{b}}\}$, $\{c_{\mathsf{b}}-Pc_{\mathsf{b}}\}$, and $\{c_{\mathsf{b}}-Pc_{\mathsf{a}}\}$.

Figure 4 presents a closed $\mathsf{RL}_{\mathcal{M}',x\tau(\varphi)y,\mathsf{b},\mathsf{a}}$-proof tree for $x[-(R_{c_{\mathsf{b}}}; R_{c_{\mathsf{a}}}^{-1}) \cup \tau(\varphi)]y$. We recall that the rule $(CD2)$ used in that proof is presented in Section 9.

14 RRA Algebras, FRA Algebras and Relational Logics

RRA is a class of algebras isomorphic to an algebra $(\mathcal{P}(1), -, \cup, \cap, ^{-1}, ;, 1, 1')$, where 1 is an equivalence relation, $1'$ is an identity on the field of 1, $-, \cup$ and $\cap$ are Boolean operations, $^{-1}$ and ; are converse and composition of binary relations, respectively. FRA is a class of algebras isomorphic to an algebra $(\mathcal{P}(U \times U), -, \cup, \cap, ^{-1}, ;, U \times U, 1')$, where U is a non-empty set, $1'$ is an identity on U and $-, \cup, \cap, ^{-1}, ;$ are as above.

The theorem below states the connection between RRA-validity and $\mathsf{RLN}^*(1, 1')$-validity:

Theorem 13
Let $R \in \mathbb{RT}_{\mathsf{RL}(1,1')}$ *and* $x, y \in \mathbb{OV}_{\mathsf{RL}(1,1')}$. *Then* xRy *is* $\mathsf{RLN}^*(1, 1')$*-valid iff* $R = 1$ *is* RRA*-valid.*

Proof

Proof of $(\rightarrow)$ Assume xRy is $\mathsf{RLN}^*(1, 1')$-valid, that is for every $\mathsf{RLN}(1, 1')$-model $\mathcal{M} = (U, m)$, $m(1) \subseteq m(R)$. Suppose $R = 1$ is not RRA-valid. Then there exist RRA-algebra $\mathfrak{A}$ and an assignment a in $\mathfrak{A}$ such that $1^{\mathfrak{A}} \not\subseteq R^{\mathfrak{A}}(a)$, where $1^{\mathfrak{A}}$ is an equivalence relation. Consider a model $\mathcal{M}_{\mathfrak{A}} =$ (field of $1^{\mathfrak{A}}, m_{\mathfrak{A}}$) such that:

– $m_{\mathfrak{A}}(P) = P^{\mathfrak{A}}(a)$ for every relational variable P;
– $m_{\mathfrak{A}}(1) = 1^{\mathfrak{A}}$;
– $m_{\mathfrak{A}}(1') = 1'^{\mathfrak{A}}$;
– $m_{\mathfrak{A}}$ extends homomorphically to all compound terms as in the definition of an RL-model.

Since $\mathfrak{A}$ is an RRA algebra, so $1'^{\mathfrak{A}}$ is an equivalence relation on the field of $1^{\mathfrak{A}}$. Therefore $\mathcal{M}_{\mathfrak{A}}$ is an $\mathsf{RLN}(1, 1')$-model. Since xRy is $\mathsf{RLN}(1, 1')$-valid, hence $m_{\mathfrak{A}}(1) \subseteq m_{\mathfrak{A}}(R)$, that is $1^{\mathfrak{A}} \subseteq R^{\mathfrak{A}}(a)$, a contradiction.

Proof of $(\leftarrow)$ Assume $R = 1$ is RRA-valid. Suppose xRy is not $\mathsf{RLN}^*(1, 1')$-valid. Then there exists an $\mathsf{RLN}^*(1, 1')$-model $\mathcal{M} = (U, m)$ such that $m(1) \not\subseteq m(R)$. Consider an algebra $\mathfrak{A}_{\mathcal{M}} = (\mathcal{P}(m(1)), \cup, \cap, -, ;, ^{-1}, m(1'), m(1))$. It is easy to see that $\mathfrak{A}$ is an RRA-algebra. Let a be an assignment in $\mathfrak{A}_{\mathcal{M}}$ such that $P^{\mathfrak{A}_{\mathcal{M}}}(a) = m(P) \cap m(1)$ for every relational variable P. Since $R = 1$ is true in $\mathfrak{A}_{\mathcal{M}}$, so $R^{\mathfrak{A}_{\mathcal{M}}}(a) = 1^{\mathfrak{A}_{\mathcal{M}}} = m(1)$. Therefore $m(R) \cap m(1) = m(1)$, hence $m(1) \subseteq m(R)$, a contradiction. □

Due to the above theorem and Theorem 5 we obtain the following:

Theorem 14
Let xRy be an $\mathsf{RL}(1,1')$*-formula. Then xRy is* $\mathsf{RL}(1,1')$*-provable iff* $R = 1$ *is* RRA*-valid.*

A non-trivial example of $\mathsf{RLN}(1,1')$-valid equation is presented in the Appendix.

Similarly we can prove the following theorem which states the connection between FRA-validity and $\mathsf{RL}(1,1')$-validity.

Theorem 15
Let $R \in \mathbb{RT}_{\mathsf{RL}(1,1')}$ *and* $x, y \in \mathbb{OV}_{\mathsf{RL}(1,1')}$*. Then xRy is* $\mathsf{RL}^*(1,1')$*-valid iff* $R = 1$ *is* FRA*-valid.*

Due to Theorem 5 the above theorems imply the following well known result:

Theorem 16
The set of equations valid in RRA *and the set of equations valid in* FRA *are equal.*

15 Conclusion and Future Work

We presented a survey of relational logics, in particular, we discussed the logics which are the counterparts to the classes RRA and FRA and the logics which enable us reasoning both about relations and about individual elements of a domain on which the relations are defined. We extensively discussed the applications of those logics to the major logical tasks: verification of validity, verification of entailment, model checking and verification of satisfaction in a model. We explained how we can perform these tasks for non-classical logics after translating them into the appropriate relational logics.

An important open problem is to modify the proof systems presented in the paper for the relational logics RL_L, where L is a modal logic, so that they become decision procedures. Another interesting problem is to establish bounds on the number of variables needed in the proofs of formulas of the relational logics presented in the paper.

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Appendix

We present a construction of a closed $\mathsf{RL}(1,1')$ proof tree of an equation which is not valid in RA, while it is valid in RRA. It has the following form:

$$\tau = 1$$

where $\tau := (1;\rho;1)$ and $\rho := (A \cup B \cup C \cup D \cup E)$ for:

- $A = -(1;R;1)$;
- $B = [R \cap -[(N;N) \cap (R;N)]]$;
- $C = (N;N \cup R;R) \cap N$;
- $D = [(R \cup R^{-1} \cup 1') \cap N]$;
- $E = -(R \cup R^{-1} \cup 1' \cup N)$.

To prove validity of $\tau = 1$ we need to prove validity of the formula $u\tau w$, for $u, w \in \mathbb{OV}_{\mathsf{RL}(1,1')}$, $u \neq w$.

It is easy to show that in $\mathsf{RL}(1,1')$-proof tree for $u\tau w$, if a formula $u\tau w$ occurs in a node of this tree, then it is possible to build a subtree of $\mathsf{RL}(1,1')$-proof tree with this formula at the root which ends with exactly one non-axiomatic node containing at least one of the following formulas: zAv, zBv, zCv, zDv and zEv, for any variables z, v. Therefore, in such cases instead of building long subtrees we will use the following abbreviations which have a form of the rules:

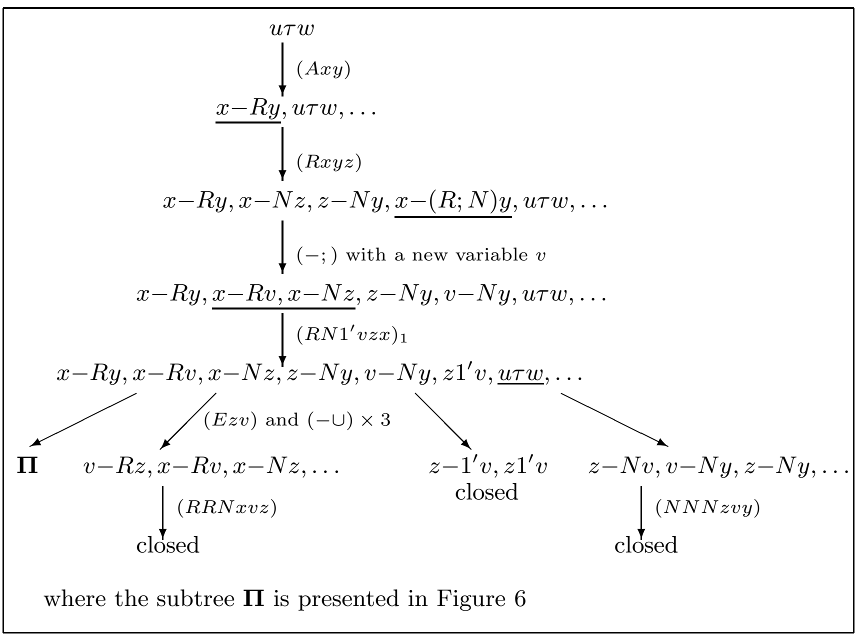

Fig. 5. $\mathsf{RL}(1,1')$-proof tree for $u\tau w$

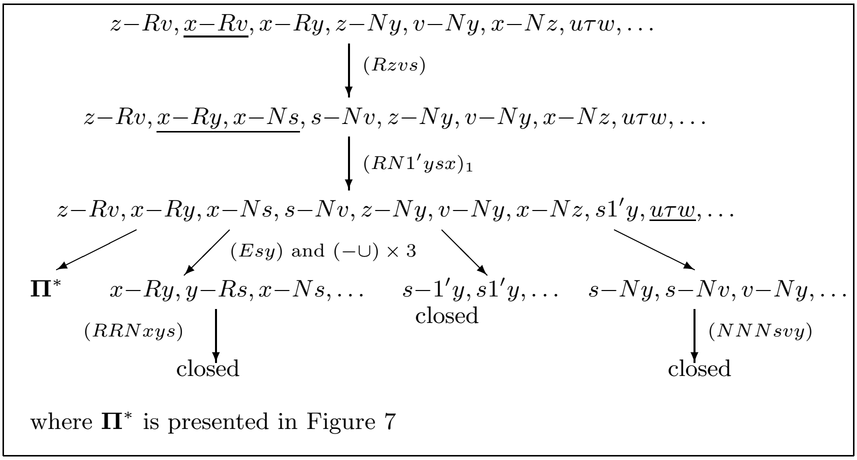

Fig. 6. The subtree **Π**

$$(Azv)\ \frac{u\tau w}{zAv, u\tau w} \qquad (Bzv)\ \frac{u\tau w}{zBv, u\tau w} \qquad (Czv)\ \frac{u\tau w}{zCv, u\tau w}$$

$$(Dzv)\ \frac{u\tau w}{zDv, u\tau w} \qquad (Ezv)\ \frac{u\tau w}{zEv, u\tau w}$$

Similarly, we can admit the following derived rules:

$$(1'^{*})\ \frac{x1'y}{y1'x} \qquad (Rxyz)\ \frac{x{-}Ry, u\tau w}{x{-}Ry, x{-}Nz, z{-}Ny, x{-}(R;N)y, u\tau w}$$

where z is a new variable,

$$(RN1'xyz)_1\ \frac{z{-}Rx, z{-}Ny, u\tau w}{z{-}Rx, z{-}Ny, x1'y, u\tau w}, \frac{z{-}Rx, z{-}Ny, u\tau w}{z{-}Rx, z{-}Ny, y1'x, u\tau w}$$

$$(RN1'xyz)_2\ \frac{x{-}Rz, y{-}Nz, u\tau w}{x{-}Rz, y{-}Nz, x1'y, u\tau w}, \frac{x{-}Rz, y{-}Nz, u\tau w}{x{-}Rz, y{-}Nz, y1'x, u\tau w}$$

$$(RRNxyz)\ \frac{x{-}Ry, y{-}Rz, x{-}Nz, u\tau w}{closed}$$

$$(NNNxyz)\ \frac{x{-}Ny, y{-}Nz, x{-}Nz, u\tau w}{closed}$$

By way of example, in Figures 8 and 9 we show how to obtain the derived rules $(Rxyz)$ and $(RN1'xyz)_1$, respectively. Similarly we may obtain the remaining derived rules. It is easy to check that the derived rule (Cxy) is needed to get $(RRNxyz)$ and $(NNNxyz)$, while (Dxy) is needed in the proofs of $(RN1'xyz)_1$ and $(RN1'xyz)_2$.

Figure 5 presents a closed $\mathsf{RL}(1, 1')$-proof tree for $u\tau w$.

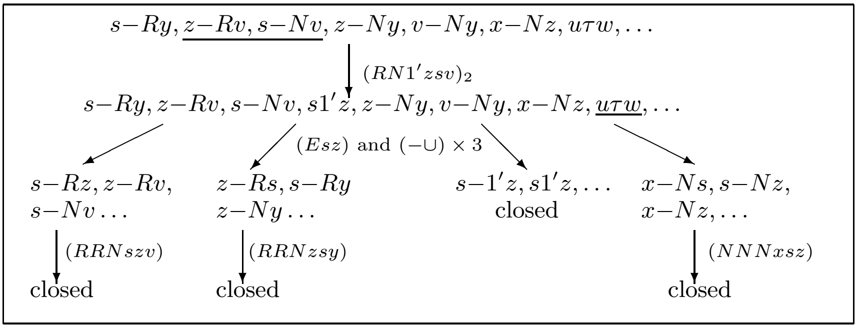

Fig. 7. The subtree $\mathbf{\Pi}^*$

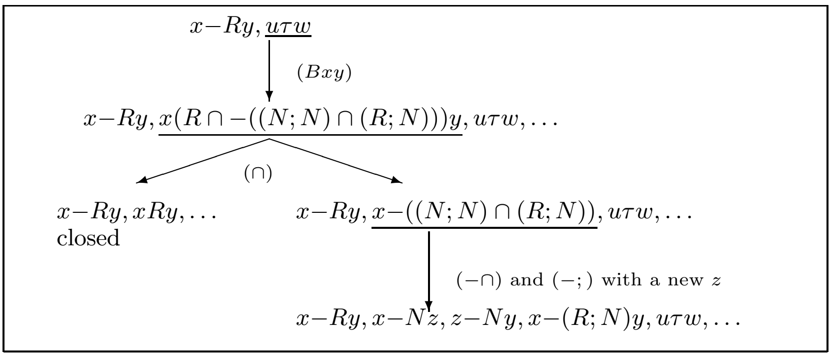

Fig. 8. A derivation of the rule $(Rxyz)$

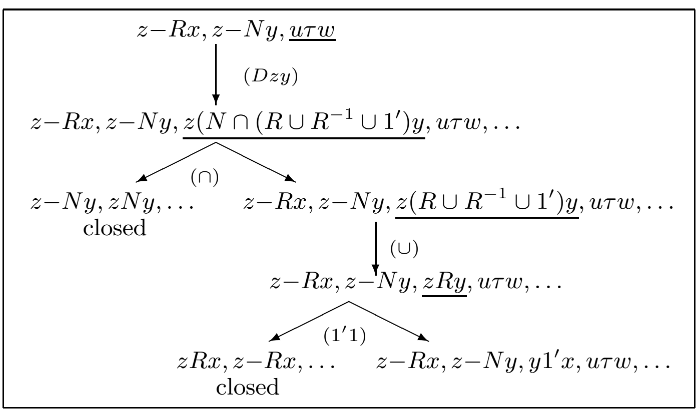

Fig. 9. A derivation of the rule $(RN1'xyz)_1$