Lattice-Based Relation Algebras II*

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Abstract. We present classes of algebras which may be viewed as weak relation algebras, where a Boolean part is replaced by a not necessarily distributive lattice. For each of the classes considered in the paper we prove a relational representation theorem.

1 Introduction

In the first paper on lattice-based relation algebras [8] we presented a class of lattices with the operators, referred to as LCP algebras, which was the abstract counterpart to the class of relation algebras with the specific operations of relative product and converse. In the present paper we expand the LCP class with new operators which model residua of relative product, relative sum, dual converse, and dual residua of relative sum. In the classical relation algebras based on Boolean algebras these operators are definable with the standard relational operations and the complement. In lattice-based algebras they should be specified axiomatically since there is no way to define them without a complement. We construct this extension in two steps. In Section 5 we introduce the class of LCPR algebras which extend the class LCP with the residua of product, and in Section 6 we present the class of LCPRS algebras which are obtained from LCPR algebras by adding sum, dual converse, and dual residua of sum. For each of these classes we prove a relational representation theorem in the style of Urquhart-Allwein-Dunn (see [1], [19]). Sections 2, 3, and 4 present an overview of Urquhart's representation theory for lattices and a survey of LCP algebras. The contributions of the paper fit, on the one hand, into the study of lattices

* This work was carried out in the framework of the European Commission's COST Action 274 *Theory and Applications of Relational Structures as Knowledge Instruments* (TARSKI).

** Ivo Düntsch gratefully acknowledges support by the Natural Sciences and Engineering Research Council of Canada.

H. de Swart et al. (Eds.): TARSKI II, LNAI 4342, pp. 267–289, 2006.

with additional operators presented in a number of papers, for example in [10], [15], [17], [18], and on the other hand, into a relational approach to modeling algebraic and logical structures. A study of lattices with operators evolved from the concept of Boolean algebras with operators originated in [13]. It is continued, among others, in the context of modeling incomplete information in [3], [5], [7], and [14].

2 Doubly Ordered Sets

In this section we recall the notions introduced in [8] and some of their properties.

Definition 1. *Let X be a non–empty set and let $\leqslant_1$ and $\leqslant_2$ be two quasi orderings in X. A structure $(X, \leqslant_1, \leqslant_2)$ is called a* ***doubly ordered set*** *iff for all $x, y \in X$, if $x \leqslant_1 y$ and $x \leqslant_2 y$ then $x = y$.* □

Definition 2. *Let $(X, \leqslant_1, \leqslant_2)$ be a doubly ordered set. We say that $A \subseteq X$ is* $\leqslant_1$***–increasing*** *(resp.* $\leqslant_2$***–increasing****) whenever for all $x, y \in X$, if $x \in A$ and $x \leqslant_1 y$ (resp. $x \leqslant_2 y$), then $y \in A$.* □

For a doubly ordered set $(X, \leqslant_1, \leqslant_2)$, we define two mappings $l, r : 2^X \to 2^X$ by: for every $A \subseteq X$,

$$l(A) = \{x \in X : (\forall y \in X)\ x \leqslant_1 y \Rightarrow y \notin A\} \quad (1)$$

$$r(A) = \{x \in X : (\forall y \in X)\ x \leqslant_2 y \Rightarrow y \notin A\}. \quad (2)$$

Observe that mappings l and r can be expressed in terms of modal operators as follows: $l(A) = [\leqslant_1](-A)$ and $r(A) = [\leqslant_2](-A)$, where $-$ is the Boolean complement and $[\leqslant_i]$, $i = 1, 2$, are the necessity operators determined by relations $\leqslant_i$. Consequently, r and l are intuitionistic–like negations.

Definition 3. *Given a doubly ordered set $(X, \leqslant_1, \leqslant_2)$, a subset $A \subseteq X$ is called* ***l–stable*** *(resp.* ***r–stable****) iff $l(r(A)) = A$ (resp. $r(l(A)) = A$).* □

The family of all l-stable (resp. r–stable) subsets of X will be denoted by $L(X)$ (resp. $R(X)$).

Recall the following notion from e.g. [4]:

Definition 4. *Let $(X, \leqslant_1)$ and $(Y, \leqslant_2)$ be partially ordered sets and let f and g be mappings $f : X \to Y$, $g : Y \to X$. We say that f and g are a* ***Galois connection*** *iff for all $x, y \in X$*

$$x \leqslant_1 g(y) \text{ iff } y \leqslant_2 f(x).$$ □

Lemma 1. [17] *For any doubly ordered set $(X, \leqslant_1, \leqslant_2)$ and for any $A \subseteq X$,*

- **(i)** *$l(A)$ is $\leqslant_1$–increasing and $r(A)$ is $\leqslant_2$–increasing*
- **(ii)** *if A is $\leqslant_1$–increasing, then $r(A) \in R(X)$*
- **(iii)** *if A is $\leqslant_2$–increasing, then $l(A) \in L(X)$*

(iv) *if* $A \in L(X)$*, then* $r(A) \in R(X)$

(v) *if* $A \in R(X)$*, then* $l(A) \in L(X)$

(vi) *if* $A, B \in L(X)$*, then* $r(A) \cap r(B) \in R(X)$*.* ■

It is well–known that the following facts hold.

Lemma 2. *The family of* $\leqslant_i$*–increasing sets,* $i = 1, 2$*, forms a distributive lattice, where join and meet are union and intersection of sets.* ■

Lemma 3. [19] *For every doubly ordered set* $(X, \leqslant_1, \leqslant_2)$*, the mappings* l *and* r *form a Galois connection between the lattice of* $\leqslant_1$*–increasing subsets of* X *and the lattice of* $\leqslant_2$*–increasing subsets of* X*.* ■

In other words, Lemma 3 implies that for any $A \in L(X)$ and for any $B \in R(X)$, $A \subseteq l(B)$ iff $B \subseteq r(A)$.

Lemma 4. [8] *For every doubly ordered set* $(X, \leqslant_1, \leqslant_2)$ *and for every* $A \subseteq X$*,*

(i) $l(r(A)) \in L(X)$ *and* $r(l(A)) \in R(X)$

(ii) *if* A *is* $\leqslant_1$*–increasing, then* $A \subseteq l(r(A))$

(iii) *if* A *is* $\leqslant_2$*–increasing, then* $A \subseteq r(l(A))$*.* ■

Lemma 4 immediately implies:

Corollary 1. *For every doubly ordered set* $(X, \leqslant_1, \leqslant_2)$ *and for every* $A \subseteq X$*,*

(i) *if* $A \in L(X)$*, then* $A \subseteq l(r(A))$

(ii) *if* $A \in R(X)$*, then* $A \subseteq r(l(A))$*.* ■

Let $(X, \leqslant_1, \leqslant_2)$ be a doubly ordered set. Define two binary operations in 2^X: for all $A, B \subseteq X$,

$$A \sqcap B = A \cap B \tag{3}$$

$$A \sqcup B = l(r(A) \cap r(B)). \tag{4}$$

Observe that $\sqcup$ is defined from $\sqcap$ resembling a De Morgan law with two different negations.

Moreover, put

$$\mathbf{0} = \emptyset. \tag{5}$$

$$\mathbf{1} = X \tag{6}$$

In [19] it was shown that for a doubly ordered set $(X, \leqslant_1, \leqslant_2)$, the system $((X), \sqcap, \sqcup, \mathbf{0}, \mathbf{1})$ is a lattice. This lattice is called the ***complex algebra of*** X.

3 Urquhart's Representation of Lattices

In this paper we are interested in studying relationships between relational structures (frames) providing Kripke–style semantics of logics, and algebras based on lattices. Therefore, we do not assume any topological structure in the frames. As a result, we have a weaker form of the representation theorems than the original Urquhart result, which requires compactness.

Let $(W, \wedge, \vee, 0, 1)$ be a non–trivial bounded lattice.

Definition 5. *A* ***filter-ideal pair*** *of a bounded lattice* $(W, \wedge, \vee, 0, 1)$ *is a pair* $x = (x_1, x_2)$ *such that* x_1 *is a filter of* W*,* x_2 *is an ideal of* W *and* $x_1 \cap x_2 = \emptyset$. □

The family of all filter–ideal pairs of a lattice W will be denoted by $FIP(W)$. Let us define the following two quasi ordering relations on $FIP(W)$: for any $(x_1, x_2), (y_1, y_2) \in FIP(W)$,

$$(x_1, x_2) \preccurlyeq_1 (y_1, y_2) \iff x_1 \subseteq y_1 \tag{7}$$

$$(x_1, x_2) \preccurlyeq_2 (y_1, y_2) \iff x_2 \subseteq y_2. \tag{8}$$

Next, define

$$(x_1, x_2) \preccurlyeq (y_1, y_2) \iff (x_1, x_2) \preccurlyeq_1 (y_1, y_2) \ \&\ (x_1, x_2) \preccurlyeq_2 (y_1, y_2).$$

We say that $(x_1, x_2) \in FIP(W)$ is ***maximal*** iff it is maximal with respect to $\preccurlyeq$. We will write $X(W)$ to denote the family of all maximal filter–ideal pairs of the lattice W.

Observe that $X(W)$ is a binary relation on 2^W.

Proposition 1. [19] *Let* $(W, \wedge, \vee, 0, 1)$ *be a bounded lattice. Then for every* $(x_1, x_2) \in FIP(W)$ *there exists* $(y_1, y_2) \in X(W)$ *such that* $(x_1, y_1) \preccurlyeq (y_1, y_2)$. ■

For any $(x_1, x_2) \in FIP(W)$, the maximal filter–ideal pair (y_1, y_2) such that $(x_1, x_2) \preccurlyeq (y_1, y_2)$ will be referred to as an *extension* of (x_1, x_2).

Definition 6. *Let* $(W, \wedge, \vee, 0, 1)$ *be a bounded lattice. The* ***canonical frame of*** W *is the structure* $(X(W), \preccurlyeq_1, \preccurlyeq_2)$. □

Lemma 5. *For every bounded lattice* W*, its canonical frame* $(X(W), \preccurlyeq_1, \preccurlyeq_2)$ *is a doubly ordered set.* ■

Consider the complex algebra $(L(X(W)), \sqcap, \sqcup, \mathbf{0}, \mathbf{1})$ of the canonical frame of a lattice $(W, \wedge, \vee, 0, 1)$. Observe that $L(X(W))$ is an algebra of subrelations of $X(W)$.

Let us define the mapping $h : W \to 2^{X(W)}$ as follows: for every $a \in W$,

$$h(a) = \{x \in X(W) : a \in x_1\}. \tag{9}$$

Theorem 1. [19] *For every lattice* $(W, \wedge, \vee, 0, 1)$ *the following assertions hold:*

(i) *For every* $a \in W$, $r(h(a)) = \{x \in X(W) : a \in x_2\}$
(ii) $h(a)$ *is* l*–stable for every* $a \in W$
(iii) h *is a lattice embedding.* ■

The following theorem is a weak version of the Urquhart result.

Theorem 2 (Representation theorem for lattices). *Every bounded lattice is isomorphic to a subalgebra of the complex algebra of its canonical frame.* ■

4 LCP Algebras and Frames

In this section we recall the class LCP of lattices with the operations of product and converse introduced in [8]. We add one more axiom, **(CP0)**, to the axioms of LCP postulated in [8] and we explain its role.

Definition 7. *An* ***LCP algebra*** *is a system* $(W, \wedge, \vee, ^{\smile}, \otimes, 0, 1, 1')$ *such that* $(W, \wedge, \vee, 0, 1)$ *is a non–trivial bounded lattice,* $^{\smile}$ *is a unary operation in* W *and* $\otimes$ *is a binary operation in* W *satisfying the following conditions for all* $a, b, c \in W$,

(CP.0) $0 \otimes a = a \otimes 0 = 0$
(CP.1) $a^{\smile\smile} = a$
(CP.2) $(a \vee b)^{\smile} = a^{\smile} \vee b^{\smile}$
(CP.3) $a \otimes 1' = 1' \otimes a = a$
(CP.4) $a \otimes (b \otimes c) = (a \otimes b) \otimes c$
(CP.5) $a \otimes (b \vee c) = (a \otimes b) \vee (a \otimes c)$
(CP.6) $(a \vee b) \otimes c = (a \otimes c) \vee (b \otimes c)$
(CP.7) $(a \otimes b)^{\smile} = b^{\smile} \otimes a^{\smile}$. □

It is worth noting that axiom **(CP.0)** does not follow from the remaining axioms. Consider, for example, a bounded lattice $(W, \wedge, \vee, 0, 1)$ and define the additional operations $\otimes$ and $^{\smile}$ as follows: for all $a, b \in W$,

$$\begin{aligned} a^{\smile} &= a \\ a \otimes b &= a \vee b \\ 1' &= 0. \end{aligned}$$

One can easily check that axioms **(CP.1)**–**(CP.7)** hold, but **(CP.0)** does not. Consequently, Lemma 24 of [8] needs repair. For its proof we refer to [1]. The crucial argument is on page 529 of [1] in the paragraph following equation (3). In line 4 of this paragraph they obtain the disjoint pair $([t), U)$, which, as they claim, can be extended to the maximal filter–ideal pair. This, however, is only possible if $t \neq 0$.

Note also that axiom **(CP.0)** follows from the relation algebra axioms and implies that $0 \neq 1'$ in every LCP algebra with at least two elements. To see that, suppose that **(CP.0)** holds and $0 = 1'$. Then $1 = 1' \otimes 1 = 0 \otimes 1 = 0$, which contradicts our hypothesis that W has at least two elements.

For any $A \subseteq W$, let us denote

$$A^{\smile} = \{a^{\smile} \in W : a \in A\}. \tag{10}$$

Lemma 6. [8] *For any LCP algebra* $(W, \wedge, \vee, ^{\smile}, \otimes, 0, 1, 1')$ *and for all subsets* $A, B \subseteq W$*,*

(i) $A \subseteq B$ *iff* $A^{\smile} \subseteq B^{\smile}$

(ii) $A^{\smile\smile} = A$. ■

Some other properties of LCP algebras can be found in [8].

Definition 8. *An* ***LCP frame*** *is a relational system* $(X, \leqslant_1, \leqslant_2, C, R, S, Q, I)$ *such that* $(X, \leqslant_1, \leqslant_2)$ *is a doubly ordered set,* C *is a mapping* $C : X \to X$*,* R*,* S*, and* Q *are ternary relations on* X *and* $I \subseteq X$ *is an unary relation on* X *satisfying the following conditions for all* $x, y \in X$*:*

Monotonicity conditions:

(MCP.1) $x \leqslant_1 y$ *implies* $C(x) \leqslant_1 C(y)$

(MCP.2) $x \leqslant_2 y$ *implies* $C(x) \leqslant_2 C(y)$

(MCP.3) $R(x, y, z)\ \&\ x' \leqslant_1 x\ \&\ y' \leqslant_1 y\ \&\ z \leqslant_1 z' \Longrightarrow R(x', y', z')$

(MCP.4) $S(x, y, z)\ \&\ x \leqslant_2 x'\ \&\ y' \leqslant_1 y\ \&\ z' \leqslant_2 z \Longrightarrow S(x', y', z')$

(MCP.5) $Q(x, y, z)\ \&\ x' \leqslant_1 x\ \&\ y \leqslant_2 y'\ \&\ z' \leqslant_2 z \Longrightarrow Q(x', y', z')$

(MCP.6) $I(x)\ \&\ x \leqslant_1 x' \Longrightarrow I(x')$

Stability conditions:

(SCP.1) $C(C(x)) = x$

(SCP.2) $R(x, y, z) \Longrightarrow \exists x'' \in X\ (x \leqslant_1 x''\ \&\ S(x'', y, z))$

(SCP.3) $R(x, y, z) \Longrightarrow \exists y'' \in X\ (y \leqslant_1 y''\ \&\ Q(x, y'', z))$

(SCP.4) $S(x, y, z) \Longrightarrow \exists z'' \in X\ (z \leqslant_2 z''\ \&\ R(x, y, z''))$

(SCP.5) $Q(x, y, z) \Longrightarrow \exists z'' \in X\ (z \leqslant_2 z''\ \&\ R(x, y, z''))$

(SCP.6) $\exists u \in X (R(x, y, u) \,\&\, Q(x', u, z)) \Longrightarrow \exists w \in X (R(x', x, w) \,\&\, S(w, y, z))$

(SCP.7) $\exists u \in X (R(x, y, u) \,\&\, S(u, z, z')) \Longrightarrow \exists w \in X (R(y, z, w) \,\&\, Q(x, w, z'))$

(SCP.8) $I(x)\ \&\ (R(x, y, z)$ *or* $R(y, x, z)) \Longrightarrow y \leqslant_1 z$

(SCP.9) $\exists u \in X (I(u)\ \&\ S(u, x, x))$

(SCP.10) $\exists u \in X (I(u)\ \&\ Q(x, u, x))$

(SCP.11) $Q(x, y, z) \Longleftrightarrow S(C(y), C(x), C(z))$. □

In [1] there was no general concept of LCP frames. The results of [1] concern canonical frames and complex algebras of the canonical frames. In our approach canonical frames are examples of a general frame.

For an LCP frame $(X, \leqslant_1, \leqslant_2, C, R, S, Q, I)$ let us define the following mappings $^{\curlyvee} : 2^X \to 2^X$ and $\otimes_S$, $\otimes_Q$, $\boxtimes : 2^X \times 2^X \to 2^X$ by: for all $A, B \subseteq X$,

$$A^{\curlyvee} = \{C(x) : x \in A\} \tag{11}$$

$$A \otimes_Q B = \{z \in X : \forall x, y \in X (Q(x, y, z)\ \&\ x \in A \Longrightarrow y \in r(B)\} \tag{12}$$

$$A \otimes_S B = \{z \in X : \forall x, y \in X (S(x, y, z)\ \&\ y \in B \Longrightarrow x \in r(A)\} \tag{13}$$

$$A \boxtimes B = l(A \otimes_Q B). \tag{14}$$

Moreover, put

$$\mathbf{1'} = l(r(I)). \tag{15}$$

The family $L(X)$ of all l–stable subsets of X is closed under the operations (11) and (14).

Lemma 7. [8] *Let $(X, \leqslant_1, \leqslant_2, C, R, S, Q, I)$ be an LCP frame. Then for all $A, B \subseteq X$,*

- **(i)** *if A is l–stable, then so is $A^{\curlyvee}$*
- **(ii)** *if A and B are l–stable, then so is $A \boxtimes B$*
- **(iii)** *$\mathbf{1'}$ is l–stable*
- **(iv)** *if A and B are l–stable, then $A \otimes_S B = A \otimes_Q B$.* ■

Definition 9. *The **complex algebra** of an LCP frame $(X, \leqslant_1, \leqslant_2, C, R, S, Q, I)$ is a system $(L(X), \sqcap, \sqcup, {}^{\curlyvee}, \boxtimes, \mathbf{0}, \mathbf{1}, \mathbf{1'})$ with the operations defined by (3)–(4), (11), (14) and the constants defined by (5), (6) and (15).* □

Theorem 3. *The complex algebra of an LCP frame is an LCP algebra.*

Proof. In [8] it was shown that any complex algebra of an LCP frame satisfies the axioms **(CP.1)**–**(CP.7)**. Then it suffices to show that **(CP.0)** also holds, i.e. $\mathbf{0} \boxtimes A = A \boxtimes \mathbf{0} = \mathbf{0}$ for every $A \in L(X)$.

First, note that $l(L(X)) = \emptyset$ and $r(\mathbf{0}) = L(X)$. Next, since for every $A \subseteq X$ and for every $x, y, z \in X$ it holds $Q(x, y, z)$ & $x \in \emptyset \Longrightarrow y \in r(A)$, whence $\mathbf{0} \otimes_Q A = L(X)$. Therefore, $\mathbf{0} \boxtimes A = l(\mathbf{0} \otimes_Q A) = \mathbf{0}$. Moreover, from the definition of $\otimes_Q$ it is easily observed that $A \otimes_Q \mathbf{0} = L(X)$. Consequently, $A \boxtimes \mathbf{0} = l(A \otimes_Q \mathbf{0}) = \mathbf{0}$. ■

Let $(W, \wedge, \vee, \smile, \odot, 0, 1, 1')$ be an LCP algebra. We will write $FIP(X)$ (resp. $X(W)$) to denote the family of all filter–ideal pairs (resp. maximal filter–ideal pairs) of the lattice reduct of W. Note that since W is non–trivial, $X(W)$ is not empty.

Let us define a mapping $C^\star : FIP(X) \to FIP(X)$ by: for $x \in FIP(X)$,

$$C^\star(x) = ({x_1}^\smile, {x_2}^\smile). \tag{16}$$

Moreover, let us define the following three ternary relations on $X(W)$ by: for all $x, y, z \in X(W)$,

$$R^\star(x, y, z) \iff (\forall a, b \in W)\; a \in x_1 \;\&\; b \in y_1 \Longrightarrow a \otimes b \in z_1 \tag{17}$$

$$S^\star(x, y, z) \iff (\forall a, b \in W)\; a \otimes b \in z_2 \;\&\; b \in y_1 \Longrightarrow a \in x_2 \tag{18}$$

$$Q^\star(x, y, z) \iff (\forall a, b \in W)\; a \otimes b \in z_2 \;\&\; a \in x_1 \Longrightarrow b \in y_2 \tag{19}$$

Also, let

$$I^\star = \{x \in X(W) : 1' \in x_1\}. \tag{20}$$

We extend the operation $\otimes$ for subsets of X in the following way: for all $A, B \subseteq W$,

$$A \otimes B = \{a \otimes b : a \in A, b \in B\}.$$

Then it is straightforward to see that for all $x, y, z \in X(W)$,

$$R^\star(x,y,z) \iff x_1 \otimes y_1 \subseteq z_1 \tag{21}$$
$$S^\star(x,y,z) \iff -x_2 \otimes y_1 \subseteq -z_2 \tag{22}$$
$$Q^\star(x,y,z) \iff x_1 \otimes -y_2 \subseteq -z_2. \tag{23}$$

In [8] we showed that for $x \in X(W)$, $C^\star(x) \in X(W)$.

Definition 10. *Let an LCP algebra* $(W, \wedge, \vee, \smile, \otimes, 0, 1, 1')$ *be given. The system* $(X(W), \preccurlyeq_1, \preccurlyeq_2, C^\star, R^\star, S^\star, Q^\star, I^\star)$ *is called the* ***canonical frame of*** W. □

The following auxiliary lemma will be useful.

Lemma 8. [8] *Let* $(W, \wedge, \vee, \otimes, ^\smile, 0, 1, 1')$ *be an LCP algebra and let* Δ *and* ∇ *be a filter and an ideal of* W*, respectively. Then the set*

$$V = \{a \in W : (\{a\} \otimes \Delta) \cap \nabla \neq \emptyset\}$$

is an ideal of W. ■

In the following theorem we show that canonical frames satisfy the postulates assumed for the LCP frames. We only give a few exemplary proofs which were not given in [8].

Theorem 4. *The canonical frame of an LCP algebra is an LCP frame.*

Proof. Let an LCP algebra $(W, \wedge, \vee, ^\smile, \otimes, 0, 1, 1')$ be given and let $(X(W), \preccurlyeq_1, \preccurlyeq_2, C^\star, R^\star, S^\star, Q^\star, I^\star)$ be its canonical frame. Proceeding as in [1] one can prove that **(MCP.3)**–**(MCP.5)** and **(SCP.2)**–**(SCP.7)** hold in the canonical frame.

We show now that **(MCP.1)** is satisfied. Let $x, y \in X(W)$ be such that $x \preccurlyeq_1 y$. This means that **(i)** $x_1 \subseteq y_1$. Also, $C^\star(x) = (x_1^\smile, x_2^\smile)$. By Lemma 6, **(i)** is equivalent with $x_1^\smile \subseteq y_1^\smile$, so $C^\star(x) \preccurlyeq_1 C^\star(y)$. In the analogous way we can show that **(MCP.2)** holds.

Next we prove that **(MCP.6)** is satisfied. Let $x, x' \in X(W)$ and assume that $I^\star(x)$ and $x \preccurlyeq_1 x'$ hold. From (20) we immediately get $1' \in x_1 \subseteq x'_1$, so $I^\star(x')$ holds.

Furthermore, we show that **(SCP.1)** holds. For every $x = (x_1, x_2) \in X(W)$, we have: $C^\star(C^\star(x)) = C^\star(x_1^\smile, x_2^\smile) = (x_1^{\smile\smile}, x_2^{\smile\smile}) = (x_1, x_2) = x$ by Lemma 6**(ii)**.

Consider now the condition **(SCP.8)**. Assume that for any $x, y, z \in X(W)$, $I^\star(x)$ holds, i.e. **(ii)** $1' \in x_1$, and $R^\star(x,y,z)$ or $R^\star(y,x,z)$. Let $R^\star(x,y,z)$ holds. Hence, by **(ii)**, we get $(\forall b \in W)\ b \in y_1 \Rightarrow 1' \otimes b \in z_1$. Since $1' \otimes b = b$, we get $y_1 \subseteq z_1$, that is $y \preccurlyeq_1 z$. If $R^\star(y,x,z)$ holds, then again by **(ii)** we get $(\forall a \in W)\ a \in y_1 \Rightarrow a \otimes 1' \in z_1$, so since $a \otimes 1' = a$, we obtain again $y_1 \subseteq z_1$, i.e. $y \preccurlyeq_1 z$.

We show now that **(SCP.9)** holds. Let $y \in X(W)$ and consider the set $V = \{a \in W : (\{a\} \otimes y_1) \cap y_2 \neq \emptyset\}$. By Lemma 8, V is an ideal of W. Let $[1')$ be the filter generated by $1'$. We show that $[1') \cap V = \emptyset$. Suppose that there exists

$a \in W$ such that **(iii)** $a \in [1')$ and **(iv)** $a \in V$. From **(iii)** it follows that **(v)** $1' \leqslant a$. Also, **(iv)** implies that there exists $b \in W$ such that **(vi)** $b \in y_1$ and **(vii)** $a \otimes b \in y_2$. Since $\otimes$ is isotone in both arguments, **(v)** implies $1' \otimes b \leqslant a \otimes b$. But $1' \otimes b = b$, so we have $b \leqslant a \otimes b$, which in view of **(vii)** and the fact that y_2 is an ideal gives $b \in y_2$ – a contradiction with **(vi)**.

Then $([1'), V)$ is a filter–ideal pair. Let $u = (u_1, u_2)$ be its extension to the maximal pair. Therefore, $[1') \subseteq u_1$ and $V \subseteq u_2$. Since $1' \in [1')$, we get $1' \in u_1$, so $I^\star(u)$ holds. We show now that $S^\star(u, y, y)$ holds. Let $a, b \in W$ be such that $a \otimes b \in y_2$ and $b \in y_1$. Then $a \in V$, so $a \in u_2$. Whence $S^\star(u, y, y)$ holds.

In the similar way one can check that **(SCP.10)** holds.

Finally we show that **(SCP.11)** holds. Using the axiom **(CP.7)** and the definition (10), we have for all $x, y, z \in X(W)$,

$$
\begin{aligned}
S^\star(C^\star(y), C^\star(x), C^\star(z)) \quad & \text{iff} \quad (\forall a, b \in W)\ a \otimes b \in z_2^{\smile}\ \&\ b \in x_1^{\smile} \Longrightarrow a \in y_2^{\smile} \\
& \text{iff} \quad (\forall a, b \in W)\ (a \otimes b)^{\smile} \in z_2\ \&\ b^{\smile} \in x_1 \Longrightarrow a^{\smile} \in y_2 \\
& \text{iff} \quad (\forall a, b \in W)\ b^{\smile} \otimes a^{\smile} \in z_2\ \&\ b^{\smile} \in x_1 \Longrightarrow a^{\smile} \in y_2 \\
& \text{iff} \quad (\forall c, d \in W)\ c \otimes d \in z_2\ \&\ c \in x_1 \Longrightarrow d \in y_2 \\
& \text{iff} \quad Q^\star(x, y, z).
\end{aligned}
$$

This completes the proof. ∎

We conclude this section by stating the representability of LCP algebras.

Theorem 5. *Every LCP algebra is isomorphic to a subalgebra of the complex algebra of its canonical frame.*

Proof. See [8]. ∎

In the axiomatization of relation algebras, apart from the axioms for Boolean algebras, the only axiom which contains complementation is

$$a \otimes -(a^{\smile} \otimes -b) \leqslant b.$$

This axiom is equivalent to the De Morgan equivalences

$$(a \otimes b) \wedge c = 0 \Longleftrightarrow (a^{\smile} \otimes c) \wedge b = 0 \Longleftrightarrow (c \otimes b^{\smile}) \wedge a = 0 \tag{24}$$

and could be added to the LCP axioms. However, we showed in [8] that adding (24) does not add anything new. An alternative is the *modular inequality*

$$(a \otimes b) \wedge c \leqslant a \otimes (b \wedge (a^{\smile} \otimes c)). \tag{25}$$

(25) is true for relation algebras and is also an axiom for rough relation algebras ([5]), i.e., relation algebras based on regular double Stone algebras. One consequence of (25) is that for every $a < 1'$ we have $a \otimes 1 < 1$ (here $a < b$ means $a \leqslant b$

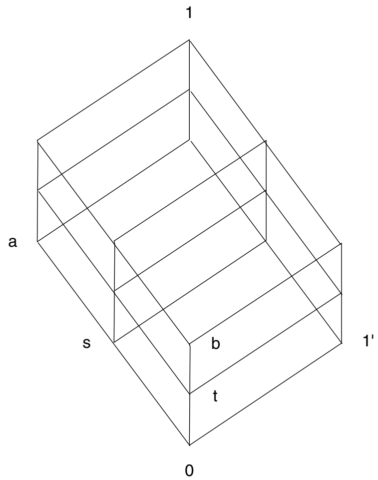

Fig. 1. An LCP–algebra where (25) fails

Table 1. Composition Table

$\otimes$	a	b	s	t
a	a	1	a	$t \vee 1' \vee a$
b	1	b	1	b
s	a	$s \vee 1' \vee b$	s	$s \vee t \vee 1'$
t	1	b	$s \vee t \vee 1'$	t

and $a \neq b$). The following example from [9] shows that not every LCP–algebra satisfies (25).

Example 1. Consider the algebra L of Fig.1. By **(CP.2)** and **(CP.5)** it is enough to define how composition and converse act on the join irreducible elements. These are $1', a, b, s, t$, and we set $a^{\smile} = b$, $s^{\smile} = t$. Composition for the non–identity irreducible elements is given in Table 1. Now consider

$$\begin{aligned}
(t \otimes a) \wedge b &= b && \text{since } t \otimes a = 1\\
&\nleqslant t \otimes s && \text{from the composition table}\\
&= t \otimes [(s \vee t \vee 1') \wedge a] && \text{from the lattice ordering}\\
&= t \otimes [(s \otimes b) \wedge a]\\
&= t \otimes [(t^{\smile} \otimes b) \wedge a].
\end{aligned}$$

So we may want the following inequality as an additional axiom of LCP algebras:

$$\textbf{(CP.8)} \quad (a \otimes b) \wedge c \leqslant a \otimes (b \wedge (a^{\smile} \otimes c)).$$

To obtain a representation theorem for LCP algebras with **(CP.8)** is still an open problem.

The next example illustrates the constructions employed in the proof of the representation theorem.

Example 2. Consider an algebra $(W, \wedge, \vee, {}^{\smile}, \otimes, 0, 1, 1')$ with $W = \{a, b, c, , 0, 1\}$, $\wedge$ and $\vee$ as on Fig.2, $a^{\smile} = a$ for every $a \in W$, $\otimes$ in given in Table 2, and $1' = c$. The maximal filter–ideal pairs of W are

$$x = ([a), (b]), \qquad y = ([b), (c]), \qquad z = ([c), (a]).$$

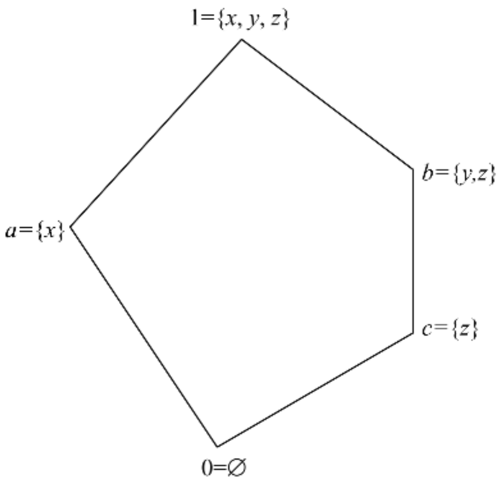

Fig. 2. The pentagon

Table 2. The product $\otimes$

$\otimes$	a	b	c	1
a	1	a	a	1
b	a	b	b	1
c	a	b	c	1
1	1	1	1	1

Let us find R, Q, and S. We can simplify the calculations by observing that $A \otimes B = B \otimes A$ for any $A, B \subseteq W$, since $\otimes$ is symmetric on W.
$R(x,y,z)$ iff $x_1 \otimes y_1 \subseteq z_1$:

$R(x,x,v)$ $x_1 \otimes x_1 = \{1\}$, and $\{1\} \subseteq v_1$ for all $v \in FIP(W)$.
$R(x,y,v)$ $x_1 \otimes y_1 = \{a,1\}$, and $\{a,1\} \subseteq v_1$ only for $v = x$.
$R(x,z,v)$ $x_1 \otimes z_1 = \{a,1\}$, and $\{a,1\} \subseteq v_1$ only for $v = x$.
$R(y,y,v)$ $y_1 \otimes y_1 = \{b,1\}$, and $\{b,1\} \subseteq v_1$ for $v \in \{y,z\}$.
$R(y,z,v)$ $y_1 \otimes z_1 = \{b,1\}$, and $\{b,1\} \subseteq v_1$ for $v \in \{y,z\}$.
$R(z,z,v)$ $z_1 \otimes z_1 = \{c,1\}$, and $\{c,1\} \subseteq v_1$ for $v = z$.

$S(x,y,z)$ iff $(-x_2 \otimes y_1) \cap z_2 = \emptyset$:

$S(x,x,v)$ $-x_2 \otimes x_1 = \{1\}$, and $\{1\} \cap v_2 = \emptyset$ for all $v \in FIP(W)$.
$S(x,y,v)$ $-x_2 \otimes y_1 = \{a,1\}$, and $\{a,1\} \cap v_2 = \emptyset$ for $v \in \{x,y\}$.
$S(x,z,v)$ $-x_2 \otimes z_1 = \{a,1\}$, and $\{a,1\} \cap v_2 = \emptyset$ for $v \in \{x,y\}$.
$S(y,x,v)$ $-y_2 \otimes x_1 = \{a,1\}$, and $\{a,1\} \cap v_2 = \emptyset$ for $v \in \{x,y\}$.

$S(y,y,v)$ $-y_2 \otimes y_1 = \{a,b,1\}$, and $\{a,1\} \cap v_2 = \emptyset$ for $v \in \{y,z\}$.
$S(y,z,v)$ $-y_2 \otimes z_1 = \{a,b,1\}$, and $\{a,1\} \cap v_2 = \emptyset$ for $v \in \{y,z\}$.
$S(z,x,v)$ $-z_2 \otimes x_1 = \{a,1\}$, and $\{a,1\} \cap v_2 = \emptyset$ for $v \in \{x,y\}$.
$S(z,y,v)$ $-z_2 \otimes y_1 = \{b,1\}$, and $\{b,1\} \cap v_2 = \emptyset$ for $v \in \{y,z\}$.
$S(z,z,v)$ $-z_2 \otimes z_1 = \{b,c,1\}$, and $\{b,c,1\} \cap v_2 = \emptyset$ only for $v = z$.

$Q(x,y,z)$ iff $(x_1 \otimes -y_2) \cap z_2 = \emptyset$:

$Q(x,x,v)$ $x_1 \otimes -x_2 = \{a,1\}$ and $\{a,1\} \cap v_2 = \emptyset$ for $v \in \{x,y\}$.
$Q(x,y,v)$ $x_1 \otimes -y_2 = \{a,1\}$ and $\{a,1\} \cap v_2 = \emptyset$ for $v \in \{x,y\}$.
$Q(x,z,v)$ $x_1 \otimes -z_2 = \{a,1\}$ and $\{a,1\} \cap m_2 = \emptyset$ for $v \in \{x,y\}$.
$Q(y,x,v)$ $y_1 \otimes -x_2 = \{a,1\}$ and $\{a,1\} \cap v_2 = \emptyset$ for $v \in \{x,y\}$.
$Q(y,y,v)$ $y_1 \otimes -y_2 = \{a,b,1\}$ and $\{a,b,1\} \cap v_2 = \emptyset$ for $v \in \{y,z\}$.
$Q(y,z,v)$ $y_1 \otimes -z_2 = \{b,1\}$ and $\{a,b,1\} \cap v_2 = \emptyset$ for $v \in \{y,z\}$.
$Q(z,x,v)$ $z_1 \otimes -x_2 = \{a,1\}$ and $\{a,1\} \cap v_2 = \emptyset$ for $m \in \{x,y\}$.
$Q(z,y,v)$ $z_1 \otimes -y_2 = \{a,b,1\}$ and $\{a,b,1\} \cap v_2 = \emptyset$ for $v \in \{y,z\}$.
$Q(z,z,v)$ $z_1 \otimes -z_2 = \{b,c,1\}$ and $\{b,c,1\} \cap v_2 = \emptyset$ only for $v = z$.

The embedding h is given by

$h(0) = \emptyset$ $h(a) = \{x\}$ $h(c) = \{z\}$.
$h(1) = \{x,y,z\}$ $h(b) = \{y,z\}$

We conclude this section with the observation that the diamond lattice of Figure 3 cannot be made into an LCP algebra. We omit the proof which is straightforward, if somewhat tedious.

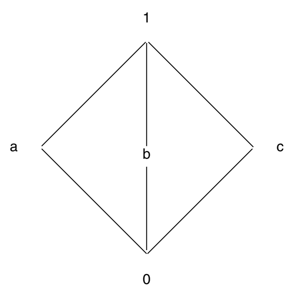

Fig. 3. The diamond lattice

5 LCPR Algebras and Frames

In this section we extend LCP algebras by adding the residuation operations. In classical relation algebras residuations are definable with composition (;), converse ($\smile$) and complement($-$) as $x/y = -(y^\smile; -x)$ and $y \backslash x = -(-x; y^\smile)$.

Definition 11. *By an* ***LCPR algebra*** *we mean a system* $(W, \wedge, \vee, ^\smile, \otimes, \rightarrow, \leftarrow, 0, 1, 1')$ *such that* $(W, \wedge, \vee, ^\smile, \otimes, 0, 1, 1')$ *is an LCP algebra and* $\rightarrow$ *and* $\leftarrow$ *are binary operations in* W *satisfying the following conditions for all* $a, b, c \in W$,

(CPR.1) $a \otimes b \leqslant c$ *iff* $b \leqslant a \rightarrow c$
(CPR.2) $a \otimes b \leqslant c$ *iff* $a \leqslant c \leftarrow b$.

The operations $\leftarrow$ *and* $\rightarrow$ *are called the left and the right residuum of* $\otimes$*, respectively.* □

Note that an LCPR algebra is an extension of a residuated lattice by the converse $\smile$ operation.

The following lemma provides some basic properties of LCPR algebras.

Lemma 9. *Let* $(W, \wedge, \vee, {}^\smile, \otimes, \rightarrow, \leftarrow, 0, 1, 1')$ *be an LCPR algebra. Then for any* $a, b, c \in W$ *and for every indexed family* $(b_i)_{i\in I}$ *of elements of* W*,*

- **(i)** *if* $a \leqslant b$*, then*
  $c \otimes a \leqslant c \otimes b$ *and* $a \otimes c \leqslant b \otimes c$
  $b \rightarrow c \leqslant a \rightarrow c$ *and* $c \rightarrow a \leqslant c \rightarrow b$
  $a \leftarrow c \leqslant b \leftarrow c$ *and* $c \leftarrow b \leqslant c \leftarrow a$
- **(ii)** $a \leqslant b$ *iff* $a^\smile \leqslant b^\smile$
- **(iii)** $(a \wedge b)^\smile = a^\smile \wedge b^\smile$
- **(iv)** $a \otimes (a \rightarrow b) \leqslant b$ **(iv')** $(b \leftarrow a) \otimes a \leqslant b$
- **(v)** $(a \rightarrow b) \otimes (b \rightarrow c) \leqslant a \rightarrow c$ **(v')** $(a \leftarrow b) \otimes (b \leftarrow c) \leqslant\ \leftarrow c$
- **(vi)** $b \leqslant a \rightarrow (a \otimes b)$ **(vi')** $a \leqslant (a \otimes b) \leftarrow b$
- **(vii)** $(a \rightarrow b)^\smile = b^\smile \leftarrow a^\smile$ **(vii')** $(a \leftarrow b)^\smile = b^\smile \rightarrow a^\smile$
- **(viii)** *if* $\sup_i b_i$ *exists, then*
  $a \otimes \sup_i b_i = \sup_i(a \otimes b_i)$
  $\sup_i b_i \otimes a = \sup_i(b_i \otimes a)$
- **(ix)** *if* $\inf_i b_i$ *exists, then*
  $a \rightarrow \inf_i b_i = \inf_i(a \rightarrow b_i)$
- **(ix')** *if* $\inf_i b_i$ *exists, then*
  $\inf_i b_i \leftarrow a = \inf_i(b_i \leftarrow a)$
- **(x)** *if* $\sup_i b_i$ *exists, then*
  $\sup_i b_i \rightarrow a = \inf_i(b_i \rightarrow a)$
- **(x')** *if* $\sup_i b_i$ *exists, then*
  $a \leftarrow \sup_i b_i = \inf_i(a \leftarrow b_i)$.

Proof. By way of example we prove **(vii)**

Let $c \in W$ such that $c \leqslant (a \rightarrow b)^\smile$. Then we have:

$c \leqslant (a \rightarrow b)^\smile$	iff	$c^\smile \leqslant (a \rightarrow b)$	by	**(ii)**, **(CP.1)**
	iff	$a \otimes c^\smile \leqslant b$	by	**(CPR.1)**
	iff	$(a \otimes c^\smile)^\smile \leqslant b^\smile$	by	**(ii)**
	iff	$c \otimes a^\smile \leqslant b^\smile$	by	**(CP.1)**, **(CP.7)**
	iff	$c \leqslant b^\smile \leftarrow a^\smile$	by	**(CPR.2)**. ∎

For the recent development of residuated lattices we refer, for example, to [2], [11], [12], and [16].

LCPR frames are the same as LCP frames defined in Section 3 (Definition 8).

Let an LCPR frame $(X, \leqslant_1, \leqslant_2, C, R, S, Q, I)$ be given. We define the following two mappings $-\!\!\rhd$, $\lhd\!\!-$: $2^X \times 2^X \rightarrow 2^X$ as follows: for all $A, B \subseteq X$,

$$A \mathrel{-\!\!\rhd} B \;=\; \{x \in X : (\forall y, z \in X)(R(y, x, z) \;\&\; y \in A \Longrightarrow z \in B)\} \quad (26)$$

$$B \mathrel{\lhd\!\!-} A \;=\; \{x \in X : (\forall y, z \in X)(R(x, y, z) \;\&\; y \in A \Longrightarrow z \in B)\}. \quad (27)$$

Lemma 10. *For any* $A, B \subseteq X$,

(i) $A \multimap B$ *and* $A \multimapinv B$ *are* $\leqslant_1$*–increasing*

(ii) *if* A *and* B *are* l*–stable, then so are* $A \multimap B$ *and* $A \multimapinv B$.

Proof.

(i) Assume that for some $A, B \subseteq X$, $A \multimap B$ is not $\leqslant_1$–increasing. Then there are $x, y \in X$ such that **(i.1)** $x \in A \multimap B$ **(i.2)** $x \leqslant_1 y$, and **(i.3)** $y \notin A \multimap B$. From **(i.3)**, by the definition (26), there exist $u, w \in X$ such that **(i.4)** $R(u, y, w)$ **(i.5)** $u \in A$ and **(i.6)** $w \notin B$. Next, by **(i.2)**, **(i.4)** and the monotonicity condition **(MCP.3)**, we get $R(u, x, w)$, which together with **(i.5)** and **(i.6)** gives $x \notin A \multimap B$ – a contradiction with **(i.1)**.

Proceeding in the similar way one can show that $A \multimapinv B$ is $\leqslant_1$–increasing.

(ii) Let $A, B \subseteq X$. We show first that $A \multimapinv B$ is l–stable.

By **(i)**, $A \multimapinv B$ is $\leqslant_1$–increasing, so from Lemma 4**(ii)**, $A \multimapinv B \subseteq l(r(A \multimapinv B))$. Then it suffices to show that $l(r(A \multimapinv B)) \subseteq A \multimapinv B$.

Let $x \in X$ be such that **(ii.1)** $x \notin A \multimapinv B$. We will show **(ii.2)** $x \notin l(r(A \multimapinv B))$. From **(ii.1)**, by the definition (27) it follows that there exist $y, z \in X$ such that **(ii.3)** $R(x, y, z)$ **(ii.4)** $y \in B$ **(ii.5)** $z \notin A$. Since B is l–stable, **(ii.5)** means that $z \notin l(r(A))$, so there exists $z' \in X$ such that **(ii.6)** $z \leqslant_1 z'$ and **(ii.7)** $z' \in r(A)$. From **(ii.3)**, **(ii.6)** and the monotonicity condition **(MCP.3)**, $R(x, y, z')$, which by the stability condition **(SCP.1)** implies that there is $x' \in X$ such that **(ii.8)** $x \leqslant_1 x'$ and **(ii.9)** $S(x', y, z')$. We show now that **(ii.10)** $x' \in r(A \multimapinv B)$. This, by **(ii.8)**, gives **(ii.2)**.

Consider an arbitrary $x'' \in X$ satisfying **(ii.11)** $x' \leqslant_2 x''$. By **(MCP.4)**, **(ii.9)** and **(ii.11)** lead to $S(x'', y, z')$, which by **(SCP.3)** gives that there exists $z'' \in X$ such that **(ii.12)** $z' \leqslant_2 z''$ and **(ii.13)** $R(x'', y, z'')$. From Lemma 1**(ii)**, $r(B)$ is $\leqslant_2$–increasing, so by **(ii.7)** and **(ii.12)** we get $z'' \in r(A)$, whence **(ii.14)** $z'' \notin A$. In view of the definition (27), **(ii.4)**, **(ii.13)** and **(ii.14)** imply **(ii.15)** $x'' \notin A \multimapinv B$. Therefore, we have shown that for any $x'' \in X$ satisfying **(ii.11)**, the condition **(ii.15)** holds, hence **(ii.10)** was proved.

Using the relation Q in place of S, in the analogous way we can show that $A \multimap B$ is l–stable. □

Definition 12. *The* ***complex algebra*** *of an LCPR frame* $(X, \leqslant_1, \leqslant_2, C, R, S, Q, I)$ *is a structure* $(L(X), \sqcap, \sqcup, ^{\curlyvee}, \boxtimes, \multimap, \multimapinv, \mathbf{0}, \mathbf{1}, \mathbf{1'})$ *with the operations defined by (3), (4), (11), (14), (26), (27) and the constants (5), (6), and (15).* □

We show that the complex algebra of an LCPR frame is an LCPR algebra. It is sufficient to show the following lemma.

Lemma 11. *For any LCPR frame* $(X, \leqslant_1, \leqslant_2, C, R, S, Q, I)$ *and for all* l*–stable subsets* $A, B, C \subseteq X$,

(i) $A \boxtimes B \subseteq C$ *iff* $B \subseteq A \multimap C$

(ii) $A \boxtimes B \subseteq C$ *iff* $A \subseteq C \multimapinv B$.

Proof.

(i) **(⇐)** Assume that **(i.1)** $A \boxtimes B \subseteq C$ and **(i.2)** $B \not\subseteq A \mathbin{-\!\!\rhd} C$. From **(i.2)**, there exists $x \in X$ such that **(i.3)** $x \in B$ and **(i.4)** $x \notin A \mathbin{-\!\!\rhd} C$. By the definition (26), **(i.4)** means that for some $y, z \in X$ it holds **(i.5)** $R(y, x, z)$, **(i.6)** $y \in A$ and **(i.7)** $z \notin C$. Next, from **(i.1)** and **(i.7)** we get **(i.8)** $x \notin A \boxtimes B$. By the definition (14), $A \boxtimes B = l(A \otimes_Q B)$, but from Lemma 7**(iv)**, $A \otimes_Q B = A \otimes_S B$. Then we get **(i.8)** implies that there exist $z' \in X$ such that **(i.9)** $z \leqslant_1 z'$ and **(i.10)** $z' \in A \otimes_S B$. Furthermore, from **(i.5)** and **(i.9)**, by **(M.3)** we get $R(y, x, z')$, which by **(S.1)** implies that there is $y' \in X$ such that **(i.11)** $y \leqslant_1 y'$ and **(i.12)** $S(y', x, z')$. Also, **(i.3)**, **(i.10)** and **(i.12)** imply $y' \in r(A)$, which together with **(i.11)** gives $y \notin l(r(A))$. Since A is l–stable, this means $y \notin A$ – a contradiction with **(i.6)**.

(⇒) Assume that **(i.13)** $A \boxtimes B \not\subseteq C$. We will show that **(i.14)** $B \not\subseteq A \mathbin{-\!\!\rhd} C$.

From **(i.13)**, there is $x \in X$ such that **(i.15)** $x \in A \boxtimes B$ and **(i.16)** $x \notin C$. Since C is l–stable, **(i.16)** gives $x \notin l(r(C))$, so there exists $x' \in X$ such that **(i.17)** $x \leqslant_1 x'$ and **(i.18)** $x' \in r(C)$. Next, from **(i.15)**, **(i.17)**, and Lemma 7**(iv)**, $x' \notin A \otimes_S B$, which means that there are $y, z \in X$ such that **(i.19)** $S(y, z, x')$, **(i.20)** $z \in B$ and **(i.21)** $y \notin r(A)$. From **(i.21)**, there is $y' \in X$ such that **(i.22)** $y \leqslant_2 y'$ and **(i.23)** $y' \in A$. By **(M.4)**, **(i.19)** and **(i.22)** imply $S(y', z, x')$. Hence, applying **(S.3)** we get that for some $x'' \in X$ such that **(i.24)** $x' \leqslant_2 x''$ it holds **(i.25)** $R(y', z, x'')$. Furthermore, by **(i.18)** and **(i.24)** it follows that $x'' \notin C$, which together with **(i.23)** and **(i.25)** gives $z' \notin A \mathbin{-\!\!\rhd} C$. Whence, in view of **(i.20)** we finally obtain **(i.14)**.

In the analogous way **(ii)** can be proved. □

Therefore, we have

Theorem 6. *The complex algebra of an LCPR frame is an LCPR algebra.* □

Since LCPR frames are just LCP frames, the above theorem implies the following

Corollary 2. *Any LCP algebra can be isomorphically embedded into an LCPR algebra.* □

Let $(W, \wedge, \vee, \otimes, \rightarrow, \leftarrow, 0, 1)$ be an LCPR algebra. For any two subsets $A, B \subseteq W$, let us define:

$$A \leftarrow B \;=\; \{a \leftarrow b : a \in A \;\;\&\;\; b \in B\}$$
$$A \rightarrow B \;=\; \{a \rightarrow b : a \in A \;\;\&\;\; b \in B\}.$$

Lemma 12. *Let* $(W, \wedge, \vee, \otimes, \rightarrow, \leftarrow, 0, 1)$ *be an LCPR algebra and let* Δ *and* Δ' *be filters of* W *and let* ∇ *be an ideal of* W*. Define the following subsets of* W*:*

$$U \;=\; \{a \in W : \Delta \cap (\nabla \leftarrow \{a\} \neq \emptyset\}$$
$$U' \;=\; \{a \in W : \Delta \cap (\{a\} \rightarrow \nabla) \neq \emptyset\}$$
$$V \;=\; \{a \in W : \Delta \cap (\{a\} \leftarrow \Delta') \neq \emptyset\}$$
$$V' \;=\; \{a \in W : \Delta \cap (\Delta' \rightarrow \{a\}) \neq \emptyset\}.$$

Then U *and* U' *are ideals of* W *and* V *and* V' *are filters of* W*.*

Proof. By way of example we show that U is an ideal of W. Let $a, b \in W$ be such that **(i)** $a \in U$ and **(ii)** $b \leqslant a$. By the definition of U, **(i)** implies that there exists $c \in \nabla$ such that **(iii)** $c \leftarrow a \in \Delta$. By Lemma 9(**i**) we get from **(ii)** that $c \leftarrow a \leqslant c \leftarrow b$. Hence, by **(iii)**, we get **(iv)** $c \leftarrow b \in \Delta$, since Δ is a filter. Therefore, for some $c \in \nabla$ **(iv)** holds, which implies $b \in U$.

Assume that **(v)** $a, b \in U$. It suffices to show that $a \vee b \in U$. From **(v)**, there are $c, d \in \nabla$ such that **(vi)** $c \leftarrow a \in \Delta$ and **(vii)** $d \leftarrow b \in \Delta$. Since $c \leqslant c \vee d$ and $d \leqslant c \vee d$, by Lemma 9(**i**) we get $c \leftarrow a \leqslant (c \vee d) \leftarrow a$ and $d \leftarrow b \leqslant (c \vee d) \leftarrow b$. Hence, by **(vi)** and **(vii)** it follows that $(c \vee d) \leftarrow a \in \Delta$ and $(c \vee d) \leftarrow b \in \Delta$, so $((c \vee d) \leftarrow a) \wedge ((c \vee d) \leftarrow b) \in \Delta$. By Lemma 9(**x'**), $((c \vee d) \leftarrow a) \wedge ((c \vee d) \leftarrow b) = (c \vee d) \leftarrow (a \vee b)$. Then $(c \vee d) \leftarrow (a \vee b) \in \Delta$. Since $c, d \in \nabla$, $c \vee d \in \nabla$. So we get that for some $e = c \vee d \in \nabla$, $e \leftarrow (a \vee b) \in \Delta$, which gives $a \vee b \in U$. ∎

The canonical frame of an LCPR algebra is the same as the canonical frame of an LCP algebra (Definition 10), i.e., it is a system $(X(W), \preccurlyeq_1, \preccurlyeq_2, C^\star, R^\star, S^\star, Q^\star, I^\star)$. Given the canonical frame of an LCPR algebra, define the following auxiliary ternary relations on $X(W)$: for all $x, y, z \in X(W)$,

$$R^\star_\leftarrow(x, y, z) \quad \textit{iff} \quad (\forall a, b \in W)\, b \leftarrow a \in x_1 \ \&\ a \in y_1 \Longrightarrow b \in z_1 \tag{28}$$

$$R^\star_\rightarrow(x, y, z) \quad \textit{iff} \quad (\forall a, b \in W)\, a \in x_1 \ \&\ a \rightarrow b \in y_1 \Longrightarrow b \in z_1. \tag{29}$$

Note that

Lemma 13. $R^\star = R^\star_\leftarrow = R^\star_\rightarrow$

Proof. We show that $R^\star = R^\star_\leftarrow$. The proof of $R^\star = R^\star_\rightarrow$ is analogous.
(⊆) Assume on the contrary that for some $x, y, z \in X(W)$, **(i)** $R^\star(x, y, z)$ and there exist $a, b \in W$ such that **(ii)** $b \leftarrow a \in x_1$ **(iiii)** $a \in y_1$ **(iv)** $b \notin z_1$. From **(i)**, **(ii)** and **(iii)** it follows that $(b \leftarrow a) \otimes a \in z_1$. By Lemma 9(**iv'**), $(b \leftarrow a) \otimes a \leqslant b$. Since z_1 is a filter, this implies $b \in z_1$ – a contradiction with **(iv)**.

(⊇) Similarly, assume that for some $x, y, z \in X(W)$, **(v)** $R^\star_\leftarrow(x, y, z)$ and there exist $a, b \in W$ such that **(vi)** $a \in x_1$, **(vii)** $b \in y_1$ and **(viii)** $a \otimes b \notin z_1$. By Lemma 9(**vi'**), $a \leqslant (a \otimes b) \leftarrow b$, so from **(vi)**, $(a \otimes b) \leftarrow b \in x_1$, since x_1 is a filter. By **(v)** this gives $a \otimes b \in z_1$ – a contradiction with **(viii)**. ∎

Theorem 7 (Representation theorem for LCPR algebras). *Any LCPR algebra is isomorphic to a subalgebra of the complex algebra of its canonical frame.*

Proof. In view of Theorem 5 it suffices to show that

(i) $h(a \leftarrow b) = h(a) \lhd\!\!- h(b)$
(ii) $h(a \rightarrow b) = h(a) -\!\!\rhd h(b)$.

(i) **(⊆)** Let $x \in h(a \leftarrow b)$. By the definition (9) of the mapping h, this means that **(i.1)** $a \leftarrow b \in x_1$. Assume that $x \notin h(a) \lhd\!\!- h(b)$. Then there are $y, z \in X(W)$ such that **(i.2)** $R^\star(x, y, z)$, **(i.3)** $y \in h(b)$ and **(i.4)** $z \notin h(a)$. From **(i.3)** we get **(i.5)** $b \in y_1$. By Lemma 13, $R^\star = R^\star_\leftarrow$, so from **(i.1)**, **(i.2)**, **(i.5)** and the definition of $R^\star_\leftarrow$, it follows $a \in z_1$, i.e. $z \in h(a)$, which contradicts **(i.4)**.

($\supseteq$) Assume that **(i.6)** $x \notin h(b \leftarrow a)$. We will show that $x \notin h(b) \lhd\!\!- h(a)$. From **(i.6)** we have **(i.7)** $b \leftarrow a \notin x_1$. Define

$$U = \{c \in W : x_1 \cap ((b] \leftarrow \{c\}) \neq \emptyset\},$$

where $(b]$ stands for the ideal generated by b. By Lemma 12, U is an ideal. Suppose that $a \in U$. Then there exists $b' \in W$ such that **(i.8)** $b' \leqslant b$ and **(i.9)** $b' \leftarrow a \in x_1$. By Lemma 9**(iii')** and **(i.8)** we get **(i.10)** $b' \leftarrow a \leqslant b \leftarrow a$. Since x_1 is a filter, **(i.9)** and **(i.10)** imply $b \leftarrow a \in x_1$, which contradicts **(i.7)**. Hence $a \notin U$. Let $[a)$ be the filter generated by a. Then $[a) \cap U = \emptyset$, so $([b), U)$ is a filter–ideal pair. Let (y_1, y_2) be its extension to the maximal filter–ideal pair. Then $[a) \subseteq y_1$ and $U \subseteq y_2$. Since $a \in y_1$, we have **(i.11)** $y \in h(a)$.
Now, consider a set:

$$V = \{c \in W : x_1 \cap (\{c\} \leftarrow y_1) \neq \emptyset\}.$$

By Lemma 12, V is a filter of W. Suppose that $b \in V$. Then there is $c' \in W$ such that **(i.12)** $c' \in y_1$ and **(i.13)** $b \leftarrow c' \in x_1$. By the definition of U, **(i.13)** implies $c' \in U \subseteq y_2$ – a contradiction with **(i.12)**. Hence $b \notin V$. Then $(V, (b])$ is a filter–ideal pair. Let (z_1, z_2) be its extension to the maximal filter–ideal pair. Then **(i.14)** $V \subseteq z_1$ and $(b] \subseteq z_2$. Since $b \in z_2$, we get $b \notin z_1$, so **(i.15)** $z \notin h(b)$.

Finally, consider $c, d \in W$ such that $c \leftarrow d \in x_1$ and $d \in y_1$. Then $c \in V$, so $c \in z_1$ by **(i.14)**. By the definition (28), $R^{\star}_{\leftarrow}(x, y, z)$ holds, and so **(i.16)** $R^{\star}(x, y, z)$ by Lemma 13. Therefore, we have shown that for some $y, z \in X(W)$, **(i.11)**, **(i.15)** and **(i.16)** hold, which means by (27) that $x \notin h(b) \lhd\!\!- h(a)$.

The proof of **(ii)** is similar ■

6 LCPRS Algebras and Frames

In the classical relation algebras relative sum is definable with composition and complement, namely we have $x \oplus y = -(-x; -y)$. In the lattice-based relation algebras sum must be added as a new independent operator. This is the purpose of the present section.

Definition 13. *An **LCPRS algebra** is a system* $(W, \wedge, \vee, \smile, \frown, \otimes, \oplus, \rightarrow, \leftarrow, \Rightarrow, \Leftarrow, 0, 1, 0', 1')$ *such that* $(W, \wedge, \vee, \smile, \otimes, \rightarrow, \leftarrow, 0, 1, 1')$ *is an LCPR algebra,* $\frown$ *is an unary operations in* W *(dual converse),* $\oplus$ *is a binary operations in* W *(sum), and* $\Rightarrow$, $\Leftarrow$ *are binary operations in* W *(dual right and dual left residua of* $\oplus$*) satisfying for all* $a, b, c \in W$,

- **(CPRS.0)** $1 \oplus a = a \oplus 1 = 1$
- **(CPRS.1)** $a^{\frown\frown} = a$
- **(CPRS.2)** $(a \wedge b)^{\frown} = a^{\frown} \wedge b^{\frown}$
- **(CPRS.3)** $a \oplus 0' = 0' \oplus a = a$
- **(CPRS.4)** $a \oplus (b \oplus c) = (a \oplus b) \oplus c$
- **(CPRS.5)** $a \oplus (b \wedge c) = (a \oplus b) \wedge (a \oplus c)$

(CPRS.6) $(a \wedge b) \oplus c = (a \oplus c) \wedge (b \oplus c)$
(CPRS.7) $(a \oplus b)^{\frown} = b^{\frown} \oplus a^{\frown}$
(CPRS.8) $a \oplus b \geqslant c$ *iff* $b \geqslant a \Rightarrow c$
(CPRS.9) $a \oplus b \geqslant c$ *iff* $a \geqslant c \Leftarrow b$
(CPRS.10) $0' \wedge 1' = 0$
(CPRS.11) $0' \vee 1' = 1$. □

Let $L = (W, \wedge, \vee, 0, 1)$ be a bounded lattice. By the *opposite lattice* we mean a lattice $L^{op} = (W, \vee, \wedge, 1, 0)$, where the meet (resp. the join) of L^{op} is the join (resp. the meet) of L and the greatest (resp. the least) element of L^{op} is the least (resp. the greatest) element of L. Observe that the algebra obtained from LCPRS algebra by deleting axioms **(CPRS.10)** and **(CPRS.11)** can be viewed as a join of an LCPR algebra based on the lattice L and an LCPR algebra based on L^{op}. In other words, we have:

Proposition 2. *Let* $(W, \wedge, \vee, ^{\smile}, ^{\frown}, \otimes, \oplus, \rightarrow, \leftarrow, \Rightarrow, \Leftarrow, 0, 1, 0', 1')$ *be an LCPRS algebra. Then* $(W, \vee, \wedge, ^{\frown}, \oplus, \Rightarrow, \Leftarrow, 1, 0, 0')$ *is an LCPR algebra.*

Proof. Straightforward from Definitions 11 and 13. ■

Remark 1. *If follows that properties of operations* $^{\frown}$, $\oplus$, $\Rightarrow$, *and* $\Leftarrow$ *can be easily obtained from the analogous properties of the operations* $^{\smile}$, $\otimes$, $\rightarrow$, $\leftarrow$, *respectively.* □

Remark 2. *Note that axioms* **(CPRS.10)** *and* **(CPRS.11)** *provide a connection between the LCPR part of an LCPRS algebra L and the LCPR part of L based on its opposite part.* □

Definition 14. *An* ***LCPRS frame*** *is a system* $(X, \leqslant_1, \leqslant_2, C, \Gamma, R, S, Q, \Theta, \Upsilon, \Omega, I, J)$ *such that* $(X, \leqslant_1, \leqslant_2, C, R, S, Q, I)$ *is an LCPR frame,* Γ *is a mapping* $\Gamma : X \rightarrow X$, Θ, Υ, Ω *are ternary relations on* X *and* $J \subseteq X$ *is a unary relation on* X *such that the following conditions are satisfied for all* $x, x', y, y', z, z' \in X$,

Monotonicity conditions:

(MCPRS.1) $x \leqslant_1 x' \Longrightarrow \Gamma(x) \leqslant_1 \Gamma(x')$
(MCPRS.2) $x \leqslant_2 x' \Longrightarrow \Gamma(x) \leqslant_2 \Gamma(x')$
(MCPRS.3) $\Theta(x, y, z)$ & $x' \leqslant_2 x$ & $y' \leqslant_2 y$ & $z \leqslant_2 z' \Longrightarrow \Theta(x', y', z')$
(MCPRS.4) $\Upsilon(x, y, z)$ & $x \leqslant_1 x'$ & $y' \leqslant_2 y$ & $z' \leqslant_1 z \Longrightarrow \Upsilon(x', y', z')$
(MCPRS.5) $\Omega(x, y, z)$ & $x' \leqslant_2 x$ & $y \leqslant_1 y'$ & $z' \leqslant_1 z \Longrightarrow \Omega(x', y', z')$
(MCPRS.6) $J(x)$ & $x \leqslant_2 x' \Longrightarrow J(x')$

Stability conditions:

(SCPRS.1) $\Gamma(\Gamma(x)) = x$
(SCPRS.2) $\Theta(x, y, z) \Longrightarrow \exists x'' \in X\ (x \leqslant_2 x''$ & $\Upsilon(x'', y, z))$
(SCPRS.3) $\Theta(x, y, z) \Longrightarrow \exists y'' \in X\ (y \leqslant_2 y''$ & $\Omega(x, y'', z))$
(SCPRS.4) $\Upsilon(x, y, z) \Longrightarrow \exists z'' \in X\ (z \leqslant_1 z''$ & $\Theta(x, y, z''))$

(SCPRS.5) $\Omega(x, y, z) \Longrightarrow \exists z'' \in X\ (z \leqslant_1 z''\ \&\ \Theta(x, y, z''))$
(SCPRS.6) $\exists u \in X(\Theta(x, y, u) \,\&\, \Upsilon(u, z, y)) \Longrightarrow \exists w \in X(\Theta(y, z', w) \,\&\, \Omega(x, w, y))$
(SCPRS.7) $\exists u \in X(\Theta(x, y, u)\ \&\ \Omega(z, u, z')) \Longrightarrow$
$\exists w \in X(\Theta(z, x, w)\ \&\ \Upsilon(w, y, z'))$
(SCPRS.8) $J(x)\ \&\ (\Theta(x, y, z)\ or\ \Theta(y, x, z)) \Longrightarrow y \leqslant_2 z$
(SCPRS.9) $\exists u \in X(J(u)\ \&\ \Upsilon(u, x, x))$
(SCPRS.10) $\exists u \in X(J(u)\ \&\ \Omega(x, u, x))$
(SCPRS.11) $\Omega(x, y, z) = \Upsilon(\Gamma(y), \Gamma(x), \Gamma(z))$
(SCPRS.12) $lr(I) \cap l(J) = \emptyset$
(SCPRS.13) $r(I) \cap rl(J) = \emptyset$. □

Let $(X, \leqslant_1, \leqslant_2)$ be a doubly ordered set. By the *opposite doubly ordered set* we mean a structure $(X, \leqslant_1^{op}, \leqslant_2^{op})$, where $\leqslant_1^{op} = \leqslant_2$ and $\leqslant_2^{op} = \leqslant_1$. Observe that the frame obtained from the LCPRS frame by deleting axioms **(SCPRS.12)** and **(SCPRS.13)** can be viewed as a join of the LCPR frame based on a doubly ordered set $(X, \leqslant_1, \leqslant_2)$ with the LCPR frame based on the opposite doubly ordered set $(X, \leqslant_1^{op}, \leqslant_2^{op})$. Therefore, we have:

Proposition 3. *Let $(X, \leqslant_1, \leqslant_2, C, \Gamma, R, S, Q, \Theta, \Upsilon, \Omega, I, J)$ be an LCPRS frame. Then $(X, \leqslant_2, \leqslant_1, \Gamma, \Theta, \Upsilon, \Omega, J)$ is an LCPR frame.*

Proof. Straightforward from the definition of LCPR frame and Definition 14. ■

Remark 3. *From the above proposition it follows easily that the properties of the relations Γ, Θ, Υ, Ω, and J can be obtained from the properties of the relations C, R, S, Q, and I, respectively, by interchanging the roles of the orderings $\leqslant_1$ and $\leqslant_2$.* □

Remark 4. *Note that axioms* **(SCPRS.12)** *and* **(SCPRS.13)** *provide a connection between the LCPR part of an LCPRS frame and its opposite part.* □

Given an LCPRS frame $(X, \leqslant_1, \leqslant_2, C, \Gamma, R, S, Q, \Theta, \Upsilon, \Omega, I, J)$, let us define the following mappings ${}^{\curlywedge} : 2^X \to 2^X$ and $\oplus_\Omega$, $\oplus_\Upsilon$, $\boxplus$, $\Rightarrow\!\!\triangleright$, $\triangleleft\!\!\Leftarrow : 2^X \times 2^X \to 2^X$ by: for all $A, B \subseteq X$,

$$A^{\curlywedge} = \{\Gamma(x) \in X : x \in A\} \tag{30}$$

$$A \oplus_\Omega B = \{z \in X : \forall x, y \in X\ (\Omega(x, y, z)\ \&\ x \in r(A) \Longrightarrow y \in B\} \tag{31}$$

$$A \oplus_\Upsilon B = \{z \in Z : \forall x, y \in X\ (\Upsilon(x, y, z)\ \&\ y \in r(B) \Longrightarrow x \in A\} \tag{32}$$

$$A \boxplus B = A \oplus_\Omega B. \tag{33}$$

$$A \Rightarrow\!\!\triangleright B = \{x \in X : (\forall y, z \in X)(\Theta(y, x, z)\ \&\ y \in A \Longrightarrow z \in B)\} \tag{34}$$

$$B \triangleleft\!\!\Leftarrow A = \{x \in X : (\forall y, z \in X)(\Theta(x, y, z)\ \&\ y \in A \Longrightarrow z \in B)\}. \tag{35}$$

Moreover, put

$$\mathbf{0'} = l(J). \tag{36}$$

Definition 15. *Let* $(X, \leqslant_1, \leqslant_2, C, \Gamma, R, S, Q, \Theta, \Upsilon, \Omega, I, J)$ *be an LCPRS frame. The* ***complex algebra of X*** *is a structure* $(L(X), \sqcap, \sqcup, ^{\curlyvee}, ^{\curlywedge}, \boxtimes, \boxplus, \rightharpoonup, \leftharpoonup, \Rightarrow, \Leftarrow, \mathbf{0}, \mathbf{1}, \mathbf{0'}, \mathbf{1'})$ *such that* $L(X)$ *is the family of all l–stable subsets of* X*, the operations* $\sqcap$, $\sqcup$, $^{\curlyvee}$, $^{\curlywedge}$, $\boxtimes$, $\boxplus$, $\rightharpoonup$, $\leftharpoonup$, $\Rightarrow$, $\Leftarrow$ *are respectively defined by (4), (3), (11), (30), (14), (33), (26), (27), (34), (35), and the constants* $\mathbf{0}$, $\mathbf{1}$, $\mathbf{0'}$, *and* $\mathbf{1'}$ *are given by (5), (6), (15) and (36), respectively.* □

We will show now that complex algebras of LCPRS frames are LCPRS algebras.

Theorem 8. *The complex algebra of an LCPRS frame is an LCPRS algebra.*

Proof. Since J is $\leqslant_2$–increasing by **(MCPRS.6)**, $L(J)$ is l–stable. From Theorem 6, Proposition 3, and Remark 3 it follows that we only need to show that the connecting axioms **(CPRS.10)** and **(CPRS.11)** hold, i.e.,

(i) $\mathbf{0'} \sqcap \mathbf{1'} = \mathbf{0}$
(ii) $\mathbf{0'} \sqcup \mathbf{1'} = \mathbf{1}$.

(i) $\mathbf{0'} \sqcap \mathbf{1'} = lr(I) \cap l(J) = \emptyset$ by **(SCPRS.12)**.

(ii) By the definitions (15), (36), and (4), $\mathbf{0'} \sqcup \mathbf{1'} = l(rlr(I) \cap rl(J))$. Also, by Lemma 4**(ii)**, $I \subseteq lr(I)$, so $rlr(I) \subseteq r(I)$. Next, $rlr(I) \cap rl(J) \subseteq r(I) \cap rl(J) = \emptyset$ by **(SCPRS.13)**. Hence we have: $rlr(I) \cap rl(J) = \emptyset$, so $l(rlr(I) \cap rl(J)) = l(\emptyset) = X(W)$ ■

Let $(W, \wedge, \vee, \smile, \frown, \otimes, \oplus, \rightarrow, \leftarrow, \Rightarrow, \Leftarrow, 0, 1, 0', 1')$ be an LCPRS algebra. As before, by $FIP(X)$ and (resp. $X(W)$) we denote the family of all filter–ideals pairs (resp. maximal filter–ideal pairs) of W.

Lemma 14. *Let* $(W, \wedge, \vee, \smile, \frown, \otimes, \oplus, \rightarrow, \leftarrow, \Rightarrow, \Leftarrow, 0, 1, 0', 1')$ *be an LCPRS algebra. Then for every* $a \in W$, $l(\{x \in X(W) : a \in x_2\}) = \{x \in X(W) : a \in x_1\}$.

Proof. ($\subseteq$) Let $a \notin x_1$. It follows that $x_1 \cap (a] = \emptyset$, so $(x_1, (a])$ is a filter–ideal pair. Let y be its extension to the maximal filter–ideal pair. Hence $x_1 \subseteq y_1$ and $a \in y_2$. It follows that $x \notin l(\{x \in X(W) : a \in x_2\})$.
($\supseteq$) Let $a \in x_1$. Take $y \in X(W)$ such that $x_1 \subseteq y_1$. Then $a \in y_1$, whence $a \notin y_2$. ■

Define a mapping $\Gamma^\star : FIP(W) \to FIP(W)$ by: for every $x \in FIP(W)$,

$$\Gamma^\star(x) = (x_1{}^\frown, x_2{}^\frown). \tag{37}$$

Furthermore, let us define the following ternary relations on $X(W)$: for all $x, y, z \in X(W)$,

$$\Theta^\star(x, y, z) \iff (\forall a, b \in W)\ a \in x_2 \ \&\ b \in y_2 \Longrightarrow a \oplus b \in z_2 \tag{38}$$

$$\Omega^\star(x, y, z) \iff (\forall a, b \in W)\ a \in x_2 \ \&\ a \oplus b \in z_1 \Longrightarrow b \in y_1 \tag{39}$$

$$\Upsilon^\star(x, y, z) \iff (\forall a, b \in W)\ b \in y_2 \ \&\ a \oplus b \in z_1 \Longrightarrow a \in x_1. \tag{40}$$

Also, put

$$J^\star = \{x \in X(W) : 0' \in x_2\} \tag{41}$$

Definition 16. *Let* $(W, \wedge, \vee, \smile, \frown, \otimes, \oplus, \rightarrow, \leftarrow, \Rightarrow, \Leftarrow, 0, 1, 0', 1')$ *be an LCPRS algebra. The* ***canonical frame of*** W *is a structure* $(X(W), \preccurlyeq_1, \preccurlyeq_2, C^\star, \Gamma^\star, R^\star, Q^\star, S^\star, \Theta^\star, \Omega^\star, \Upsilon^\star, I^\star, J^\star)$ *such that* $(X(W), \preccurlyeq_1, \preccurlyeq_2, C^\star, R^\star, Q^\star, S^\star, I^\star)$ *is the canonical frame of the LCPR part* $(W, \wedge, \vee, ^\smile, \otimes, \rightarrow, \leftarrow, 0, 1, 1')$ *of* W *and* $\Gamma^\star$, $\Theta^\star$, $\Omega^\star$, $\Upsilon^\star$, *and* $J^\star$ *are defined by (37)–(41).* □

Theorem 9. *The canonical frame of an LCPRS algebra is an LCPRS frame.*

Proof. We have to show that the conditions **(SCPRS.12)** and **(SCPRS.13)** hold in the canonical frame of an LCPRS algebra. The remaining conditions follow from Theorem 6, Proposition 3, and Remark 1.

We show that $lr(I^\star) \cap l(J^\star) = \emptyset$. Note that

$$
\begin{aligned}
& lr(I^\star) \cap l(J^\star) \\
&\quad = lr(\{x \in X(W) : 1' \in x_1\}) \cap l(\{x \in X(W) : 0' \in x_2\}) \\
&\quad = l(\{x \in X(W) : 1' \in x_2\}) \cap l(\{x \in X(W) : 0' \in x_2\}) && \text{by Theorem 1}\textbf{(i)} \\
&\quad = \{x \in X(W) : 1' \in x_1\} \cap \{x \in X(W) : 0' \in x_1\} && \text{by Lemma 14} \\
&\quad = \{x \in X(W) : 1' \in x_1 \ \&\ 0' \in x_1\} \\
&\quad \subseteq \{x \in X(W) : 1' \wedge 0' \in x_1\}
\end{aligned}
$$

However, by **(CPRS.10)**, $1' \wedge 0' = 0$. Since x_1 is a proper filter, $0 \notin x_1$, so we have $\{x \in X(W) : 1' \wedge 0' \in x_1\} = \emptyset$, and consequently $lr(I^\star) \cap l(J^\star) = \emptyset$.

Now we prove that $r(I) \cap rl(J) = \emptyset$. Observe:

$$
\begin{aligned}
& r(I) \cap rl(J) \\
&\quad = r(\{x \in X(W) : 1' \in x_1\}) \cap rl(\{x \in X(W) : 0' \in x_2\}) \\
&\quad = r(\{x \in X(W) : 1' \in x_1\}) \cap r(\{x \in X(W) : 0' \in x_1\}) && \text{by Lemma 14} \\
&\quad = \{x \in X(W) : 1' \in x_2\} \cap \{x \in X(W) : 0' \in x_2\} && \text{Theorem 1}\textbf{(i)} \\
&\quad = \{x \in X(W) : 1' \in x_2 \ \&\ 0' \in x_2\} \\
&\quad \subseteq \{x \in X(W) : 1' \vee 0' \in x_2\}.
\end{aligned}
$$

Since $1' \vee 0' = 1 \notin x_2$, it follows that $\{x \in X(W) : 1' \vee 0' \in x_2\} = \emptyset$. In conclusion, $r(I) \cap rl(J) = \emptyset$. ■

We conclude this section by the following representation theorem.

Theorem 10 (Representation theorem for LCPRS algebras)
Any LCPRS algebra is isomorphic to a subalgebra of the complex algebra of its canonical frame.

Proof Taking into account Propositions 2, 3, and Remarks 1, 3 the proof is analogous to the proof of Theorem 7. ■

7 Conclusion

In this paper we have studied not necessarily distributive lattices with operators that are the abstract counterparts to the converse and composition of binary relations. On the algebraic side, we have presented relational representation theorems for these classes of algebras. These theorems are obtained by a suitable extensions of Urquhart's representation theorem for lattices [19]; here, we have stressed the relational aspect of representability and have omitted the topological aspect.

On the logical side, with every class of algebras studied in the paper we have associated an appropriate class of frames. These frames constitute a basis of a Kripke-style semantics for the logics whose algebraic semantics is determined by the classes of algebras presented in the paper. The representation theorems would enable us to prove completeness of the logics. For a detailed elaboration of the respective relational logics one can follow the developments in [1] and [17].

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