Relational Representation Theorems for Lattices with Negations: A Survey

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Abstract. Relational representation theorems are presented in a unified framework for general (including non-distributive) lattices endowed with various negation operations.

1 Introduction

We present relational representation theorems in a unified framework for the lattice based algebras of logics with various negation operations, for both general (including non-distributive) lattices and for distributive lattices.

The negation operations include sufficiency or negative necessity as negation, Heyting negation, pseudo-complement, De Morgan negation and ortho-negation. Part of the results are carried out within the framework of Urquhart's representation theorem for lattices [17] and Allwein–Dunn developments on Kripke semantics for linear logic [1] which we jointly call Urquhart–Allwein–Dunn – framework, generalized to a duality between the algebras and abstract frames (relational systems). In order to have it in the same unified framework, we also include representations of distributive lattices with relative pseudo-complement, with relative pseudo-complement and minimal negation (of Johansson), with De Morgan negation, and Boolean algebras with sufficiency (negative necessity) operator. The distributive lattice cases contain known results, but we include them to present all results together in the unified framework.

Our framework, based on a generalization of the Urquhart–Allwein–Dunn representation, requires the following steps:

Step 1. A class of algebras is given. Its signature is that of lattices extended by a unary operation corresponding to negation.

Step 2. We define a class of relational structures (frames) that provide a Kripke-style semantics for the logic whose algebraic semantics is determined by the class of algebras in question.

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Step 3. For any algebra $\boldsymbol{W}$ of the given class we define its canonical frame. The universe $X(\boldsymbol{W})$ of this frame consists of all pairs (x_1, x_2) such that x_1 is a filter and x_2 is an ideal of the lattice reduct of $\boldsymbol{W}$ and (x_1, x_2) is a maximal disjoint pair. Relations are defined on $X(\boldsymbol{W})$ which correspond in an appropriate way to the operations of the algebra.

Step 4. For any frame $\boldsymbol{X}$ we define its complex algebra. The universe of the complex algebra is a family $L(\boldsymbol{X})$ of special subsets of X referred to as ℓ-stable sets.

Step 5. We prove a representation theorem saying that every algebra $\boldsymbol{W}$ is embeddable into the complex algebra of its canonical frame, i.e., $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W}))$. The universe of the representation algebra consists of subrelations of $X(\boldsymbol{W})$.

Below we list several well known examples of classical representations giving, in particular, the algebras, frames, complex algebras and canonical frames.

The class of Boolean algebras has the class of sets as its class of frames which can be seen as relational systems with the empty family of relations. A canonical frame is the set of ultrafilters of a given algebra. A complex algebra is the powerset algebra of the set of ultrafilters. The Stone representation theorem says that a given Boolean algebra is embeddable into this powerset algebra.

The class of distributive lattices has the class of partial orders as its class of frames. A canonical frame is the set of prime filters of a given distributive lattice with set inclusion. A complex algebra of a frame is a family of $\leq$-increasing subsets with the set union and intersection. The representation theorem says that a given distributive lattice is embeddable into the complex algebra of the canonical frame.

The class of ortholattices has the class of orthogonality spaces (sets with orthogonality relations, i.e., irreflexive and symmetric relations $\perp$, first defined by Foulis and Randall) as its class of frames. A canonical frame is the set of proper filters of a given ortholattice with the set inclusion and ortho-negation defined by orthogonality relation $\perp$: for two proper filters x and y, $x \perp y$ iff there is an element a such that $-a \in x$ and $a \in y$. A complex algebra of a frame is a family of regular subsets of this frame defined as follows: first $a \perp Y$ iff for all $b \in Y$, $a \perp b$ and $Y^* = \{a : a \perp Y\}$; now Y is $\perp$-*regular* iff $Y = Y^{**}$. The representation theorem of Goldblatt [9] says that a given ortholattice is embeddable into the lattice of regular subsets of the orthogonality space.

The framework described above serves, on the one hand, as a tool for investigation of classes of lattices with negation operations and, on the other hand, as a means for developing Kripke-style semantics for the logics whose algebraic semantics is given. Then representation theorems play an essential role in proving completeness of the logics with respect to a Kripke-style semantics determined by a class of frames associated with a given class of algebras. In this paper we deal mainly with the algebraic aspects of lattices with negation. The framework presented above has been used in [13] and [7] in the context of lattice-based modal logics. It has been applied to lattice-based relation algebras in [6] and to double residuated lattices in [11] and [12]. In our relational representations we will provide definitions of abstract relational systems or frames such that particular properties of the relations in frames correspond to particular types of negations.

2 Negations

We follow J.M. Dunn's analysis of negations, also known as "Dunn's Kite of Negations". Dunn's study of negation in non-classical logics as a negative modal operator is an application of his gaggle theory, cf. [5], which is a generalization of the Jonsson-Tarski Theorem. In gaggle theory, negation $\neg$ is treated as a Galois connective on an underlying poset or bounded lattice. This treatment requires the Galois condition:

(Gal) $\quad a \leq \neg b \;\Leftrightarrow\; b \leq \neg a$

Further analysis of negation on a bounded lattice leads to the following conditions for $\neg$ (we always assume that 0 is the least element and 1 the greatest):

(Suff1)	$\neg(a \vee b) = \neg a \wedge \neg b$	(Sufficiency 1)
(Suff2)	$\neg 0 = 1$	(Sufficiency 2)
(WCon)	$a \leq b \Rightarrow \neg b \leq \neg a$	(Weak Contrapositive, Preminimal)
(Weak$\neg\neg$)	$a \leq \neg\neg a$	(Weak Double Negation)
(Abs)	$a \wedge \neg a = 0$	(Absurdity, Intuitionistic)
(DeM)	$\neg\neg a \leq a$	(De Morgan, Strong Double Negation)

Lemma 2.1. *In any bounded lattice with an operation $\neg$ the following implications hold:*

(a) (Suff1) $\Rightarrow$ (WCon)
(b) (Gal) $\Rightarrow$ (Suff2)
(c) (Gal) $\Leftrightarrow$ (Suff1) and (Weak$\neg\neg$)
(d) (Gal) $\Leftrightarrow$ (WCon) and (Weak$\neg\neg$)

Proof. We show only the implication (Gal) $\Rightarrow$ (Suff1) of (c). By (Weak$\neg\neg$), $a \leq a \vee b \leq \neg\neg(a \vee b)$, hence by (Gal), $\neg(a \vee b) \leq \neg a$. Similarly, $\neg(a \vee b) \leq \neg b$, so we have $\neg(a \vee b) \leq \neg a \wedge \neg b$. By (Gal), $a \leq \neg(\neg a \wedge \neg b)$ and $b \leq \neg(\neg a \wedge \neg b)$, so $a \vee b \leq \neg(\neg a \wedge \neg b)$ hence $\neg a \wedge \neg b \leq \neg(a \vee b)$.

As noted in (b), one may derive $\neg 0 = 1$ from (Gal) or its equivalents. If one has, in addition, either (Abs) or (DeM) then one may also derive $\neg 1 = 0$. Also note that from (WCon) one may derive $\neg a \vee \neg b \leq \neg(a \wedge b)$. Lastly, by (c), note that the class of bounded lattices with negation satisfying the Galois condition (Gal) is a variety (i.e., an equational class) with equational axioms (Suff1) and (Weak$\neg\neg$).

We shall consider five types of negation on bounded (non-distributive) lattices. In each case, the negation satisfies (Suff1) and (Suff2); in the first and weakest case we consider just these two axioms. The next case is Heyting negation in which the negation satisfies (WCon), (Weak$\neg\neg$) (or, equivalently, just (Gal)) and (Abs); such algebras are also called 'weakly pseudo-complemented lattices'. Thereafter, we consider 'pseudo-complemented lattices' which satisfy, in addition, the following pseudo-complement quasi-identity:

(Pcq) $a \wedge b = 0 \quad \Rightarrow \quad a \leq \neg b.$

Its converse is derivable from the identity (Abs). In the case of De Morgan negation the identities (Gal) and (DeM) are assumed giving the class of 'De Morgan lattices'. Finally, ortho-negation is considered which satisfies (Gal) and both (Abs) and (DeM); these algebras are known as 'ortholattices'.

In the distributive lattice case, we consider 'relatively pseudo-complemented lattices', that is, where the 'residuum' (or relative pseudo-complement) of $\wedge$ exists, denoted $\rightarrow$. One may induce a negation by choosing any element ∂ in the lattice and defining $\neg x = x \rightarrow \partial$. The negation induced in this way is a minimal negation in the sense of Johansson and Rasiowa. This negation satisfies (Gal) (hence also (WCon) and (Weak$\neg\neg$)) but not necessarily (Abs) (unless the chosen element ∂ is the least element, in which case we have Heyting algebras). It does not necessarily satisfy (DeM) either, so we also consider distributive lattices in which (DeM) is added, namely, 'De Morgan algebras'. Adding both (Abs) and (DeM) to (Gal) results in the class of Boolean algebras. At the end we consider Boolean algebras with sufficiency (or negative necessity) operator.

Part I Non-distributive Lattices

3 Preliminaries

We give here the necessary background on the relational representation of non-distributive lattices in the style of Urquhart [17] (see also [6] and [13]). The representations of non-distributive lattices with negations is built on top of this framework.

Let X be a non-empty set and let $\leqslant_1$ and $\leqslant_2$ be two quasi-orders on X. The structure $\langle X, \leqslant_1, \leqslant_2\rangle$ is called a *doubly ordered set* if it satisfies:

$$(\forall x, y)((x \leqslant_1 y \text{ and } x \leqslant_2 y) \Rightarrow x = y). \tag{1}$$

For a doubly ordered set $\boldsymbol{X} = \langle X, \leqslant_1, \leqslant_2\rangle$, $A \subseteq X$ is $\leqslant_1$*–increasing* (resp., $\leqslant_2$*–increasing*) if, for all $x, y \in X$, $x \in A$ and $x \leqslant_1 y$ (resp., $x \leqslant_2 y$) imply $y \in A$. We define two mappings $\ell, r : 2^X \rightarrow 2^X$ by

$$\ell(A) = \{x \in X : \forall y(x \leqslant_1 y \Rightarrow y \notin A)\} \tag{2}$$

$$r(A) = \{x \in X : \forall y(x \leqslant_2 y \Rightarrow y \notin A)\}. \tag{3}$$

Then $A \subseteq X$ is called ℓ*–stable* (resp., r*–stable*) if $\ell(r(A)) = A$ (resp., $r(\ell(A)) = A$). The set of all ℓ-stable subsets of X will be denoted by $L(\boldsymbol{X})$.

Lemma 3.1. [6],[13] *If $\langle X, \leqslant_1, \leqslant_2\rangle$ is a doubly ordered set then, for all $A \subseteq X$,*

(a) $\ell(A)$ is $\leqslant_1$–increasing and $r(A)$ is $\leqslant_2$–increasing,
(b) if A is $\leqslant_1$–increasing, then $A \subseteq \ell(r(A))$,
(c) if A is $\leqslant_2$–increasing, then $A \subseteq r(\ell(A))$.

Lemma 3.2. [17] *Let $\langle X, \leqslant_1, \leqslant_2\rangle$ be a doubly ordered set. Then the mappings ℓ and r form a Galois connection between the lattices of $\leqslant_1$–increasing and $\leqslant_2$–increasing subsets of X. In particular, for every $\leqslant_1$–increasing set A and $\leqslant_2$–increasing set B,*

$$A \subseteq \ell(B) \text{ iff } B \subseteq r(A).$$

Let $\boldsymbol{X} = \langle X, \leqslant_1, \leqslant_2\rangle$ be a doubly ordered set. Define two binary operations $\wedge^C$ and $\vee^C$ on 2^X and two constants 0^C and 1^C as follows: for all $A, B \subseteq X$,

$$A \wedge^C B = A \cap B \tag{4}$$

$$A \vee^C B = \ell(r(A) \cap r(B)) \tag{5}$$

$$0^C = \emptyset \tag{6}$$

$$1^C = X. \tag{7}$$

Observe that the definition of $\vee^C$ in terms of $\wedge^C$ resembles a De Morgan law with two different negations. In [17], $\boldsymbol{L}(\boldsymbol{X}) = \langle L(\boldsymbol{X}), \wedge^C, \vee^C, 0^C, 1^C\rangle$ is shown to be a bounded lattice; it is called the ***complex algebra*** of $\boldsymbol{X}$.

Let $\boldsymbol{W} = \langle W, \wedge, \vee, 0, 1\rangle$ be a bounded lattice. By a *filter-ideal pair* of $\boldsymbol{W}$ we mean a pair (x_1, x_2) such that x_1 is a filter of $\boldsymbol{W}$, x_2 is an ideal of $\boldsymbol{W}$ and $x_1 \cap x_2 = \emptyset$. The family of all filter-ideal pairs of $\boldsymbol{W}$ will be denoted by $FIP(\boldsymbol{W})$. Define the following three quasi-ordering relations: for any (x_1, x_2), $(y_1, y_2) \in FIP(\boldsymbol{W})$,

$$\begin{aligned}
(x_1, x_2) &\leqslant_1 (y_1, y_2) \quad \text{iff} \quad x_1 \subseteq y_1 \\
(x_1, x_2) &\leqslant_2 (y_1, y_2) \quad \text{iff} \quad x_2 \subseteq y_2 \\
(x_1, x_2) &\leqslant (y_1, y_2) \quad \text{iff} \quad (x_1, x_2) \leqslant_1 (y_1, y_2) \text{ and } (x_1, x_2) \leqslant_2 (y_1, y_2).
\end{aligned}$$

We say that $(x_1, x_2) \in FIP(\boldsymbol{W})$ is *maximal* if it is maximal with respect to $\leqslant$. We denote by $X(\boldsymbol{W})$ the set of all maximal filter-ideal pairs of $\boldsymbol{W}$. Note that $X(\boldsymbol{W})$ is a binary relation on 2^W. In the sequel, if we write $x \in X(\boldsymbol{W})$, we shall assume that $x = (x_1, x_2)$ where x_1 denotes the filter and x_2 denotes the ideal. The same convention holds for y, z, etc. It was shown in [17] that for any $x \in FIP(\boldsymbol{W})$ there exists $y \in X(\boldsymbol{W})$ such that $x \leqslant y$; in this case, we say that x has been *extended* to y.

If $\boldsymbol{W} = \langle W, \wedge, \vee, 0, 1\rangle$ is a bounded lattice then the ***canonical frame*** of $\boldsymbol{W}$ is defined as the relational structure $\boldsymbol{X}(\boldsymbol{W}) = \langle X(\boldsymbol{W}), \leqslant_1, \leqslant_2\rangle$.

Consider the complex algebra $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W}))$ of the canonical frame of a bounded lattice $\boldsymbol{W}$. Note that $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W}))$ is an algebra of subrelations of $X(\boldsymbol{W})$. Define a mapping $h : W \to 2^{X(\boldsymbol{W})}$ by

$$h(a) = \{x \in X(\boldsymbol{W}) : a \in x_1\}.$$

Then h is a map from $\boldsymbol{W}$ to $L(\boldsymbol{X}(\boldsymbol{W}))$ and, moreover, we have the following result.

Proposition 3.1. [17] *For every bounded lattice $\boldsymbol{W}$, h is a lattice embedding of $\boldsymbol{W}$ into $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W}))$.*

The following theorem is a weak version of Urquhart's result.

Theorem 3.1 (Representation theorem for lattices). *Every bounded lattice is embeddable into the complex algebra of its canonical frame.*

4 Lattices with Sufficiency (Negative Necessity) Operator

By a *lattice with a sufficiency operator* we mean an algebra $\boldsymbol{W} = \langle W, \wedge, \vee, \neg, 0, 1\rangle$ which is a bounded lattice with a unary operation $\neg$, called a *sufficiency operator*, satisfying:

(Suff1) $\neg(a \vee b) = \neg a \wedge \neg b$
(Suff2) $\neg 0 = 1$.

Such operators are also called 'negative necessity'. (Note that such operators are antitone.) The name is due to its modal interpretation (cf. Orłowska, E., Vakarelov, D. [13]). The operator $[R]\neg$, which is the composition of the classical necessity operator $[R]$ with the classical negation, is a sufficiency operator. Recall that, given a Kripke frame $\langle X, R\rangle$, where R is a binary relation on X and $A \subseteq X$, the classical necessity is defined by

$$[R]A = \{x \in X : \forall y(xRy \Rightarrow y \in A)\}.$$

Let $\mathcal{LS}$ denote the variety of all lattices with a sufficiency operator. The following definitions and results are based on the treatment of sufficiency in [13].

Let $\mathcal{R}_{LS}$ denote the class of all *sufficiency frames*, i.e., relational structures of the type $\boldsymbol{X} = \langle X, \leqslant_1, \leqslant_2, R, S\rangle$, where $\langle X, \leqslant_1, \leqslant_2\rangle$ is a doubly ordered set (i.e., $\leqslant_1$ and $\leqslant_2$ are quasi-orders satisfying (1)) and R and S are binary relations on X such that the following hold:

(Mono R) $(x' \leqslant_1 x \text{ and } xRy \text{ and } y \leqslant_2 y') \Rightarrow x'Ry'$
(Mono S) $(x \leqslant_2 x' \text{ and } xSy \text{ and } y' \leqslant_1 y) \Rightarrow x'Sy'$
(SC $\mathrm{R_S}$) $xRy \Rightarrow (\exists x' \in X)(x \leqslant_1 x' \text{ and } x'Sy)$
(SC $\mathrm{S_R}$) $xSy \Rightarrow (\exists y' \in X)(y \leqslant_1 y' \text{ and } xRy')$.

The conditions (Mono R) and (Mono S) are called sufficiency monotonicity conditions, and (SC $\mathrm{R_S}$) and (SC $\mathrm{S_R}$) are called sufficiency stability conditions.

Unary operators $[R]$ and $\langle S\rangle$ are defined on 2^X as follows. For all $A \subseteq X$,

$[R]A = \{x \in X : \forall y(xRy \Rightarrow y \in A)\}$,
$\langle S\rangle A = \{x \in X : \exists y(xSy \text{ and } y \in A)\}$.

For each $\boldsymbol{W} \in \mathcal{LS}$ we define the **canonical frame** of $\boldsymbol{W}$ as the relational structure $\boldsymbol{X}(\boldsymbol{W}) = \langle X(\boldsymbol{W}), \leqslant_1, \leqslant_2, R^c, S^c\rangle$, where $X(\boldsymbol{W})$ is the set of all maximal disjoint filter-ideal pairs of $\boldsymbol{W}$ and, for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in X(\boldsymbol{W})$,

$$x \leqslant_1 y \text{ iff } x_1 \subseteq y_1$$
$$x \leqslant_2 y \text{ iff } x_2 \subseteq y_2$$
$$xR^cy \text{ iff } \forall a(\neg a \in x_1 \Rightarrow a \in y_2)$$
$$xS^cy \text{ iff } \forall a(a \in y_1 \Rightarrow \neg a \in x_2).$$

Lemma 4.1. [13] *If* $\boldsymbol{W} \in \mathcal{LS}$ *then* $\boldsymbol{X}(\boldsymbol{W}) \in \mathcal{R}_{LS}$.

Let $\boldsymbol{X} = \langle X, \leqslant_1, \leqslant_2, R, S \rangle \in \mathcal{R}_{LS}$. Then $\langle X, \leqslant_1, \leqslant_2 \rangle$ is a doubly ordered set hence we may consider its complex algebra $\langle L(\boldsymbol{X}), \wedge^C, \vee^C, 0^C, 1^C \rangle$, where $L(\boldsymbol{X})$ is the set of ℓ-stable sets (see definitions (2) and (3)) and the operations are defined as in (4–7). We extend this definition to define the ***complex algebra*** of $\boldsymbol{X}$ as $\boldsymbol{L}(\boldsymbol{X}) = \langle L(\boldsymbol{X}), \wedge^C, \vee^C, \neg^C, 0^C, 1^C \rangle$ where, for all $A \subseteq X$,

$$\neg^C A = [R]r(A).$$

Lemma 4.2. [13] *If* $\boldsymbol{X} \in \mathcal{R}_{LS}$ *then* $\boldsymbol{L}(\boldsymbol{X}) \in \mathcal{LS}$.

Let $\boldsymbol{W} = \langle W, \wedge, \vee, \neg, 0, 1 \rangle \in \mathcal{LS}$. By the above lemmas, we have $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W})) \in \mathcal{LS}$ as well. Recall that the function $h : W \to L(\boldsymbol{X}(\boldsymbol{W}))$ defined by

$$h(a) = \{x \in X(\boldsymbol{W}) : a \in x_1\}$$

is an embedding of the lattice part of $\boldsymbol{W}$ into $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W}))$. Moreover, h also preserves negation, hence we have the following result.

Theorem 4.1. [13] *Each* $\boldsymbol{W} \in \mathcal{LS}$ *is embeddable into* $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W}))$.

5 Lattices with Heyting Negation

A *weakly pseudo-complemented lattice* is an algebra $\boldsymbol{W} = \langle W, \wedge, \vee, \neg, 0, 1 \rangle$ which is a bounded lattice with a unary operation $\neg$ satisfying:

(WCon)	$a \leq b \Rightarrow \neg b \leq \neg a$
(Weak¬¬)	$a \leq \neg\neg a$
(Abs)	$a \wedge \neg a = 0$

We denote by $\mathcal{W}$ the variety of all weakly pseudo-complemented lattices. By Lemma 2.1, $\mathcal{W}$ also satisfies (Gal), (Suff1) and (Suff2), as well as $\neg 1 = 0$ and $\neg a \vee \neg b \leq \neg(a \wedge b)$.

We shall need the following lemma. We use $(X]$ to denote the downward closure of a subset X of a lattice and $[X)$ for the upward closure. Also, for any subset X of a a weakly pseudo-complemented lattice, we define

$$\neg X = \{\neg b : b \in X\}.$$

Lemma 5.1. *Let* F *be a proper filter of* $\boldsymbol{W} \in \mathcal{W}$. *Then the following hold.*

(a) $(\neg F]$ *is an ideal.*
(b) $F \cap (\neg F] = \emptyset$.
(c) For all $a \in W$, $\neg a \in F$ *iff* $a \in (\neg F]$.

Proof. (a) Note that $(\neg F]$ is downward closed. Suppose that $a, b \in (\neg F]$. Then $a \leq \neg c$ and $b \leq \neg d$ for some $c, d \in F$. Since F is a filter, $c \wedge d \in F$ so $\neg(c \wedge d) \in \neg F$. Since $a \vee b \leq \neg c \vee \neg d \leq \neg(c \wedge d)$, we have $a \vee b \in (\neg F]$. Thus, $(\neg F]$ is an ideal.

(b) Suppose there is some $a \in F \cap (\neg F]$. Then $a \leq \neg b$ for some $b \in F$, so $b \leq \neg a$. Thus, $\neg a \in F$ hence $0 = a \wedge \neg a \in F$, which is a contradiction.

(c) If $\neg a \in F$ then $\neg\neg a \in (\neg F]$ hence $a \in (\neg F]$ since $a \leq \neg\neg a$. If $a \in (\neg F]$ then $a \leq \neg b$ for some $b \in F$, so $b \leq \neg a$ hence $\neg a \in F$.

We will denote by $\mathcal{R}_W$ the class of all relational structures of type $\boldsymbol{X} = \langle X, \leqslant_1, \leqslant_2, C\rangle$, where $\langle X, \leqslant_1, \leqslant_2\rangle$ is a doubly ordered set and C is a binary relation on X such that the following hold:

(FC1) $(\forall x, y, z)((xCy \text{ and } z \leqslant_1 x) \Rightarrow zCy)$
(FC2) $(\forall x, y, z)((xCy \text{ and } y \leqslant_2 z) \Rightarrow xCz)$
(FC3) $(\forall x)(\exists y)(xCy \text{ and } x \leqslant_1 y)$
(FC4) $(\forall x, y)(xCy \Rightarrow \exists z(yCz \text{ and } x \leqslant_1 z))$
(FC5) $(\forall s, t, y)[(yCs \text{ and } s \leqslant_2 t) \Rightarrow \exists z(y \leqslant_1 z \text{ and } \forall u(z \leqslant_2 u \Rightarrow tCu))]$.

For each $\boldsymbol{W} \in \mathcal{W}$ we define the **canonical frame** of $\boldsymbol{W}$ as the relational structure $\boldsymbol{X}(\boldsymbol{W}) = \langle X(\boldsymbol{W}), \leqslant_1, \leqslant_2, C\rangle$, where $X(\boldsymbol{W})$ is the set of all maximal disjoint filter-ideal pairs of $\boldsymbol{W}$ and, for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in X(\boldsymbol{W})$,

$$x \leqslant_1 y \text{ iff } x_1 \subseteq y_1$$
$$x \leqslant_2 y \text{ iff } x_2 \subseteq y_2$$
$$xCy \text{ iff } \forall a(\neg a \in x_1 \Rightarrow a \in y_2).$$

Lemma 5.2. *If* $\boldsymbol{W} \in \mathcal{W}$ *then* $\boldsymbol{X}(\boldsymbol{W}) \in \mathcal{R}_W$.

Proof. We know that $\langle X(\boldsymbol{W}), \leqslant_1, \leqslant_2\rangle$ is a doubly ordered set. Properties (FC1) and (FC2) are straightforward to prove. For (FC3), suppose $x \in X(\boldsymbol{W})$. By Lemma 5.1, $\langle x_1, (\neg x_1]\rangle$ is a disjoint filter-ideal pair, so we can extend it to a maximal one, say y. If $\neg a \in x_1$ then $a \in (\neg x_1]$ (by Lemma 5.1(c)) hence $a \in y_2$. Thus, xCy. Also, $x_1 \subseteq y_1$, i.e., $x \leqslant_1 y$, so we have found the required y.

For (FC4), suppose $x, y \in X(\boldsymbol{W})$ and xCy. By Lemma 5.1(a), $(\neg y_1]$ is an ideal. If $a \in x_1 \cap (\neg y_1]$ then $a \in x_1$ implies $\neg\neg a \in x_1$, which implies $\neg a \in y_2$. But $a \in (\neg y_1]$ implies $\neg a \in y_1$ (by Lemma 5.1(c)), which contradicts the fact that $y_1 \cap y_2 = \emptyset$. Thus, $x_1 \cap (\neg y_1] = \emptyset$. Thus, we can extend $\langle x_1, (\neg y_1]\rangle$ to a maximal disjoint filter-ideal pair, say z. If $\neg a \in y_1$ then $a \in (\neg y_1]$ hence $a \in z_2$, so yCz. Also, $x \leqslant_1 z$, so we have proved (FC4).

For (FC5), suppose that $s, t, y \in X(\boldsymbol{W})$ such that yCs and $s \leqslant_2 t$. First, we show that $y_1 \cap (\neg t_1] = \emptyset$. Suppose $a \in y_1 \cap (\neg t_1]$. Then, $\neg\neg a \in y_1$ hence $\neg a \in s_2$. Since $s \leqslant_2 t$ we have $\neg a \in t_2$. Also, $a \leq \neg b$ for some $b \in t_1$, so $\neg a \geq \neg\neg b \geq b$ hence $\neg a \in t_1$. This contradicts the fact that t_1 and t_2 are disjoint.

We therefore have that $\langle y_1, (\neg t_1]\rangle$ is a disjoint filter-ideal pair, so we may extend it to a maximal one, say z. Then, $y_1 \subseteq z_1$, i.e., $y \leqslant_1 z$. Suppose $z \leqslant_2 w$ and $\neg a \in t_1$. Then $\neg\neg a \in \neg t_1$ so $a \in (\neg t_1] \subseteq z_2 \subseteq w_2$ hence $a \in w_2$. Thus, we have proved (FC5).

Let $\boldsymbol{X} = \langle X, \leqslant_1, \leqslant_2, C\rangle \in \mathcal{R}_W$. Since $\langle X, \leqslant_1, \leqslant_2\rangle$ is a doubly ordered set we may consider its complex algebra $\langle L(\boldsymbol{X}), \wedge^C, \vee^C, 0^C, 1^C\rangle$, where $L(\boldsymbol{X})$ is the set of ℓ-stable sets with operations defined as in (4–7). Extending this definition we define the ***complex algebra*** of $\boldsymbol{X}$ as $\boldsymbol{L}(\boldsymbol{X}) = \langle L(\boldsymbol{X}), \wedge^C, \vee^C, \neg^C, 0^C, 1^C\rangle$, where, for $A \in L(X)$,

$$\neg^C A = \{x \in X : \forall y(xCy \Rightarrow y \notin A)\}.$$

Lemma 5.3. *If A is ℓ-stable then so is $\neg^C A$.*

Proof. We have $\neg^C A = \{x : \forall y(xCy \Rightarrow y \notin A)\}$ and

$$\ell r(\neg^C A) = \{x : \forall s(x \leqslant_1 s \Rightarrow \exists t(s \leqslant_2 t \text{ and } \forall u(tCu \Rightarrow u \notin A)))\}.$$

Let $x \in \neg^C A$ and suppose that $x \leqslant_1 s$ for some s. We claim that $t = s$ satisfies the required properties. Clearly, $s \leqslant_2 s$. If sCu, then xCu since $x \leqslant_1 s$, by (FC1) hence $u \notin A$. Thus, $x \in \ell r(\neg^C A)$ so $\neg^C A \subseteq \ell r(\neg^C A)$.

For the reverse inclusion, note that, since A is ℓ-stable, we have

$$\neg^C A = \neg^C \ell r(A) = \{x : \forall y(xCy \Rightarrow \exists z(y \leqslant_1 z \text{ and } \forall u(z \leqslant_2 u \Rightarrow u \notin A)))\}.$$

Let $x \in \ell r(\neg^C A)$ and suppose that xCy for some y. By (FC4), there exists s such that

$$x \leqslant_1 s \text{ and } yCs.$$

Then, since $x \in \ell r(\neg^C A)$ and $x \leqslant_1 s$, there exists t such that

$$s \leqslant_2 t \text{ and } \forall u(tCu \Rightarrow u \notin A).$$

Since yCs and $s \leqslant_2 t$, by (FC5) there exists z such that

$$y \leqslant_1 z \text{ and } \forall u(z \leqslant_2 u \Rightarrow tCu).$$

Thus, $\forall u(z \leqslant_2 u \Rightarrow u \notin A)$, so we have found the required z, so $x \in \neg^C \ell r(A) = \neg^C A$.

Lemma 5.4. *If $\boldsymbol{X} \in \mathcal{R}_W$ then $\boldsymbol{L}(\boldsymbol{X}) \in \mathcal{W}$.*

Proof. To see that (WCon) holds, suppose A, B are ℓ-stable sets and $A \subseteq B$. Let $x \in \neg^C B$. Then, for all y, xCy implies $y \notin B$ hence also $y \notin A$, so $x \in \neg^C A$.

To see that (Weak$\neg\neg$) holds, note that

$$\neg^C \neg^C A = \{x : \forall y(xCy \Rightarrow \exists z(yCz \text{ and } z \in A))\}.$$

Let $x \in A$ and suppose that xCy for some y. By (FC4), there exists z such that yCz and $x \leqslant_1 z$. Since A is $\leqslant_1$–increasing and $x \in A$, we have $z \in A$. Thus, the required z exists, showing that $x \in \neg^C \neg^C A$.

To see that (Abs) holds, let A be an ℓ-stable set and suppose there exists $x \in A \cap \neg^C A$. By (FC3), there exists a y such that xCy and $x \leqslant_1 y$. Since $x \in \neg^C A$ and xCy we have $y \notin A$. But $x \in A$ and A is ℓ-stable, hence $\leqslant_1$–increasing, so $x \leqslant_1 y$ implies $y \in A$, a contradiction.

The above lemmas show that if $\boldsymbol{W} \in \mathcal{W}$ then so is $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W}))$. Recall that the function $h : W \to L(\boldsymbol{X}(\boldsymbol{W}))$ defined by

$$h(a) = \{x \in X(\boldsymbol{W}) : a \in x_1\}$$

is an embedding of the lattice part of $\boldsymbol{W}$ into $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W}))$. We show that h also preserves negation.

Theorem 5.1. [8] *Each* $\boldsymbol{W} \in \mathcal{W}$ *is embeddable into* $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W}))$.

Proof. We need only show that $h(\neg a) = \neg^C h(a)$ for all $a \in W$, where

$$h(\neg a) = \{x : \neg a \in x_1\}$$

and

$$\neg^C h(a) = \{x : \forall y(xCy \Rightarrow a \notin y_1)\}.$$

First, let $x \in h(\neg a)$ and suppose that xCy for some y. Then $\neg a \in x_1$ so $a \in y_2$ hence $a \notin y_1$, as required.

Next, let $x \in \neg^C h(a)$ and suppose that $\neg a \notin x_1$. Then $a \notin (\neg x_1]$ (by Lemma 5.1(c)) so $\langle [a), (\neg x_1] \rangle$ forms a disjoint filter-ideal pair which we can extend to a maximal one, say y. If $\neg c \in x_1$ then $c \in (\neg x_1]$ so xCy hence $a \notin y_1$, a contradiction since $[a) \subseteq y_1$.

6 Pseudo-complemented Lattices

A *pseudo-complemented lattice* is an algebra $\boldsymbol{W} = \langle W, \wedge, \vee, \neg, 0, 1\rangle$ which is a bounded lattice with a unary operation $\neg$ satisfying:

$$a \wedge b = 0 \quad \Leftrightarrow \quad a \leq \neg b.$$

The class of all pseudo-complemented lattices is denoted $\mathcal{P}$. Note that (Gal) is derivable by

$$a \leq \neg b \;\Leftrightarrow\; a \wedge b = 0 \;\Leftrightarrow\; b \wedge a = 0 \;\Leftrightarrow\; b \leq \neg a.$$

Thus, (Suff1), (Suff2), (WCon) and (Weak¬¬) are derivable and, from $a \leq \neg\neg a$, we get $a \wedge \neg a = 0$, so (Abs) is derivable hence also $\neg 1 = 0$. The class $\mathcal{W}$ of weakly pseudo-complemented lattices is easily seen to satisfy the quasi-identity

$$a \leq \neg b \quad \Rightarrow \quad a \wedge b = 0,$$

hence $\mathcal{P}$ is a subclass of $\mathcal{W}$ defined by the quasi-identity

(Pcq) $\qquad a \wedge b = 0 \quad \Rightarrow \quad a \leq \neg b.$

As an example that shows that $\mathcal{P}$ is a proper subclass of $\mathcal{W}$ consider the lattice with 6 elements $1, 0, a, b, c, d$, where 1 is the top, 0 is the bottom and a, b, c, d are incomparable. Let $\neg a = b$, $\neg b = a$, $\neg c = d$ and $\neg d = c$. This example is in $\mathcal{W}$ but not in $\mathcal{P}$ since $a \wedge c = 0$ but $a \not\leq \neg c$.

We will denote by $\mathcal{R}_P$ the class of all relational structures of type $\boldsymbol{X} = \langle X, \leqslant_1, \leqslant_2, C\rangle$, where $\langle X, \leqslant_1, \leqslant_2\rangle$ is a doubly ordered set and C is a binary relation on X such that (FC1–FC5) hold as well as

(FC6) $\qquad (\forall x, y)(xCy \Rightarrow \exists z(x \leqslant_1 z \text{ and } y \leqslant_1 z)).$

That is, $\mathcal{R}_P$ is the subclass of $\mathcal{R}_W$ defined by (FC6).

If $\boldsymbol{W} \in \mathcal{P}$ then $\boldsymbol{W} \in \mathcal{W}$ as well hence its **canonical frame** is the relational structure $\boldsymbol{X}(\boldsymbol{W}) = \langle X(\boldsymbol{W}), \leqslant_1, \leqslant_2, C\rangle$, where $X(\boldsymbol{W})$ is the set of all maximal disjoint filter-ideal pairs of $\boldsymbol{W}$ and, for all x, $y \in X(\boldsymbol{W})$,

$$x \leqslant_1 y \text{ iff } x_1 \subseteq y_1$$
$$x \leqslant_2 y \text{ iff } x_2 \subseteq y_2$$
$$xCy \text{ iff } \forall a(\neg a \in x_1 \Rightarrow a \in y_2).$$

Lemma 6.1. *If* $\boldsymbol{W} \in \mathcal{P}$ *then* $\boldsymbol{X}(\boldsymbol{W}) \in \mathcal{R}_P$.

Proof. We need only show that (FC6) holds. So, let $x, y \in X(\boldsymbol{W})$ such that xCy. Consider the filter generated by $x_1 \cup y_1$, denoted $Fi(x_1 \cup y_1)$. We claim that $0 \notin Fi(x_1 \cup y_1)$. If we suppose otherwise, then there exist $a_1, \ldots, a_n \in x_1$ and $b_1, \ldots, b_m \in y_1$ such that

$$(\textstyle\bigwedge_{i=1}^{n} a_i) \wedge (\bigwedge_{j=1}^{m} b_j) = 0.$$

If we set $a = \bigwedge_{i=1}^{n} a_i$ and $b = \bigwedge_{j=1}^{m} b_j$, then $a \in x_1$ and $b \in y_1$ such that $a \wedge b = 0$. But this implies that $a \leq \neg b$, by (Pcq), hence $\neg b \in x_1$. Finally, since xCy and $\neg b \in x_1$, we have $b \in y_2$. Thus, $b \in y_1 \cap y_2$, a contradiction.

This shows that $0 \notin Fi(x_1 \cup y_1)$ so $\langle Fi(x_1 \cup y_1), \{0\}\rangle$ is a disjoint filter-ideal pair. This can be extended to a maximal disjoint filter-ideal pair, say z. Then $x \leqslant_1 z$ and $y \leqslant_1 z$, as required.

Let $\boldsymbol{X} = \langle X, \leqslant_1, \leqslant_2, C\rangle \in \mathcal{R}_P$ (so $\boldsymbol{X}$ satisfies (FC1–FC6)). Then $\boldsymbol{X}$ is also in $\mathcal{R}_W$ hence we may consider its complex algebra $\boldsymbol{L} = \langle L(\boldsymbol{X}), \wedge^C, \vee^C, \neg^C, 0^C, 1^C\rangle$, where $L(\boldsymbol{X})$ is the set of ℓ-stable sets, the lattice operations are defined as in (4–7) and, for $A \in L(\boldsymbol{X})$,

$$\neg^C A = \{x \in X : \forall y(xCy \Rightarrow y \notin A)\}.$$

Lemma 6.2. *If* $\boldsymbol{X} \in \mathcal{R}_P$ *then* $\boldsymbol{L}(\boldsymbol{X}) \in \mathcal{P}$.

Proof. We need only show that $\boldsymbol{L}(\boldsymbol{X})$ satisfies the quasi-identity (Pcq), i.e., for $A, B \in L(\boldsymbol{X})$,

$$A \cap B = \emptyset \quad \Rightarrow \quad A \subseteq \neg^C B = \{x \in X : \forall y(xCy \Rightarrow y \notin B)\}.$$

Suppose that $A \cap B = \emptyset$ and let $x \in A$. Let $y \in X$ such that xCy. By (FC6), there exists $z \in X$ such that $x \leqslant_1 z$ and $y \leqslant_1 z$. Since $x \in A$ and A is $\leqslant_1$–increasing, we have $z \in A$ as well. If $y \in B$ then, since B is $\leqslant_1$–increasing, it would follow that $z \in B$ and hence that $z \in A \cap B$, contradicting our assumption that $A \cap B = \emptyset$. Thus, $y \notin B$ hence $x \in \neg^C B$, as required.

Thus, we have shown that if $\boldsymbol{W} \in \mathcal{P}$ then so is $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W}))$. Moreover, from the previous section we know that h is an embedding of $\boldsymbol{W}$ into $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W}))$, hence we have the following result.

Theorem 6.1. [8] *Each* $\boldsymbol{W} \in \mathcal{P}$ *is embeddable into* $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W}))$.

7 Lattices with De Morgan Negation

By a *De Morgan lattice* we mean an algebra $\boldsymbol{W} = \langle W, \wedge, \vee, \neg, 0, 1\rangle$ which is a bounded lattice with a unary operation $\neg$ satisfying:

(Gal) $a \leq \neg b \Rightarrow b \leq \neg a$
(DeM) $\neg\neg a \leq a$

Let $\mathcal{M}$ denote the variety of all De Morgan lattices. Recall that from (Gal) and (DeM) one may derive (Suff1), (Suff2), (WCon), (Weak¬¬) and $\neg 1 = 0$. The following are also derivable in $\mathcal{M}$:

$\neg\neg a = a$
$\neg(a \wedge b) = \neg a \vee \neg b$
$\neg a = \neg b \;\Rightarrow\; a = b.$

We will denote by $\mathcal{R}_M$ the class of all relational structures of type $\boldsymbol{X} = \langle X, \leqslant_1, \leqslant_2, N\rangle$, where $\langle X, \leqslant_1, \leqslant_2\rangle$ is a doubly ordered set, $N : X \to X$ is a function and, for all $x, y \in X$,

(M1) $N(N(x)) = x$,
(M2) $x \leqslant_1 y \Rightarrow N(x) \leqslant_2 N(y)$,
(M3) $x \leqslant_2 y \Rightarrow N(x) \leqslant_1 N(y)$.

The representation in this section essentially comes from [1], where the function N is called a 'generalized Routley-Meyer star operator'. We give full details here and in the next section show how the method may be extended to ortholattices.

For each $\boldsymbol{W} \in \mathcal{M}$, define the ***canonical frame*** of $\boldsymbol{W}$ as the relational structure $\boldsymbol{X}(\boldsymbol{W}) = \langle X(\boldsymbol{W}), \leqslant_1, \leqslant_2, N\rangle$, where $X(\boldsymbol{W})$ is the set of all maximal disjoint filter-ideal pairs of $\boldsymbol{W}$ and, for x, $y \in X(\boldsymbol{W})$,

$x \leqslant_1 y$ iff $x_1 \subseteq y_1$,
$x \leqslant_2 y$ iff $x_2 \subseteq y_2$,
$N(x) = (\neg x_2, \neg x_1)$, where $\neg A = \{\neg a : a \in A\}$ for any $A \subseteq W$.

Lemma 7.1. *If $\boldsymbol{W} \in \mathcal{M}$ then $\boldsymbol{X}(\boldsymbol{W}) \in \mathcal{R}_M$.*

Proof. We have already observed that $\langle X(\boldsymbol{W}), \leqslant_1, \leqslant_2\rangle$ is a doubly ordered set. Condition (M1) follows from (DeM) and conditions (M2) and (M3) are immediate. Thus, we need only show that N is a function from $X(\boldsymbol{W})$ to $X(\boldsymbol{W})$. That is, if $x \in X(\boldsymbol{W})$, we must show that $N(x)$ is a maximal disjoint filter-ideal pair. (This is also done by Allwein and Dunn.) Let $a_1, a_2 \in x_2$ hence $\neg a_1, \neg a_2 \in \neg x_2$. Then $\neg a_1 \wedge \neg a_2 = \neg(a_1 \vee a_2)$ and $a_1 \vee a_2 \in x_2$, hence $\neg x_2$ is closed under $\wedge$. If $\neg a_1 \leq b$ then $\neg b \leq \neg\neg a_1 = a_1$, so $\neg b \in x_2$. Then $b = \neg\neg b \in \neg x_2$, so $\neg x_2$ is upward closed. Thus, $\neg x_2$ is a filter. Similarly, $\neg x_1$ is an ideal. Also, $\neg x_1$ and $\neg x_2$ can be shown disjoint using the implication: $\neg b = \neg c \Rightarrow b = c$ and the fact that x_1 and x_2 are disjoint. To show maximality, suppose $y \in X(\boldsymbol{W})$ and $\neg x_1 \subseteq y_1$ and $\neg x_2 \subseteq y_2$. Then $\neg\neg x_1 \subseteq \neg y_1$, i.e., $x_1 \subseteq \neg y_1$ and also $x_2 \subseteq \neg y_2$. Since $(\neg y_2, \neg y_1)$ is a disjoint filter-ideal pair, the maximality of x implies $x_1 = \neg y_2$ and $x_2 = \neg y_1$. Thus, $\neg x_1 = y_2$ and $\neg x_2 = y_1$ so $N(x)$ is maximal.

If $\boldsymbol{X} = \langle X, \leqslant_1, \leqslant_2, N\rangle \in \mathcal{R}_M$, then $\langle X, \leqslant_1, \leqslant_2\rangle$ is a doubly ordered set, so we may consider its complex algebra $\langle L(\boldsymbol{X}), \wedge^C, \vee^C, 0^C, 1^C\rangle$, where $L(\boldsymbol{X})$ is the set of ℓ-stable sets and the operations are as in (4–7). We extend this definition to define the ***complex algebra*** of $\boldsymbol{X}$ as $\boldsymbol{L}(\boldsymbol{X}) = \langle L(\boldsymbol{X}), \wedge^C, \vee^C, \neg^C, 0^C, 1^C\rangle$ where, for $A \in L(\boldsymbol{X})$,

$$\neg^C A = \{x \in X : N(x) \in r(A)\}.$$

Lemma 7.2. *If $\boldsymbol{X} \in \mathcal{R}_M$ then $\boldsymbol{L}(\boldsymbol{X}) \in \mathcal{M}$.*

Proof. We need to show that $\neg^C A$ is ℓ-stable, i.e., $\ell r(\neg^C A) = \neg^C A$, and that $\boldsymbol{L}(\boldsymbol{X})$ satisfies (Gal) and (DeM). Since ℓ and r form a Galois connection, by Lemma 3.2, we have $\neg^C A \subseteq \ell r(\neg^C A)$ iff $r(\neg^C A) \subseteq r(\neg^C A)$. For the converse, suppose that for every y, if $x \leqslant_1 y$ then $y \notin r(\neg^C A)$ and assume, to the contrary, that $x \notin \neg^C A$. Then $N(x) \notin r(A)$ and there is z such that $N(x) \leqslant_2 z$ and $z \in A$. It follows by (M3) and (M1) that $x \leqslant_1 N(z)$ and hence, by the above assumption, $N(z) \notin r(\neg^C A)$. Thus, there is t such that $N(z) \leqslant_2 t$ and $t \in \neg^C A$. By application of N and (M3) and (M1), we have that $z \leqslant_1 N(t)$ and $N(t) \in r(A)$, in particular $N(t) \notin A$. But $z \in A$ and A is $\leqslant_1$–increasing, as $A = \ell r(A)$, hence $N(t) \in A$, a contradiction.

To prove (Gal), suppose that $A \subseteq \neg^C B$. Then, for every x, if $x \in A$ then $N(x) \in r(B)$. Suppose that $x \in B$ and, to the contrary, that $x \notin \neg^C A$, i.e., $N(x) \notin r(A)$, in which case $N(x) \leqslant_2 y$ and $y \in A$, for some y. By (M3) and (M1), $x \leqslant_1 N(y)$ hence $N(y) \in B$ since $B = \ell r(B)$ is $\leqslant_1$–increasing. But also $y \in \neg^C B$, by the assumption, and $N(y) \in r(B)$, a contradiction since $B \cap r(B) = \emptyset$.

To prove (DeM), let $x \in \neg^C\neg^C A$, hence $N(x) \in r(\neg^C A)$. We show that $x \in \ell(r(A))$ which equals A since A is ℓ-closed. Let $x \leq_1 w$. Then $N(x) \leq_2 N(w)$, by (M2), hence $N(w) \in r(\neg^C A)$ since $r(\neg^C A)$ is $\leq_2$–increasing. Thus, $N(w) \notin \neg^C A$, i.e., $w = N(N(w)) \notin r(A)$. Thus, $x \in \ell(r(A)) = A$.

The above lemmas imply that if $\boldsymbol{W} \in \mathcal{M}$, then $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W})) \in \mathcal{M}$ as well. Recall that the function $h : \boldsymbol{W} \to \boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W}))$ defined by

$$h(a) = \{x \in X(\boldsymbol{W}) : a \in x_1\}$$

is an embedding of the lattice part of $\boldsymbol{W}$ into $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W}))$. As in the case of Heyting negation, we shall show that h also preserves De Morgan negation.

Theorem 7.1. [8] *Each $\boldsymbol{W} \in \mathcal{M}$ is embeddable into $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W}))$.*

Proof. We need only show that $h(\neg a) = \neg^C h(a)$ for all $a \in W$, where

$$h(\neg a) = \{x \in X(\boldsymbol{W}) : \neg a \in x_1\}$$

and

$$\begin{aligned}\neg^C h(a) &= \{x \in X(\boldsymbol{W}) : N(x) \in r(h(a))\}\\ &= \{x \in X(\boldsymbol{W}) : (\forall y \in X(\boldsymbol{W}))(\neg x_1 \subseteq y_2 \Rightarrow a \notin y_1)\}.\end{aligned}$$

First, let $x \in h(\neg a)$. Then $\neg a \in x_1$, hence $a = \neg\neg a \in \neg x_1$. Suppose that $\neg x_1 \subseteq y_2$. Then $a \notin y_1$, since y_1 and y_2 are disjoint.

Next, let $x \in \neg^C h(a)$. Suppose, to the contrary, that $\neg a \notin x_1$. Then $a \notin (\neg x_1]$ and so $\langle [a), (\neg x_1] \rangle$ is a disjoint filter-ideal pair, which can be extended to a maximal one, say y. Thus, $(\neg x_1] \subseteq y_1$, so $a \notin y_1$, but $[a) \subseteq y_1$, a contradiction.

8 Lattices with Ortho-negation (Ortholattices)

An *ortholattice* is an algebra $\boldsymbol{W} = \langle W, \wedge, \vee, \neg, 0, 1 \rangle$ which is a bounded lattice with a unary operation $\neg$ which satisfies (Gal), (DeM) and (Abs). That is, the negation in an ortholattice is both De Morgan and Intuitionistic. Let $\mathcal{O}$ denote the variety of all ortholattices. Since $\mathcal{O}$ is a subclass of both $\mathcal{W}$ and $\mathcal{M}$, it satisfies all the identities satisfied by either class. We extend the relational representation for De Morgan lattices to ortholattices.

We will denote by $\mathcal{R}_O$ the class of all relational structures of type $\boldsymbol{X} = \langle X, \leqslant_1, \leqslant_2, N \rangle$, where $\langle X, \leqslant_1, \leqslant_2 \rangle$ is a doubly ordered set and $N : X \to X$ is a function such that, for all $x, y \in X$,

(M1) $N(N(x)) = x$
(M2) $x \leqslant_1 y \Rightarrow N(x) \leqslant_2 N(y)$
(M3) $x \leqslant_2 y \Rightarrow N(x) \leqslant_1 N(y)$
(O) $(\forall x)(\exists y)(x \leqslant_1 y$ and $N(x) \leqslant_2 y)$

That is, $\mathcal{R}_O$ is the subclass of $\mathcal{R}_M$ defined by (O). If $\boldsymbol{W} \in \mathcal{O}$, then $\boldsymbol{W} \in \mathcal{M}$ hence its canonical frame is the relational structure $\boldsymbol{X}(\boldsymbol{W}) = \langle X(\boldsymbol{W}), \leqslant_1, \leqslant_2, N \rangle$, where $X(\boldsymbol{W})$ is the set of all maximal disjoint filter-ideal pairs of $\boldsymbol{W}$ and, for x, $y \in X(\boldsymbol{W})$,

$x \leqslant_1 y$ iff $x_1 \subseteq y_1$
$x \leqslant_2 y$ iff $x_2 \subseteq y_2$
$N(x) = (\neg x_2, \neg x_1)$, where $\neg A = \{\neg a : a \in A\}$ for $A \subseteq W$.

Lemma 8.1. *If $\boldsymbol{W} \in \mathcal{O}$ then $\boldsymbol{X}(\boldsymbol{W}) \in \mathcal{R}_{(O)}$.*

Proof. We need only show that $\boldsymbol{X}(\boldsymbol{W})$ satisfies (O). Let $x \in X(\boldsymbol{W})$. Observe that x_1 and $\neg x_1$ are disjoint, for if $a \in x_1 \cap (\neg x_1)$ then $a \in x_1$ and $a \in \neg x_1$, so $\neg a \in \neg\neg x_1 = x_1$, hence $a \wedge \neg a \in x_1$. But, by (Abs), $a \wedge \neg a = 0$, so $x_1 = W$, a contradiction. Thus, we may extend $(x_1, \neg x_1)$ to a maximal disjoint filter-ideal pair y. Then $x_1 \subseteq y_1$ and $\neg x_1 \subseteq y_2$, so we have found a y that satisfies the required conditions of (O).

If $\boldsymbol{X} = \langle X, \leqslant_1, \leqslant_2, N \rangle \in \mathcal{R}_O$, then $\boldsymbol{X} \in \mathcal{R}_M$ so it has a canonical algebra $\boldsymbol{L}(\boldsymbol{X}) = \langle L(X), \wedge^C, \vee^C, \neg^C, 0^C, 1^C \rangle$ defined as in the De Morgan negation case.

Lemma 8.2. *If $\boldsymbol{X} \in \mathcal{R}_O$ then $\boldsymbol{L}(\boldsymbol{X}) \in \mathcal{O}$.*

Proof. We need only show that $\boldsymbol{L}(\boldsymbol{X})$ satisfies $A \wedge^C (\neg^C A) = 0^C$. Suppose, to the contrary, that there exists $A \in L(\boldsymbol{X})$ such that $A \cap (\neg^C A) \neq \emptyset$, and let

$x \in A \cap (\neg^C A)$. By (O), there exists y such that $x \leqslant_1 y$ and $N(x) \leqslant_2 y$. Since A is $\leqslant_1$–increasing, $y \in A$. Since $x \in \neg^C A$, $N(x) \in r(A)$. But then $N(x) \leqslant_2 y$ implies $y \notin A$, a contradiction.

Thus, the above lemmas imply that if $\boldsymbol{W} \in \mathcal{O}$, then $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W})) \in \mathcal{O}$ as well. Since the map h is an embedding of De Morgan lattices, we have the following result.

Theorem 8.1. [8] *Each $\boldsymbol{W} \in \mathcal{O}$ is embeddable into $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W}))$.*

Part II Distributive Lattices

9 Relatively Pseudo-complemented Lattices

A *relatively pseudo-complemented lattice* is an algebra $\boldsymbol{W} = \langle W, \wedge, \vee, \rightarrow\rangle$ where $\langle W, \wedge, \vee\rangle$ is a lattice and $\rightarrow$ is a binary operation on W satisfying:

$$a \wedge c \leq b \quad \Leftrightarrow \quad c \leq a \rightarrow b.$$

The operation $\rightarrow$ is the 'residuum' of $\wedge$. For properties of relatively pseudo-complemented lattices, see [15] or [2]). It is known that every relatively pseudo-complemented lattice is distributive and has a constant 1 definable by $1 = a \rightarrow a$, which is the greatest element of the lattice. We include 1 in the language so that $\boldsymbol{W} = \langle W, \wedge, \vee, \rightarrow, 1\rangle$. It is known that all relatively pseudo-complemented lattices form a variety and we denote this variety by $\mathcal{RP}$. $\mathcal{RP}$ satisfies the following:

$a \rightarrow b = 1 \Leftrightarrow a \leq b$
$1 \rightarrow b = b, \quad a \rightarrow 1 = 1$
$a \rightarrow b = 1$ and $a = 1 \ \Rightarrow b = 1$
$a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c)$
$a \wedge (a \rightarrow b) = a \wedge b$
$b \leq a \rightarrow b$
$a \leq b \Rightarrow c \rightarrow a \leq c \rightarrow b.$

In the case of distributive lattices such as $\mathcal{RP}$ the relational representation is built on the set of prime ideals of the lattice rather than the maximal disjoint filter-ideal pairs used in the non-distributive cases. The underlying relational structures are of the type $\langle X, \leq\rangle$, where X is a set and $\leq$ a quasi-order on X. The class of all such relational structures is denoted by $\mathcal{R}_{RP}$.

For each $\boldsymbol{W} \in \mathcal{RP}$ we define the ***canonical frame*** of $\boldsymbol{W}$ as the relational structure $\boldsymbol{X}(\boldsymbol{W}) = \langle X(\boldsymbol{W}), \leq^C\rangle$, where $X(\boldsymbol{W})$ is the set of all prime filters of $\boldsymbol{W}$ and $\leq^C = \subseteq$.

Lemma 9.1. *If $\boldsymbol{W} \in \mathcal{RP}$ then $\boldsymbol{X}(\boldsymbol{W}) \in \mathcal{R}_{RP}$.*

For each $\langle X, \leq\rangle \in \mathcal{RP}$, we define the operation $[\leq] : 2^X \rightarrow 2^X$ by

$$[\leq]A = \{x \in X : \forall y(x \leq y \Rightarrow y \in A)\}.$$

Observe that $[\leq]A$ is the largest upward closed subset of A. Note also that $[\leq]$ is monotonic and, for any $A \subseteq X$, $[\leq]A = A$ iff A is upward closed, and $[\leq][\leq]A = [\leq]A$.

If $\boldsymbol{X} = \langle X, \leq \rangle \in \mathcal{R}_{RP}$ we define the ***complex algebra*** of $\boldsymbol{X}$ as $\boldsymbol{L}(\boldsymbol{X}) = \langle L(\boldsymbol{X}), \wedge^C, \vee^C, \rightarrow^C, 1^C \rangle$ where $L(\boldsymbol{X}) = \{A \subseteq X : [\leq]A = A\}$ and, for all $A, B \in L(\boldsymbol{X})$,

$A \wedge^C B = A \cap B,$
$A \vee^C B = A \cup B,$
$A \rightarrow^C B = [\leq](-A \cup B),$ where $-A$ is the set complement of A in X,
$1^C = X.$

Lemma 9.2. *If $\boldsymbol{X} \in \mathcal{R}_{RP}$ then $\boldsymbol{L}(\boldsymbol{X}) \in \mathcal{RP}$.*

Proof. It is clear that $L(\boldsymbol{X})$ is closed under $\wedge^C$ and $\vee^C$ and that these operations describe a distributive lattice with greatest element 1^C. We need only show that $\rightarrow^C$ is the residuum of $\cap$, i.e., for all $A, B, C \in L(\boldsymbol{X})$,

$$A \cap C \subseteq B \quad \text{iff} \quad C \subseteq A \rightarrow^C B = [\leq](-A \cup B).$$

Assume that $A \cap C \subseteq B$ and let $x \in C$. Take any $y \in X$ such that $x \leq y$. Then $y \in C$ since C is a filter. If $y \in A$ then $y \in A \cap C$ hence $y \in B$ so $y \in -A \cup B$. If $y \notin A$ then, trivially, $y \in -A \cup B$. Conversely, assume $C \subseteq [\leq](-A \cup B)$ and let $x \in A \cap C$. Then $x \in C$ hence $x \in [\leq](-A \cup B)$. Since $x \leq x$, we have $x \in -A \cup B$, but $x \in A$, so we must have $x \in B$, as required.

The above lemmas show that if $\boldsymbol{W} \in \mathcal{RP}$, then so is $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W}))$. To show that $\boldsymbol{W}$ embeds into $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W}))$ we define the map $f : W \rightarrow L(\boldsymbol{X}(\boldsymbol{W}))$ by

$$f(a) = \{F \in X(\boldsymbol{W}) : a \in F\}.$$

For the proof of next theorem we need the following observations. Let F be a (lattice) filter of a relatively pseudo-complemented lattice $\boldsymbol{W}$. Then the following hold for all $a, b \in W$:

$a \in F$ and $a \rightarrow b \in F \Rightarrow b \in F$;
if $b \notin F$, then there is a prime filter F' such that $F \subseteq F'$ and $b \notin F'$.

Theorem 9.1. *Each $\boldsymbol{W} \in \mathcal{RP}$ is embeddable into $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W}))$.*

Proof. That the map f is a lattice embedding follows by standard arguments of M.H. Stone [16] (see also [2]). We need only show the preservation of relative pseudo-complement by f, i.e., that $f(a \rightarrow b) = f(a) \rightarrow^C f(b) = [\leq^C](-f(a) \cup f(b))$. Let $F \in f(a \rightarrow b)$, i.e., $a \rightarrow b \in F$. It follows that $a \notin F$ or $b \in F$, hence $F \notin f(a)$ or $F \in f(b)$, i.e., $F \in -f(a) \cup f(b)$, so $f(a \rightarrow b) \subseteq -f(a) \cup f(b)$. Since for every $a \in W$, $f(a) = [\leq^C]f(a)$ we have, by monotonicity of $[\leq^C]$, that $f(a \rightarrow b) = [\leq^C]f(a \rightarrow b) \subseteq [\leq^C](-f(a) \cup f(b))$. For the converse inclusion, suppose $F \in [\leq^C](-f(a) \cup f(b))$. Then, for all G,

$$F \subseteq G \Rightarrow a \notin G \text{ or } b \in G. \tag{8}$$

In particular, $a \notin F$ or $b \in F$. If $b \in F$ then, since $b \leq a \to b$, we have $a \to b \in F$. If $b \notin F$, then $a \notin F$. We show that also in this case $a \to b \in F$. Suppose, to the contrary, that $a \to b \notin F$. Set $H = \{c : a \to c \in F\}$. Since $a \to (c \wedge d) = (a \to c) \wedge (a \to d)$, it follows that H is closed under meets. Since $c \leq d$ implies $a \to c \leq a \to d$, H is upward closed. Thus, H is a filter of $\boldsymbol{W}$. Moreover, $F \subseteq H$, $a \in H$ and $b \notin H$. Thus, we may extend H to a prime filter H' such that $b \notin H'$, but $F \subseteq H'$ and $a \in H'$, contradicting (8).

10 Relatively Pseudo-complemented Lattices with Minimal Negation

Now we consider relatively pseudo-complemented lattices with minimal negation, also called minimal negation of Johansson [10], (cf. Dunn and Hardegree [5]) or contrapositional negation, (cf. Rasiowa [14]). This is a relatively pseudo-complemented lattice enriched with an operation corresponding to minimal negation, (i.e., minimal negation of Johansson, or contrapositional negation).

By a *relatively pseudo-complemented lattice with minimal negation* we mean an algebra $\boldsymbol{W} = \langle W, \wedge, \vee, \to, \neg, \partial, 1\rangle$, where $\langle W, \wedge, \vee, \to, 1\rangle$ is a relatively pseudo-complemented lattice, $\partial \in W$ (not necessarily the smallest element) and $\neg$ is a unary operator satisfying:

(RPM1) $\quad a \to \neg b \leq b \to \neg a$,
(RPM2) $\quad \neg 1 = \partial$.

Let $\mathcal{RPM}$ denote the variety of all relatively pseudo-complemented lattices with minimal negation. Note that (RPM1) is equivalent to $a \to \neg b = b \to \neg a$ and corresponds to the condition for quasi-minimal, or Galois, negation (Gal): $a \leq \neg b \Rightarrow b \leq \neg a$.

Lemma 10.1

(a) If $\boldsymbol{W} \in \mathcal{RPM}$, then $\neg a = a \to \partial$ for all $a \in W$.
(b) Let $\boldsymbol{W} \in \mathcal{RP}$ and let ∂ be any element of W. If we define a unary operation $\neg$ by $\neg a = a \to \partial$ for all $a \in W$, then $\neg$ is a minimal negation.

Proof. (a) For all $a \in W$ we have $\neg a = 1 \to \neg a = a \to \neg 1 = a \to \partial$. (b) For (RPM1), for all $a, b \in W$ we have $a \to \neg b = a \to (b \to \partial) = b \to (a \to \partial) = b \to \neg a$. For (RPM2), we have $\neg 1 = 1 \to \partial = \partial$.

We will denote by $\mathcal{R}_{RPM}$ the class of all relational structures of type $\boldsymbol{X} = \langle X, \leq, D\rangle$, where $\leq$ is a quasi-order on X and $D \subseteq X$.

For each $\boldsymbol{W} \in \mathcal{RPM}$ we define the ***canonical frame*** of $\boldsymbol{W}$ as $\boldsymbol{X}(\boldsymbol{W}) = \langle X(\boldsymbol{W}), \leq^C, D^C\rangle$, where $X(\boldsymbol{W})$ is the set of all prime filters of $\boldsymbol{W}$, $\leq^C = \subseteq$ and

$$D^C = \{F \in X(\boldsymbol{W}) : \partial \in F\}.$$

Lemma 10.2. *If* $\boldsymbol{W} \in \mathcal{RPM}$ *then* $\boldsymbol{X}(\boldsymbol{W}) \in \mathcal{R}_{RPM}$*.*

If $\boldsymbol{X} = \langle X, \leq, D\rangle \in \mathcal{R}_{RPM}$, then $\langle X, \leq\rangle \in \mathcal{R}_{RP}$ hence it has a complex algebra $\langle L(\boldsymbol{X}), \wedge^C, \vee^C, \rightarrow^C, 1^C\rangle$ as defined in the previous section. The ***complex algebra*** of $\boldsymbol{X}$, denoted $\boldsymbol{L}(\boldsymbol{X})$, is the extension of this algebra by the constant ∂^C and the operation $\neg^C$ defined by

$$\partial^C = [\leq]D,$$
$$\neg^C A = A \rightarrow^C \partial^C \text{ for } A \in L(\boldsymbol{X}).$$

Lemma 10.3. *If* $\boldsymbol{X} \in \mathcal{R}_{RPM}$ *then* $\boldsymbol{L}(\boldsymbol{X}) \in \mathcal{RPM}$*.*

Proof. Since $[\leq][\leq]D = [\leq]D$, we have $\partial^C \in L(\boldsymbol{X})$ and hence $L(\boldsymbol{X})$ is also closed under $\neg^C$. Since $\boldsymbol{L}(\boldsymbol{X})$ is a relatively pseudo-complemented lattice, (RPM1) follows from properties of $\rightarrow$. (RPM2) follows from $\neg^C 1 = [\leq](-X \cup [\leq]D) = [\leq][\leq]D = [\leq]D = \partial^C$.

Thus, if $\boldsymbol{W} \in \mathcal{RPM}$ so is $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W}))$.

Theorem 10.1. *Each* $\boldsymbol{W} \in \mathcal{RPM}$ *is embeddable into* $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W}))$*.*

Proof. From the previous section we know that the function $f : W \rightarrow L(\boldsymbol{X}(\boldsymbol{W}))$ defined by

$$f(a) = \{F \in X(\boldsymbol{W}) : a \in F\}$$

is an embedding on the reduct $\langle W, \wedge, \vee, \rightarrow, 1\rangle$. We have $f(\partial) = \{F \in X(\boldsymbol{W}) : \partial \in F\} = D^C$ and $f(\partial)$ is an upward closed subset of $X(\boldsymbol{W})$ so $f(\partial) = [\leq]D^C = \partial^C$. Since $\rightarrow$ is preserved it follows that $\neg$ is too.

11 Distributive Lattices with De Morgan Negation

Now we consider distributive lattices with negation operation corresponding to De Morgan negation (i.e., satisfying (Gal) and (DeM)). We will see the difference in techniques of representation between the previous non-distributive case and the distributive case. The representation theorem below is a modification of the result of Białynicki-Birula and Rasiowa [3] to the unified framework.

By a *De Morgan algebra* (also called a *distributive lattice with involution*) we mean a De Morgan lattice $\langle W, \wedge, \vee, \neg, 0, 1\rangle$ whose lattice reduct is distributive. Let $\mathcal{DM}$ denote the variety of all De Morgan algebras. Thus, $\mathcal{DM}$ satisfies (Gal) and (DeM), as well as (Suff1), (Suff2), (WCon), (Weak¬¬), $\neg 1 = 0$ and

$$\neg\neg a = a$$
$$\neg(a \wedge b) = \neg a \vee \neg b$$
$$\neg a = \neg b \;\Rightarrow\; a = b.$$

For $\boldsymbol{W} \in \mathcal{DM}$ and $A \subseteq W$, let $\neg A = \{\neg a : a \in A\}$. Then the following hold:

(A1) $\neg A = \{a : \neg a \in A\}$
(A2) $\neg(W - A) = W - (\neg A)$
(A3) $\neg\neg A = A$
(A4) A is a prime filter iff $\neg A$ is a prime ideal.

We will denote by $\mathcal{R}_{DM}$ the class of all relational structures of type $\boldsymbol{X} = \langle X, \leq, N\rangle$, where $\leq$ is a quasi-order on X, $N : X \to X$ is a function and, for all $x, y \in X$,

(DM1) $x \leq y \Rightarrow N(y) \leq N(x)$,
(DM2) $N(N(x)) = x$.

Compare these with (M1–M3). If we let $N(A) = \{N(x) : x \in A\}$, for $A \subseteq X$, then the following hold:

(A5) $N(A) = \{x : N(x) \in A\}$
(A6) $N(X - A) = X - N(A)$
(A7) $N(A \cup B) = N(A) \cup N(B)$
(A8) $NN(A) = A$.

The only non-trivial property is (A6), but this follows since: $x \in N(X - A)$ iff $N(x) \in X - A$ iff $N(x) \notin A$ iff $x \notin N(A)$.

For each $\boldsymbol{W} \in \mathcal{DM}$ we define the ***canonical frame*** of $\boldsymbol{W}$ as the relational structure $\boldsymbol{X}(\boldsymbol{W}) = \langle X(\boldsymbol{W}), \leq^C, N^C\rangle$, where $X(\boldsymbol{W})$ is the set of all prime filters of $\boldsymbol{W}$, $\leq^C = \subseteq$ and, for $F \in X(\boldsymbol{W})$,

$$N^C(F) = W - (\neg F).$$

Lemma 11.1. *If $\boldsymbol{W} \in \mathcal{DM}$ then $\boldsymbol{X}(\boldsymbol{W}) \in \mathcal{R}_{DM}$.*

Proof. We first show that N is a function from $X(\boldsymbol{W})$ to $X(\boldsymbol{W})$. Let $F \in X(\boldsymbol{W})$, so F is a prime filter. It is routine to check that $N^C(F)$ is a filter. For primeness, suppose that $a \vee b \in N^C(F) = W - (\neg F)$. Then $a \vee b \notin \neg F$ so $\neg(a \vee b) = \neg a \wedge \neg b \notin F$. Thus, either $\neg a \notin F$ or $\neg b \notin F$, so $a \notin \neg F$ or $b \notin \neg F$, hence $a \in W - (\neg F)$ or $b \in W - (\neg F)$.

For (DM1), suppose $F, G \in X(\boldsymbol{W})$ and $F \subseteq G$. Now, by (A2), (A1) and definitions we have $a \in N^C(G)$ iff $a \in W - (\neg G)$ iff $a \notin \neg G$ iff $\neg a \notin G$ hence, by the assumption, $\neg a \notin F$ iff $a \notin \neg F$ iff $a \in W - (\neg F)$ iff $a \in N^C(F)$.

For (DM2), by (A6) and (A7) we have $N^C(N^C(F)) = W - (\neg N^C(F)) = W - (\neg(W - (\neg F))) = W - (W - \neg\neg F) = \neg\neg F = F$.

If $\boldsymbol{X} = \langle X, \leq\rangle \in \mathcal{R}_{DM}$ we define the ***complex algebra*** of $\boldsymbol{X}$ as $\boldsymbol{L}(\boldsymbol{X}) = \langle L(\boldsymbol{X}), \wedge^C, \vee^C, \neg^C, 0^C, 1^C\rangle$ where $L(\boldsymbol{X}) = \{A \subseteq X : [\leq]A = A\}$ and, for all $A, B \in L(\boldsymbol{X})$,

$A \wedge^C B = A \cap B$,
$A \vee^C B = A \cup B$,
$\neg^C A = X - N(A)$,
$1^C = X$,
$0^C = \emptyset$.

Recall that, for $A \subseteq X$,

$$[\leq]A = \{x \in X : \forall y(x \leq y \Rightarrow y \in A)\}.$$

Lemma 11.2. *If* $\boldsymbol{X} \in \mathcal{R}_{DM}$ *then* $\boldsymbol{L}(\boldsymbol{X}) \in \mathcal{DM}$.

Proof. We show that if $A \in L(\boldsymbol{X})$, then $\neg^C A \in L(\boldsymbol{X})$, that is $\neg^C A = [\leq]\neg^C A$. Let $x \in \neg^C A$, so $N(x) \notin A$. Suppose that $x \notin [\leq]\neg^C A$. Then there is y such that $x \leq y$ and $y \notin \neg^C A$, thus $N(y) \in A = [\leq]A$, that is $\forall z(N(y) \leq z \Rightarrow z \in A)$. Since $x \leq y$, we have $N(y) \leq N(x)$, and taking $z = N(x)$ we get $N(x) \in A$, a contradiction. For the converse, let $x \in [\leq]\neg^C A$. Then $\forall y(x \leq y \Rightarrow N(y) \notin A)$; suppose that $x \notin \neg^C A$, hence $N(x) \in A$. Taking $y = x$ we get a contradiction.

Now we show that $\neg^C\neg^C A = A$. Using (A6) and (A8) we have $X - N(\neg^C A) = X - N(X - N(A)) = X - (X - NN(A)) = NN(A) = A$. This proves (DeM) and (Weak¬¬) hence (Gal) follows by Lemma 2.1. Next we show (Suff1), i.e., that $\neg^C(A \cup B) = \neg^C A \cap \neg^C B$. By (A7) we have $x \in X - N(A \cup B)$ iff $x \notin N(A \cup B)$ iff $N(x) \notin A$ and $N(x) \notin B$ iff $x \in \neg^C A \cap \neg^C B$.

The above lemmas imply that if $\boldsymbol{W} \in \mathcal{DM}$, then $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W})) \in \mathcal{DM}$ as well. Recall that the function $f : W \to L(\boldsymbol{X}(\boldsymbol{W}))$ defined by

$$f(a) = \{F \in X(\boldsymbol{W}) : a \in F\}$$

is an embedding of the lattice parts of $\boldsymbol{W}$ and $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W}))$. We show that it preserves negation as well.

Theorem 11.1. *Each* $\boldsymbol{W} \in \mathcal{DM}$ *is embeddable into* $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W}))$.

Proof. We need only show the preservation of negation. We have, by definition,

$$\begin{aligned} \neg^C f(a) &= X(\boldsymbol{W}) - (N^C(f(a)) \\ &= X(\boldsymbol{W}) - \{N^C(F) : F \in f(a)\} \\ &= X(\boldsymbol{W}) - \{W - (\neg F) : a \in F\} \end{aligned}$$

and

$$f(\neg a) = \{G : \neg a \in G\}.$$

Note that $a \in F$ iff $\neg a \in \neg F$ iff $\neg a \notin W - (\neg F)$. Thus, $\{W - (\neg F) : a \in F\}$ consists of all $G \in X(\boldsymbol{W})$ for which $\neg a \notin G$. Therefore $X(\boldsymbol{W}) - \{W - (\neg F) : a \in F\}$ consists of all $G \in X(\boldsymbol{W})$ such that $\neg a \in G$, i.e., $\neg^C f(a) = f(\neg a)$.

12 Boolean Algebras with Sufficiency Operator

By a *Boolean algebra with sufficiency* (or *negative necessity*) *operator* we mean an algebra $\boldsymbol{W} = \langle \boldsymbol{W}', \neg \rangle$, where $\boldsymbol{W}' = \langle W, \wedge, \vee, -, 0, 1 \rangle$ is a Boolean algebra, and $\neg$ a unary operation satisfying:

(Suff1) $\neg(a \vee b) = \neg a \wedge \neg b$
(Suff2) $\neg 0 = 1$.

Let $\mathcal{SUA}$ denote the variety of all Boolean algebras with sufficiency operator. We extend the relational representation to Boolean algebras with sufficiency operator.

A *frame* is a relational structure of type $\boldsymbol{X} = \langle X, R \rangle$, where $R \subseteq X \times X$. Let $\mathcal{R}$ be a class of all frames.

For each $\boldsymbol{W} \in \mathcal{SUA}$ we define the ***canonical frame*** of $\boldsymbol{W}$ as the relational structure $\boldsymbol{X}(\boldsymbol{W}) = \langle X(\boldsymbol{W}), R^C \rangle$, where $X(\boldsymbol{W})$ is the set of all prime filters of $\boldsymbol{W}$ and, for $F, G \in X(\boldsymbol{W})$,

$$FR^C G \text{ iff } \neg G \cap F \neq \emptyset$$

where $\neg A = \{a \in W : \neg a \in A\}$ for each $A \subseteq X$.

Given a frame $\boldsymbol{X} = \langle X, R \rangle$, we define the ***complex algebra*** of $\boldsymbol{X}$ as $\boldsymbol{L}(\boldsymbol{X}) = \langle \mathcal{P}(X), \neg^C \rangle$, where $\mathcal{P}(X)$ is the powerset Boolean algebra of X and, for $A \in \mathcal{P}(X)$,

$$\neg^C A = \{x \in X : A \subseteq R(x)\} = \{x \in X : \forall y (y \in A \Rightarrow xRy)\}.$$

Lemma 12.1. *If $\boldsymbol{W} \in \mathcal{SUA}$, then $\boldsymbol{X}(\boldsymbol{W}) \in \mathcal{R}$. If $\boldsymbol{X} \in \mathcal{R}$ then $\boldsymbol{L}(\boldsymbol{X}) \in \mathcal{SUA}$.*

Theorem 12.1. *Each $\boldsymbol{W} \in \mathcal{SUA}$ is embeddable into $\boldsymbol{L}(\boldsymbol{X}(\boldsymbol{W}))$.*

Proof. The embedding is defined in a standard way:

$$f(a) = \{G \in X(\boldsymbol{W}) : a \in G\}.$$

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