General Representation Theorems for Fuzzy Weak Orders

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Abstract. The present paper gives a state-of-the-art overview of general representation results for fuzzy weak orders. We do not assume that the underlying domain of alternatives is finite. Instead, we concentrate on results that hold in the most general case that the underlying domain is possibly infinite. This paper presents three fundamental representation results: (i) score function-based representations, (ii) inclusion-based representations, (iii) representations by decomposition into crisp linear orders and fuzzy equivalence relations.

1 Introduction

Weak orders are among the most fundamental concepts in preference modeling. A binary relation $\lesssim$ on a given non-empty domain X is called a *weak order* if it has the following three properties for all $x, y, z \in X$:

$$\begin{array}{ll} x \lesssim x & \text{(reflexivity)} \\ \text{if } x \lesssim y \text{ and } y \lesssim z \text{ then } x \lesssim z & \text{(transitivity)} \\ x \lesssim y \text{ or } y \lesssim x & \text{(completeness)} \end{array}$$

Obviously the only difference between weak orders and linear orders is that weak orders need not be antisymmetric, i.e., a weak order $\lesssim$ is a linear order if and only if the additional property

$$\text{if } x \lesssim y \text{ and } y \lesssim x \text{ then } x = y \text{ (antisymmetry)}$$

holds for all $x, y \in X$. It is easy to see that the ranking of linearly ordered properties of objects constitutes a weak order, e.g., ranking cars by their maximum speed, ranking persons by their height or weight, ranking products by their price, and so forth. This basic fact is not only a fundamental construction principle, but a fundamental representation of weak orders.

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Theorem 1. *A binary relation $\lesssim$ on a non-empty domain X is a weak order if and only if there exists a linearly ordered non-empty set Y and a mapping $f : X \to Y$ such that $\lesssim$ can be represented in the following way for all $x, y \in X$:*

$$x \lesssim y \quad \text{if and only if} \quad f(x) \leq f(y) \tag{1}$$

The proof that a relation defined as in Eq. (1) is a weak order is straightforward. To prove the existence of a set Y and a mapping f such that representation (1) holds for a given weak order $\lesssim$, one has to follow the following steps: (a) define an equivalence relation $\sim$ as the symmetric kernel of $\lesssim$, (b) define Y as the factor set $X_{/\sim}$, (c) define f as the projection $f(x) = \langle x \rangle_\sim$, (d) prove that the projection of $\lesssim$ onto $X_{/\sim}$ is a linear order on $X_{/\sim}$, (e) prove that representation (1) holds. From this perspective, we can view weak orders as linear orders of equivalence classes. In the context of Theorem 1, the equivalence classes contain exactly those elements that share the same property, i.e., those elements for which f yields the same value.

Note that there is an alternative construction of Y and f. Let us define the *foreset* of an element $x \in X$, denoted $C(x)$, as the set of elements smaller than or equivalent to x, i.e., $C(x) = \{y \in X \mid y \lesssim x\}$. Then define Y as the set of all foresets, i.e., $Y = \{C(x) \mid x \in X\}$. It is straightforward to prove that $x \lesssim y$ if and only $C(x) \subseteq C(y)$, and it follows directly from the completeness of $\lesssim$ that Y is linearly ordered with respect to ordinary set inclusion. Thus, we can also conclude that weak orders on X can be represented by embedding into linearly ordered subsets of the partially ordered set $(\mathcal{P}(X), \subseteq)$.

In the case that X is at most countable, Theorem 1 can be strengthened in the following way: it is always possible to choose $Y = [0, 1]$, i.e., for each weak order, we can find a mapping $f : X \to [0, 1]$ such that representation (1) holds. In other words, weak orders on countable domains can always be embedded into the linear order on the unit interval. This is a classic result that goes back to Cantor [7, 17, 21].

Weak orders are not only simple and fundamental concepts (as the above examples illustrate), they are the basis for representing other fundamental concepts in preference modeling and order theory: it is known that *preorders*, i.e., reflexive and transitive binary relations, are uniquely characterized as intersections of weak orders.

In analogy to the crisp case, fuzzy weak orders are fundamental concepts in fuzzy preference modeling [8, 11, 12, 19]. Given a non-empty set of alternatives X, a fuzzy relation $R : X^2 \to [0, 1]$ is a *fuzzy weak order* if it has the following three properties for all $x, y, z \in X$, where T denotes a left-continuous t-norm:

$$\begin{array}{ll} R(x,x) = 1 & \text{(reflexivity)} \\ T(R(x,y), R(y,z)) \leq R(x,z) & \text{(T-transitivity)} \\ R(x,y) = 1 \text{ or } R(y,x) = 1 & \text{(strong completeness)} \end{array}$$

The goal of this paper is to provide an overview of representation results for fuzzy weak orders. We concentrate on those results that hold for all possible domains X. Results holding only for finite and/or countable domains will not be considered. Consequently, this paper is organized as follows. After providing some preliminaries in Section 2, we discuss score function-based representations in depth in Section 3 that will be complemented by inclusion-based representations in Section 4. Section 5 is devoted to decomposing fuzzy weak orders into crisp linear orders and fuzzy equivalence relations—in direct analogy to the factor set representation discussed above.

Note that this paper is a state-of-the-art review that mainly integrates results from previously published papers on similarity-based fuzzy orders [3, 4, 5]. This paper consistently views the results from the perspective of fuzzy weak orders.

2 Preliminaries

In this paper, we solely use values from the unit interval to express degrees of order/preference. This is not a serious restriction from a practical point of view, and it is also the standard setting widely used in fuzzy preference modeling. Correspondingly, we use left-continuous triangular norms as standard models for fuzzy conjunctions [16].

Definition 1. An associative, commutative, and non-decreasing binary operation on the unit interval (i.e. a $[0,1]^2 \to [0,1]$ mapping) which has 1 as neutral element is called *triangular norm*, short *t-norm*. A t-norm T is called *left-continuous* if the equality

$$T(\sup_{i \in I} x_i, y) = \sup_{i \in I} T(x_i, y)$$

holds for all families $(x_i)_{i \in I} \in [0,1]^I$ and all $y \in [0,1]$.

The three basic t-norms are denoted as $T_{\mathbf{M}}(x, y) = \min(x, y)$, $T_{\mathbf{P}}(x, y) = x \cdot y$, and $T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)$. Further assume that

$$\vec{T}(x, y) = \sup\{u \in [0,1] \mid T(x, u) \leq y\}$$

denotes the unique residual implication of T. For the sake of completeness, let us list the following fundamental properties (valid for all $x, y, z \in [0,1]$) [13, 15, 16]:

(I1) $x \leq y$ if and only if $\vec{T}(x, y) = 1$
(I2) $T(x, y) \leq z$ if and only if $x \leq \vec{T}(y, z)$
(I3) $T(\vec{T}(x, y), \vec{T}(y, z)) \leq \vec{T}(x, z)$
(I4) $\vec{T}(1, y) = y$
(I5) $T(x, \vec{T}(x, y)) \leq y$
(I6) $y \leq \vec{T}(x, T(x, y))$

Furthermore, $\vec{T}$ is non-increasing and left-continuous in the first argument and non-decreasing and right-continuous in the second argument.

If T is a continuous t-norm, then the following holds for all $x, y, z \in [0,1]$:

(I7) if $z \geq x$ then $\vec{T}(x, y) = \vec{T}(\vec{T}(z, x), \vec{T}(z, y))$

The biimplication of T is defined as $\overleftrightarrow{T}(x,y) = T(\vec{T}(x,y), \vec{T}(y,x))$ and fulfills the following assertions for all $x, y, z \in [0,1]$, see [13, 15]:

(B1) $\overleftrightarrow{T}(x,y) = 1$ if and only if $x = y$
(B2) $\overleftrightarrow{T}(x,y) = \overleftrightarrow{T}(y,x)$
(B3) $\overleftrightarrow{T}(x,y) = \min(\vec{T}(x,y), \vec{T}(y,x))$
(B4) $T(\overleftrightarrow{T}(x,y), \overleftrightarrow{T}(y,z)) \leq \overleftrightarrow{T}(x,z)$
(B5) $\overleftrightarrow{T}(x,y) = \vec{T}(\max(x,y), \min(x,y))$

In this paper, uppercase letters will be used synonymously for fuzzy sets/relations and their corresponding membership functions. The fuzzy power set of X will be denoted with $\mathcal{F}(X) = \{A \mid A : X \to [0,1]\}$.

A binary fuzzy relation $R : X^2 \to [0,1]$ is called

- *reflexive* if $R(x,x) = 1$ for all $x \in X$,
- *symmetric* if $R(x,y) = R(y,x)$ for all $x, y \in X$,
- *T-transitive* if $T(R(x,y), R(y,z)) \leq R(x,z)$ for all $x, y, z \in X$,
- *strongly complete* if $\max(R(x,y), R(y,x)) = 1$ for all $x, y \in X$.

Fuzzy relations that are reflexive and T-transitive are called *fuzzy preorders* with respect to T, short *T-preorders*. Symmetric T-preorders are called *fuzzy equivalence relations* with respect to T, short *T-equivalences*. As mentioned in Section 1 already, strongly complete T-preorders are called *fuzzy weak orders* with respect to T, short *weak T-orders*. Given a T-equivalence $E : X^2 \to [0,1]$, a binary fuzzy relation $L : X^2 \to [0,1]$ is called a *fuzzy order* with respect to T and E, short *T-E-order*, if it is T-transitive and additionally has the following two properties:

- *E-reflexivity:* $E(x,y) \leq L(x,y)$ for all $x, y \in X$
- *T-E-antisymmetry:* $T(L(x,y), L(y,x)) \leq E(x,y)$ for all $x, y \in X$

Given a binary fuzzy relation $R : X^2 \to [0,1]$ and an $x \in X$, analogously to the crisp case (cf. Section 1), the *foreset* of x is defined as the fuzzy set $C(x) \in \mathcal{F}(X)$ that expresses the degree to which a given value $y \in X$ is smaller than or equivalent to x, i.e., $C(x)(y) = R(y,x)$ [2].

3 Score Function-Based Representations

The starting point of this section is Theorem 1. It is natural to first ask the question whether there is a straightforward generalization of this theorem to the case of fuzzy weak orders.

Theorem 2. *A binary fuzzy relation $R : X^2 \to [0,1]$ is a weak T-order if and only if there exist a non-empty domain Y, a T-equivalence $E : Y^2 \to [0,1]$, a strongly complete T-E-order $L : Y^2 \to [0,1]$, and a mapping $f : X \to Y$ such that the following equality holds for all $x, y \in X$:*

$$R(x,y) = L(f(x), f(y)) \tag{2}$$

Theorem 2 can be viewed from two different angles. On the one hand, it is a nice straightforward generalization of Theorem 1 and demonstrates the smooth interplay between fuzzy weak orders and strongly complete fuzzy orders (analogously to the crisp case). On the other hand, fuzzy weak orders and strongly complete fuzzy orders are basically the same concepts. From this point of view, Theorem 2 does not provide us with a new construction method or any new insight. More insight would potentially be obtained if we could restrict the choice of Y or E to certain standard cases that could be utilized for constructions in an easier way.

One interesting question is, for instance, whether Y, L, and f can be chosen such that Theorem 2 holds for E being the crisp equality (i.e., with L being a so-called T-order [4, 12, 13, 14], which, in the case that $T = T_{\mathbf{M}}$, is nothing else but a fuzzy partial order in the sense of Zadeh [18, 19, 26]). The answer is quick and negative: as demonstrated in [4, Subsection 2.3], strongly complete fuzzy orders with respect to some t-norm T and the crisp equality can only be crisp orders. Thus, it is never possible to embed a non-crisp weak order into a strongly complete T-order, so it is impossible to strengthen Theorem 2 by fixing E as the crisp equality.

So the question remains whether there is any standard choice Y, E, L, f into which we can embed all, or at least a subclass of, weak T-orders. As shown by Ovchinnikov, it is possible to embed a weak T-order into a continuous weak T-order on the real numbers $\mathbb{R}$, but it is necessary to restrict to strict t-norms and finite domains X [20]. Since this is outside the scope of this paper, we turn our attention to a different investigation. The standard crisp case consists of the unit interval $[0, 1]$ equipped with its natural linear order. Given a left-continuous t-norm T, the canonical fuzzification of the natural linear order on $[0, 1]$ consists in the residual implication $\vec{T}$ [4, 13, 15]. The following proposition, therefore, provides us with a construction that can be considered a straightforward counterpart of (1).

Proposition 1. *Given a function $f : X \to [0, 1]$, the relation defined by*

$$R(x, y) = \vec{T}\big(f(x), f(y)\big) \tag{3}$$

is a weak T-order.

The function f in Proposition 1 can also be understood as a fuzzy set on X. In this section, we rather leave this aspect aside and adopt the classical interpretation as a *score function.*

Note that the simple construction of Proposition 1 is not a unique characterization, i.e., there are weak T-orders that cannot be represented by means of a single score function. In order to demonstrate that, let us consider a set X with at least three elements. We choose an arbitrary linear order of the elements of X (which always exists due to basic results from order theory [22, 23]) and define R as the crisp linear order itself:

$$R(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Clearly, R is a fuzzy weak order with respect to every t-norm T. Now assume that there exists a score function $f : X \to [0,1]$ such that representation (3) holds. Let us choose an arbitrary chain of three distinct elements $x < y < z$. Then it clearly follows that $R(z,x) = \vec{T}(f(z), f(x)) = 0$ and $R(z,y) = \vec{T}(f(z), f(y)) = 0$. Since the monotonicity of $\vec{T}$ and (I4) imply $\vec{T}(x,y) \geq \vec{T}(1,y) = y$, it trivially follows that $\vec{T}(x,y) = 0$ can hold only if $y = 0$. Thus, we obtain that $f(x) = f(y) = 0$. This entails

$$R(y,x) = \vec{T}(f(y), f(x)) = \vec{T}(0,0) = 1,$$

which is a contradiction. Hence, we obtain that the most basic fuzzy weak orders—crisp linear orders—are never representable as in Proposition 1, no matter which t-norm we choose. It is, therefore, justified to introduce the representability according to Proposition 1 as a distinct notion.

Definition 2. Consider a weak T-order $R : X^2 \to [0,1]$. R is called *representable* if there exists a function $f : X \to [0,1]$, called *generating (score) function*, such that Eq. (3) holds.

Example 1. Let us consider $X = [0,5]$ and the following two score functions $f_1, f_2 : X \to [0,1]$:

$$f_1(x) = \min\big(1, \max(0, x-2)\big)$$

$$f_2(x) = \begin{cases} 0 & \text{if } x \in [0,1[\\ 0.4 \cdot (x-1) & \text{if } x \in]1,2[\\ 0.7 + 0.3 \cdot (x-2) & \text{if } x \in [2,3[\\ 1 & \text{if } x \in [3,5] \end{cases}$$

Figure 1 depicts six fuzzy weak orders defined according to Proposition 1:

$$R_1(x,y) = \vec{T}_{\mathbf{M}}(f_1(x), f_1(y)) \qquad R_2(x,y) = \vec{T}_{\mathbf{M}}(f_2(x), f_2(y))$$
$$R_3(x,y) = \vec{T}_{\mathbf{P}}(f_1(x), f_1(y)) \qquad R_4(x,y) = \vec{T}_{\mathbf{P}}(f_2(x), f_2(y))$$
$$R_5(x,y) = \vec{T}_{\mathbf{L}}(f_1(x), f_1(y)) \qquad R_6(x,y) = \vec{T}_{\mathbf{L}}(f_2(x), f_2(y))$$

The fuzzy relations plotted in Figure 1 have one common feature: the lower right edge always corresponds to the generating score function. More specifically, all fuzzy weak orders in the left column fulfill $R(5,y) = f_1(y)$, while $R(5,y) = f_2(y)$ holds for the fuzzy weak orders in the right column. Note that this is true independent of the t-norm chosen (at least for the three basic t-noVrms). The question arises whether this is a coincidence or whether there is a principle behind. The following theorem tells us that the latter is the case, but even more than that, we obtain a unique characterization of representable fuzzy weak orders (at least for continuous t-norms).

Theorem 3. *Assume that T is continuous. Then a weak T-order R is representable if and only if the following function is a generating function of R:*

$$\bar{f}(x) = \inf_{z \in X} R(z,x)$$

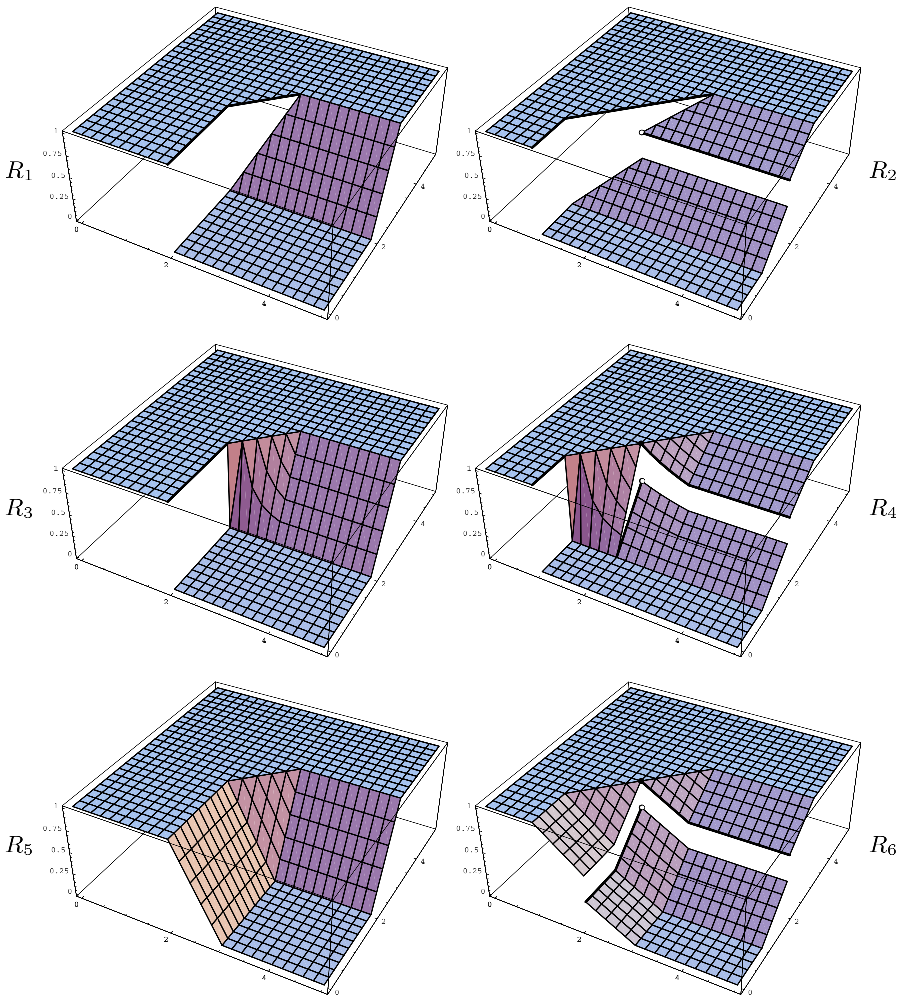

Fig. 1. Fuzzy weak orders constructed from the two score functions f_1 and f_2 by means of Proposition 1 using the three basic t-norms

Theorem 3 provides us with an easy-to-use tool for checking whether a fuzzy weak order is representable—we only have to check whether one specific function is a generating score function. Note, however, that the generating function need not be unique, i.e., it may happen that a fuzzy weak order R is generated by some score function f that does not coincide with $\bar{f}$ defined as in Theorem 3. Let us shortly consider this issue and ask ourselves under which condition $\bar{f}$ coincides with some generating score function f. So assume that R is representable as $R(x,y) = \vec{T}(f(x), f(y))$, then we obtain the following:

$$\bar{f}(x) = \inf_{z\in X} R(z,x) = \inf_{z\in X} \vec{T}(f(z), f(x)) = \vec{T}\big(\sup_{z\in X} f(z), f(x)\big)$$

Then $\sup_{z\in X} f(z) = 1$ is a sufficient criterion for f and $\bar{f}$ to coincide:

$$\bar{f}(x) = \vec{T}\big(\sup_{z\in X} f(z), f(x)\big) = \vec{T}(1, f(x)) \overset{\text{(I4)}}{=} f(x)$$

It should be clear now that by far not all fuzzy weak orders are representable by single score functions—for all left-continuous t-norms, there exist non-representable fuzzy weak orders. What has not been answered so far is the question whether fuzzy weak orders can be represented by more than one score function. The following well-known theorem provides us with a starting point to this investigation.

Theorem 4. [24] *Consider a binary fuzzy relation $R : X^2 \to [0,1]$. Then the following two statements are equivalent:*

(i) R is a T-preorder.
(ii) There exists a non-empty family of $X \to [0,1]$ score functions $(f_i)_{i\in I}$ such that the following representation holds:

$$R(x,y) = \inf_{i\in I} \vec{T}(f_i(x), f_i(y)) \tag{4}$$

Theorem 4 is essential for two main reasons: (1) it shows that every T-preorder is an intersection of representable weak T-orders, (2) as weak T-orders are a special kind of T-preorders, we know for sure that, for each weak T-order R, there exists a family of score functions such that R can be represented as in Eq. (4). Be aware, however, that this is only a representation of theoretical nature. We do not know yet how to choose a family of score functions $(f_i)_{i\in I}$ such that fuzzy relation defined as in Eq. (4) is guaranteed to fulfill strong completeness. The following theorem provides us with a unique characterization of weak T-orders.

Theorem 5. *Consider a binary fuzzy relation $R : X^2 \to [0,1]$. Then the following two statements are equivalent:*

(i) R is a weak T-order.
(ii) There exists a crisp weak order $\lesssim$ and a non-empty family of $X \to [0,1]$ score functions $(f_i)_{i\in I}$ that are non-decreasing with respect to $\lesssim$ such that representation (4) *holds.*

If we want to use Theorem 5 to construct fuzzy weak orders on the real numbers (or a subset of them), one can start from the natural linear order of real numbers, since this order is a crisp weak order, of course. The question arises whether each fuzzy weak order can be represented by a family of score functions that are monotonic with respect to a linear order. The following theorem gives a positive answer and characterizes weak T-orders as intersections of representable weak T-orders that are generated by score functions that are monotonic at the same time with respect to the same crisp linear order.

Theorem 6. *Consider a binary fuzzy relation $R : X^2 \to [0,1]$. Then the following two statements are equivalent:*

(i) R is a weak T-order.
(ii) There exists a linear order $\preceq$ and a non-empty family of $X \to [0,1]$ score functions $(f_i)_{i\in I}$ that are non-decreasing with respect to $\preceq$ such that representation (4) *holds.*

Example 2. We consider $X = [0,5]$ again and a family of five functions that are defined as follows:

$$\begin{aligned} g_1(x) &= \min(1, x) \\ g_2(x) &= \min\big(1, \max(0, x-1)\big) \\ g_3(x) &= \min\big(1, \max(0, x-2)\big) \\ g_4(x) &= \min\big(1, \max(0, x-3)\big) \\ g_5(x) &= \min\big(1, \max(0, x-4)\big) \end{aligned}$$

It is immediate that all five functions are non-decreasing with respect to the natural order of real numbers. Figure 2 depicts six fuzzy weak orders defined in accordance with Theorem 6:

$$\begin{aligned} R_7(x,y) &= \min_{i\in\{1,3,5\}} \vec{T}_{\mathbf{M}}(g_i(x), g_i(y)) & R_8(x,y) &= \min_{i\in\{1,\ldots,5\}} \vec{T}_{\mathbf{M}}(g_i(x), g_i(y)) \\ R_9(x,y) &= \min_{i\in\{1,3,5\}} \vec{T}_{\mathbf{P}}(g_i(x), g_i(y)) & R_{10}(x,y) &= \min_{i\in\{1,\ldots,5\}} \vec{T}_{\mathbf{P}}(g_i(x), g_i(y)) \\ R_{11}(x,y) &= \min_{i\in\{1,3,5\}} \vec{T}_{\mathbf{L}}(g_i(x), g_i(y)) & R_{12}(x,y) &= \min_{i\in\{1,\ldots,5\}} \vec{T}_{\mathbf{L}}(g_i(x), g_i(y)) \end{aligned}$$

Example 2 uses the natural linear order of real numbers and rather simple monotonic score functions. The next example constructs some more complicated weak $T_{\mathbf{L}}$-orders on the basis of a non-trivial order on the real numbers.

Example 3. Let us consider the following transformation function:

$$\varphi(x) = \begin{cases} 4 - x & \text{if } x \in [1,3] \\ x & \text{otherwise} \end{cases}$$

It is immediate that φ is a bijective $\mathbb{R} \to \mathbb{R}$ mapping that equals the identity in $]-\infty, 1[\cup]3, \infty[$ and flips the values in $[1,3]$. It is clear, therefore, that the binary relation

$$x \preceq y \text{ if and only if } \varphi(x) \leq \varphi(y)$$

is a linear order on the real numbers. Taking the score functions $g_1, \ldots, g_5$ from Example 2, we can define another family of score functions $h_1, \ldots, h_5$ as

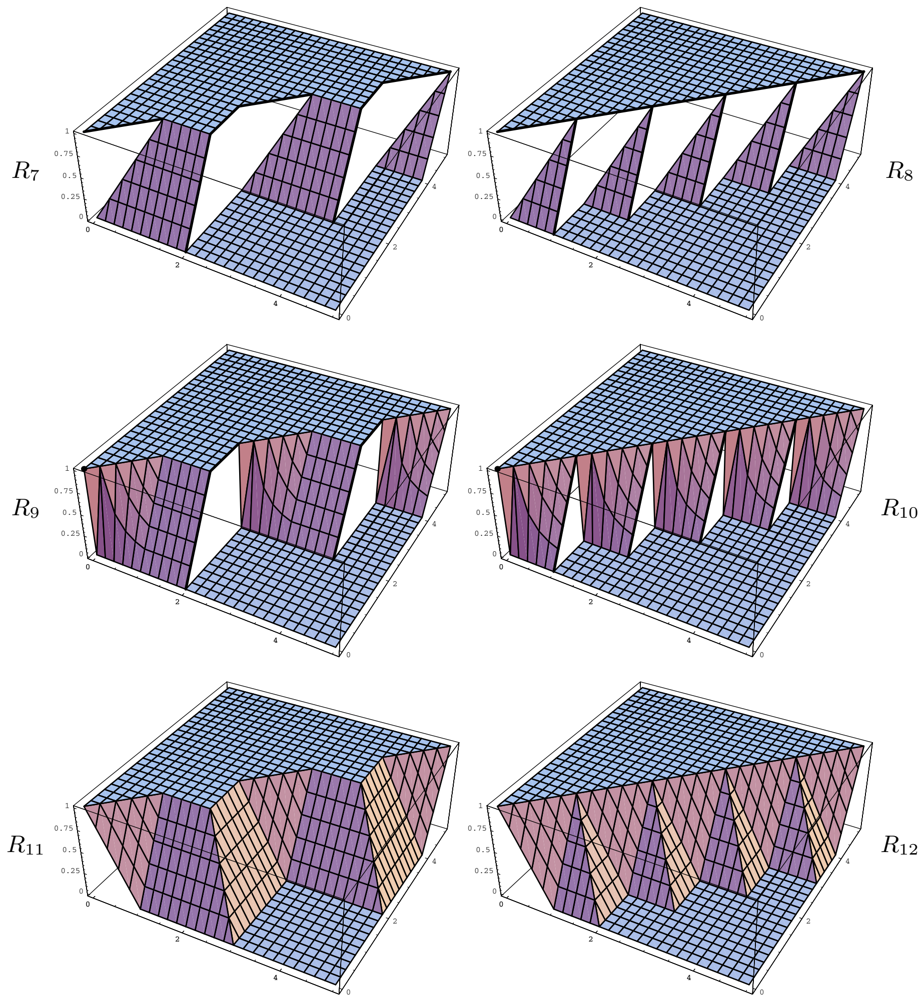

Fig. 2. Six fuzzy weak orders constructed by means of Theorem 6 using the three basic t-norms

$h_i(x) = g_i(\varphi(x))$ (for all $x \in [0, 5]$). It is easy to see that all functions h_i are non-decreasing with respect to $\preceq$. Thus, we can use them to define fuzzy weak orders. Figure 3 shows three weak $T_{\mathbf{L}}$-orders defined as follows:

$$\begin{aligned} R_{13}(x, y) &= \vec{T}_{\mathbf{L}}(h_3(x), h_3(y)) \\ R_{14}(x, y) &= \min_{i \in \{1,3,5\}} \vec{T}_{\mathbf{L}}(h_i(x), h_i(y)) \\ R_{15}(x, y) &= \min_{i \in \{1,\ldots,5\}} \vec{T}_{\mathbf{L}}(h_i(x), h_i(y)) \end{aligned}$$

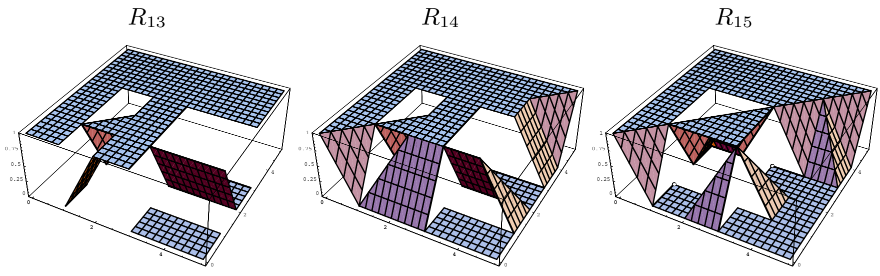

Fig. 3. Three weak $T_{\mathbf{L}}$-orders with a non-trivial underlying crisp linear order

4 Inclusion-Based Representations

As mentioned in Section 1, Theorem 1 can also be proved by embedding the given weak order into the partially ordered set $(\mathcal{P}(X), \subseteq)$. Technically, this is done by mapping the elements $x \in X$ to their foresets $C(x)$. The question arises whether an analogous technique works for fuzzy weak orders as well. This section is devoted to this topic.

Consider the fuzzy power set $\mathcal{F}(X)$. Then the well-known crisp inclusion of fuzzy sets

$$A \subseteq B \text{ if and only if } A(x) \leq B(x) \text{ for all } x \in X$$

is a crisp partial order on $\mathcal{F}(X)$ [25]. Given a left-continuous t-norm T, we can define the following two binary fuzzy relations on $\mathcal{F}(X)$ [1, 4, 13]:

$$\mathrm{INCL}_T(A, B) = \inf_{x \in X} \vec{T}(A(x), B(x))$$
$$\mathrm{SIM}_T(A, B) = \inf_{x \in X} \overleftrightarrow{T}(A(x), B(x))$$

It was proved in [4] that SIM_T is a T-equivalence on $\mathcal{F}(X)$ and that INCL_T is a T-SIM_T-order on $\mathcal{F}(X)$. Moreover, it is easy to see from elementary properties of residual (bi)implications that $\mathrm{INCL}_T(A, B) = 1$ if and only if $A \subseteq B$ and that $\mathrm{SIM}_T(A, B) = 1$ if and only if $A = B$.

The following theorem provides us with a unique characterization of fuzzy weak orders that is based on an embedding of the given fuzzy weak order into the fuzzy power set.

Theorem 7. *Consider a binary fuzzy relation $R : X^2 \to [0, 1]$. Then the following two statements are equivalent:*

(i) R is a weak T-order.
(ii) There exists a non-empty family of fuzzy sets $S \subseteq \mathcal{F}(X)$ that are linearly ordered with respect to the inclusion relation $\subseteq$ and a mapping $\varphi : X \to S$ such that the following representation holds for all $x, y \in X$:

$$R(x, y) = \mathrm{INCL}_T(\varphi(x), \varphi(y)) \tag{5}$$

We can formulate an equivalent result that appears a bit more appealing than Theorem 7.

Corollary 1. *Consider a binary fuzzy relation* $R : X^2 \to [0,1]$*. Then the following two statements are equivalent:*

(i) R *is a weak* T*-order.*
(ii) There exists a mapping $\varphi : X \to \mathcal{F}(X)$ *fulfilling* $\varphi(x) \subseteq \varphi(y)$ *or* $\varphi(y) \subseteq \varphi(x)$ *for all* $x, y \in X$ *such that representation* (5) *holds.*

If we omit the linearity conditions in Theorem 7 and Corollary 1, a unique representation of T-preorders is obtained: a fuzzy relation R is a T-preorder if and only if there exists a mapping $\varphi : X \to \mathcal{F}(X)$ such that Eq. (5) holds [5]. In this sense, the T-preorder INCL_T on $\mathcal{F}(X)$ "contains" all T-preorders that can be defined on X. Weak T-orders are then the sub-class that is obtained by restricting to linearly ordered subsets of $\mathcal{F}(X)$.

The proof of Theorem 7 (and Corollary 1) is based on mapping each $x \in X$ to its foreset. However, there is no restriction to only use foresets in (5), as long as the range of the embedding mapping $\varphi(X)$ is linearly ordered. Thus, Theorem 7 and Corollary 1 give rise to potentially interesting constructions. For infinite domains, however, $\mathrm{INCL}_T(A, B)$ is mostly difficult to compute, as an infimum over an infinite set has to be determined. Only under very restrictive assumptions, for instance, that all membership functions of the fuzzy sets $\varphi(x)$ are piecewise linear or differentiable, practically feasible constructions are imaginable. One can overall conclude that Theorem 7 and Corollary 1 provide us with nice theoretical insight, but they do not have much practical value. That is why we do not provide an example in this section.

5 Decompositions into Crisp Linear Orders and *T*-Equivalences

The standard proof of Theorem 1 is based on the factorization with respect to the symmetric kernel of a given weak order (cf. Section 1). One can state, in other words, that a crisp weak order can always be decomposed into a crisp linear order and an equivalence relation. This section follows this idea and presents corresponding results for fuzzy weak orders. Before coming to the main result, let us shortly introduce an important prerequisite.

Definition 3. Let $\preceq$ be a crisp order on X and let $E : X^2 \to [0,1]$ be a fuzzy equivalence relation (regardless of the underlying t-norm T). E is called *compatible with* $\preceq$ if and only if the following inequality holds for all ascending three-element chains $x \preceq y \preceq z$ in X:

$$E(x, z) \leq \min(E(x, y), E(y, z))$$

Compatibility of a crisp order and a fuzzy equivalence relation can be understood as follows: the two outer elements of an ascending three-element chain are at most as similar as any two elements of this chain.

Theorem 8. *Consider a binary fuzzy relation $R : X^2 \to [0,1]$. Then the following two statements are equivalent:*

(i) R is a weak T-order.
(ii) There exists a crisp linear order $\preceq$ and a T-equivalence E that is compatible with $\preceq$ such that R can be represented as follows:

$$R(x,y) = \begin{cases} 1 & \text{if } x \preceq y \\ E(x,y) & \text{otherwise} \end{cases} \tag{6}$$

Representation (6) simply says the following: weak T-orders are characterized as unions of crisp linear orders and compatible T-equivalences. In other words, we can say that weak T-orders are a fuzzification of crisp linear orders, and the fuzzy component can solely be attributed to a T-equivalence.

To utilize Theorem 8 for constructing weak T-orders, we have to know more about how to construct T-equivalences that are compatible with a given crisp linear order. Let us start with a well-known result on T-equivalences.

Theorem 9. [24] *Consider a binary fuzzy relation $E : X^2 \to [0,1]$. Then the following two statements are equivalent:*

(i) E is a T-equivalence.
(ii) There exists a non-empty family of $X \to [0,1]$ functions $(f_i)_{i \in I}$ such that the following representation holds:

$$E(x,y) = \inf_{i \in I} \overleftrightarrow{T}(f_i(x), f_i(y)) \tag{7}$$

The following theorem finally provides a unique characterization of T-equivalences that are compatible with a given crisp linear order.

Theorem 10. *Consider a crisp linear order $\preceq$ on X and a binary fuzzy relation $E : X^2 \to [0,1]$. Then the following two statements are equivalent:*

(i) E is a T-equivalence that is compatible with $\preceq$.
(ii) There exists a non-empty family of $X \to [0,1]$ functions $(f_i)_{i \in I}$ that are non-decreasing with respect to $\preceq$ such that representation (7) holds.

Note that Theorem 10 remains valid if we replace "non-decreasing" in (ii) by "non-increasing".

Example 4. It is easy to see that $E_1(x,y) = \exp(-|x-y|)$ is a $T_{\mathbf{P}}$-equivalence on the real numbers $X = \mathbb{R}$ that is compatible with the natural order $\leq$ and that $E_2(x,y) = \max(1 - |x-y|, 0)$ is a $T_{\mathbf{L}}$-equivalence on the real numbers $X = \mathbb{R}$ that is also compatible with $\leq$ [6, 9, 10]. Hence, Theorem 8 entails that

$$\begin{aligned} R_{16}(x,y) &= \begin{cases} 1 & \text{if } x \leq y \\ \exp(-|x-y|) & \text{otherwise} \end{cases} \\ &= \min(1, \exp(y-x)) \end{aligned}$$

is a weak $T_{\mathbf{P}}$-order and

$$R_{17}(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ \max(1 - |x-y|, 0) & \text{otherwise} \end{cases}$$
$$= \min(1, \max(1 - x + y, 0))$$

is a weak $T_{\mathbf{L}}$-order. Figure 4 shows these two fuzzy weak orders (where the plots are restricted to $[0,5]^2$).

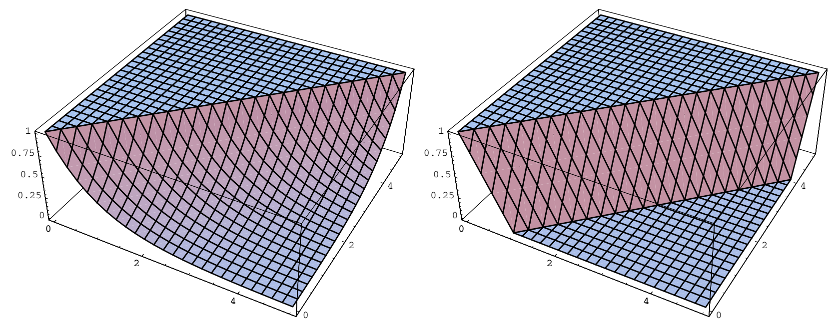

Fig. 4. Two fuzzy weak orders constructed from the absolute distance of real numbers

6 Concluding Remarks

In this contribution, we have highlighted various representations of fuzzy weak orders. Score function-based representations and the decomposition of fuzzy weak orders into crisp linear orders and fuzzy equivalence relations also provided us with practically feasible construction methods. Unlike most of the existing literature, we have not assumed that the underlying domain is finite.

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References

1. W. Bandler and L. J. Kohout. Fuzzy power sets and fuzzy implication operators. *Fuzzy Sets and Systems*, 4:183–190, 1980.
2. W. Bandler and L. J. Kohout. Fuzzy relational products as a tool for analysis and synthesis of the behaviour of complex natural and artificial systems. In S. K. Wang and P. P. Chang, editors, *Fuzzy Sets: Theory and Application to Policy Analysis and Information Systems*, pages 341–367. Plenum Press, New York, 1980.
3. U. Bodenhofer. Representations and constructions of strongly linear fuzzy orderings. In *Proc. EUSFLAT-ESTYLF Joint Conference*, pages 215–218, Palma de Mallorca, September 1999.
4. U. Bodenhofer. A similarity-based generalization of fuzzy orderings preserving the classical axioms. *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems*, 8(5):593–610, 2000.
5. U. Bodenhofer. Representations and constructions of similarity-based fuzzy orderings. *Fuzzy Sets and Systems*, 137(1):113–136, 2003.
6. U. Bodenhofer and J. Küng. Fuzzy orderings in flexible query answering systems. *Soft Computing*, 8(7):512–522, 2004.
7. G. Cantor. Beiträge zur Begründung der transfiniten Mengenlehre. *Math. Ann.*, 46:481–512, 1895.
8. B. De Baets, J. Fodor, and E. E. Kerre. Gödel representable fuzzy weak orders. *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems*, 7(2):135–154, 1999.
9. B. De Baets and R. Mesiar. Pseudo-metrics and T-equivalences. *J. Fuzzy Math.*, 5(2):471–481, 1997.
10. B. De Baets and R. Mesiar. Metrics and T-equalities. *J. Math. Anal. Appl.*, 267:331–347, 2002.
11. J. Fodor and S. V. Ovchinnikov. On aggregation of T-transitive fuzzy binary relations. *Fuzzy Sets and Systems*, 72:135–145, 1995.
12. J. Fodor and M. Roubens. *Fuzzy Preference Modelling and Multicriteria Decision Support*. Kluwer Academic Publishers, Dordrecht, 1994.
13. S. Gottwald. *Fuzzy Sets and Fuzzy Logic*. Vieweg, Braunschweig, 1993.
14. S. Gottwald. *A Treatise on Many-Valued Logics*. Studies in Logic and Computation. Research Studies Press, Baldock, 2001.
15. P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer Academic Publishers, Dordrecht, 1998.
16. E. P. Klement, R. Mesiar, and E. Pap. *Triangular Norms*, volume 8 of *Trends in Logic*. Kluwer Academic Publishers, Dordrecht, 2000.
17. D. H. Krantz, R. D. Luce, P. Suppes, and A. Tversky. *Foundations of Measurement*. Academic Press, San Diego, CA, 1971.
18. S. V. Ovchinnikov. Similarity relations, fuzzy partitions, and fuzzy orderings. *Fuzzy Sets and Systems*, 40(1):107–126, 1991.
19. S. V. Ovchinnikov. An introduction to fuzzy relations. In D. Dubois and H. Prade, editors, *Fundamentals of Fuzzy Sets*, volume 7 of *The Handbooks of Fuzzy Sets*, pages 233–259. Kluwer Academic Publishers, Boston, 2000.
20. S. V. Ovchinnikov. Numerical representation of transitive fuzzy relations. *Fuzzy Sets and Systems*, 126:225–232, 2002.
21. F. S. Roberts. *Measurement Theory*. Addison-Wesley, Reading, MA, 1979.
22. J. G. Rosenstein. *Linear Orderings*, volume 98 of *Pure and Applied Mathematics*. Academic Press, New York, 1982.

23. E. Szpilrajn. Sur l'extension de l'ordre partiel. *Fund. Math.*, 16:386–389, 1930.
24. L. Valverde. On the structure of F-indistinguishability operators. *Fuzzy Sets and Systems*, 17(3):313–328, 1985.
25. L. A. Zadeh. Fuzzy sets. *Inf. Control*, 8:338–353, 1965.
26. L. A. Zadeh. Similarity relations and fuzzy orderings. *Inform. Sci.*, 3:177–200, 1971.