Social Software for Coalition Formation*

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Abstract. This paper concerns an interdisciplinary approach to coalition formation. We apply the MacBeth software, relational algebra, the RelView tool, graph theory, bargaining theory, social choice theory, and consensus reaching to a model of coalition formation. A feasible government is a pair consisting of a coalition of parties and a policy supported by this coalition. A feasible government is stable if it is not dominated by any other feasible government. Each party evaluates each government with respect to certain criteria. MacBeth helps to quantify the importance of the criteria and the attractiveness and repulsiveness of governments to parties with respect to the given criteria. Feasibility, dominance, and stability are formulated in relation-algebraic terms. The RelView tool is used to compute the dominance relation and the set of all stable governments. In case there is no stable government, i.e., in case the dominance relation is cyclic, we apply graph-theoretical techniques for breaking the cycles. If the solution is not unique, we select

* Co-operation for this paper was supported by European COST Action 274 "Theory and Applications of Relational Structures as Knowledge Instruments" (TARSKI). We thank Gunther Schmidt for his most valuable contributions to this paper.

H. de Swart et al. (Eds.): TARSKI II, LNAI 4342, pp. 1–30, 2006.

the final government by applying bargaining or appropriate social choice rules. We describe how a coalition may form a government by reaching consensus about a policy.

Keywords: stable government, MacBeth, relational algebra, RelView, graph theory, bargaining, social choice rule, consensus.

1 Introduction

This paper presents an overview of the results on coalition formation obtained from cooperation within the European COST Action 274: TARSKI (Theory and Applications of Relational Structures as Knowledge Instruments). The authors were connected to two different Work Areas of the COST Action, namely Work Area WA2 (Mechanization and Relational Reasoning) and Work Area WA3 (Relational Scaling and Preferences). This cooperation, which was not foreseen but gradually evolved over the years, resulted in an interdisciplinary approach to coalition formation. The MacBeth technique, relational algebra, the RelView tool, graph theory, bargaining theory, social choice theory, and consensus reaching were applied to the basic model of coalition formation described in Rusinowska et al. [44].

Coalition formation is one of the more interesting and at the same time more popular topics, and consequently a lot of work has already been done in this field. There are several ways to distinguish different coalition formation theories: one may talk, for instance, about power-oriented versus policy-oriented theories, one-dimensional versus multi-dimensional models, or actor-oriented versus non-actor oriented theories. The power-oriented theories, where the motivation for political parties to join a coalition is based only on their personal gains, are the earliest theories of coalition formation. One may mention here the theory of minimal winning coalitions (von Neuman and Morgenstern [55]), the minimum size theory (Riker [40]), and the bargaining proposition (Leiserson [35]). In policy-oriented theories, the process of coalition formation is determined by both policy and power motivations. Some of the most important early policy-oriented theories were the minimal range theory (Leiserson [34]), conflict of interest theory (Axelrod [2]), and the policy distance theory (de Swaan [21]). Actor-oriented theories, like the dominant player theory (Peleg [38], [39]) and the center player theory (van Deemen [53]), select an actor that has a more powerful position in the process of coalition formation. Also a lot of work has been done on spatial coalition formation theories, especially with respect to multi-dimensional policy-oriented theories. A main assumption in such models is that policy positions of parties are very important in the coalition formation process. One must mention here the political heart solution (Schofield [48], [49], [50]), the protocoalition formation (Grofman [29]), the winset theory (Laver and Shepsle [32], [33]), and the competitive solution (McKelvey, Ordeshook and Winer [36]). Many authors also considered institutional theories of coalition formation. One of the first theorists who acknowledged the important role of institutions was Shepsle [52], followed, in particular, by Austen-Smith and Banks [1], Laver and Schofield [31], and

Baron [6]. For an overview of coalition formation models we also like to refer to van Deemen [54], de Vries [24], Kahan and Rapoport [30].

The point of departure in this paper is a multi-dimensional model of coalition formation (see Rusinowska et al. [44]) in which the notion of stable government is central. In the model, the approach we use to represent party preferences allows us to include both rent-seeking and idealistic (policy-seeking) motivations. Moreover, a policy space does not have to be a Euclidean space, as is assumed frequently in coalition formation models, but may be any kind of space. The policy space is assumed to be multi-dimensional, which allows us to consider many political issues at the same time.

A government is defined as a pair consisting of a coalition and a policy supported by that coalition. It has a value (utility) to each party with respect to every given issue. In order to determine these values in practice, we propose to use the MacBeth approach; see also Roubens et al. [41]. MacBeth, which stands for *Measuring Attractiveness by a Categorical Based Evaluation Technique*, is an interactive approach to quantify the attractiveness of each alternative, such that the measurement scale constructed is an interval scale. For an overview and some applications of the software, we refer to the web site (www.m-macbeth.com), Bana e Costa and Vansnick [3]; Bana e Costa et al. [5]. The notion of absolute judgement has also been used in Saaty's Analytical Hierarchy Process (AHP); see Saaty [45], [46]. In the MacBeth technique, the absolute judgements concern differences of attractiveness, while in Saaty's method they concern ratios of priority, or of importance. One of the advantages of using the MacBeth approach is related to ensuring consistency. In case of any inconsistency of the initial evaluations, the MacBeth software indicates to the user what is the cause of the inconsistency and how to reach consistency. For a critical analysis of the AHP, see Bana e Costa and Vansnick [4].

Another application to the coalition formation model we propose here concerns Relational Algebra and the RelView tool which helps us to calculate stable governments; see also Berghammer et al. [11]. The RelView system, which has been developed at Kiel University, is a computer system for the visualization and manipulation of relations and for relational prototyping and programming. The tool is written in the C programming language, uses reduced ordered binary decision diagrams for implementing relations, and makes full use of the X-windows graphical user interface. For details and applications see, for instance, Berghammer et al. [14], Behnke et al. [7], Berghammer et al. [10], and Berghammer et al. [13].

In this paper, we also present an application of Graph Theory to the model of coalition formation in question; see Berghammer et al. [12]. We present a graph-theoretical procedure for choosing a government in case there is no stable government. If, on the other hand, more than one stable government exists, we may apply Social Choice Theory to choose one government. For an overview and comparison of social choice rules see, for instance, Brams and Fishburn [16], and de Swart et al. [23]. Another natural application is based on Bargaining Theory. We use a strategic approach to bargaining; see Rubinstein [42], Fishburn and

Rubinstein [27], Osborne and Rubinstein [37]. We formulate several bargaining games in which parties bargain over the choice of one stable government, and next we look for refinements of Nash equilibria called subgame perfect equilibria (Selten [51]) of these games; see also Rusinowska and de Swart [43].

We describe a procedure for a coalition to choose a policy in order to propose a government, based on consensus reaching, by combining some ideas from Carlsson et al. [18] and Rusinowska et al. [44]. It has been first proposed in Eklund et al. [25], where the authors consider consensus reaching in a committee, and next in Eklund et al. [26], where a more complicated model, i.e., consensus reaching in coalition formation, is presented.

The paper is structured as follows. Section 2 introduces the model of coalition formation. In Section 4, the basic notions of relational algebra are presented. In Sections 3 and 5, we present applications of the MacBeth and RelView tools, respectively, to the model in question. Section 6 concerns applications of Social Choice Theory and Bargaining Theory to the model, in order to choose a stable government in the case there exists more than one. Next, an application of Graph Theory to the model of coalition formation is proposed in Section 7, in order to choose a 'rather stable' government in the case that there exists no stable one. Section 8 describes how a coalition may reach consensus about a policy in order to propose a government. In Section 9, we present our conclusions.

2 The Model of Coalition Formation

In this section we recapitulate a model of coalition formation, first introduced in Rusinowska et al. [44], and further refined, in particular, in Eklund et al. [26].

2.1 Description of the Model

Let $N = \{1, \ldots, n\}$ be the set of political parties in a parliament, and let w_i denote the number of seats received by party $i \in N$. Moreover, let W denote the *set of all winning coalitions*. The model concerns the creation of a government by a winning coalition. It is assumed that there are some independent policy issues on which a government has to decide. Let P denote the *set of all policies*.

A *government* is defined as a pair $g = (S, p)$, where S is a winning coalition and p is a policy. Hence, the *set G of all governments* is defined as

$$G := \{(S, p) \mid S \in W \;\wedge\; p \in P\}. \tag{1}$$

Each party has preferences concerning all policies and all (winning) coalitions. A coalition is called *feasible* if it is acceptable to all its members. A policy is *feasible for a given coalition* if it is acceptable to all members of that coalition. A government (S, p) is *feasible* if both, S and p, are acceptable to each party belonging to S. By G^* we denote the *set of all feasible governments*, and by G^*_i the *set of all feasible governments containing party i*, i.e., for each $i \in N$,

$$G^*_i := \{(S, p) \in G^* \mid i \in S\}. \tag{2}$$

A *decision maker* is a party involved in at least one feasible government, i.e., the *set DM of all decision makers* is equal to

$$DM := \{i \in N \mid G_i^* \neq \emptyset\}. \quad (3)$$

Moreover, let the subset W^* of W be defined as

$$W^* := \{S \in W \mid \exists\, p \in P : (S, p) \in G^*\}. \quad (4)$$

A feasible government is evaluated by each decision maker with respect to the given policy issues and with respect to the issue concerning the coalition. Let C^* be the finite *set of criteria*. The criteria do not have to be equally important to a party, and consequently, each decision maker evaluates the importance of the criteria. Formally, for each $i \in DM$, we assume a function $\alpha_i : C^* \to [0, 1]$, such that the following property holds:

$$\forall\, i \in DM : \sum_{c \in C^*} \alpha_i(c) = 1. \quad (5)$$

The number $\alpha_i(c)$ is i's evaluation of criterion c. Moreover, each decision maker evaluates each feasible government with respect to all the criteria. Hence, for each $i \in DM$, we assume $u_i : C^* \times G^* \to \mathbb{R}$ where the real number $u_i(c, g)$ is called the *value of government $g \in G^*$ to party $i \in DM$ with respect to criterion $c \in C^*$*. Moreover, for each $i \in DM$, we define $U_i : G^* \to \mathbb{R}$ such that

$$(U_i(g))_{g \in G^*} = (\alpha_i(c))_{c \in C^*} \cdot (u_i(c, g))_{c \in C^*, g \in G^*}, \quad (6)$$

where $(\alpha_i(c))_{c \in C^*}$ is the $1 \times |C^*|$ matrix representing the evaluation (comparison) of the criteria by party i, $(u_i(c, g))_{c \in C^*, g \in G^*}$ is the $|C^*| \times |G^*|$ matrix containing party i's evaluation of all governments in G^* with respect to each criterion in C^*, and $(U_i(g))_{g \in G^*}$ is the $1 \times |G^*|$ matrix containing party i's evaluation of each government in G^*.

In order to determine in practice the values of $\alpha_i(c)$ and $u_i(c, g)$ for all parties $i \in DM$, criteria $c \in C^*$ and governments $g \in G^*$, we can use the MacBeth technique. We do so in Section 3.

The central notion of the model introduced in Rusinowska et al. [44] is the notion of *stability*. A feasible government $h = (S, p) \in G^*$ *dominates* a feasible government $g \in G^*$ (denoted as $h \succ g$) if the property

$$(\forall\, i \in S : U_i(h) \geq U_i(g)) \wedge (\exists\, i \in S : U_i(h) > U_i(g)) \quad (7)$$

holds. A feasible government is said to be *stable* if it is dominated by no feasible government. By

$$SG^* := \{g \in G^* \mid \neg \exists\, h \in G^* : h \succ g\} \quad (8)$$

we denote the *set of all (feasible) stable governments*. In Rusinowska et al. [44], necessary and sufficient conditions for the existence and the uniqueness of a stable government are investigated. Moreover, the authors introduce some alternative definitions of 'stability', and establish the relations between the new notions of 'stability' and the chosen one. In the present paper, we decide for the definition of a stable government given by (8), which we find the most natural definition of stability.

2.2 A Running Example

Let us consider a very small parliament consisting of only three parties. We assume each coalition consisting of at least two parties is winning and there are only two policy issues and four policies, i.e., we have

$$N = \{A, B, C\}, \quad W = \{AB, AC, BC, ABC\}, \quad P = \{p_1, p_2, p_3, p_4\}.$$

As a consequence, we have 16 governments. Assume that the grand coalition is not feasible, but all two-party coalitions are feasible. Further, assume both policies p_1 and p_2 are acceptable to all three parties, policy p_3 is not acceptable to party C, while policy p_4 is not acceptable to party B. Hence, policies p_1 and p_2 are feasible for coalitions AB, AC, and BC, policy p_3 is feasible for coalition AB, and p_4 is feasible for coalition AC.

Consequently, there are eight feasible governments, i.e.,

$$G^* = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8\},$$

which are given as

$$\begin{aligned} g_1 &= (AB, p_1), \quad & g_2 &= (AC, p_1), \quad & g_3 &= (BC, p_1), \quad & g_4 &= (AB, p_2), \\ g_5 &= (AC, p_2), \quad & g_6 &= (BC, p_2), \quad & g_7 &= (AB, p_3), \quad & g_8 &= (AC, p_4) \end{aligned}$$

and therefore obtain the governments containing the parties as

$$\begin{aligned} G^*_A &= \{g_1, g_2, g_4, g_5, g_7, g_8\}, \\ G^*_B &= \{g_1, g_3, g_4, g_6, g_7\}, \\ G^*_C &= \{g_2, g_3, g_5, g_6, g_8\}. \end{aligned}$$

Moreover, we have

$$DM = N, \quad W^* = \{AB, AC, BC\}, \quad C^* = \{1, 2, 3\},$$

where the criteria 1 and 2 refer to the first and the second policy issue, while criterion 3 concerns the (attractiveness of the) 'coalition'. In order to determine $\alpha_i(c)$ and $u_i(c, g)$ for each $i \in DM$, $c \in C^*$, and $g \in G^*$, we will use the MacBeth technique in the next section.

3 Applying MacBeth to Coalition Formation

When applying the coalition formation model described in Section 2 in practice, the question arises how to determine the $\alpha_i(c)$ and the $u_i(c, g)$ for $i \in DM$. The answer to this question will be given in this section, where we propose to use the MacBeth software to determine these values. In Subsection 3.1, we show how the utilities of governments to parties may be calculated using the MacBeth technique (see also [41]), while in Subsection 3.2 the application is illustrated by an example. It is assumed here that each party judges only a finite number of governments differently, even if there is an infinite number of possible governments.

3.1 Computing the Utilities by MacBeth

Given a party $i \in DM$ and a criterion $c \in C^*$, in order to determine the values $u_i(c, g)$ for each feasible government $g \in G^*$, we will use the MacBeth approach. For each criterion $c \in C^*$, each party ranks in a non-increasing order all feasible governments taking into account the attractiveness of these governments with respect to the given criterion. In particular, for each criterion $c \in C^*$, each party $i \in DM$ specifies two particular references:

- $neutral_i^c$ (*'a for party i neutral government with respect to criterion c'*) defined as a for i neither satisfying nor unsatisfying government wrt. c,
- $good_i^c$ (*'a for party i good government with respect to criterion c'*) defined as a for i undoubtedly satisfying government wrt. c.

These references may be fictitious. We need to add that $neutral_i^c$ and $good_i^c$ are only related to the component of the government concerning the given criterion c, which is either the policy on issue c or the coalition forming the government. For each $c \in C^*$ the remaining 'components' do not matter. Define for all $c \in C^*$ and $i \in DM$ the set

$$G_i^c = G^* \cup \{neutral_i^c, good_i^c\}.$$

For each $c \in C^*$, each party $i \in DM$ judges verbally the difference of attractiveness between each two governments $g, h \in G_i^c$, where g is at least as attractive to i as h. When judging, a party chooses one of the following categories:

D_0 : *no* difference of attractiveness,
D_1 : *very weak* difference of attractiveness,
D_2 : *weak* difference of attractiveness,
D_3 : *moderate* difference of attractiveness,
D_4 : *strong* difference of attractiveness,
D_5 : *very strong* difference of attractiveness,
D_6 : *extreme* difference of attractiveness.

(Formally, the categories are relations.) A party may also choose the union of several successive categories among these above or a *positive* difference of attractiveness in case the party is not sure about the difference of attractiveness.

Given a party $i \in DM$ and a criterion $c \in C^*$, a non-negative number $u_i(c, g)$ is associated to each $g \in G_i^c$. If there is no hesitation about the difference of attractiveness, the following rules are satisfied; see Bana e Costa and Vansnick [3], Bana e Costa et al. [5]. First, for all $g, h \in G_i^c$

$$u_i(c, g) > u_i(c, h) \iff g \text{ more attractive to } i \text{ wrt. } c \text{ than } h. \tag{9}$$

Second, for all $k, k' \in \{1, 2, 3, 4, 5, 6\}$ with $k \geq k' + 1$ and all $g, g', h, h' \in G_i^c$ with $(g, g') \in D_k$ and $(h, h') \in D_{k'}$

$$u_i(c, g) - u_i(c, g') > u_i(c, h) - u_i(c, h'). \tag{10}$$

The numerical scale, called the MacBeth basic scale, is obtained by linear programming, and it exists if and only if it is possible to satisfy rules (9) and (10).

In that case the matrix of judgements is called consistent. If it is impossible to satisfy rules (9) and (10), a message appears on the screen ('inconsistent judgements'), inviting the party to revise the judgements, and the MacBeth tool gives suggestions how to obtain a consistent matrix of judgements.

The basic MacBeth scale, which is still a *pre-cardinal scale*, is presented both in a numerical way and in a graphical way ('thermometer'). In order to obtain a *cardinal scale*, and the final utilities $u_i(c, g)$ for party i of the governments g with respect to the given criterion c, the party uses the thermometer. When a party selects with the mouse a government, an interval appears around this government. By moving the mouse, the position of the selected government within this interval is modified, by which the party obtains a new positioning of the governments such that both conditions (9) and (10) are satisfied. We obtain the *cardinal scale* and the (final, agreed) utilities of the governments with respect to the given criterion, when the party agrees that the scale adequately represents the relative difference of attractiveness with respect to the given criterion between any two governments.

Using the MacBeth software, we can also calculate the coefficients or weights $(\alpha_i(c))_{c\in C^*}$ of criterion c for party i. Let us assume that $C^* = \{1, 2, \ldots, m\}$. For each party $i \in N$, we consider the following reference profiles:

$$\begin{aligned}
[neutral_i] &= (neutral_i^1, neutral_i^2, \ldots, neutral_i^m) \\
[Crit._i^1] &= (good_i^1, neutral_i^2, \ldots, neutral_i^m) \\
[Crit._i^2] &= (neutral_i^1, good_i^2, \ldots, neutral_i^m) \\
&\vdots \\
[Crit._i^m] &= (neutral_i^1, neutral_i^2, \ldots, good_i^m)
\end{aligned}$$

For each $c \in C^*$, the difference in attractiveness between $[Crit._i^c]$ and $[neutral_i]$ corresponds to the added value of the 'swing' from $neutral_i^c$ to $good_i^c$. A party ranks the reference profiles in decreasing order of attractiveness and, using categories D_0 to D_6, judges the difference of attractiveness between each two reference profiles, where the first one is more attractive than the second one. After the adjustment of the MacBeth scale proposed by the software, an interval scale is obtained, which measures the overall attractiveness of the reference profiles, and leads to obtaining the coefficients $(\alpha_i(c))_{c\in C^*}$.

3.2 Example (Continued)

In order to determine for our running example (introduced in Subsection 2.2) the utilities to each party of all governments with respect to each criterion, and the coefficients concerning the importance of the criteria for each party, we will use the MacBeth approach. First, each party expresses its preferences. Note that since g_1, g_2, and g_3 have the same policy p_1, they must be equally attractive to each party with respect to the first and the second policy issue. The same holds for governments g_4, g_5, and g_6 which have the same policy p_2. Moreover, governments formed by the same coalition are equally attractive to each party with respect to the third issue, the one concerning the coalition.

In the following three tables we show for each party A, B, and C of our example the non-increasing order of all eight feasible governments $g_1, \ldots, g_8$ with respect to the first, the second, and the third (coalition) criterion. By the symbol $\sim_i$ we denote the equivalence relation for party $i \in DM$.

Table 1. Non-increasing order of all governments wrt. issue 1

party	order
A	$good^1_A \quad g_1 \sim_A g_2 \sim_A g_3 \quad g_4 \sim_A g_5 \sim_A g_6 \quad g_8 \quad g_7 = neutral^1_A$
B	$good^1_B \quad g_4 \sim_B g_5 \sim_B g_6 \quad g_1 \sim_B g_2 \sim_B g_3 \quad g_7 \quad g_8 = neutral^1_B$
C	$g_7 \quad good^1_C = g_8 \quad g_1 \sim_C g_2 \sim_C g_3 \quad g_4 \sim_C g_5 \sim_C g_6 \quad neutral^1_C$

Table 2. Non-increasing order of all governments wrt. issue 2

party	order
A	$good^2_A \quad g_1 \sim_A g_2 \sim_A g_3 \quad g_4 \sim_A g_5 \sim_A g_6 \quad g_7 \quad g_8 = neutral^2_A$
B	$good^2_B \quad g_1 \sim_B g_2 \sim_B g_3 \quad g_4 \sim_B g_5 \sim_B g_6 \quad g_7 \quad g_8 = neutral^2_B$
C	$good^2_C = g_8 \quad g_7 \quad g_1 \sim_C g_2 \sim_C g_3 \quad g_4 \sim_C g_5 \sim_C g_6 \quad neutral^2_C$

Table 3. Non-increasing order of all governments wrt. issue 3

party	order
A	$good^3_A = g_1 \sim_A g_4 \sim_A g_7 \quad g_2 \sim_A g_5 \sim_A g_8 \quad g_3 \sim_A g_6 = neutral^3_A$
B	$good^3_B = g_1 \sim_B g_4 \sim_B g_7 \quad g_3 \sim_B g_6 \quad g_2 \sim_B g_5 \sim_B g_8 = neutral^3_B$
C	$good^3_C = g_2 \sim_C g_5 \sim_C g_8 \quad g_3 \sim_C g_6 \quad g_1 \sim_C g_4 \sim_C g_7 = neutral^3_C$

Each party $i \in DM$ also has to judge the difference of attractiveness between each two reference profiles. Here we obtain the following values:

$$[neutral_i] = (neutral^1_i, neutral^2_i, neutral^3_i)$$
$$[Crit._i^1] = (good^1_i, neutral^2_i, neutral^3_i)$$
$$[Crit._i^2] = (neutral^1_i, good^2_i, neutral^3_i)$$
$$[Crit._i^3] = (neutral^1_i, neutral^2_i, good^3_i)$$

Let us assume that Table 4 shows the decreasing orders of these reference profiles for all parties. Then we obtain:

Table 4. Decreasing order of the reference profiles

party	order of the profiles			
A	$[Crit._A^1]$	$[Crit._A^2]$	$[Crit._A^3]$	$[neutral_A]$
B	$[Crit._B^3]$	$[Crit._B^2]$	$[Crit._B^1]$	$[neutral_B]$
C	$[Crit._C^3]$	$[Crit._C^1]$	$[Crit._C^2]$	$[neutral_C]$

First, we consider party A which has to judge the difference of attractiveness for all the governments with respect to each issue. The following Tables 5, 6 and 7 show the matrices of judgements for this party.

Table 5. Judgements of the attractiveness for party A and issue 1

	$good_A^1$	g_1	g_4	g_8	$neutral_A^1$
$good_A^1$	**no**	very weak	weak	strong	extreme
g_1		**no**	weak	strong	extreme
g_4			**no**	strong	extreme
g_8				**no**	extreme
$neutral_A^1$					**no**

Table 6. Judgements of the attractiveness for party A and issue 2

	$good_A^2$	g_1	g_4	g_7	$neutral_A^2$
$good_A^2$	**no**	weak	moderate	strong	very strong
g_1		**no**	moderate	strong	very strong
g_4			**no**	strong	very strong
g_7				**no**	very strong
$neutral_A^2$					**no**

Table 7. Judgements of the attractiveness for party A and issue 3

	$good_A^3$	g_2	$neutral_A^3$
$good_A^3$	**no**	weak	extreme
g_2		**no**	extreme
$neutral_A^3$			**no**

Based on the above tables, next, the MacBeth tool proposes the basic scale for party A – using the thermometer – discusses the scale, and after that the final values (utilities) are calculated. The following Table 8 shows the results $u_A(c, g)$ for c ranging over the three issues and g ranging over all eight feasible governments $g_1, \ldots, g_8$ of our example:

Table 8. Values of the governments wrt. each issue for party A

$g =$	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
$u_A(1, g) =$	93.0	93.0	93.0	82.3	82.3	82.3	0.0	53.5
$u_A(2, g) =$	93.0	93.0	93.0	78.6	78.6	78.6	57.0	0.0
$u_A(3, g) =$	100.0	75.0	0.0	100.0	75.0	0.0	100.0	75.0

In a similar way, the values for parties B and C may be calculated. Tables 9 and 10 present the values for these parties.

Table 9. Values of the governments wrt. each issue for party B

$g =$	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
$u_B(1, g) =$	80.0	80.0	80.0	95.0	95.0	95.0	55.0	0.0
$u_B(2, g) =$	96.5	96.5	96.5	93.0	93.0	93.0	53.5	0.0
$u_B(3, g) =$	100.0	0.0	57.0	100.0	0.0	57.0	100.0	0.0

Table 10. Values of the governments wrt. each issue for party C

$g =$	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
$u_C(1, g) =$	90.0	90.0	90.0	60.0	60.0	60.0	110.0	100.0
$u_C(2, g) =$	64.2	64.2	64.2	53.5	53.5	53.5	92.8	100.0
$u_C(3, g) =$	0.0	100.0	96.5	0.0	100.0	96.5	0.0	100.0

Moreover, using the MacBeth technique, we can calculate the coefficients $\alpha_i(c)$ for all decision makers $i \in DM$ (in the case of the example, hence, for all parties A, B, and C) and all three issues $c \in C^*$. These numbers are summarized in the following Table 11.

Table 11. The scaling constants

$i \in DM$	$\alpha_i(1)$	$\alpha_i(2)$	$\alpha_i(3)$
A	0.6	0.3	0.1
B	0.1	0.3	0.6
C	0.3	0.1	0.6

Finally, based on all the values, the utilities of all governments are calculated by means of formula (6). The results are presented in Table 12. This table will be the base for obtaining the input of the RelView tool in order to compute the stable governments, as described in the next section.

Table 12. The utilities of all feasible governments

$g \in G^*$	$U_A(g)$	$U_B(g)$	$U_C(g)$
$g_1 = (AB, p_1)$	93.7	97.0	33.4
$g_2 = (AC, p_1)$	91.2	37.0	93.4
$g_3 = (BC, p_1)$	83.7	71.2	91.3
$g_4 = (AB, p_2)$	82.7	97.4	23.4
$g_5 = (AC, p_2)$	80.5	37.4	83.4
$g_6 = (BC, p_2)$	73.0	71.6	81.3
$g_7 = (AB, p_3)$	27.1	81.6	42.3
$g_8 = (AC, p_4)$	39.6	0.0	100.0

4 Relational Algebraic Preliminaries

In this section we recall the basics of relational algebra and some further relational constructions, which are used in this paper later on. For more details on relations and relational algebra, see Schmidt et al. [47] or Brink et al. [17] for example.

4.1 Relational Algebra

If X and Y are sets, then a subset R of the Cartesian product $X \times Y$ is called a (binary) relation with *domain* X and *range* Y. We denote the set (in this context also called type) of all relations with domain X and range Y by $[X \leftrightarrow Y]$ and write $R : X \leftrightarrow Y$ instead of $R \in [X \leftrightarrow Y]$. If X and Y are finite sets of size m and n respectively, then we may consider a relation $R : X \leftrightarrow Y$ as a Boolean matrix with m rows and n columns. In particular, we write $R_{x,y}$ instead of $\langle x, y\rangle \in R$. The Boolean matrix interpretation of relations is used as one of the graphical representations of relations within the RelView tool.

The basic operations on relations are R^T (*transposition*), $\overline{R}$ (*complement*), $R \cup S$ (*union*), $R \cap S$ (*intersection*), $R; S$ (*composition*), R^* (*reflexive-transitive closure*), and the special relations O (*empty relation*), L (*universal relation*), and I (*identity relation*). If R is included in S we write $R \subseteq S$, and equality of R and S is denoted as $R = S$.

A *vector* v is a relation v with $v = v; \mathsf{L}$. For v being of type $[X \leftrightarrow Y]$ this condition means: Whatever set Z and universal relation $\mathsf{L} : Y \leftrightarrow Z$ we choose, an element $x \in X$ is either in relationship $(v; \mathsf{L})_{x,z}$ to no element $z \in Z$ or to all elements $z \in Z$. As for a vector, therefore, the range is irrelevant, we consider in the following mostly vectors $v : X \leftrightarrow \mathbf{1}$ with a specific singleton set $\mathbf{1} := \{\bot\}$ as range and omit in such cases the second subscript, i.e., write v_x instead of $v_{x,\bot}$.

Analogously to linear algebra we use in the following lower-case letters to denote vectors. A vector $v : X \leftrightarrow \mathbf{1}$ can be considered as a Boolean matrix with exactly one column, i.e., as a Boolean column vector, and *describes* (or is a description of) the subset $\{x \in X \mid v_x\}$ of X.

As a second way to model sets we will use the relation-level equivalents of the set-theoretic symbol "$\in$", i.e., *membership-relations* $\varepsilon : X \leftrightarrow 2^X$. These specific relations are defined by $\varepsilon_{x,Y}$ if and only if $x \in Y$, for all $x \in X$ and $Y \in 2^X$. A Boolean matrix representation of the ε relation requires exponential space. However, in Berghammer et al. [10] an implementation of ε using reduced ordered binary decision diagrams is presented, the number of vertices of which is linear in the size of X.

4.2 Relational Products and Sums

Given a Cartesian product $X \times Y$ of two sets X and Y, there are two projection functions which decompose a pair $u = \langle u_1, u_2\rangle$ into its first component u_1 and its second component u_2. For a relation-algebraic approach it is useful to consider instead of these functions the corresponding *projection relations* $\pi : X \times Y \leftrightarrow X$

and $\rho : X \times Y \leftrightarrow Y$ such that for all $u \in X \times Y$, $x \in X$, and $y \in Y$ we have $\pi_{u,x}$ if and only if $u_1 = x$ and $\rho_{u,y}$ if and only if $u_2 = y$. Projection relations enable us to describe the well-known pairing operation of functional programming relation-algebraically as follows. For relations $R : Z \leftrightarrow X$ and $S : Z \leftrightarrow Y$ we define their *pairing* (frequently also called *fork* or *tupling*) $[R, S] : Z \leftrightarrow X \times Y$ by

$$[R, S] := R; \pi^{\mathsf{T}} \cap S; \rho^{\mathsf{T}}. \tag{11}$$

Using (11), for all $z \in Z$ and $u \in X \times Y$ a simple reflection shows that $[R, S]_{z,u}$ if and only if R_{z,u_1} and S_{z,u_2}. As a consequence, the *exchange relation*

$$E := [\rho, \pi] = \rho; \pi^{\mathsf{T}} \cap \pi; \rho^{\mathsf{T}} \tag{12}$$

of type $[X \times Y \leftrightarrow X \times Y]$ exchanges the components of a pair. This means that for all $u, v \in X \times Y$ the relationship $E_{u,v}$ holds if and only if $u_1 = v_2$ and $u_2 = v_1$.

Analogously to the Cartesian product, the disjoint union (or direct sum) $X + Y := (X \times \{1\}) \cup (Y \times \{2\})$ of two sets X and Y leads to the two *injection relations* $\imath : X \leftrightarrow X+Y$ and $\kappa : Y \leftrightarrow X+Y$ such that for all $u \in X + Y$, $x \in X$, and $y \in Y$ we have $\imath_{x,u}$ if and only if $u = \langle x, 1\rangle$ and $\kappa_{y,u}$ if and only if $u = \langle y, 2\rangle$. In this case the counter-part of pairing is the *sum* $R + S : X+Y \leftrightarrow Z$ of two relations $R : X \leftrightarrow Z$ and $S : Y \leftrightarrow Z$, defined by

$$R + S := \imath^{\mathsf{T}}; R \cup \kappa^{\mathsf{T}}; S. \tag{13}$$

From specification (13) we obtain for all $u \in X + Y$ and $z \in Z$ that $(R + S)_{u,z}$ if and only if there exists $x \in X$ such that $u = \langle x, 1\rangle$ and $R_{x,z}$ or there exists $y \in Y$ such that $u = \langle y, 2\rangle$ and $S_{y,z}$.

The representation of a relation $R : X \leftrightarrow Y$ by a vector $vec(R) : X \times Y \leftrightarrow \mathbf{1}$ means that for all $x \in X$ and $y \in Y$ the properties $R_{x,y}$ and $vec(R)_{\langle x,y\rangle,\perp}$, or $vec(R)_{\langle x,y\rangle}$ for short, are equivalent. To obtain a relation-algebraic specification of $vec(R)$, i.e., an expression which does not use element relationships, but only the constants and operations of relational algebra, we assume $x \in X$ and $y \in Y$ and calculate as follows.

$$\begin{array}{rcll} R_{x,y} & \iff & \exists\, a : \pi_{\langle x,y\rangle,a} \wedge R_{a,y} & \pi : X \times Y \leftrightarrow X \text{ projection} \\ & \iff & (\pi; R)_{\langle x,y\rangle,y} & \\ & \iff & \exists\, b : (\pi; R)_{\langle x,y\rangle,b} \wedge \rho_{\langle x,y\rangle,b} & \rho : X \times Y \leftrightarrow Y \text{ projection} \\ & \iff & \exists\, b : (\pi; R \cap \rho)_{\langle x,y\rangle,b} \wedge \mathsf{L}_b & \mathsf{L} : Y \leftrightarrow \mathbf{1} \\ & \iff & ((\pi; R \cap \rho); \mathsf{L})_{\langle x,y\rangle}. & \end{array}$$

An immediate consequence of the last expression of this calculation and the equality of relations is the relation-algebraic specification

$$vec(R) = (\pi; R \cap \rho); \mathsf{L} \tag{14}$$

of the vector $vec(R) : X \times Y \leftrightarrow \mathbf{1}$; see also Schmidt et al. [47].

Later we also will consider a list $R^{(1)}, R^{(2)}, \ldots, R^{(n)}$ of relations $R^{(i)} : X \leftrightarrow Y$ and compute from these a new relation as follows. Let $N := \{1, \ldots, n\}$. If we identify this set with the disjoint union of n copies of $\mathbf{1}$, then

$$C := vec(R^{(1)})^{\mathsf{T}} + \ldots + vec(R^{(n)})^{\mathsf{T}} \tag{15}$$

defines a relation of type $[N \leftrightarrow X \times Y]$ such that, using Boolean matrix terminology, for all $i \in N$ the i^{th} row of C equals the transpose of the vector $vec(R^{(i)})$. Hence, from the above considerations we obtain for all $i \in N, x \in X$, and $y \in Y$ the equivalence of $C_{i,\langle x,y \rangle}$ and $R^{(i)}_{x,y}$.

5 Applying RelView to Coalition Formation

In this section we recapitulate the application of the RelView tool to the model of a stable government (see Berghammer et al. [11]). The main purpose of the RelView tool is the evaluation of relation-algebraic expressions. These are constructed from the relations of its workspace using pre-defined operations and tests, user-defined relational functions, and user-defined relational programs. A relational program is much like a function procedure in the programming languages Pascal or Modula 2, except that it only uses relations as data type. It starts with a head line containing the program name and the formal parameters. Then the declaration of the local relational domains, functions, and variables follows. Domain declarations can be used to introduce projection relations and pairings of relations in the case of Cartesian products, and injection relations and sums of relations in the case of disjoint unions, respectively. The third part of a program is the body, a while-program over relations. As a program computes a value, finally, its last part consists of a return-clause, which is a relation-algebraic expression whose value after the execution of the body is the result. RelView makes the results visible in the form of graphs or matrices.

5.1 Computing the Dominance Relation by RelView

In the following we step-wisely develop relation-algebraic specifications of the notions presented in Section 2, such as feasible governments, the dominance relationship, and stable governments. As we will demonstrate, these can be translated immediately into the programming language of the RelView tool and, hence, the tool can be applied to deal with concrete examples.

In order to develop a relation-algebraic specification of feasible governments, we need two 'acceptability' relations A and B. We assume $A : DM \leftrightarrow P$ such that for all $i \in DM$ and $p \in P$

$$A_{i,p} \iff \text{party } i \text{ accepts policy } p,$$

and $B : DM \leftrightarrow W$ such that for all $i \in DM$ and $S \in W$

$$B_{i,S} \iff \text{party } i \text{ accepts coalition } S.$$

Next we consider the following three relations:

- A relation $isFea(A) : W \leftrightarrow P$ such that a coalition $S \in W$ and a policy $p \in P$ are in relationship $isFea(A)_{S,p}$ if and only if p is feasible for S. A formal predicate logic definition of this is

$$isFea(A)_{S,p} \iff \forall\, i : i \in S \to A_{i,p}. \tag{16}$$

- A vector $\mathit{feaC}(B) : W \leftrightarrow \mathbf{1}$ which describes the set of all feasible coalitions. For all $S \in W$ the predicate logic definition is

$$\mathit{feaC}(B)_S \iff \forall i : i \in S \rightarrow B_{i,S}. \tag{17}$$

- A relation $\mathit{feaG}(A,B) : W \leftrightarrow P$ which coincides with the set G^* of feasible governments. Here we have for all coalitions $S \in W$ and policies $p \in P$ the predicate logic description

$$\mathit{feaG}(A,B)_{S,p} \iff \mathit{feaC}(B)_S \ \wedge \ \mathit{isFea}(A)_{S,p}\ . \tag{18}$$

Our goal is to obtain from the predicate logic definitions (16), (17), and (18) of the relations $\mathit{isFea}(A)$, $\mathit{feaC}(B)$, and $\mathit{feaG}(A,B)$ equivalent relation-algebraic specifications. In Berghammer et al. [11] it is shown that

$$\mathit{isFea}(A) = \overline{\varepsilon^{\mathsf{T}};\overline{A}}\,, \tag{19}$$

$$\mathit{feaC}(B) = \overline{(\varepsilon \cap \overline{B})^{\mathsf{T}};\mathsf{L}}\,, \tag{20}$$

$$\mathit{feaG}(A,B) = \overline{\varepsilon^{\mathsf{T}};\overline{A}} \cap \overline{(\varepsilon \cap \overline{B})^{\mathsf{T}};\mathsf{L}};\mathsf{L}\,, \tag{21}$$

where $\varepsilon : DM \leftrightarrow W$ is the membership-relation between decision makers and winning coalitions. Note that $W \subseteq 2^{DM}$. Using matrix terminology, the relation ε is obtained from the ordinary membership-relation of type $[DM \leftrightarrow 2^{DM}]$ by removing from the latter all columns not corresponding to a set of W.

Next, we develop a relation-algebraic specification of the dominance relationship between feasible governments. To this end, we suppose a relational description of government membership to be given, that is, a relation $M : DM \leftrightarrow G^*$ such that for all $i \in DM$ and $g \in G^*$ the equivalence

$$M_{i,g} \iff \text{party } i \text{ is a member of government } g$$

holds. Moreover, for each party $i \in DM$, we introduce a utility (or comparison) relation $R^{(i)} : G^* \leftrightarrow G^*$ such that for all $g, h \in G^*$

$$R^{(i)}_{g,h} \iff U_i(g) \geq U_i(h).$$

Based on these relations, we introduce a global utility (or comparison) relation $C : DM \leftrightarrow G^* \times G^*$ as follows. For all $i \in DM$ and $g, h \in G^*$ we define

$$C_{i,\langle g,h\rangle} \iff R^{(i)}_{g,h}.$$

An immediate consequence of (15) is the equation

$$C = \mathit{vec}(R^{(1)})^{\mathsf{T}} + \ldots + \mathit{vec}(R^{(n)})^{\mathsf{T}}.$$

Next, we consider the dominance relationship, and we get for all $g, h \in G^*$

$$g \succ h \iff (\forall i : M_{i,g} \rightarrow C_{i,\langle g,h\rangle}) \wedge (\exists i : M_{i,g} \wedge \overline{C}_{i,\langle h,g\rangle}). \tag{22}$$

Since $\overline{C}_{i,\langle h,g\rangle} \iff (\overline{C};E)_{i,\langle g,h\rangle}$, where $E : G^*\times G^* \leftrightarrow G^*\times G^*$ is the exchange relation $[\rho,\pi]$, we have the following description of dominance:

$$g \succ h \iff (\forall\, i : M_{i,g} \rightarrow C_{i,\langle g,h\rangle}) \wedge (\exists\, i : M_{i,g} \wedge (\overline{C};E)_{i,\langle g,h\rangle}). \qquad (23)$$

In Berghammer et al. [11], the following fact is proved: Let $\pi : G^*\times G^* \leftrightarrow G^*$ and $\rho : G^*\times G^* \leftrightarrow G^*$ be the projection relations and $E : G^*\times G^* \leftrightarrow G^*\times G^*$ the exchange relation. If we define

$$\mathit{dominance}(M,C) = \overline{(\pi;M^{\mathsf{T}} \cap \overline{C}^{\mathsf{T}});\mathsf{L}} \cap (\pi;M^{\mathsf{T}} \cap E;\overline{C}^{\mathsf{T}});\mathsf{L}\,, \qquad (24)$$

then we have for all $u = \langle g,h\rangle \in G^* \times G^*$ that $\mathit{dominance}(M,C)_u$ if and only if $g \succ h$, i.e., g dominates h.

The relation-algebraic specification $\mathit{dominance}(M,C)$ of the vector describing the dominance relationship between feasible governments immediately leads to the following RelView-program.

```
dominance(M,C)
  DECL Prod = PROD(M^*M,M^*M);
       pi, rho, E
  BEG  pi = p-1(Prod);
       rho = p-2(Prod);
       E = [rho,pi]
       RETURN -dom(pi*M^ & -C^) & dom(pi*M^ & E*-C^)
  END.
```

In this program the first declaration introduces `Prod` as a name for the direct product $G^* \times G^*$. Using the relational product domain `Prod`, the two projection relations and the exchange relation are then computed by the three assignments of the body and stored as `pi`, `rho`, and `E`, respectively. The return-clause of the program consists of a direct translation of (24) into RelView-notation, where `^`, `-`, `&`, and `*` denote transposition, complement, intersection, and composition, and, furthermore, the pre-defined operation `dom` computes for a relation $R : X \leftrightarrow Y$ the vector $R;\mathsf{L} : X \leftrightarrow \mathbf{1}$.

Finally, we consider stability of feasible governments. Due to the original definition of stability and the above result concerning dominance we have for all $g \in G^*$ the equivalence

$$\mathit{stable}(M,C)_g \iff \neg\,\exists\, h : \mathit{dominance}(M,C)_{\langle h,g\rangle}. \qquad (25)$$

In Berghammer et al. [11], it is shown how to transform this specification into the relation-algebraic specification

$$\mathit{stable}(M,C) = \overline{\rho^{\mathsf{T}};\mathit{dominance}(M,C)}. \qquad (26)$$

Also a translation of the relation-algebraic specification of $\mathit{stable}(M,C)$ into RelView-code is straightforward.

5.2 Example (Continued)

The above RelView-program `dominance` expects two relations as inputs. In the following, we show for our running example how these can be obtained from the hitherto results, and also how then the dominance relation can be computed and visualized with the aid of the RelView tool.

The first input `M` of the RelView-program `dominance` is a description of government membership in the form of a relation of type $[DM \leftrightarrow G^*]$ that column-wisely enumerates the governments. In the case of our running example, it immediately is obtained from the list of governments of Subsection 2.2. Its RelView-representation as 3×8 Boolean matrix is shown in Figure 1, where we additionally have labeled the rows and columns of the matrix with the parties and governments, respectively, for explanatory purposes.

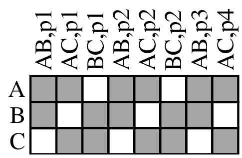

Fig. 1. Relational description of government membership

The second input is the global utility relation of type $[DM \leftrightarrow G^* \times G^*]$. It is constructed from the three utility relations $R^{(A)}, R^{(B)}, R^{(C)} : G^* \leftrightarrow G^*$ of the parties A, B, and C, respectively. The latter three relations are obtained immediately from Table 12 and the labeled 8×8 Boolean matrices representations look in RelView as given in the following Figure 2.

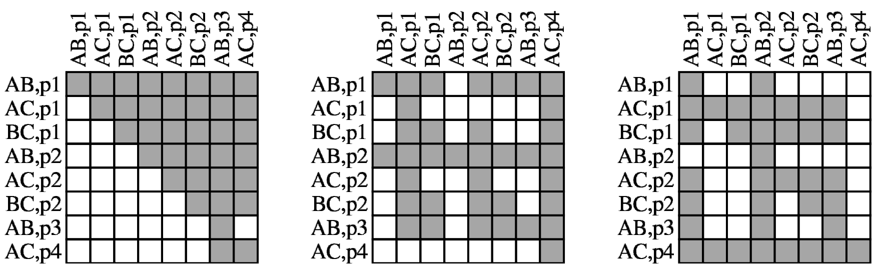

Fig. 2. The parties' Utility Relations

We renounce the RelView-picture for the global utility relation, since the explanatory power of this 3×64 Boolean matrix is rather small. The same holds for the vector description (24) of the dominance relation. Instead we show in the following Figure 3 the dominance relation of the example as a labeled 8×8 Boolean matrix. For obtaining this matrix we used that the relation

$$R := \pi^{\mathsf{T}}; (\rho \cap v; \mathsf{L})$$

describes a vector $v : X \times Y \leftrightarrow \mathbf{1}$ as relation of type $[X \leftrightarrow Y]$, i.e., $v_{\langle x,y \rangle}$ and $R_{x,y}$ are equivalent for all $x \in X$ and $y \in Y$ (where π and ρ are the projection relations of the direct product $X \times Y$). See Schmidt et al. [47] for details.

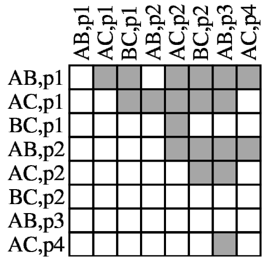

Fig. 3. The Dominance Relation

A representation of this relation as directed graph is shown in the following Figure 4. For drawing this graph, the RelView tool used the specific layout algorithm of Gansner et al. [28].

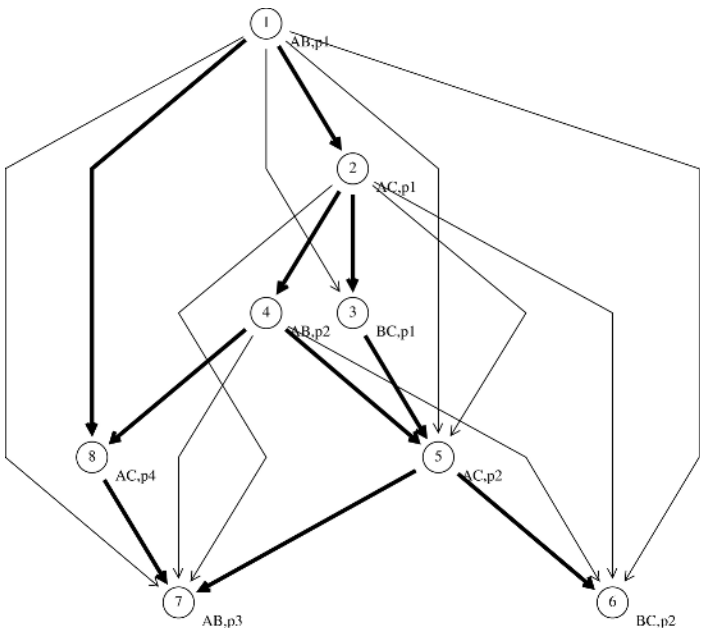

Fig. 4. Graphical Representation of the Dominance Relation

In this *dominance graph* we additionally have marked the immediate neighbourhood relationships as boldface arcs to make things more clear. From Figure 4 we immediately obtain $g_1 = (AB, p_1)$ as the only stable government of our example, since, by definition, a government is stable if and only if it is a source of the dominance graph.

6 Applying Social Choice Theory and Bargaining Theory

We will now address the question of how to proceed in cases where multiple stable governments exist. In such cases, social choice rules or bargaining theory may be applied.

6.1 Selection of Governments Via Social Choice Rules

The input for an application of social choice theory consists of: (at least two) selected governments (from which we have to choose one), parties forming these governments, and preferences of the parties over the governments. Moreover, for each government each party either accepts (approves of) or does not accept (disapproves of) it. We consider the following rules; see Subsection 7.2 for an illustration.

- *Plurality Rule*: Under this rule only the first preference of a party is considered. A government g is collectively preferred to a government h if the number of parties that prefer g most is greater than the number of parties that prefer h most. The government chosen under the plurality rule is the government which is put first by most parties.
- *Majority Rule*: This rule is based on the majority principle. A government g is collectively preferred to h if g defeats h, i.e., the number of parties that prefer g to h is greater than the number of parties that prefer h to g. If there is a government that defeats every other government in a pairwise comparison, this government is chosen, and it is called a Condorcet winner; see also Condorcet [19].
- *Borda Rule*: Here weights are given to all the positions of the governments in the individual preferences. For n governments, every party gives n points to its most preferred government, $n-1$ points to its second preference, etc., and 1 point to its least preferred government. A decision is made based on the total score of every government in a given party profile; see also [20] for more details.
- *Approval Voting Rule*: Under Approval Voting (Brams and Fishburn [15]), each party divides the governments into two classes: the governments it approves of and the ones it disapproves of. Each time a government is approved of by a party is good for one point. The government chosen is the one that receives most points.

6.2 Selection of Governments Via Bargaining

Apart from the application of social choice rules, we may propose an alternative method for choosing a government. If there is more than one stable government, bargaining theory may be applied in order to choose one government. In Rusinowska and de Swart [43] (see also Berghammer et al. [12]), the authors define several bargaining games in which parties belonging to stable governments (assuming that there are at least two stable ones) bargain over the choice of one stable government. Subgame perfect equilibria of the games are investigated. Also a procedure for choosing the order of parties for a given game is proposed.

We define three kinds of bargaining games, denoted here as Games I, II, and III, in which parties involved in at least one stable government bargain over the choice of one government. The order of the parties in which they bargain is according to the number of seats in the parliament. The common assumptions for the bargaining games are as follows.

- A party, when submitting an offer, may propose only one government.
- The same offers are not repeated: a party cannot propose a government which has already been proposed before.
- It is assumed that choosing no government is the worst outcome for each party.

The differences between the three bargaining games are specified by the following four rules.

- In Game I, a party, when submitting an offer, may propose only a government the party belongs to. Each party involved in a proposed government either accepts of rejects the proposal. The acceptance of the offer by all parties involved causes the government to be formed. Rejection leads to proposing a government by the rejecting party.
- In Game II, a party does not have to belong to the government it proposes, and all parties have to react to each offer.
- In Game III, only the strongest party may submit an offer, and the other parties forming the proposed government have to react.

Our bargaining games differ from each other with respect to the bargaining procedures. We consider games in which a party prefers to form a government it likes most with a delay, rather than to form immediately (with no delay) a less preferred government. We refer to Subsection 7.2 for an illustration.

7 Applying Graph Theory to Coalition Formation

In this section we consider the case that there exists no stable government. Using graph-theoretical terminology this means that the computed dominance graph has no source. As we will show in the following, a combination of social choice rules, bargaining, and techniques from graph theory can be applied to select a government that can be considered as 'rather stable'.

7.1 Graph-Theoretical Procedure for Choosing a Government

First, we use *strongly connected components* (SCCs). A SCC of a directed graph is defined as a maximal set of vertices such that each pair of vertices is mutually reachable. In particular, we are interested in SCCs without arcs leading from outside into them. These SCCs are said to be *initial.* We also apply the concept of a minimum feedback vertex set, where a *feedback vertex set* (FVS) is a set of vertices that contains at least one vertex from every cycle of the graph. For computing the initial strongly connected components and minimum feedback vertex sets one may use the RelView tool again, see Berghammer and Fronk [8], [9], and Berghammer et al. [12] for details.

We propose the following procedure for choosing a government in case there is no stable government (see also Berghammer et al. [12]):

(1) Compute the set $\mathfrak{I}$ of all initial SCCs of the dominance graph.
(2) For each SCC C from $\mathfrak{I}$ do:
   (a) Compute the set $\mathfrak{F}$ of all minimum FVSs of the subgraph generated by the vertices of C.
   (b) Select from all sets of $\mathfrak{F}$ with a maximal number of ingoing arcs one with a minimal number of outgoing arcs. We denote this one by F.
   (c) Break all cycles of C by removing the vertices of F from the dominance graph.
   (d) Select an un-dominated government from the remaining graph. If there is more than one candidate, use social choice rules or bargaining in order to choose one.
(3) If there is more than one set in $\mathfrak{I}$, select the final stable government from the results of the second step by applying social choice rules or bargaining again.

An outgoing arc of the dominance graph denotes that a government dominates another one and an ingoing arc denotes that a government is dominated by another one. Hence, the governments of an initial SCC can be seen as a cluster which is not dominated from outside. The application of the second step to such a set of 'candidates' corresponds to a removal of those candidates which are 'least attractive', because they are most frequently dominated and they dominate other governments least frequently.

According to the procedure just mentioned, if the application of steps (1) and (2) does not give a unique solution, we select the final government from among the 'graph-theoretical' results by applying again social choice rules or bargaining games.

7.2 Example (Continued)

The computation of the dominance graph of Figure 4 is based upon the values of columns 2 to 4 of Table 12. By changing our running example a little bit (viz. by rounding each value to the next natural number being a multiple of 5) the situation changes drastically. We obtain the dominance graph of Figure 5, that does not possess a source. In this RelView-picture the subgraph induced by the only initial SCC (corresponding to the set $\{g_1, g_2, g_3, g_4, g_5\}$ of governments) is emphasized by black vertices and boldface arcs.

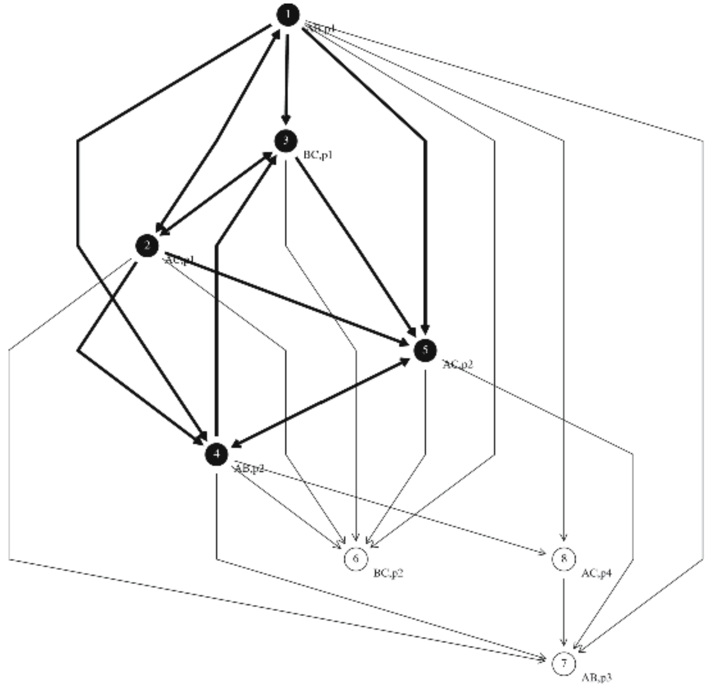

Fig. 5. Dominance relation and initial SCC after rounding

We have applied the procedure of Subsection 7.1 to obtain a government that can be considered as an approximation of a stable one. The next figure shows the two minimum FVSs of the initial SCC as presented on the RelView screen:

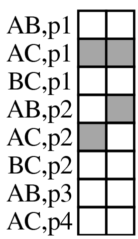

Fig. 6. Minimum feedback vertex sets of the initial SCC

Each of the initial components possesses 3 ingoing arcs and the number of their outgoing arcs is also 3. If we select the minimum FVS represented by the first column of the matrix of Figure 6 in step (b) of our procedurs, then step (c) yields vertex 1 as source. A selection of the second column yields the same result. This shows that the stable government g_1 of the original example is rather 'robust' with respect to modifying the parties' utilities to a certain extent.

To demonstrate an application of the concepts of Section 6, we have changed our example again and used a still coarser scale for the utilities. It divides the values of Table 12 into four categories, viz. small (0 to 25), medium (26 to 50), large (51 to 75), and very large (76 to 100). Such a quatrigrade scale leads to the dominance graph depicted in Figure 7; in this RelView-drawing again the only initial SCC is emphasized.

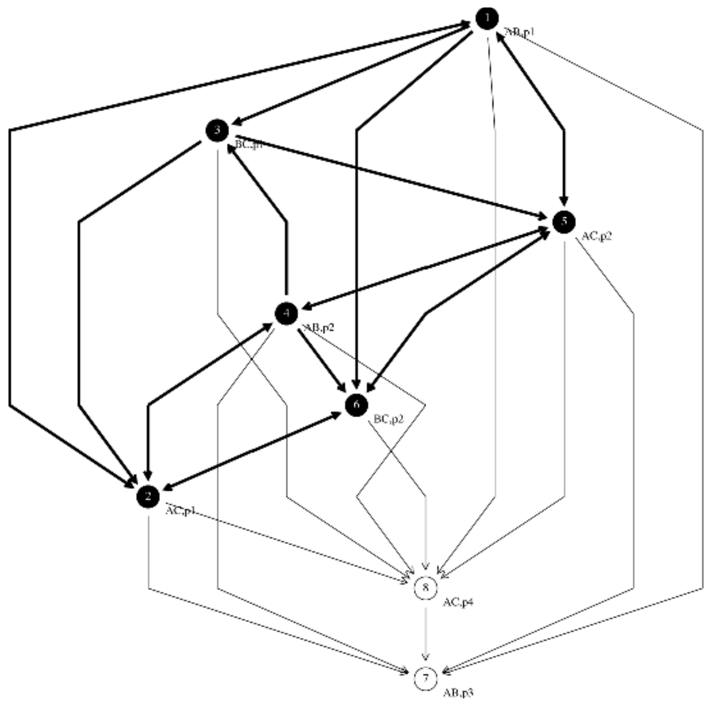

Fig. 7. Dominance relation and initial SCC after a coarser rounding

If we apply the procedure of Subsection 7.1 to this graph, we obtain vertices 2 and 5 as the only minimum FVS and their removal converts vertices 1 and 4 to sources. Hence, besides government g_1 now also government g_4 is a candidate for being selected as rather stable.

Let us apply the Plurality rule for the final selection. From the utility relations $R^{(A)}, R^{(B)}$, and $R^{(C)}$ of Figure 2 we obtain for the three parties A, B, and C the following preferences:

$$A : g_1 \text{ before } g_4, \qquad B : g_4 \text{ before } g_1, \qquad C : g_1 \text{ before } g_4.$$

Hence, government g_1 is put first by two parties whereas government g_4 is put first by one party only. This means that again g_1 is selected.

Alternatively, we can apply the bargaining games to this example. Since both governments g_1 and g_4 are formed by coalition AB, only parties A and B are involved in bargaining. Consequently, both Games I and II are the same. In Game I/II, with the order of parties (A, B), there is only one subgame perfect equilibrium, and it leads to the choice of government g_4 already in the first period of the game. Game I/II with the order of parties (B, A) has also one subgame perfect equilibrium, but it leads to the choice of government g_1 in the first period of the game. Let us note that being the first proposer in bargaining may be disadvantageous: when party A is the first proposer, the subgame perfect equilibrium gives g_4 which is worse for party A than government g_1. The same holds for party B being the first proposer: the subgame perfect equilibrium leads to government g_1 which is less attractive for party B than government g_4. When applying Game III, if party A is stronger than B (i.e., for instance, A has more seats in parliament than B), we get the same result as in Games I and II with the order (A, B). If party B is stronger than A, Game III gives the same result as in Games I and II with the order (B, A).

8 Consensus Reaching

In this section, we describe a procedure for a winning coalition to reach consensus on a policy in order to form a feasible government.

8.1 Consensus Reaching Within a Coalition

In what follows, we assume a kind of mediator, called the *chairman*, who does not belong to any party and is indifferent between all the parties. First of all, this chairman chooses the parties that should adjust their preferences if needed, and gives suggestions to the parties how they should change their preferences. Moreover, in case of any non-uniqueness, the chairman chooses one solution. Also, if a coalition seems to be unable to reach consensus, the chairman decides when to stop the process of consensus reaching within that coalition. If the attempts to reach consensus within a coalition fail, this means that the given coalition does not propose any government.

We propose the following procedure for consensus reaching within a winning coalition S; see also Eklund et al. [26]. Let G^*_S denote the *set of all feasible governments with $S \in W^*$ as coalition*. Each party $i \in S$ evaluates each government from G^*_S with respect to all the criteria. The notations here are similar to the ones presented in Subsection 2.1, except that we add the lower index S, since now the parties of coalition S only consider the governments formed by S. For each $i \in S$, we assume $u_{i,S} : C^* \times G^*_S \to [0, 1]$ such that

$$\forall\, c \in C^* : \sum_{g \in G^*_S} u_{i,S}(c, g) = 1. \tag{27}$$

The real number $u_{i,S}(c, g)$ is called the *value of government $g \in G^*_S$ to party $i \in S$ with respect to criterion $c \in C^*$*. Moreover, for each $i \in S$, we define $U_{i,S} : G^*_S \to [0, 1]$ such that

$$(U_{i,S}(g))_{g \in G^*_S} = (\alpha_i(c))_{c \in C^*} \cdot (u_{i,S}(c, g))_{c \in C^*, g \in G^*_S}, \tag{28}$$

where $(\alpha_i(c))_{c \in C^*}$ is the $1 \times |C^*|$ matrix representing the evaluation (comparison) of the criteria by party i, $(u_{i,S}(c, g))_{c \in C^*, g \in G^*_S}$ is the $|C^*| \times |G^*_S|$ matrix containing party i's evaluation (comparison) of all governments in G^*_S with respect to each criterion in C^*, and $(U_{i,S}(g))_{g\in G^*_S}$ is the $1 \times |G^*_S|$ matrix containing party i's evaluation of each government in G^*_S. Because of property (5) (with the set DM replaced by S) and (27) we have that

$$\sum_{g\in G^*_S} U_{i,S}(g) = 1. \tag{29}$$

Reaching consensus within a coalition means that the preferences of the parties from this coalition, as well as their evaluation of the importance of all criteria from C^*, should be relatively 'close' to each other. We specify this in detail. We define an assessment or 'distance' function $d_S : S \times S \to [0, 1]$ satisfying the conditions $d_S(i,i) = 0$ and $d_S(i,j) = d_S(j,i)$ for all $i, j \in S$. In Eklund et al. [26], the authors consider the specific assessment function

$$d_S(i,j) = \sqrt{\frac{1}{|G^*_S|}\sum_{g\in G^*_S}(U_{i,S}(g) - U_{j,S}(g))^2}$$

but one may apply other assessment functions as well. Moreover, the *consensus degree between decision makers i and j in coalition S* is given by

$$\delta_S(i,j) = 1 - d_S(i,j). \tag{30}$$

The higher the consensus (degree), the smaller the 'distance' between pairs of decision makers, i.e., between i and j. In particular, if $d_S(i,j) = 0$, then we say that i and j are in *complete consensus in coalition S*. If $d_S(i,j) = 1$, then we say that i and j are in *complete disagreement in coalition S*. Moreover, we define

$$d^*_S = \max\{d_S(i,j) \mid i,j \in S\}, \tag{31}$$

and a *generalized consensus degree for coalition S* as

$$\delta^*_S = 1 - d^*_S, \tag{32}$$

which concerns the consensus reached by all the decision makers from S.

A certain consensus degree $0 < \widetilde{\delta} < 1$ is required in the model. We say that coalition S reaches consensus if $\delta^*_S \geq \widetilde{\delta}$. If $\delta^*_S < \widetilde{\delta}$, then the chairman will ask at least one party to adjust its preferences. Any change of preferences leads to a new generalized consensus degree for the coalition.

Now, let D^*_S denote the set of all parties from S with most different preferences, that is, we have

$$D^*_S = \{i \in S \mid \exists j \in S\ [d_S(i,j) = d^*_S]\}. \tag{33}$$

The chairman decides which party from D^*_S will be advised to change its evaluation(s) regarding some government(s) and/or the importance of the criteria. The party $i^D_S \in D^*_S$ asked to adjust its preferences is a party such that

$$i^D_S = \arg\max_{i\in D^*_S}\sum_{j\in S} d_S(i,j). \tag{34}$$

If this party does not agree to adjust its evaluations according to the chairman's advice, the chairman may propose another change to the same party or a change to another party. Of course, this procedure of consensus reaching may consist of several steps.

Assuming that w_i is the weight of decision maker $i \in S$, we define the weighted value $U_S(g)$ of government $g \in G_S^*$ as

$$U_S(g) = \sum_{i \in S} w_i' \cdot U_{i,S}(g), \tag{35}$$

where

$$w_i' = \frac{w_i}{\sum_{j \in S} w_j}. \tag{36}$$

Finally, if the generalized (final) consensus degree is not smaller than $\widetilde{\delta}$, the consensus government g_S^* formed by coalition S is chosen such that

$$g_S^* = \arg\max_{g \in G_S^*} U_S(g), \tag{37}$$

Of course, there may be more than one such government g_S^*. As noticed in Eklund et al. [26], any government g_S^* chosen by consensus reaching within coalition S is stable in G_S^*.

8.2 Example (Continued)

Consider coalition AB which has to choose from three policies p_1, p_2, p_3; p_4 is not acceptable to B. So AB has to choose from governments $\{g_1, g_4, g_7\}$; see Subsection 2.2. Suppose the weights of the three criteria for A are $\alpha_A =$ (1/3, 1/3, 1/3) and for B, $\alpha_B =$ (1/2, 1/4, 1/4) respectively. Also suppose that the matrices u_A and u_B of the utilities for A, respectively B, of the different governments with respect to the three criteria look as follows:

$$u_A = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \text{ and } u_B = \begin{pmatrix} 1/4 & 1/2 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

Then $U_A = \alpha_A \cdot u_A =$ (1/3, 1/3, 1/3) and $U_B = \alpha_B \cdot u_B =$ (4/16, 7/16, 5/16). Hence,

$$d_{AB}^* = d_{AB}'(A, B) = \sqrt{\frac{1}{3}[(\frac{1}{3} - \frac{1}{4})^2 + (\frac{1}{3} - \frac{7}{16})^2 + (\frac{1}{3} - \frac{5}{16})^2]} = \frac{1}{48}\sqrt{2}.$$

Supposing that the required (generalized) consensus degree is $\frac{15}{16}$, the (generalized) consensus degree δ_{AB}^* for coalition AB, being $1 - \frac{1}{48}\sqrt{14}$, is too small. So, the chairman comes into play and suppose that after discussion he is able to convince party B to adjust its utilities as follows:

$$u_B' = \begin{pmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

Then $U'_B = \alpha_B \cdot u'_B = (5/16,\ 6/16,\ 5/16)$ and consequently

$$d^*_{AB} = d'_{AB}(A,B) = \sqrt{\frac{1}{3}[(\frac{1}{3}-\frac{5}{16})^2+(\frac{1}{3}-\frac{6}{16})^2+(\frac{1}{3}-\frac{5}{16})^2]} = \frac{1}{48}\sqrt{2}.$$

Hence, the generalized consensus degree δ^*_{AB} becomes $1-\frac{1}{48}\sqrt{2}$, which is larger than the required consensus degree of $\frac{15}{16}$. So, coalition AB reaches consensus. Assuming that each party has equal weight, we compute the utilities $U_{AB}(g)$ for coalition AB of each government $g \in \{g_1, g_4, g_7\}$ and we find that $U_{AB}(g_1) = \frac{1}{2}U_A(g_1)+\frac{1}{2}U_B(g_1) = \frac{1}{2}(1/3+5/16) = 31/96$, $U_{AB}(g_4) = \frac{1}{2}(1/3+6/16) = 34/96$ and $U_{AB}(g_7) = \frac{1}{2}(1/3+5/16) = 31/96$. Consequently, coalition AB will propose government g_4. Of course, it may happen that there is more than one government with a maximal utility for a given coalition, in which case the coalition may propose all these governments with maximal utility.

9 Conclusions

We used the MacBeth software in order to determine the utilities of policies to parties. Based on these utilities one can determine the feasible governments. Next we used the RelView tool in order to calculate the stable governments. If there is more than one stable government we showed how social choice rules or bargaining may result in a particular choice. In case there is no stable government we used techniques from graph theory in order to choose a government which is as close as possible to being stable. We also indicated a procedure for a coalition to reach consensus about a policy, in order to propose a government.

Due to the MacBeth and RelView software, our model of coalition formation seems to be applicable in practice. It could be helpful in the real world in order to form a stable government after elections in a rational way. It would be interesting to test the model in practice and to compare the outcome of the model with the actual outcome.

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