Strong and Correlated Strong Equilibria in Monotone Congestion Games

Ola Rozenfeld and Moshe Tennenholtz

Technion–Israel Institute of Technology
Haifa 32000, Israel

Abstract. The study of congestion games is central to the interplay between computer science and game theory. However, most work in this context does not deal with possible deviations by coalitions of players, a significant issue one may wish to consider. In order to deal with this issue we study the existence of strong and correlated strong equilibria in monotone congestion games. Our study of strong equilibrium deals with monotone-increasing congestion games, complementing the results obtained by Holzman and Law-Yone on monotone-decreasing congestion games. We then present a study of correlated-strong equilibrium for both decreasing and increasing monotone congestion games.

Keywords: Congestion Games, Strong Equilibrium.

1 Introduction and Overview of Results

A *congestion game* (Rosenthal, [7]) is defined as follows: A finite set of players[1], $N = \{1, ..., n\}$; A finite non-empty set of facilities, M; For each player $i \in N$ a non-empty set $A_i \subseteq 2^M$, which is the set of actions available to player i (an action is a subset of the facilities). We denote by A the set of all possible action profiles ($A = \prod_{i \in N} A_i$). With every facility $m \in M$ and integer number $1 \leq k \leq n$ a real number $v_m(k)$ is associated, having the following interpretation: $v_m(k)$ is the utility to each user of m if the total number of users of m is k. Let $a \in A$; the ($|M|$ dimensional) congestion vector corresponding to a is $\sigma(a) = (\sigma_m(a))_{m \in M}$ where $\sigma_m(a) = |\{i | m \in a_i\}|$. The utility function of player i, $u_i : A \to \mathrm{R}$ is defined as follows: $u_i(a) = \sum_{m \in a_i} v_m(\sigma_m(a))$. It is assumed that all players try to maximize their utility. Therefore, equilibrium analysis is typically used for the study of these settings.

Congestion games have become a central topic of study in the interplay between computer science and game theory (see e.g. [1,9,8,6]). Congestion games possess some interesting properties. In particular, Rosenthal [7] showed that every congestion game possesses a pure strategy Nash equilibrium. In this paper we would like to explore the possibility of replacing Nash equilibrium with stronger solution concepts.

[1] We will use the terms player and agent interchangeably.

P. Spirakis et al. (Eds.): WINE 2006, LNCS 4286, pp. 74–86, 2006.

One particular weakness of Nash equilibrium is its vulnerability to deviations by coalitions of players. This issue is addressed in the solution concept known as strong equilibrium (Aumann, [2]): Let us denote the projection of $a \in A$ on the set of players $S \subseteq N$ (resp. on $N \backslash S$) by a_S (resp. by a_{-S}). We say that a profile of actions $a^* \in A$ is a *strong equilibrium* (SE) if for no non-empty coalition $S \subseteq N$ there is a choice of actions $a_i \in A_i, i \in S$ such that $\forall i \in S \;\; u_i(a_S, a^*_{-S}) > u_i(a^*)$.

Such profiles are indeed much more stable than simple Nash equilibria, and therefore their existence is a very desirable property; however, simple examples show that congestion games in general need not possess a strong equilibrium (in fact, the well-known Prisoner's Dilemma may be obtained as a congestion game).

The above definition applies to the case where the players may use only pure strategies. A natural extension of Aumann's definition of strong equilibrium to settings where mixed strategies are available is to apply the original definition to the mixed extension of the original game. Formally, we say that a profile of actions $a^* \in \prod_{i \in N} \Delta(A_i)$ is a *mixed strong equilibrium* (MSE) if for no non-empty coalition $S \subseteq N$ there is a choice of actions $a_i \in \Delta(A_i), i \in S$ such that $\forall i \in S \;\; U_i(a_S, a^*_{-S}) > U_i(a^*)$. Here, by $\Delta(A_i)$ we mean the set of all probability distributions overA_i, and U_i denotes the expected utility of player i.

There are two important things to note when considering the definition of MSE. First, notice that unlike the extension of Nash equilibrium to mixed strategies, this definition yields a stronger solution concept even when applied to pure strategy profiles; i.e., a pure profile of actions may be a strong equilibrium, but not a mixed strong equilibrium. A second point to notice is that in the definition of MSE we assume that the players cannot use *correlated* mixed strategies, i.e. choose their actions using a joint probability distribution. However, in many settings this assumption is too restrictive: if we assume that a coalition of players has the means to choose a coordinated profile of actions, it is natural to assume that they have means of communication that would also allow them to coordinate their actions using joint coin flips. The above leads us to the following definition: we say that $a^* \in \Delta(A)$ is a *correlated strong equilibrium* (CSE) if for no non-empty coalition $S \subseteq N$ there is a choice of actions $a_S \in \Delta(\prod_{i \in S} A_i)$, such that $\forall i \in S \;\; U_i(a_S, a^*_{-S}) > U_i(a^*)$. This definition is strictly stronger than the previous one: every CSE is also an MSE, but not vice versa.[2]

The aim of this article is to explore the conditions for existence of strong and correlated strong equilibria within two most interesting and central subclasses of congestion games:

We call a congestion game *monotone-increasing* (or simply *increasing*) if $\forall m \in M, 1 \leq k < n \; v_m(k) \leq v_m(k+1)$. These games model settings where congestion

[2] Notice that although we allow a^* to be a correlated profile, CSE doesn't extend the notion of correlated Nash equilibrium [3] to the context of deviations by coalitions: our solution concept is weaker, since we assume that the deviators cannot see the "signals" that result from the current realization. However, in the scope of this article, generalizing Aumann's definition would yield the same results.

has a positive effect on the players, e.g. settings in which the cost of using a facility is shared between its users.

We call a congestion game *monotone-decreasing* (or simply *decreasing*) if $\forall m \in M, 1 \leq k < n\ v_m(k) \geq v_m(k+1)$. These games model settings where congestion has a negative effect on the players, e.g. routing games, where cost represents latency.

Ron Holzman and Nissan Law-Yone [5,4] explored the conditions for existence of *strong equilibria* in *monotone decreasing* congestion games. They start by observing that a strong equilibrium always exists in the case where all strategies are singletons. Following that, they explore the *structural properties* of the strategy sets that are necessary and sufficient to guarantee the existence of strong equilibria. These structural properties may, for example, refer to the underlying graph structure in route selection games.

In this paper we first explore the conditions for existence of strong equilibria in *monotone increasing* congestion games. Then, we extend the study of both the decreasing and increasing settings to the solution concept of correlated strong equilibrium. Our contributions can therefore be described by the following table:

	Pure deviations	Correlated deviations
Increasing	*This work*	*This work*
Decreasing	*Holzman & Law-Yone*	*This work*

Main results

Throughout this paper, when we refer to strong equilibrium, we present the results of Holzman and Law-Yone [5] for the decreasing setting alongside our results for the increasing setting. This is done for the sake of viewing the complete picture and ease of comparing between the two settings.

In section 2 we explore the case of singleton strategies, i.e. resource selection games where each player should choose a single resource from a set of resources available to him. In the decreasing setting Holzman and Law-Yone observe that every Nash equilibrium of the game is, in fact, a strong equilibrium. For the increasing setting, we present an efficient algorithm for constructing a strong equilibrium; however, unlike in the decreasing setting, we show that not every Nash equilibrium of the game is strong.

In section 3 we develop a notation, *congestion game forms*, that allows us to speak about the underlying structure of congestion games; using this notation we will be able to formalize statements such as "a certain structural property is necessary and sufficient for the existence of SE in all games with that underlying structure". We define two substructures, which we call *d-bad configuration* and *i-bad configuration* and prove some simple properties of strategy spaces that avoid bad configurations. These properties will serve as a technical tool in some of our proofs.

Section 4 explores the conditions for existence of strong equilibrium. In the decreasing setting, [5] shows that a SE always exists if and only if d-bad configurations are avoided. In the increasing setting we show that a SE always exists if and only if i-bad configurations are avoided, in which case the equilibrium can be efficiently computed. As we will show, our results imply that avoiding i-bad configurations makes the games essentially isomorphic to the case of singleton strategies.

Section 5 deals with the concept of correlated strong equilibrium. We show that a CSE might not be achievable even in simple (two players, two strategies) examples of the decreasing setting. In the increasing setting, though, we show that all our results regarding SE still hold with CSE, namely that a CSE always exists if and only if i-bad configurations are avoided, in which case it can be efficiently computed. Moreover, we show that in this case every SE of the game is also a CSE (a claim which doesn't hold if i-bad configurations are not avoided).

Together, our results provide full characterization for the connection between the underlying game structure and the existence of SE and CSE for both the decreasing and the increasing cases.

2 SE: The Case of Singleton Strategies

Here we investigate the case in which only singleton strategies are allowed, i.e. resource selection games where each player should choose a facility from among a set of facilities available to him.[3]

First, recall the result for the decreasing case:

Theorem 1. *[5] Let G be a monotone decreasing congestion game in which all strategies are singletons. Then G possesses a strong equilibrium; moreover, every Nash equilibrium of G is SE.*

We now address the existence of SE in monotone-increasing congestion games:

Theorem 2. *Let G be a monotone increasing congestion game in which all strategies are singletons. Then G possesses a strong equilibrium; moreover, a SE can be efficiently computed.*

Proof (sketch): Consider the following algorithm for computing a strong equilibrium: at each step, we assign a facility to a non empty subset of the remaining players, in the following way: for each facility $m \in M$, we compute $v_m(k)$, where k is the maximal number of the remaining players that can choose $\{m\}$ as their strategy. We choose m for which such $v_m(k)$ is maximal, and assign $\{m\}$ to all the players that can choose it. We continue in the same fashion until all players are assigned a facility.

[3] In particular, this classical setting can model simple route selection games. In a simple route selection game each player has to select a link for reaching from source to target in a graph consisting of several parallel links. In general, each player may have a different subset of the links that he may use.

We claim that the resulting strategy profile is a SE. We prove by induction on the steps of the algorithm, that no player can belong to a deviating coalition in which his payoff strictly increases: in the first step of the algorithm, this is obvious because each assigned player gets the highest possible payoff in the game (due to monotonicity); at subsequent steps, we use the induction hypothesis and assume that all players from the previous steps don't belong to the deviating coalition, i.e. all of them use the facilities they were assigned; but this means that the game is effectively reduced to the remaining players and the remaining facilities, so the same reasoning applies: due to monotonicity, each assigned player gets the highest possible payoff in the (new) game. Regarding the complexity of the algorithm, it is trivial to verify that the most straightforward implementation runs in $O(m^2n^2)$; it is also a simple exercise to construct an implementation that runs in $O(mn)$. □

Unlike in the decreasing setting, not every Nash equilibrium (NE) of the decreasing game is a SE. Consider, for example, an instance with two facilities $\{m_1, m_2\}$ and two players, where the cost of a facility is shared equally between its users. The cost of m_1 is 2, and the cost of m_2 is 1. Both facilities are available to both players. Then, the profile (m_1, m_1) is a NE, since each player cannot decrease his cost of 1 by deviating alone; but it is not a SE, since if both players deviate to m_2, their cost decreases to 0.5.

We will now illustrate why our proof of Thm. 2 wouldn't hold in the general case (where the strategies don't have to be singletons). The situation is best illustrated by an example. Fig.1 presents a graph of an instance of a network

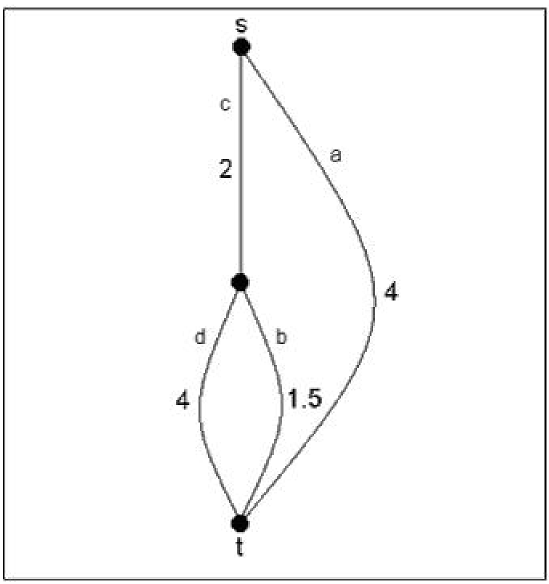

Fig. 1. SE doesn't exist

design game in the increasing setting: there are two agents, who both need to construct a path from s to t, using the edges available in the graph. The construction cost of each edge (the number near the edge) is shared equally between the agents. Each agent wants to minimize his construction cost; however, agent 1 cannot use edge a, and agent 2 cannot use edge b.

Our algorithm assigns $\{c, d\}$ to 1 and $\{c, d\}$ to 2, with a payoff of 3 each. This however is not an SE; in fact there is no SE in this game; to see this observe that playing $\{c, b\}$ is dominant for agent 1, and given that playing $\{a\}$ is dominant for agent 2, which leads to a payoff of (4,3.5), which is smaller than (3,3). Therefore, a SE does not exist in this instance.

3 Congestion Game Forms, Bad Configurations and Tree Representations

In this section we extend upon the definitions and notations introduced in [5] in order to provide some basic tools that will be useful for our characterization results.

A *congestion game form* is a tuple $F = (M, N, A)$ where M is the set of facilities, $N = \{1, ..., n\}$ is the set of players, and $A \subseteq 2^M$. A congestion game $G = (M, N, \{A_i\}, \{v_m(k)\})$ is said to be *derived* from F if $A = \bigcup_{i \in N} A_i$. Given a congestion game form F, one can derive from it a whole family of (monotone, increasing or decreasing) congestion games by assigning (monotone, increasing or decreasing) utility levels to the facilities and assigning specific strategy sets to the players. The congestion game form represents the underlying structure of the strategy spaces; for example, in the network design setting, it is the game graph. We say that a congestion game form F is *strongly consistent* if every monotone congestion game derived from F possesses a SE (we will always specify which setting, increasing or decreasing, is under discussion). We say that a congestion game form F is *strong-Nash equivalent* if in every monotone congestion game derived from F every NE is a SE. Similarly, we say that a congestion game form F is *correlated strongly consistent* if every monotone congestion game derived from F possesses a CSE; F is *correlated-strong equivalent* if in every monotone congestion game derived from F every SE is a CSE.

In this terminology, the results of section 2 state: if $F = (M, N, A)$ is a congestion game form in the decreasing setting in which A contains only singletons, then F is strong-Nash equivalent; if $F = (M, N, A)$ is a congestion game form in the increasing setting in which A contains only singletons, then F is strongly consistent.

We are interested in a property of A which is both necessary and sufficient for F to be strongly consistent.

Let $A \subseteq 2^M$. A *d-bad configuration* in A is a tuple $(x, y; X, Y, Z)$ where:

$$x, y \in M$$
$$X, Y, Z \in A$$

and the following relations hold:

$$\begin{array}{ll} x \in X & y \notin X \\ x \notin Y & y \in Y \\ x \in Z & y \in Z \end{array}$$

Thus, two facilities x, y give rise to a d-bad configuration if there is a strategy that uses both x and y, there is a strategy that uses x without y, and there is a strategy that uses y without x. We call $A \subseteq 2^M$ *d-good* if it does not contain a d-bad configuration.

An *i-bad configuration* in A is a tuple $(x, y; X, Y, Z)$ where:

$$\begin{array}{l} x, y \in M \\ X, Y, Z \in A \end{array}$$

and the following relations hold:

$$\begin{array}{ll} x \in X & y \notin X \\ x \notin Y & \\ x \in Z & y \in Z \end{array}$$

Thus, two facilities x, y give rise to an i-bad configuration if there is a strategy that uses both x and y, there is a strategy that uses x without y, and there is a strategy that avoids x (with, or without using y). In Fig.1, for example, the edges c, d give rise to a i-bad configuration. We call $A \subseteq 2^M$ *i-good* if it does not contain an i-bad configuration. In particular, a d-bad configuration is an i-bad configuration, so A is i-good implies that A is d-good.

By an *M-tree*, we shall mean the following:

- a tree with a root r
- a labeling of the nodes of the tree (except r) by elements of M; not all elements of M must appear, but each can appear at most once
- a designated subset D of the nodes, which contains all terminal nodes (and possibly other nodes as well).

An example of an M-tree appears in Fig. 2.

Given an M-tree T, we associate with it a set A of strategies on M, as follows: to each node in D there corresponds a strategy in A consisting of the labels which appear on the path from r to that node. For instance, if T is the M-tree depicted in Fig. 2, then $A = \{\{a, b\}, \{a, b, c\}, \{a, d\}, \{a, e, f\}, \{a, e, g\}, \{h\}, \{h, i\}, \{h, j, k\}\}$. If $r \in D$, it means that $\emptyset \in A$. If A is obtained from T in this way, we say that T is a *tree-representation* of A.

Lemma 1. *[5] Let A be a nonempty set of strategies on M. Then A is d-good if and only if it has a tree-representation.*

Given a congestion game form $F = (M, N, A)$, a tree representation of A gives us a convenient method of reasoning about equilibria, since in this case any congestion game derived from F is isomorphic to a tree-game: a game where

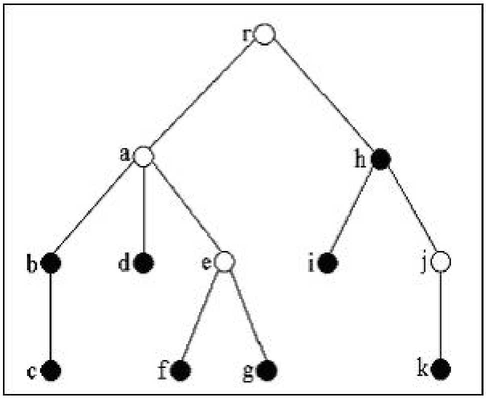

Fig. 2. An M-tree. The labels appear to the left of the nodes; the nodes in D are blackened.

given an M-tree, players must build a path from r to one of the nodes in D, and the strategies of each player can be represented by the subset of D that he is allowed to use.

Given a tree representation of A, a non-leaf node v is called *split* if $v \in D$ or v has more than one child (intuition: a path from r that reaches v has more than one way to be extended to a path leading to a node in D). A tree representation of A is called *simple* if no path from r to a node in D contains more than one split node. The general case of a simple tree representation is depicted in Fig. 3.

We can now prove:

Lemma 2. *Let A be a nonempty set of strategies on M. Then A is i-good if and only if it has a simple tree-representation.*

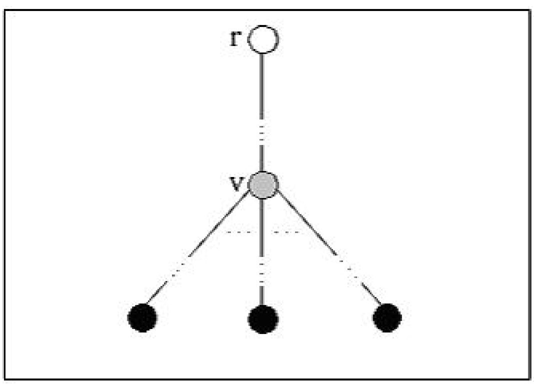

Fig. 3. The general case of a simple M-tree. The grayed node v can belong to D and can be outside of D; The dots represent chains of nodes (could be empty), where no intermediate node belongs to D.

Proof: Suppose A has a simple tree representation, and suppose, for contradiction, that A also has an i-bad configuration $(x, y; X, Y, Z)$. Since $x, y \in Z$, both x and y appear on the path from r to a node in D that corresponds to Z; also, x must occur above y on this path, since a path that corresponds to X contains x, but not y. This means that a split node v must exist between x and y on the path of Z; but since the path corresponding to Y doesn't include x, this means another split node v' must exist above x as well. So, the path of Z contains two split nodes – contradiction.

Suppose now that A is i-good. Then, it is also d-good, so by Lemma 1 A has a tree representation. Suppose, for contradiction, that this tree representation is not simple; i.e. there exists a path (corresponding to some strategy Z in A) with two split nodes, x and x'. W.l.o.g., suppose x' is above x. Since x is split, it has a child, y. Since x' is split and is above x, there exists a path (corresponding to some strategy Y) that doesn't contain x. Since x is split, there exists a path (corresponding to some strategy X) that contains x, but not y. Thus, $(x, y; X, Y, Z)$ is an i-bad configuration – contradiction. □

4 Structural Characterization of Existence of SE

Recall the following:

Theorem 3. *[5] Consider the monotone decreasing setting, and let F be a congestion game form with $n \geq 2$. Then, F is strongly consistent if and only if A is d-good.*

We now show:

Theorem 4. *Consider the monotone increasing setting, and let $F = (M, N, A)$ be a congestion game form, with $n \geq 2$. Then, F is strongly consistent if and only if A is i-good; moreover, if A is i-good, a SE can be efficiently computed.*

Proof: Let $F = (M, N, A)$ be a congestion game form, and suppose A is i-good. As we know from Lemma 2, A has a simple tree representation. In the general case, a simple M-tree has the form depicted in Fig. 3: a single chain descending from r to a single split node v, from which descend several chains to nodes in D. Each such chain (including the one from r to v) might be empty. What it means in terms of strategies in A, is that: $\exists C \subseteq M$ s.t. $\forall S_1 \neq S_2 \in A : S_1 \cap S_2 = C$; i.e. except one common subset of facilities that all players have to choose, their allowed strategies are either equal or disjoint. We claim that this case is strategically isomorphic to the case of singleton strategies. First, since all users must choose all the facilities in C, these facilities don't influence the game and can be removed. Then, A becomes pair wise disjoint collection of subsets of facilities; therefore, each such subset $S \in A$ can be replaced by a single new facility m_S, with $v_{m_S}(k) = \sum_{m \in S} v_m(k)$ for every k. Now, we have an equivalent game with only singleton strategies allowed; as we know from Thm. 2, such game has a strong equilibrium which can be efficiently computed.

Now suppose $F = (M, N, A)$ is a congestion game form where $n \geq 2$ and A contains an i-bad configuration $(x, y; X, Y, Z)$. We must show that F is not strongly consistent; i.e. there exists a monotone increasing congestion game G derived from F which doesn't possess a SE. To construct such game, we must specify the exact strategy spaces $A_1, ..., A_n$ so that $A = \bigcup_{i \in N} A_i$, and specify monotone increasing $v_m(k)$ for each $m \in M$. We can express A as a union of four disjoint sets $A = A_X \cup A_Y \cup A_Z \cup A_\emptyset$, where:

$$\begin{aligned} A_X &= \{S \in A | S \cap \{x, y\} = \{x\}\}, \\ A_Y &= \{S \in A | S \cap \{x, y\} = \{y\}\}, \\ A_Z &= \{S \in A | S \cap \{x, y\} = \{x, y\}\}, \\ A_\emptyset &= \{S \in A | S \cap \{x, y\} = \emptyset\} \end{aligned}$$

From the i-bad configuration definition, we know that $A_X,\ A_Z$ and $A_Y \cup A_\emptyset$ are not empty (since $X \in A_X,\ Z \in A_Z$ and $Y \in A_Y \cup A_\emptyset$). We consider two distinct cases:

1. $A_\emptyset = \emptyset$. In this case, G is specified as follows:

$$A_1 = A_X \cup A_Z,\ A_2 = A_Y \cup A_Z,\ A_3 = ... = A_n = A_Z$$

$$v_m(k) = \begin{cases} -3,\ m \in \{x, y\},\ k < n \\ -1,\ m \in \{x, y\},\ k = n \\ 0,\ m \notin \{x, y\} \end{cases}$$

Since both facilities x,y have negative utility no matter how many players choose them, it is a strictly dominant strategy for players 1,2 to choose a subset that contains only one facility among x,y. Therefore, in any NE of the game (pure or mixed) player 1 will choose a strategy in A_X and player 2 will choose a strategy in A_Y, so both will gain -3. However, if both players deviate to a strategy in A_Z, both will gain -2. Therefore, any NE of the game is not strong, i.e. SE does not exist.

2. $A_\emptyset \neq \emptyset$. In this case, G is specified as follows:

$$A_1 = A_X \cup A_Z,\ A_2 = A_Y \cup A_Z \cup A_\emptyset,\ A_3 = ... = A_n = A_\emptyset$$

$$v_m(k) = \begin{cases} 2,\ m = x,\ k < 2 \\ 4,\ m = x,\ k \geq 2 \\ \text{-}5,\ m = y,\ k < 2 \\ \text{-}1,\ m = y,\ k \geq 2 \\ 0,\ m \neq x, y \end{cases}$$

Since the facility y always yields a negative utility, it is strictly dominant for player 1 to choose a strategy in A_X. Therefore, in any NE player 2 will choose a strategy in $A_\emptyset$; so, in any NE (pure or mixed) they will gain 2 and 0 respectively. But then, if both players deviate to a strategy in A_Z, they will gain 3; So in this case too a SE does not exist, which completes our proof. □

The above results suggest that in the monotone increasing setting there are (in a sense) strictly less games which possess SE than in the monotone decreasing setting (unless we consider the symmetric case). In the setting where congestion has a negative effect, the whole class of "tree games" is guaranteed to have a SE; in the increasing setting, where congestion has a positive effect on the players, SE is guaranteed to exist only in a strict subset of the corresponding structures. As shown in the proof of Thm. 4, this set of structures is strategically isomorphic to the singletons setting. This result is (perhaps) a bit surprising, since it contradicts the intuition – the players "help" each other instead of "harming" each other, but despite of that the setting is less stable, in the sense that there are less strong equilibria. Nevertheless, as we will later see, the decreasing case is not more stable than the increasing case when we consider CSE.

5 Structural Characterization of Existence of CSE

When we attempt to replace the notion of SE with the much stronger notion of CSE, many of the previous results no longer hold. It is easy to see that in the monotone decreasing setting even the following simple example with two players in a symmetric singleton strategies game doesn't possess a CSE. Consider two facilities $\{m_1, m_2\}$ with $v_1(m_1) = -2$, $v_2(m_1) = -4$, $v_1(m_2) = -5$, $v_2(m_2) = -10$. Both facilities are available to both players. Here, playing m_1 is a strictly dominant strategy for both players; however, (m_1, m_1) is not a CSE, since a deviation to the correlated profile $\{\frac{1}{2}(m_1, m_2), \frac{1}{2}(m_2, m_1)\}$ strictly increases the payoff of both players (each player will suffer a cost of 3.5 instead of a cost of 4). Therefore, a CSE doesn't exist in this example (which is a variant of the Prisoner's Dilemma). In fact, we can generalize this example to the following statement:

Proposition 1. *Consider the monotone decreasing setting, and let $F = (M, N, A)$ be a congestion game form with $n \geq 2$ and $|A| \geq 2$. Then, F is not correlated strongly consistent.*

Proof (sketch): The proof is in the same spirit as the proof of Theorem 4. It is therefore omitted due to lack of space.
In the increasing setting, however, we see that our results still hold; moreover, we can prove the following strong claim:

Theorem 5. *Consider the monotone increasing setting, and let $F = (M, N, A)$ be a congestion game form. Suppose A is i-good. Then F is correlated-strong equivalent.*

Proof: From our previous observations we know that if A is i-good, we can assume w.l.o.g. that A has only singleton strategies. So we must show that any SE of a monotone increasing congestion game where all strategies are singletons is also a CSE. Suppose, for contradiction, that $a^* \in A$ is a SE of a monotone increasing congestion game with singleton strategies, and it is not a CSE. Therefore, there exists a non-empty coalition $S \subseteq N$ and a correlated mixed strategy $a_S \in \Delta(\prod_{i \in S} A_i)$ such that $\forall i \in S \;\; U_i(a_S, a^*_{-S}) > U_i(a^*)$. Let

i be a player in S with maximal utility in a^*: $\forall j \in S \quad u_i(a^*) \geq u_j(a^*)$. Since $U_i(a_S, a^*_{-S}) > U_i(a^*)$, there must be a realization $b_S \in \prod_{j \in S} A_j$ of a_S such that $u_i(b_S, a^*_{-S}) > u_i(a^*)$. Since the game contains only singleton strategies, $u_i(b_S, a^*_{-S}) = v_m(\sigma_m(b_S, a^*_{-S}))$ for a resource m such that $b_i = \{m\}$. Let $T = \{j \in S | b_j = b_i\}$. T is non-empty, since $i \in T$. From the definition of T and since $T \subseteq S$ it holds that $\sigma_m(b_S, a^*_{-S}) \leq \sigma_m(b_T, a^*_{-T})$; therefore, since the game is monotone-increasing, $u_i(b_T, a^*_{-T}) \geq u_i(b_S, a^*_{-S}) > u_i(a^*)$. Since $\forall j \in T, u_j(b_T, a^*_{-T}) = u_i(b_T, a^*_{-T})$, we have that $\forall j \in T, u_j(b_T, a^*_{-T}) = u_i(b_T, a^*_{-T}) \geq u_i(b_S, a^*_{-S}) > u_i(a^*) \geq u_j(a^*)$, which contradicts our assumption that a^* is a SE. □

This brings us to the following result:

Theorem 6. *Consider the monotone increasing setting, and let $F = (M, N, A)$ be a congestion game form, with $n \geq 2$. Then, F is correlated strongly consistent if and only if A is i-good; moreover, if A is i-good, a CSE can be efficiently computed.*

Proof: $\Leftarrow$ Follows from Thms. 4 and 5.

$\Rightarrow$ The proof is similar to the proof of this direction in Thm. 4, observing that the counter examples given there are solved via elimination of strictly dominated strategies, and therefore don't posses a CSE. □

Notice that while the set of congestion game forms that are strongly consistent in the increasing case is a strict subset of the set of congestion game forms that are strongly consistent in the decreasing case, we get inclusion in the other direction when considering correlated-strong consistency.

6 Further Work

One interesting question is whether further common restrictions, e.g. linearity, on the utility functions may have significant effects on the existence of SE and CSE. A related aspect has to do with restrictions on the utility functions to be only positive or only negative. Our initial study suggests that using such assumptions (in addition to monotonicity) one can slightly expand the set of situations where SE and/or CSE exist, but only in a very esoteric manner. Other aspects of SE and CSE, such as uniqueness and Pareto-optimality are also under consideration.

References

1. E. Anshelevich, Dasgupta, J. Kleinberg, E. Tardos, T. Wexler, and T. Roughgarden. The price of stability for network design with fair cost allocation. In *Proc. 45th IEEE Symp. on Foundations of Computer Science*, pages 59–73, 2004.
2. R.J. Aumann. Acceptable points in general cooperative n-person games. In A.W. Tucker and R.D. Luce, editors, *Contribution to the Theory of Games, Vol. IV, Annals of Mathematics Studies, 40*, pages 287–324. 1959.

3. R.J. Aumann. Subjectivity and correlation in randomized strategies. *Journal of Mathematical Economics*, 1:67–96, 1974.
4. R. Holzman and N. Law-Yone. Network structure and strong equilibrium in route selection games. *Mathematical Social Sciences*, 46:193–205, 1997.
5. R. Holzman and N. Law-Yone. Strong equilibrium in congestion games. *Games and Economic Behavior*, 21:85–101, 1997.
6. C.H. Papadimitriou. Algorithms, Games, and the Internet. In *Proc. 33rd ACM Symp. on Theory of Computing*, pages 749–753, 2001.
7. R.W. Rosenthal. A class of games possessing pure-strategy nash equilibria. *International Journal of Game Theory*, 2:65–67, 1973.
8. T. Roughgarden. Selfish routing with atomic players. In *Proc. The 16th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), Vancouver, Canada*, pages 973–974, 2005.
9. T. Roughgarden and E. Tardos. How bad is selfish routing? *Journal of the ACM*, 49(2):236–259, April 2002.