On the Complexity of Pure-Strategy Nash Equilibria in Congestion and Local-Effect Games
—Extended Abstract—

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Abstract. Congestion games are a fundamental class of noncooperative games possessing pure-strategy Nash equilibria. In the network version, each player wants to route one unit of flow on a path from her origin to her destination at minimum cost, and the cost of using an arc only depends on the total number of players using that arc. A natural extension is to allow for players sending different amounts of flow, which results in so-called weighted congestion games. While examples have been exhibited showing that pure-strategy Nash equilibria need not exist, we prove that it actually is strongly NP-hard to determine whether a given weighted network congestion game has a pure-strategy Nash equilibrium. This is true regardless of whether flow is unsplittable (has to be routed on a single path for each player) or not.

A related family of games are local-effect games, where the disutility of a player taking a particular action depends on the number of players taking the same action and on the number of players choosing related actions. We show that the problem of deciding whether a bidirectional local-effect game has a pure-strategy Nash equilibrium is NP-complete, and that the problem of finding a pure-strategy Nash equilibrium in a bidirectional local-effect game with linear local-effect functions (for which the existence of a pure-strategy Nash equilibrium is guaranteed) is PLS-complete. The latter proof uses a tight PLS-reduction, which implies the existence of instances and initial states for which any sequence of selfish improvement steps needs exponential time to reach a pure-strategy Nash equilibrium.

1 Introduction

Game theory in general and the concept of Nash equilibrium in particular have lately come under increased scrutiny by theoretical computer scientists. Computing a mixed Nash equilibrium is a case in point. Goldberg and Papadimitriou (2006) showed only recently that finding a mixed Nash equilibrium in a game of any constant number of players is reducible to solving a 4-player game. Daskalakis, Goldberg, and Papadimitriou (2006) showed in turn that the latter problem is PPAD-complete. Subsequently, Chen and Deng (2005) and Daskalakis

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and Papadimitriou (2005) proved that computing mixed Nash equilibria in games with three players is PPAD-complete as well. Eventually, Chen and Deng (2006) established the same result for the two-player case.

While Nash (1951) showed that mixed Nash equilibria do exist in any finite noncooperative game, it is well known that pure-strategy Nash equilibria are in general not guaranteed to exist. It is therefore natural to ask which games have pure-strategy Nash equilibria and, if applicable, how difficult is it to find one. In this article, we study these questions for certain classes of weighted congestion and local-effect games.

Congestion games were introduced by Rosenthal (1973), who showed that they are guaranteed to possess pure-strategy Nash equilibria. In a congestion game, a player's strategy consists of a subset of resources, and her disutility only depends on the number of players choosing the same resources. An important subclass of congestion games can be represented by means of networks. Each player wants to route one unit of flow from her origin to her destination, at minimal cost. The network arcs are the resources, and a player's set of pure strategies consists of the sets of arcs corresponding to paths connecting her origin-destination pair. Fabrikant, Papadimitriou, and Talwar (2004) studied the computational complexity of finding pure-strategy Nash equilibria in congestion games. For symmetric network congestion games, where all players have the same origin-destination pair, they presented a polynomial-time algorithm for computing a pure-strategy Nash equilibrium. On the other hand, they proved that this problem is PLS-complete for symmetric congestion games as well as for asymmetric network congestion games. A simpler proof of the latter result was given by Ackermann, Röglin, and Vöcking (2006a), who also showed that this result still holds if the cost functions are affine-linear.

In (unweighted) network congestion games, each player routes exactly one unit of flow along a single path. In weighted congestion games, players can have different amounts of flow. Depending on whether players are allowed to split their flows or not, a player's strategy consists of a set of paths with corresponding integer flow values between her origin-destination pair, or of a single path.

Fotakis, Kontogiannis, and Spirakis (2005) studied weighted network congestion games with unsplittable flows. They constructed simple examples of symmetric instances that do not possess a pure-strategy Nash equilibrium. On the other hand, they proved that for the special case of affine cost functions, a pure-strategy Nash equilibrium is always guaranteed to exist. Awerbuch, Azar, and Epstein (2005) derived a tight bound of $(\sqrt{5}+3)/2$ on the pure price of anarchy for this special case. They also considered the case when the cost functions are polynomials of fixed degree greater than 1. However, Goemans, Mirrokni, and Vetta (2005) showed that a pure-strategy Nash equilibrium need not exist for instances with cost functions that are polynomials of degree at most 2. Milchtaich (1996) had earlier shown that weighted congestion games with player-specific disutility functions on networks consisting of parallel arcs only do not always have a pure-strategy Nash equilibrium.

In this article, we show that the problem of deciding whether a weighted network congestion game with simple, non-linear cost functions possesses a pure-strategy Nash equilibrium is strongly NP-hard, regardless of whether we consider splittable or unsplittable flows. In the unsplittable case, the problem remains NP-hard even if all players have the same origin and the same destination. The same is true for weighted congestion games with affine player-specific cost functions in networks consisting of parallel arcs only.

Leyton-Brown and Tennenholtz (2003) introduced local-effect games as a tool to model situations in which the use of one resource can affect the cost of other resources. Local-effect games are in general not guaranteed to possess pure-strategy Nash equilibria. However, Leyton-Brown and Tennenholtz showed that so-called bidirectional local-effect games with linear local-effect functions belong to the class of exact potential games, and therefore always have pure-strategy Nash equilibria. The question of whether there exists a polynomial-time algorithm for finding a pure-strategy Nash equilibrium for these games was left open.

We prove that computing a pure-strategy Nash equilibrium is PLS-complete. Because the proof uses a tight PLS-reduction, our result implies the existence of instances of these games that have exponentially long shortest improvement paths. It also implicates that the problem of finding a pure-strategy Nash equilibrium that is reachable from a given strategy state via selfish improvement steps is PSPACE-hard. In addition, we show that, given an initial strategy profile for a bidirectional local-effect game with linear local-effect functions and a positive integer k (unarily encoded), it is NP-complete to decide whether there is a sequence of at most k selfish steps that transforms the initial state into a pure-strategy Nash equilibrium. We also prove that the problem of deciding whether a bidirectional local-effect game with general, strictly increasing local-effect functions has a pure-strategy Nash equilibrium is NP-complete.

For bidirectional local-effect games with linear local-effect functions (for which a pure-strategy Nash equilibrium is guaranteed to exist), we also study the pure price of stability w.r.t. the social objective that is given by the sum of the costs of all players. In the case of linear cost functions, in which the worst-possible ratio of the social cost of a pure-strategy Nash equilibrium to that of a social optimum (i.e., the pure price of anarchy) is unbounded, we obtain a bound of 2 on the pure price of stability. Thus, there always exists a pure-strategy Nash equilibrium whose cost is at most twice that of a socially optimal solution. For the case of quadratic cost functions and linear local-effect functions we derive a bound of 3 on the pure price of stability.

Before we present the details of our results on weighted congestion games and local-effect games in Sections 2 and 3, respectively, let us end this introduction by briefly discussing additional related work on the computational complexity of pure-stratgey Nash equilibria. Gottlob, Greco, and Scarcello (2005) considered restrictions of strategic games intended to capture certain aspects of bounded rationality. Among other results, they proved that even in the setting where each player's payoff function depends on the actions of at most three other players and where each player is allowed to play at most three actions, the problem of determining whether a strategic game has a pure-strategy Nash equilibrium is NP-complete. This result was strengthened by Fischer, Holzer, and Katzenbeisser (2006) who showed that this problem remains hard even if each player has only two actions to choose from and her payoff depends on the actions of at most two other players. Àlvarez, Gabarró, and Serna (2005) studied how various representations of a strategic game influence the computational complexity of deciding the existence of a pure-strategy Nash equilibrium. They showed that this problem is NP-complete when the number of players is large and the number of strategies for each player is constant, while the problem is $\sum_2^p$-complete when the number of players is constant and the size of the sets of strategies is exponential (with respect to the length of the strategies). Schoenebeck and Vadhan (2006) analyzed the computational complexity of deciding whether a pure-strategy Nash equilibrium exists in graph games and circuit games. Brandt, Fischer, and Holzer (2006) studied the impact of various notions of symmetry in strategic games on the computational complexity of finding pure-strategy Nash equilibria. Expanding on a line of research started by Ieong et al. (2005), who considered singleton congestion games, Ackermann, Röglin, and Vöcking (2006a) proved that the lengths of all best-response sequences are polynomially bounded in the number of players and resources in congestion games where the strategy space of each player consists of the bases of a matroid over the set of resources. This especially implies that pure-strategy Nash equilibria for congestion games with the matroid property can be computed in polynomial time, even in the case of player-specific costs (Ackermann, Röglin, and Vöcking 2006b). In the latter paper, Ackermann et al. also showed the existence of pure-strategy Nash equilibria in weighted congestion games with the same matroid property.

Due to space limitations, proofs are only sketched or omitted completely from this extended abstract. Most details can be found in Dunkel (2005). A journal version is forthcoming.

2 Weighted Congestion Games

An *unweighted congestion game* is a tuple $\langle N, E, (S_i)_{i \in N}, (f_e)_{e \in E} \rangle$, where $N = \{1, 2, \ldots, n\}$ is the set of players, and E is a set of resources. For each player $i \in N$, her set S_i of available strategies is a collection of subsets of the resources; that is, $S_i \subseteq 2^E$. A cost function $f_e : \mathbb{N} \rightarrow \mathbb{R}_+$ is associated with each resource $e \in E$. Given a strategy profile $s = (s_1, s_2, \ldots, s_n) \in S = S_1 \times S_2 \times \cdots \times S_n$, the cost (disutility) of player i is $c_i(s) = \sum_{e \in s_i} f_e\big(n_e(s)\big)$, where $n_e(s)$ denotes the number of players using resource e in s. In other words, in a congestion game each player chooses a subset of resources that are available to her, and the cost to a player is the sum of the costs of the resources used by her, where the cost of a resource only depends on the total number of players using this resource.

A *network congestion game* is a congestion game where the arcs of an underlying directed network represent the resources. Each player $i \in N$ has an origin-destination pair (a_i, b_i), where a_i and b_i are nodes of the network, and the set S_i of pure strategies available to player i is the set of directed (simple) paths from a_i

to b_i. A symmetric network congestion game is also called a *single-commodity* network congestion game because all players have the same origin-destination pair.

In a *weighted network congestion game* $\langle N, E, (w_i)_{i\in N}, (S_i)_{i\in N}, (f_e)_{e\in E}\rangle$, each player $i \in N$ has a positive integer weight w_i, which constitutes the amount of flow that player i wants to ship from a_i to b_i. In the case of unsplittable flows, the cost of player i adopting strategy s_i in a strategy profile $s = (s_1, s_2, \ldots, s_n) \in S$ is given by $c_i(s) = \sum_{e\in s_i} f_e\bigl(\theta_e(s)\bigr)$, where $\theta_e(s) = \sum_{i:e\in s_i} w_i$ denotes the total flow on arc e in s. In integer-splittable network congestion games, a player with weight greater than one can choose a subset of paths on which to route her flow simultaneously; that is, player i's strategy consists of the specification of the a_i-b_i-paths used and the (integer) amounts of flow routed on them, which sum up to w_i.

In terms of the input size of a weighted network congestion game, we assume that the cost functions are explicitly specified; that is, for each $0 \le x \le \sum_{i\in N} w_i$ and each arc e, the value $f_e(x)$ is given in binary encoding.

While every unweighted congestion game possesses a pure-strategy Nash equilibrium (Rosenthal 1973), this is not true for weighted congestion games; see, e.g., Fig. 1 in Fotakis, Kontogiannis, and Spirakis (2005). We can actually turn their instance into a gadget to derive the following result.

Theorem 1. *The problem of deciding whether a weighted symmetric network congestion game with unsplittable flows possesses a pure-strategy Nash equilibrium is strongly NP-complete.*

The proof is by a reduction from 3-Partition, and it is omitted from this extended abstract. While the NP-hardness of the corresponding decision problem for weighted network congestion games with player-specific payoff functions follows immediately, we can actually strengthen this result.

Theorem 2. *The problem of deciding whether a weighted network congestion game with parallel arcs and affine player-specific disutility functions possesses a pure-strategy Nash equilibrium is strongly NP-complete.*

For network congestion games with integer-splittable flows, we obtain the following result.

Theorem 3. *The problem of deciding whether a weighted network congestion game with integer-splittable flows possesses a pure-strategy Nash equilibrium is strongly NP-hard. Hardness even holds if there is only one player with weight* 2*, and all other players have unit weights.*

Proof. Consider an instance of Monotone3Sat with set of variables $X = \{x_1, x_2, \ldots, x_n\}$ and set of clauses $C = \{c_1, c_2, \ldots, c_m\}$. We construct a game that has one player p_x for every variable $x \in X$ with weight $w_x = 1$, origin x and destination $\bar{x}$. Moreover, each clause $c \in C$ gives rise to a player p_c with weight $w_c = 1$, origin c, and destination $\bar{c}$. There are three more players p_1, p_2, and p_3 with weights $w_1 = 1, w_2 = 2, w_3 = 1$ and origin-destination pairs $(s,t_1),(s,t_2),(s,t_3)$, respectively. For every variable $x \in X$ there are two disjoint paths P_x^1, P_x^0 from x to $\bar{x}$ in the network. Path P_x^0 consists of $2\,|\{c \in C \mid x \in c\}| + 1$ arcs and P_x^1 has $2\,|\{c \in C \mid \bar{x} \in c\}| + 1$ arcs with cost functions as shown in Fig. 1. For each pair $(c,\bar{c})$, we construct two disjoint paths P_c^1, P_c^0 from c to $\bar{c}$. Path P_c^1 consists of only two arcs. The paths P_c^0 will have seven arcs each and are constructed for $c = c_j$ in the order $j = 1, 2, \ldots, m$ as follows. For a positive clause $c = c_j = (x_{j_1} \vee x_{j_2} \vee x_{j_3})$ with $j_1 < j_2 < j_3$, path P_c^0 starts with the arc connecting c to the first inner node v_1 on path $P_{x_{j_1}}^1$ that has only two incident arcs so far. The second arc is the unique arc (v_1, v_2) of path $P_{x_{j_1}}^1$ that has v_1 as its start vertex. The third arc connects v_2 to the first inner node v_3 on path $P_{x_{j_2}}^1$ that has only two incident arcs so far. The fourth arc is the only arc (v_3, v_4) on $P_{x_{j_2}}^1$ with start vertex v_3. From v_4, there is an arc to the first inner node v_5 on $P_{x_{j_3}}^1$ that has only two incident arcs so far, followed by (v_5, v_6) of $P_{x_{j_3}}^1$. The last arc of P_c^0 connects v_6 to $\bar{c}$ (see Fig. 1). For a negative clause $c = c_j = (\bar{x}_{j_1} \vee \bar{x}_{j_2} \vee \bar{x}_{j_3})$ we proceed in the same way, except that we choose the inner nodes v_i from the upper variable paths $P_{x_{j_1}}^0, P_{x_{j_2}}^0, P_{x_{j_3}}^0$. The strategy set of player p_x is $\{P_x^1, P_x^0\}$. We will interpret it as setting the

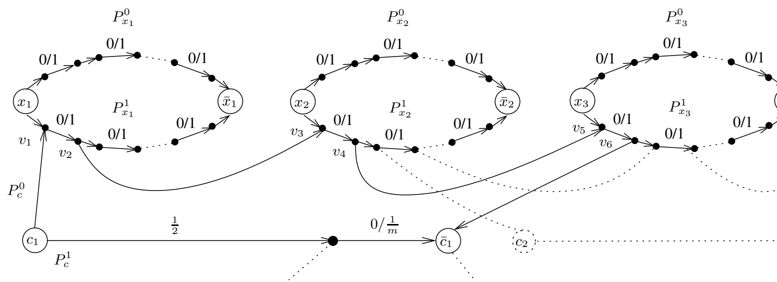

Fig. 1. Part of the constructed network corresponding to a positive clause $c_1 = (x_1 \vee x_2 \vee x_3)$. The notation a/b defines a cost per unit flow of value a for load 1 and b for load 2. Arcs without specified values have zero cost.

variable x to *true* (*false*) if p_x sends her unit of flow over P_x^1 (P_x^0). Note that player p_c can only choose between the paths P_c^1 and P_c^0. This is due to the order in which the paths $P_{c_j}^0$ are constructed. Depending on whether player p_c sends her unit of flow over path P_c^1 or P_c^0, the clause c will be satisfied or not.

The second part of the network consists of all origin-destination pairs and paths for players p_1, p_2, p_3 (see Fig. 2). Player p_1 can choose between paths $U_1 = \{(s,t_2),(t_2,t_1)\}$ and $L_1 = \{(s,t_1)\}$. Player p_2 is the only player who can split her flow; that is, she can route her two units either both over path $U_2 = \{(s,t_2)\}$, both over path $L_2 = \{(s,t_1),(t_1,t_2\}$, or one unit on the upper and the other unit on the lower path; i.e., $S_2 = \{L_2, U_2, LU_2\}$. Finally, player p_3 has three possible paths to choose from. The upper path U_3 shares an arc with each clause path P_c^1 and has some additional arcs to connect these. The paths $M_3 = \{(s,t_2),(t_2,t_3)\}$

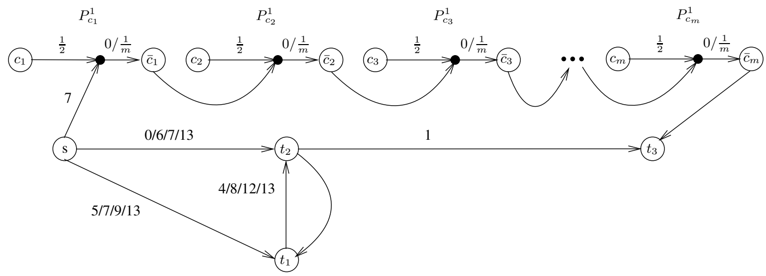

Fig. 2. Part of the constructed network that is used by players p_1, p_2, and p_3. A single number a on an arc defines a constant cost a per unit flow for this arc.

and $L_3 = \{(s, t_1), (t_1, t_2), (t_2, t_3)\}$ have only arcs with the paths of p_1 and p_2 in common. The cost functions are defined in Fig. 2.

Given a satisfying truth assignment, we define a strategy state s of the game by setting the strategy of player p_x to be P^1_x for a true variable x, and P^0_x otherwise. Each player p_c plays P^1_c. Furthermore, $s_1 = L_1$, $s_2 = U_2$, and $s_3 = M_3$. It is easy to show that no player can decrease her cost by unilaterally switching to another strategy; i.e., the defined strategy configuration is a pure-strategy Nash equilibrium.

For the other direction, we first observe that any pure-strategy Nash equilibrium s of the game has the following properties: (a) player p_3 does not use path U_3, (b) for the cost of player p_3 we have $c_3(s) \geq 8$, and (c) each player p_c routes her unit flow over path P^1_c. Property (a) follows from the fact that the subgame shown in Fig. 3 with players p_1 and p_2 only does not have a pure-strategy Nash equilibrium. Property (a) implies (b), and property (c) can be proved by contradiction assuming (a) and (b). We claim that the truth assignment that

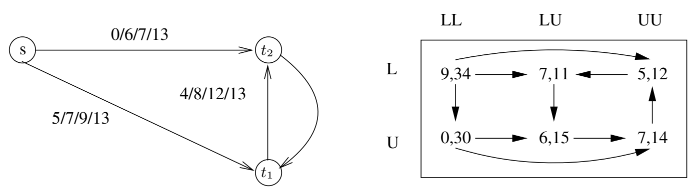

Fig. 3. Sub-game with two players without pure-strategy Nash equilibrium (Papadimitriou 2003)

sets a variable x to *true* if the corresponding player uses P^1_x, and x to *false* otherwise, satisfies all clauses. Suppose for a positive clause $c = (x_1 \vee x_2 \vee x_3)$ that all variables are *false*; i.e., $s_{x_i} = P^0_{x_i}$ for $i = 1, 2, 3$. By property (c), player p_c uses P^1_c. Because of (a), her current cost is $c_c(s) = \frac{1}{2}$. Choosing path P^0_c instead would decrease the cost to zero, which contradicts the assumption of s being a Nash equilibrium. A similar argument holds for a negative clause. $\square$

3 Local-Effect Games

A *local-effect game* is a tuple $\langle N, A, \mathcal{F} \rangle$ where $N = \{1, 2, \ldots, n\}$ is the set of players, A is a common set of actions (strategies) available to each player, and $\mathcal{F}$ is a set of cost functions. For each pair of actions $a, a' \in A$, the function $F_{a',a} : \mathbb{Z}_+ \to \mathbb{R}_+$ expresses the impact of action a' on the cost of action a, which depends only on the number of players that choose action a'. For $a, a' \in A$ with $a \neq a'$, $F_{a',a}$ is called a *local-effect function*, and it is assumed that $F_{a',a}(0) = 0$. Moreover, the local-effect function $F_{a',a}$ is either strictly increasing or identical zero. If $F_{a',a}$ is not identical zero, then this is also the case for $F_{a,a'}$. In other words, if action a has an effect on action a', then the converse is also true. For a given strategy state $s = (s_1, s_2, \ldots, s_n) \in A^n$, n_a denotes the number of players playing action a in s. The cost to a player $i \in N$ for playing action s_i in strategy state s is given by $c_i(s) = F_{s_i,s_i}(n_{s_i}) + \sum_{a \in A, a \neq s_i} F_{a,s_i}(n_a)$. If the local-effect functions $F_{a',a}$ are zero for all $a \neq a'$, the local-effect game is equivalent to a symmetric network congestion game with only parallel arcs.

A local-effect game is called a *bidirectional* local-effect game if for all $a, a' \in A$, $a \neq a'$, and for all $x \in \mathbb{Z}_+$, $F_{a',a}(x) = F_{a,a'}(x)$. Leyton-Brown and Tennenholtz (2003) gave a characterization of local-effect games that have an exact potential function and which are therefore guaranteed to possess pure-strategy Nash equilibria. One of these subclasses are bidirectional local-effect games with linear local-effect functions. However, without linear local-effect functions, deciding the existence is hard.

3.1 Computational Complexity

Theorem 4. *The problem of deciding whether a bidirectional local-effect game has a pure-strategy Nash equilibrium is NP-complete.*

The proof will be given in the full version of this paper. The next result implies that computing a pure-strategy Nash equilibrium for a bidirectional local-effect game with linear local-effect functions is as least as hard as finding a local optimum for several combinatorial optimization problems with efficiently searchable neighborhoods.

Theorem 5. *The problem of computing a pure-strategy Nash equilibrium for a bidirectional local-effect game with linear local-effect functions is PLS-complete.*

Proof. We reduce from PosNae3Flip (Schäffer and Yannakakis 1991): Given not-all-equal clauses with at most three literals, (x_1, x_2, x_3) or (x_1, x_2), where x_i is either a variable or a constant (0 or 1), and a weight for each clause, find a truth assignment such that the total weight of satisfied clauses cannot be improved by flipping a single variable.

For simplicity, we assume that we are given an instance of PosNae3Flip with set $C = C_2 \dot{\cup} C_3$ of clauses containing two or three variables but no constants, a positive integer weight w_c for each clause $c \in C$, and set of variables $\{x_1, \ldots, x_n\}$. We construct a bidirectional local-effect game with linear local-effect functions as follows: There are n players with common action set A that contains two actions a_i and $\overline{a}_i$ for each variable x_i, $i = 1, 2, \ldots, n$. Let $M = 2n \sum_{c \in C} w_c + 1$. For each action $a \in A$, $F_{a,a}(x) = 0$ if $x \leq 1$, and $F_{a,a}(x) = M$ otherwise. If $C_i = \{c \in C \mid x_i \in c\}$ denotes the subset of clauses containing variable x_i, the local-effect functions are given for $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$, by

$$F_{a_i,a_j}(x) = F_{\overline{a}_i,\overline{a}_j}(x) = \left(2 \sum_{c \in C_2 \cap C_i \cap C_j} w_c + \sum_{c \in C_3 \cap C_i \cap C_j} w_c \right) x \ .$$

However, the local-effect functions F_{a_i,a_j} and $F_{\overline{a}_i,\overline{a}_j}$ are zero if there is no clause containing both x_i and x_j. Furthermore, $F_{a_i,\overline{a}_i}(x) = F_{\overline{a}_i,a_i}(x) = M\,x$ for all $i \in \{1, 2, \ldots, n\}$. All local-effect functions not defined so far are identical zero. For any solution $s = (s_1, s_2, \ldots, s_n)$, $s_i \in A$, of the game, we define the corresponding truth assignment to the variables x_i of the PosNae3Flip instance by $x_i = 1$ if $|\{j \mid s_j = a_i\}| \geq 1$, and $x_i = 0$ otherwise.

Now we show that for any pure-strategy Nash equilibrium $s = (s_1, s_2, \ldots, s_n)$ of the game, the corresponding truth assignment is indeed a local optimum of the PosNae3Flip instance. The proof is demonstrated only for the case of flipping a positive variable $x_i = 1$ to $x_i' = 0$. First, observe that for all $i \in \{1, 2, \ldots, n\}$

$$\big|\{j \mid s_j = a_i\}\big| + \big|\{j \mid s_j = \overline{a}_i\}\big| = 1 \ , \tag{1}$$

since otherwise, because of the choice of M, there is always a player who can decrease her cost by choosing another action.

Let X and X' denote the truth assignments before and after flipping variable x_i. Let the set of clauses that contain variable x_i and are satisfied by truth assignment X, X' be C_i^X, $C_i^{X'}$, respectively. Further, let $C_i^{X \setminus X'}$ ($C_i^{X' \setminus X}$) be the set of clauses containing x_i that are satisfied by truth assignment X (X'), but not by X' (X). Then the difference in the total weight of satisfied clauses by X' and X can be written as

$$\varDelta W = \sum_{c \in C_2 \cap C_i^{X' \setminus X}} w_c + \sum_{c \in C_3 \cap C_i^{X' \setminus X}} w_c - \sum_{c \in C_2 \cap C_i^{X \setminus X'}} w_c - \sum_{c \in C_3 \cap C_i^{X \setminus X'}} w_c \ . \tag{2}$$

For a clause $c = (x_i, x_j) \in C_i^{X' \setminus X}$, it follows because of $x_i = 1$ that $x_j = 1$. Then, by definition of X and by (1), $n_{a_j} = 1$ and $n_{\overline{a}_j} = 0$. If $c = (x_i, x_j) \in C_i^{X \setminus X'}$, we have $x_j = 0$, $n_{a_j} = 0$, and $n_{\overline{a}_j} = 1$. Similarly, for a three-variable clause $c = (x_i, x_j, x_k) \in C_i^{X' \setminus X}$, $x_i = 1$ implies $x_j = x_k = 1$, $n_{a_j} = n_{a_k} = 1$, and $n_{\overline{a}_j} = n_{\overline{a}_k} = 0$. If $c = (x_i, x_j, x_k) \in C_i^{X \setminus X'}$, then $x_j = x_k = 0$, $n_{a_j} = n_{a_k} = 0$, and $n_{\overline{a}_j} = n_{\overline{a}_k} = 1$. Thus we can rewrite (2) as

$$\begin{aligned} \varDelta W = &\sum_{j=1, j \neq i}^{n} \left[\left(\sum_{c \in C_2 \cap C_i^{X' \setminus X} \cap C_j} w_c \right) n_{a_j} - \left(\sum_{c \in C_2 \cap C_i^{X \setminus X'} \cap C_j} w_c \right) n_{\overline{a}_j} \right] \\ &+ \frac{1}{2} \sum_{j=1, j \neq i}^{n} \left[\left(\sum_{c \in C_3 \cap C_i^{X' \setminus X} \cap C_j} w_c \right) n_{a_j} - \left(\sum_{c \in C_3 \cap C_i^{X \setminus X'} \cap C_j} w_c \right) n_{\overline{a}_j} \right] \ . \end{aligned} \tag{3}$$

By the above observations on the numbers n_{a_j} and $n_{\overline{a}_j}$ we have

$$\sum_{j=1,j\neq i}^{n} \sum_{c\in C_2\cap C_i^{X\setminus X'}\cap C_j} w_c n_{a_j} = -\sum_{j=1,j\neq i}^{n} \sum_{c\in C_2\cap C_i^{X'\setminus X}\cap C_j} w_c n_{\overline{a}_j} = 0 \ , \tag{4}$$

$$\sum_{j=1,j\neq i}^{n} \sum_{c\in C_3\cap C_i^{X\setminus X'}\cap C_j} w_c n_{a_j} = -\sum_{j=1,j\neq i}^{n} \sum_{c\in C_3\cap C_i^{X'\setminus X}\cap C_j} w_c n_{\overline{a}_j} = 0 \ . \tag{5}$$

Now consider clauses $c = (x_i, x_j, x_k) \in (C_3 \cap C_i) \setminus (C_i^{X\setminus X'} \cup C_i^{X'\setminus X})$. Since the case of clause c not being satisfied by both X and X' cannot happen, we have $c \in C_i^X \cap C_i^{X'}$. Then, either $x_j = 1, x_k = 0$ or $x_j = 0, x_k = 1$, and therefore $n_{a_j} = 1, n_{a_k} = 0$ or $n_{a_j} = 0, n_{a_k} = 1$. By (1), we have in both cases $n_{a_j} + n_{a_k} = n_{\overline{a}_j} + n_{\overline{a}_k} = 1$; i.e., $w_c(n_{a_j} + n_{a_k}) - w_c(n_{\overline{a}_j} + n_{\overline{a}_k}) = 0$. Thus

$$\sum_{j=1,j\neq i}^{n} \left(\sum_{c\in C_3\cap C_i^X\cap C_i^{X'}\cap C_j} w_c \right) n_{a_j} - \sum_{j=1,j\neq i}^{n} \left(\sum_{c\in C_3\cap C_i^X\cap C_i^{X'}\cap C_j} w_c \right) n_{\overline{a}_j} = 0 \ . \tag{6}$$

Adding the terms in (4), (5), and (6) to (3), we obtain

$$\begin{aligned} \Delta W = &\sum_{j=1,j\neq i}^{n} \left[\left(\sum_{c\in C_2\cap C_i\cap C_j} w_c \right) n_{a_j} - \left(\sum_{c\in C_2\cap C_i\cap C_j} w_c \right) n_{\overline{a}_j} \right] \\ &+ \frac{1}{2} \sum_{j=1,j\neq i}^{n} \left[\left(\sum_{c\in C_3\cap C_i\cap C_j} w_c \right) n_{a_j} - \left(\sum_{c\in C_3\cap C_i\cap C_j} w_c \right) n_{\overline{a}_j} \right] \leq 0 \ . \end{aligned}$$

Here, the last inequality follows from the fact that the player i with action $s_i = a_i$ cannot decrease her cost by switching to action $\overline{a}_i$. The described construction is indeed a PLS-reduction. □

Since the reduction actually is a *tight* PLS-reduction, we obtain the following results.

Corollary 6. *There are instances of bidirectional local-effect games with linear local-effect functions that have exponentially long shortest improvement paths.*

Corollary 7. *For a bidirectional local-effect game with linear local-effect functions, the problem of finding a pure-strategy Nash equilibrium that is reachable from a given strategy state via selfish improvement steps is PSPACE-complete.*

The following result underlines that finding a pure Nash equilibrium for bidirectional local-effect games with linear local-effect functions is indeed hard.

Theorem 8. *Given an instance of a bidirectional local-effect games with linear local-effect functions, a pure-strategy profile s_0, and an integer $k > 0$ (unarily encoded), it is NP-complete to decide whether there exists a sequence of at most k selfish steps that transforms s_0 to a pure-strategy Nash equilibrium.*

3.2 Pure Price of Stability for Bidirectional Local-Effect Games

We derive bounds on the pure-price of stability for games with linear local-effect functions where the social objective is the sum of the costs of all players.

Theorem 9. *The pure price of stability for bidirectional local-effect games with only linear cost functions is bounded by* 2*.*

The proof is based on a technique suggested by Anshelevich et al. (2004) using the potential function introduced by Leyton-Brown and Tennenholtz (2003). By the same technique, we can derive the following bound for the case of quadratic cost-functions and linear local-effect functions.

Theorem 10. *The pure price of stability for bidirectional local-effect games with* $F_{a,a}(x) = m_a\, x^2 + q_a\, x$*,* $q_a \geq 0$ *for all* $a \in A$ *and linear local-effect functions is bounded by* 3*.*

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