Pure Nash Equilibria in Player-Specific and Weighted Congestion Games*

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Abstract. Unlike standard congestion games, weighted congestion games and congestion games with player-specific delay functions do not necessarily possess pure Nash equilibria. It is known, however, that there exist pure equilibria for both of these variants in the case of *singleton congestion games*, i. e., if the players' strategy spaces contain only sets of cardinality one. In this paper, we investigate how far such a property on the players' strategy spaces guaranteeing the existence of pure equilibria can be extended. We show that both weighted and player-specific congestion games admit pure equilibria in the case of *matroid congestion games*, i. e., if the strategy space of each player consists of the bases of a matroid on the set of resources. We also show that the matroid property is the maximal property that guarantees pure equilibria without taking into account how the strategy spaces of different players are interweaved. In the case of player-specific congestion games, our analysis of matroid games also yields a polynomial time algorithm for computing pure equilibria.

1 Introduction

Congestion games are a natural model for resource allocation in large networks like the Internet. It is assumed that n players share a set $\mathcal{R}$ of m resources. Players are interested in subsets of resources. For example, the resources may correspond to the edges of a graph, and each player may want to allocate a spanning tree of this graph. The delay (cost, negative payoff) of a resource depends on the number of players that allocate the resource, and the delay of a set of allocated resources corresponds to the sum of the delays of the resources in the set. A well known potential function argument of Rosenthal [11] shows that congestion games always possess Nash equilibria[1], i. e., allocations of resources from which no player wants to deviate unilaterally.

The existence of Nash equilibria gives a natural solution concept for congestion games. Unfortunately, this property does not hold anymore if we slightly extend the class of considered games towards congestion games with player-specific delay

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[1] In this paper, the term *Nash equilibrium* always refers to a pure equilibrium.

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functions, i. e., a variant of congestion games in which different players might have different delay functions, and weighted congestion games, i. e., a variant of congestion games in which the delay of a resource depends on a weighted number of players. For both of these classes one can easily construct examples of games that do not possess Nash equilibria (cf. Fotakis et al. [4] in the case of weighted network congestion games). In this paper, we study which conditions on the strategy spaces of individual players guarantee the existence of Nash equilibria. We only consider games with non-decreasing delay functions since otherwise one can construct examples of weighted and player-specific *singleton congestion games*, i. e., games in which the players' strategy spaces contain only sets of cardinality one, that do not possess Nash equilibria.

It is known, however, that there exist pure equilibria for both of these variants in the case of singleton congestion games with non-decreasing delay functions [10,2]. We extend these results and show that both player-specific and weighted congestion games admit pure equilibria in the case of *matroid congestion games*, i. e., if the strategy space of each player consists of the bases of a matroid on the set of resources. We also show that the matroid property is the maximal condition on the players' strategy spaces that guarantees Nash equilibria without taking into account how the strategy spaces of different players are interweaved. In the case of player-specific matroid congestion games, our analysis also yields a polynomial time algorithm for computing pure equilibria. Let us remark that the best response dynamics may cycle for player-specific singleton congestion games [10]. For weighted matroid congestion games we do not have an efficient algorithm for computing a Nash equilibrium, but we show that players playing "lazy best responses" converge to a Nash equilibrium.

Related Work. Milchtaich [10] considers player-specific singleton congestion games and shows that every such game possesses at least one Nash equilibrium. Additionally, he shows that players iteratively playing best responses in such games do not necessarily reach a Nash equilibrium, that is, the best response dynamics may cycle. However, he implicitly describes an algorithm for computing an equilibrium. Our work generalizes Milchtaich's analysis from singleton congestion games towards matroid congestion games. Gairing et al. [6] consider player-specific singleton congestion games with linear delay functions without offsets and show that the best response dynamics of these games do not cycle anymore. Milchtaich [10] also addresses the existence of Nash equilibria in congestion games which are both player-specific and weighted. In this case, a Nash equilibrium does not necessarily exist in singleton congestion games. However, Georgiou et al. [7] and Garing et al. [6] conjecture that these games possess Nash equilibria in the case of linear player-specific delay functions without offsets.

Even-Dar et al. [2] consider a load balancing scenario with weighted jobs. They show that in this scenario at least one Nash equilibrium always exists and that players iteratively playing best responses converge to such an equilibrium. A similar result can also be found in [10] and [3]. Our proof that every weighted matroid congestion game possesses at least one Nash equilibrium reworks the proof in [2]. Even-Dar et al. [2] also consider the convergence time in the case of unrelated, related, and identical machines, and different types of job weights. They show that players do not necessarily converge quickly in any of these scenarios. Fotakis et al. [4] consider weighted network congestion games in which the strategy space of each player corresponds to the set of all paths between possibly different sources and sinks in a network. First they show that a Nash equilibrium does not necessarily exist. However, they are able to show that in the case of l-layered networks with delays equal to the congestion every weighted network congestion game possesses at least one Nash equilibrium. This shows that if we consider more than the combinatorial structure of the strategy spaces of the players, then one can identify larger classes of weighted congestion games possessing Nash equilibria.

It is interesting to relate the results about the existence of Nash equilibria in player-specific and weighted matroid congestion games to our recent work about the convergence time of standard congestion games: In [1] we characterize the class of congestion games that admit polynomial time convergence to a Nash equilibrium. Motivated by the fact that in singleton congestion games players converge quickly [9], we show that if the strategy space of each player consists of the bases of a matroid on the set of resources, then players iteratively playing best responses reach a Nash equilibrium quickly. Furthermore, we show that the matroid property is a necessary and sufficient condition on the players' strategy spaces for guaranteeing polynomial time convergence to a Nash equilibrium if one does not take into account the global structure of the game.

Formal Definition of Congestion Games. A *congestion game* Γ is a tuple $(\mathcal{N}, \mathcal{R}, (\Sigma_i)_{i\in\mathcal{N}}, (d_r)_{r\in\mathcal{R}})$ where $\mathcal{N} = \{1, \ldots, n\}$ denotes the set of players, $\mathcal{R} = \{1, \ldots, m\}$ the set of resources, $\Sigma_i \subseteq 2^{\mathcal{R}}$ the strategy space of player i, and $d_r : \mathbb{N} \to \mathbb{N}$ a delay function associated with resource r. We call a congestion game *symmetric* if all players share the same set of strategies, otherwise we call it *asymmetric*. We denote by $S = (S_1, \ldots, S_n)$ the *state of the game* where player i plays strategy $S_i \in \Sigma_i$. Furthermore, we denote by $S \oplus S'_i$ the state $S' = (S_1, \ldots, S_{i-1}, S'_i, S_{i+1}, \ldots, S_n)$, i. e., the state S except that player i plays strategy S'_i instead of S_i. For a state S, we define the *congestion* $n_r(S)$ on resource r by $n_r(S) = |\{i \mid r \in S_i\}|$, that is, $n_r(S)$ is the number of players sharing resource r in state S. Players act selfishly and like to play a strategy $S_i \in \Sigma_i$ minimizing their individual delay. The delay $\delta_i(S)$ of player i in state S is given by $\delta_i(S) = \sum_{r\in S_i} d_r(n_r(S))$. Given a state S, we call a strategy S^*_i a *best response* of player i to S if, for all $S'_i \in \Sigma_i$, $\delta_i(S \oplus S^*_i) \leq \delta_i(S \oplus S'_i)$. Furthermore, we call a state S a *Nash equilibrium* if no player can decrease her delay by changing her strategy, i. e., for all $i \in \mathcal{N}$ and for all $S'_i \in \Sigma_i$, $\delta_i(S) \leq \delta_i(S \oplus S'_i)$. Rosenthal [11] shows that every congestion game possesses at least one Nash equilibrium by considering the potential function $\phi : \Sigma_1 \times \cdots \times \Sigma_n \to \mathbb{N}$ with $\phi(S) = \sum_{r\in\mathcal{R}} \sum_{i=1}^{n_r(S)} d_r(i)$.

There are two well known extensions of congestion games, namely *player-specific congestion games* and *weighted congestion games*. In a player-specific congestion game every player i has its own delay function $d^i_r : \mathbb{N} \to \mathbb{N}$ for every resource $r \in \mathcal{R}$. Given a state S, the delay of player i is defined as $\delta_i(S) = \sum_{r \in S_i} d_r^i(n_r(S))$. In a weighted congestion game every player $i \in \mathcal{N}$ has a weight $\omega_i \in \mathbb{N}$. Given a state S, we define the congestion on resource r by $n_r(S) = \sum_{i:r \in S_i} \omega_i$, that is, $n_r(S)$ is the weight of all players sharing resource r in state S.

Matroids and Matroid Congestion Games. We now introduce *matroid congestion games*. Before we give a formal definition of such games we shortly introduce matroids. For a detailed discussion we refer the reader to [12].

Definition 1. *A tuple $\mathcal{M} = (\mathcal{R}, \mathcal{I})$ is a matroid if $\mathcal{R} = \{1, \ldots, m\}$ is a finite set of resources and $\mathcal{I}$ is a nonempty family of subsets of $\mathcal{R}$ such that, if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$, and, if $I, J \in \mathcal{I}$ and $|J| < |I|$, then there exists an $i \in I \setminus J$ with $J \cup \{i\} \in \mathcal{I}$.*

Let $I \subseteq \mathcal{R}$. If $I \in \mathcal{I}$, then we call I an *independent set*, otherwise we call it *dependent*. It is well known that all maximal independent sets of $\mathcal{I}$ have the same cardinality. The *rank* $rk(\mathcal{M})$ of the matroid is the cardinality of the maximal independent sets. A maximal independent set B is called a *basis* of $\mathcal{M}$. In the case of a weight function $w : \mathcal{R} \to \mathbb{N}$, we call a matroid *weighted*, and seek to find a basis of minimum weight, where the weight of an independent set I is given by $w(I) = \sum_{r \in I} w(r)$. It is well known that such a basis can be found by a greedy algorithm. Now we are ready to define matroid congestion games.

Definition 2. *We call a congestion game $\Gamma = (\mathcal{N}, \mathcal{R}, (\Sigma_i)_{i \in \mathcal{N}}, (d_r)_{r \in \mathcal{R}})$ a* matroid congestion game *if for every player $i \in \mathcal{N}$, $\mathcal{M}_i := (\mathcal{R}, \mathcal{I}_i)$ with $\mathcal{I}_i = \{I \subseteq S \mid S \in \Sigma_i\}$ is a matroid and Σ_i is the set of bases of M_i. Additionally, we denote by $rk(\Gamma) = \max_{i \in \mathcal{N}} rk(\mathcal{M}_i)$ the* rank of a matroid congestion game *Γ.*

The obvious application of matroid congestion games are network design problems in which players compete for the edges of a graph in order to build a spanning tree [13]. There are quite a few more interesting applications as even simple matroid structures like uniform matroids, that are rather uninteresting from an optimization point of view, lead to rich combinatorial structures when various players with possibly different strategy spaces are involved. Illustrative examples based on uniform matroids are market sharing games with uniform market costs [8] and scheduling games in which each player has to injectively allocate a given set of tasks (services) to a given set of machines (servers).

Let us remark that, in the case of matroid congestion games, the assumption that all delays are positive is not a restriction. Since all strategies have the same size, one can easily shift all delays by the same value in order to obtain positive delays without changing the better and best response dynamics.

2 Player-Specific Matroid Congestion Games

In this section, we consider player-specific matroid congestion games with non-decreasing player-specific delay functions and prove that every such game possesses at least one Nash equilibrium. Moreover, the proof we present implicitly describes an efficient algorithm to compute an equilibrium.

Theorem 3. *Every player-specific matroid congestion game Γ with non-decreasing delay functions possesses at least one Nash equilibrium.*

Proof. Recall that since the strategy space of player i corresponds to the set of bases of a matroid $\mathcal{M}_i$, all strategies of player i have the same size $rk(\mathcal{M}_i)$. In the following, we represent a strategy of player i by $rk(\mathcal{M}_i)$ tokens that the player places on the resources she allocates. Suppose that we reduce the number of tokens of some of the players, that is, player i has $k_i \leq rk(\mathcal{M}_i)$ tokens that she places on the resources of an independent set of cardinality k_i. Observe that the independent sets of cardinality k_i form the bases of a matroid $\mathcal{M}'_i$ whose independent sets correspond to those independent sets of $\mathcal{M}_i$ with cardinality at most k_i. Hence, a game in which some of the players have a reduced number of tokens is also a matroid congestion game.

We prove the theorem by induction on the total number of tokens $\tau = \sum_{i\in\mathcal{N}} rk(\mathcal{M}_i)$ that the players are allowed to place, that is, we prove the existence of Nash equilibria for a sequence of games $\Gamma_0, \Gamma_1, \ldots \Gamma_\tau$, where $\Gamma_{\ell+1}$ is obtained from Γ_ℓ by giving one more token to one of the players. Γ_0 is the game in which each player has only the empty strategy. Obviously, Γ_0 has only one state and this state is a Nash equilibrium.

As induction hypothesis assume that player i has placed $k_i \geq 0$ tokens, for $1 \leq i \leq n$, and this placement corresponds to a Nash equilibrium of the player-specific matroid congestion game $\Gamma_\ell = (\mathcal{N}, \mathcal{R}, (\Sigma_i^{k_i})_{i\in\mathcal{N}}, (d_r^i)_{i\in\mathcal{N},r\in\mathcal{R}})$ with $\ell = \sum_{i\in\mathcal{N}} k_i$, in which the set of strategies $\Sigma_i^{k_i}$ coincides with the set of independent sets of size k_i of the matroid $\mathcal{M}_i$.

Now assume that some player i_0 has to place an additional token t_0. We show how to compute a Nash equilibrium for the game $\Gamma_{\ell+1}$ obtained from a Nash equilibrium of Γ_ℓ by changing i_0's strategy space to the set of independent sets of size $k_{i_0} + 1$. Due to the greedy property of matroids, there exists a resource r_0 such that placing the token t_0 on r_0 gives an independent set of size $k_{i_0} + 1$ with minimum delay among all independent sets of the same size. Thus, assuming that the tokens of the other players are fixed, an optimal strategy for player i_0 is to place t_0 on r_0 and leave all other tokens unchanged. However, as the congestion on r_0 is increased by one, other players might want to move their tokens from r_0 in order to obtain a better independent set. We now use matroid properties to show that a Nash equilibrium of $\Gamma_{\ell+1}$ can be reached with only $n \cdot m \cdot rk(\Gamma)$ moves of tokens.

Lemma 4. *Let $\mathcal{M}$ be a weighted matroid and B_{opt} be a basis of $\mathcal{M}$ with minimum weight. If the weight of a single resource $r_{opt} \in B_{opt}$ is increased such that B_{opt} is no longer of minimum weight, then, in order to obtain a minimum weight basis again, it suffices to exchange r_{opt} with a resource r^* of minimum weight such that $B_{opt} \cup \{r^*\} \setminus \{r_{opt}\}$ is a basis.*

Proof. In order to prove the lemma we use the following property of a matroid $\mathcal{M} = (\mathcal{R}, \mathcal{I})$. Let $I, J \in \mathcal{I}$ with $|I| = |J|$ be independent sets. Consider the bipartite graph $G(I \Delta J) = (V, E)$ with $V = (I \setminus J) \cup (J \setminus I)$ and $E = \{\{i, j\} \mid$

$i \in I \setminus J, j \in J \setminus I : I \cup \{j\} \setminus \{i\} \in \mathcal{I}\}$. It is well known that $G(I \Delta J)$ contains a perfect matching (cf. Lemma 39.12(a) from [12]).

Let B'_{opt} be a minimum weight basis w. r. t. the increased weight of r_{opt}. Let P be a perfect matching of the graph $G(B_{opt} \Delta B'_{opt})$ and denote by e the edge from P that contains r_{opt}. For every edge $\{r, r'\} \in P \setminus \{e\}$, it holds $w(r) \leq w(r')$ as, otherwise, if $w(r) > w(r')$, the basis $B_{opt} \cup \{r'\} \setminus \{r\}$ would have smaller weight than B_{opt}.

Now denote by r'_{opt} the resource that is matched with r_{opt}, i. e., the resource such that $e = \{r_{opt}, r'_{opt}\} \in P$. As $w(r) \leq w(r')$ for every $\{r, r'\} \in P \setminus \{e\}$, the weight of $B_{opt} \setminus \{r_{opt}\}$ is bounded from above by the weight of $B'_{opt} \setminus \{r'_{opt}\}$. By the definition of the matching P, $B_{opt} \cup \{r'_{opt}\} \setminus \{r_{opt}\}$ is a basis. By our arguments above, the weight of this basis is bounded from above by the weight of B'_{opt} . Hence, this basis is optimal w. r. t. the increased weight of r_{opt}. □

After placing token t_0 of player i_0 on resource r_0, resource r_0 has one additional token in comparison to the initial Nash equilibrium S of the game Γ_ℓ. Since we assume non-decreasing delay functions, only the players with a token on r_0 might now have an incentive to change their strategies. Let i_1 be one of these players. It follows from Lemma 4 that i_1 has a best response in which she moves a token t_1 from resource r_0 to another resource that we call r_1. Now r_1 is the only resource with one additional token in comparison to S. Suppose we have not yet reached a Nash equilibrium. Only those players with a token on r_1 might have an incentive to change their strategies. Again applying Lemma 4, we can identify a player i_2 that has a best response in which she moves a token t_2 from r_1 to a resource r_2, which then is the only resource with one additional token.

The token migration process described above can be continued in the same way until it reaches a Nash equilibrium of the game $\Gamma_{\ell+1}$. The correctness of the process is ensured by the following invariant.

Invariant 1. *For every $j \geq 0$, after player i_j moves token t_j onto resource r_j,*

a) only players with a token on r_j might violate the Nash equilibrium condition,
b) the Nash equilibrium condition of all players would be satisfied if one ignores the additional token on r_j, that is, if each player calculates the delay on r_j as if there would be one token less on this resource.

The invariant follows by induction on j: For player i_j the invariant is satisfied as this player plays a best response according to Lemma 4. Thus she satisfies the Nash equilibrium condition even without virtually reducing the congestion on r_j. For all other players, the validity of the invariant for j follows directly from the validity of the invariant for $j - 1$ as these players do not move their tokens.

Thus, in order to show the existence of a Nash equilibrium for $\Gamma_{\ell+1}$, it suffices to show that the token migration process is finite. Consider an arbitrary token t of any player i. For a resource r, let $D_i(r)$ denote the delay of i on r if r has one more token than in the initial state S. Observe, whenever t is moved by the migration process from a resource r to a resource r' then $D_i(r) > D_i(r')$. Hence, the token t can visit each resource at most once during the token migration process. As there are at most $n \cdot rk(\Gamma)$ tokens, the migration process terminates after at most $n \cdot m \cdot rk(\Gamma)$ steps in a Nash equilibrium of $\Gamma_{\ell+1}$. □

The proof of Theorem 3 implicitly describes an efficient algorithm to compute a Nash equilibrium with at most $n^2 \cdot m \cdot rk^2(\Gamma)$ moves of tokens.

Corollary 5. *There exists a polynomial time algorithm to compute a Nash equilibrium of a player-specific matroid congestion game with non-decreasing player-specific delay functions.*

3 Weighted Matroid Congestion Games

In this section we consider weighted matroid congestion games with non-decreasing delay functions and show that every such game possesses at least one Nash equilibrium. Moreover, we show that players find such an equilibrium if they iteratively play *"lazy best responses"*. Formally, given a state S we call a best response S_i^* of player i *lazy* if it can be decomposed into a sequence of strategies $S_i = S_i^0, S_i^1, \ldots, S_i^k = S_i^*$ with $|S_i^{j+1} \setminus S_i^j| = 1$ and $\gamma_i(S \oplus S_i^{j+1}) < \gamma_i(S \oplus S_i^j)$ for $0 \leq j < k$. The existence of such a best response is guaranteed since given a weighted matroid $\mathcal{M} = (\mathcal{R}, \mathcal{I})$, a basis $B \in \mathcal{I}$ is an optimal basis of $\mathcal{M}$ if and only if there exists no basis $B^* \in \mathcal{I}$ with $|B \setminus B^*| = 1$ and $w(B^*) < w(B)$ (cf. Lemma 39.12(b) from [12]). In particular, a best response which exchanges the least number of resources compared to the current strategy S_i is a lazy best response.

Theorem 6. *Every weighted matroid congestion game Γ with non-decreasing delay functions possesses at least one Nash equilibrium which is reached after a finite number of lazy best responses.*

Proof. Let S be a state of Γ. With each resource r, we associate a pair $z_r(S) = (d_r(n_r(S)), n_r(S))$ consisting of the delay and the congestion of r in state S. For two resources r and r' and states S and S', let $z_r(S) \geq z_{r'}(S')$ iff $d_r(n_r(S)) > d_{r'}(n_{r'}(S'))$ or $d_r(n_r(S)) = d_{r'}(n_{r'}(S'))$ and $n_r(S) \geq n_{r'}(S')$. Let $z_r(S) > z_{r'}(S')$ iff $z_r(S) \geq z_{r'}(S')$ and $z_r(S) \neq z_{r'}(S')$. Let $\bar{z}(S)$ denote a vector containing the pairs $z_r(S)$ of all resources $r \in \mathcal{R}$ in non-increasing order, that is, $\bar{z}_j(S) \geq \bar{z}_{j+1}(S)$, where $\bar{z}_j(S)$ denotes the j-th component of $\bar{z}$, for $1 \leq j < |\mathcal{R}|$.

We denote by $\leq_{\text{lex}}$ the lexicographic order among the vectors $\bar{z}(S)$, i. e., $\bar{z}(S_1) \leq_{\text{lex}} \bar{z}(S_2)$ if there exists an index l such that $\bar{z}_k(S_1) = \bar{z}_k(S_2)$, for all $k \leq l$, and $\bar{z}_l(S_1) \leq \bar{z}_l(S_2)$. Additionally, we define $\bar{z}(S_1) <_{\text{lex}} \bar{z}(S_2)$ if $\bar{z}(S_1) \leq_{\text{lex}} \bar{z}(S_2)$ and $\bar{z}(S_1) \neq \bar{z}(S_2)$.

Now given a state S, let player i play a lazy best response S_i^*. Since S_i^* is a lazy best response, there exists a sequence of strategies $S_i = S_i^0, \ldots, S_i^k = S_i^*$ such that, for every $0 \leq j < k$, $|S_i^{j+1} \setminus S_i^j| = 1$ and

$$\gamma_i(S) = \gamma_i(S \oplus S_i^0) > \gamma_i(S \oplus S_i^1) > \ldots > \gamma_i(S \oplus S_i^k) = \gamma_i(S \oplus S_i^*) \ .$$

We claim that $\bar{z}(S \oplus S_i^{j+1}) <_{\text{lex}} \bar{z}(S \oplus S_i^j)$, for every $0 \leq j < k$. Let r_j be the unique resource in S_i^j that is not contained in S_i^{j+1} and let r_j^* be the resource that is contained in S_i^{j+1} but not in S_i^j. Since the delay decreases strictly with the exchange, we have

$$d_{r_j}(n_{r_j}(S \oplus S_i^j)) > d_{r_j^*}(n_{r_j^*}(S \oplus S_i^{j+1})) \ .$$

Additionally, since we assume non-decreasing delay functions,

$$d_{r_j}(n_{r_j}(S \oplus S_i^j)) \geq d_{r_j}(n_{r_j}(S \oplus S_i^{j+1})) = d_{r_j}(n_{r_j}(S \oplus S_i^j) - \omega_i) \ .$$

Furthermore, $n_{r_j}(S \oplus S_i^j) > n_{r_j}(S \oplus S_i^{j+1})$. Combining these inequalities implies $z_{r_j}(S \oplus S_i^j) > z_{r_j}(S \oplus S_i^{j+1})$ and $z_{r_j}(S \oplus S_i^j) > z_{r_j^*}(S \oplus S_i^{j+1})$. Combined with the observation that $z_{r_j}(S \oplus S_i^j) > z_{r_j^*}(S \oplus S_i^j)$, this yields $\bar{z}(S \oplus S_i^j) >_{\text{lex}} \bar{z}(S \oplus S_i^{j+1})$, that is, the lexicographic order decreases with every exchange and, hence, with every lazy best response. This concludes the proof of the theorem. □

In the full version of this paper we show that playing lazy best responses is a necessary assumption in order to obtain convergence to a Nash equilibrium, that is, we present a weighted matroid congestion game in which the best response dynamic cycles if players are not restricted to lazy best responses. The delay functions in this congestion game are non-decreasing but not strictly increasing. We leave open the questions whether players playing arbitrary best responses converge to a Nash equilibrium if each delay function is strictly increasing and whether there is an efficient algorithm for computing a Nash equilibrium in weighted matroid congestion games in general. To the best of our knowledge the only positive result is known in the case of weighted singleton matroid congestion games with identical resources, i. e., all resources have identical, non-decreasing delay functions. In this case, Gairing et al. [5] show how to compute a Nash equilibrium in polynomial time. If additionally the players are symmetric, Even-Dar et al. [2] show that if one assigns the players in non-increasing order of their weights to the resources, then the resulting assignment is a Nash equilibrium.

Finally, we like to comment on the convergence time. Theorem 6 implies that players iteratively playing lazy best responses reach a Nash equilibrium after at most $\min\left\{\left(\sum_{i=1}^n \omega_i\right)^m, \binom{m}{rk(\Gamma)}^n\right\}$ strategy changes. The first term is an upper bound on the maximal number of different vectors $\bar{z}(S)$ and the second one bounds the number of different states of a matroid congestion game. Even-Dar et al. [2] establish an exponential lower bound in the case of weighted singleton congestion games with symmetric players and identical resources. However, they use exponentially large weights to show this. In the full version of this paper we present an infinite family of weighted singleton congestion games possessing superpolynomially long best response sequences although every player has either weight one or two and all delays are polynomially bounded in the number of players and resources. This immediately implies that players do not necessarily reach a Nash equilibrium in pseudopolynomial time.

4 Non-matroid Strategy Spaces

In this section, we show that the matroid property is the maximal property on the individual players' strategy spaces that guarantees the existence of Nash equilibria in player-specific and weighted congestion games with non-decreasing (player-specific) delay functions. For this, let Σ be a set system over a set $\mathcal{R}$ of resources. We call Σ *inclusion-free* if for every $X \in \Sigma$, no proper superset $Y \supset X$ belongs to Σ. Moreover, we call Σ a *non-matroid set system* if the tuple $(\mathcal{R}, \{X \subseteq S \mid S \in \Sigma\})$ is not a matroid. In [1] we show that every inclusion-free, non-matroid set system possesses the $(1, 2)$-exchange property. Here we need a variant of this property with positive (instead of non-negative) delays.

Definition 7 ($(1, 2)$-exchange property). *Let Σ be an inclusion-free set system over a set of resources $\mathcal{R}$. We say that Σ satisfies the* $(1, 2)$-exchange property *if we can identify three distinct resources $a, b, c \in \mathcal{R}$ with the property that for any given $k \in \mathbb{N}$ with $k > |\mathcal{R}|$, we can choose a delay $d(r) \in \{1, k + |\mathcal{R}|\}$ for every $r \in \mathcal{R} \setminus \{a, b, c\}$ such that for every choice of the delays of a, b, and c with $|\mathcal{R}| \le d(a), d(b), d(c) \le k$, the following property is satisfied: If $d(a) + |\mathcal{R}| \le d(b) + d(c)$, then for every set $S \in \Sigma$ with minimum delay, $a \in S$ and $b, c \notin S$. If $d(a) \ge d(b) + d(c) + |\mathcal{R}|$, then for every set $S \in \Sigma$ with minimum delay, $a \notin S$ and $b, c \in S$.*

Lemma 8. *Let Σ be an inclusion-free set system over a set of resources $\mathcal{R}$. Furthermore, let $\mathcal{I} = \{X \subseteq S \mid S \in \Sigma\}$, and assume that $(\mathcal{R}, \mathcal{I})$ is not a matroid, i. e., that Σ is not the set of bases of some matroid. Then Σ possesses the* $(1, 2)$-*exchange property.*

Proof. Since $(\mathcal{R}, \mathcal{I})$ is not a matroid, there exist two sets $X, Y \in \Sigma$ and a resource $x \in X \setminus Y$ such that for every $y \in Y \setminus X$, the set $X \setminus \{x\} \cup \{y\}$ is not contained in Σ (cf. Theorem 39.6 from [12]).

Let X and Y be such sets and let $x \in X$ be such a resource. Consider all subsets Y' of the set $X \cup Y \setminus \{x\}$ with $Y' \in \Sigma$. Every such set Y' can be written as $Y' = X \setminus \{x = x_1, \ldots, x_l\} \cup \{y_1, \ldots, y_{l'}\}$ with $x_i \in X \setminus Y$ and $y_i \in Y \setminus X$ and $l + l' > 2$. This is true since l as well as l' are both larger than 0 as Σ is inclusion-free. Furthermore l and l' cannot both equal 1 as otherwise we obtain a contradiction to the choice of X, Y, and x. Among all these sets Y', let $Y_{\min}$ denote one set for which l' is minimal. Observe that we can replace Y by $Y_{\min}$ without changing the aforementioned properties of X, Y, and x. Hence, in the following, we assume that $Y = Y_{\min}$, that is, we assume that $Y \setminus X = Y' \setminus X$ for all of the aforementioned sets Y'.

We claim that we can always identify resources $a, b, c \in X \cup Y$ such that either $a \in X \setminus Y$ and $b, c \in Y \setminus X$ or $a \in Y \setminus X$ and $b, c \in X \setminus Y$ with the property that for every $Z \subseteq X \cup Y$ with $Z \in \Sigma$, if $a \notin Z$, then $b, c \in Z$. In order to see this, we distinguish between the cases $l' = 1$ and $l' \ge 2$:

1. Let $Y \setminus X = \{y_1\}$ and hence $X \setminus Y = \{x = x_1, \ldots, x_l\}$ with $l \ge 2$. Then we set $a = y_1$, $b = x_1$, and $c = x_2$. Consider a set $Z \subseteq X \cup Y$ with $Z \in \Sigma$ and $a \notin Z$. Then $Z = X$ since Σ is inclusion-free, and hence $b, c \in Z$.

2. Let $Y \setminus X = \{y_1, \dots, y_{l'}\}$ with $l' \geq 2$. Then we set $a = x$, $b = y_1$, and $c = y_2$. Consider a set $Z \subseteq X \cup Y$ with $Z \in \Sigma$ and $a \notin Z$. Since we assumed that $Y = Y_{\min}$, it must be $b, c \in Z$ as otherwise $Z \setminus X \neq Y \setminus X$.

Now we define delays for the resources in $\mathcal{R} \setminus \{a, b, c\}$ such that the properties in Definition 7 are satisfied. Let $k \in \mathbb{N}$ be chosen as in Definition 7, that is, $d(a), d(b), d(c) \in \{|\mathcal{R}|, \dots, k\}$. We set $d(r) = k + |\mathcal{R}|$ for every resource $r \notin X \cup Y$ and $d(r) = 1$ for every resource $r \in (X \cup Y) \setminus \{a, b, c\}$. First of all, observe that in the first case the delay of Y equals $d(a) + |Y| - 1 < k + |\mathcal{R}|$ and that in the second case the delay of X equals $d(a) + |X| - 1 < k + |\mathcal{R}|$. Hence, a set $Z \in \Sigma$ that contains a resource $r \notin X \cup Y$ can never have minimum delay as its delay is at least $k + |\mathcal{R}|$. Thus, only sets $Z \in \Sigma$ with $Z \subseteq X \cup Y$ can have minimum delay. Since for such sets, $a \notin Z$ implies $b, c \in Z$, we know that every set with minimum delay must contain a or it must contain b and c.

Consider the case $d(a) + |\mathcal{R}| \leq d(b) + d(c)$ and assume for contradiction that there exists an optimal set Z^* with $a \notin Z^*$. Due to the choice of a, b, and c, the set Z^* must then contain b and c. Hence $d(Z^*) \geq d(b) + d(c)$. Furthermore, again due to the choice of a, b, and c, there exists a set $Z' \subseteq X \cup Y$ with $a \in Z'$ and $b, c \notin Z'$. The delay of Z' is $d(Z') = d(a) + |Z'| - 1 < d(a) + |\mathcal{R}| \leq d(b) + d(c) \leq d(Z^*)$, contradicting the assumption that Z^* has minimum delay. Hence every optimal set Z^* must contain a. If Z^* additionally contains b or c, then its delay is at least $d(a) + |\mathcal{R}| > d(Z')$. Hence, in the case $d(a) + |\mathcal{R}| \leq d(b) + d(c)$ every optimal set Z^* contains a but it does not contain b and c.

Consider the case $d(a) \geq d(b) + d(c) + |\mathcal{R}|$ and assume for contradiction that there exists an optimal set Z^* with $b \notin Z^*$ or $c \notin Z^*$. Then Z^* must contain a and hence its delay is at least $d(a)$. Due to the choice of a, b, and c, there exists a set $Z' \subseteq X \cup Y$ with $a \notin Z'$ and $b, c \in Z'$. The delay of Z' is $d(Z') = d(b) + d(c) + |Z'| - 2 < d(b) + d(c) + |\mathcal{R}| \leq d(a) \leq d(Z^*)$, contradicting the assumption that Z^* has minimum delay. Hence every optimal set Z^* must contain b and c. If Z^* additionally contains a, then its delay is at least $d(b) + d(c) + |\mathcal{R}| > d(Z')$. Hence, in the case $d(a) \geq d(b) + d(c) + |\mathcal{R}|$ every optimal set Z^* contains b and c but it does not contain a. □

Theorem 9. *For every inclusion-free, non-matroid set system Σ over a set of resources $\mathcal{R}$ there exists a weighted congestion game Γ with two players whose strategy spaces are isomorphic to Σ that does not possess a Nash equilibrium. The delay functions in Γ are positive and non-decreasing.*

Proof. Given an inclusion-free, non-matroid set system we describe how to construct a weighted congestion game with the properties stated in the theorem. We will first describe how the strategy spaces are defined and then how the delay functions are chosen.

Let Σ_1 and Σ_2 be two set systems over sets of resources $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. In the following we assume that both sets are isomorphic to Σ and that Σ_i is the strategy space of player i, for $i = 1, 2$. Due to the $(1, 2)$-exchange property we can, for every player i, identify three distinct resources $a_i, b_i, c_i \in \mathcal{R}_i$ with the properties as in Definition 7. Since we have not made any assumption on the global structure of the resources, we can arbitrarily decide which resources from $\mathcal{R}_1$ and $\mathcal{R}_2$ coincide. The resources $\mathcal{R}_i \setminus \{a_i, b_i, c_i\}$ are exclusively used by player i. Hence, we can assume that their delays are chosen such that the $(1, 2)$-exchange property is satisfied. Thus, to simplify matters we can assume that

$$\Sigma_1 = \{\underbrace{\{a_1\}}_{S_1^1}, \underbrace{\{b_1, c_1\}}_{S_1^2}\} \text{ and } \Sigma_2 = \{\underbrace{\{a_2\}}_{S_2^1}, \underbrace{\{b_2, c_2\}}_{S_2^2}\} .$$

In the following, we assume that $a_1 = b_2$, $b_1 = a_2$ and $c_1 = c_2$. Thus we can rewrite the strategy spaces as follows: $\Sigma_1 = \{\{x\}, \{y, z\}\}$ and $\Sigma_2 = \{\{y\}, \{x, z\}\}$.

We set $\omega_1 = 2$ and $\omega_2 = 1$ and define the following non-decreasing delay functions for the resources x, y and z, where $m = |\mathcal{R}|$:

	$n_r = 1$	$n_r = 2$	$n_r = 3$
$d_x(n_x)$	m	$20 \cdot m$	$21 \cdot m$
$d_y(n_y)$	$5 \cdot m$	$12 \cdot m$	$15 \cdot m$
$d_z(n_z)$	$3 \cdot m$	$4 \cdot m$	$10 \cdot m$

One can easily verify that $|\delta_i(S \oplus S_i^1) - \delta_i(S \oplus S_i^2)| \geq m$, for $i = 1, 2$, regardless of the choice of the other player. Hence, for every player, one of the inequalities in Definition 7 is always satisfied. This game does not possess a Nash equilibrium since player 1 prefers to play strategy S_1^2 if player 2 plays strategy S_2^1, and S_1^1 if player 2 plays strategy S_2^2. Additionally, player 2 prefers to play strategy S_2^2 if player 1 plays strategy S_1^2, and S_2^1 if player 1 plays strategy S_1^1. □

Theorem 10. *For every inclusion-free, non-matroid set system Σ over a set of resources $\mathcal{R}$ there exists a player-specific congestion game Γ with two players whose strategy spaces are isomorphic to Σ that does not possess a Nash equilibrium. The delay functions in Γ are positive and non-decreasing.*

Proof. The proof is similar to the proof of Theorem 9. In particular, the construction of the strategy spaces of the players is identical. The player-specific delay functions are obtained from the delay functions in the proof of Theorem 9 as follows: For the first player $d_r^1(n_r) = d_r(n_r + 1)$, for every resource $r \in \{x, y, z\}$ and every congestion $n_r \in \{1, 2\}$. For the second player $d_r^2(1) = d_r(1)$ and $d_r^2(2) = d_r(3)$, for every resource $r \in \{x, y, z\}$. □

Summarizing, every inclusion-free non-matroid set system can be used to construct a player-specific or weighted congestion game with positive delay functions that does not posses a Nash equilibrium. Observe that this result also holds if the system is not inclusion-free but the *pruned set system*, i. e., the set system obtained after removing all supersets, is not the set of bases of a matroid because supersets cannot occur in a Nash equilibrium in the case of positive delay functions. Correspondingly, our results presented in Theorems 3 and 6 show that a player-specific or weighted congestion game in which all players' strategy spaces correspond to the bases of a matroid after pruning the supersets possesses a Nash equilibrium with respect to the pruned and, hence, also with respect to the original strategy spaces as supersets are weakly dominated by subsets in the case of non-negative delay functions. Thus, the matroid property (applied to the pruned strategy spaces) is necessary and sufficient to show the existence of Nash equilibria.

Corollary 11. *The matroid property is the maximal property on the pruned strategy spaces of the individual players that guarantees the existence of Nash equilibria in weighted and player-specific congestion games with non-negative, non-decreasing delay functions.*

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