Selfish Service Installation in Networks
(Extended Abstract)

Jean Cardinal[1] and Martin Hoefer[2,⋆]

[1] Computer Science Department, Université Libre de Bruxelles,
CP 212, 1050 Brussels, Belgium
jcardin@ulb.ac.be
[2] Department of Computer & Information Science, Konstanz University,
Box D 67, 78457 Konstanz, Germany
hoefer@inf.uni-konstanz.de

Abstract. We consider a scenario of distributed service installation in privately owned networks. Our model is a non-cooperative *vertex cover game* for k players. Each player owns a set of edges in a graph G and strives to cover each edge by an incident vertex. Vertices have costs and must be purchased to be available for the cover. Vertex costs can be shared arbitrarily by players. Once a vertex is bought, it can be used by any player to fulfill the covering requirement of her incident edges. Despite its simplicity, the model exhibits a surprisingly rich set of properties. We present a cumulative set of results including tight characterizations for prices of anarchy and stability, NP-hardness of equilibrium existence, and polynomial time solvability for important subclasses of the game. In addition, we consider the task of finding approximate Nash equilibria purchasing an approximation to the optimum social cost, in which each player can improve her contribution by selfish defection only by at most a certain factor. A variation of the primal-dual algorithm for minimum weighted vertex cover yields a guarantee of 2, which is shown to be tight.

1 Introduction

In this paper we consider a simple model for service installation in networks, e.g. highway or communication networks like the internet. Many networks including the internet are built and maintained by a number of different agents with relatively limited goals whereas others are centrally planned and operated – e.g. the system of interstate highways in some countries is centrally owned and planned whereas in other countries certain roads are owned privately. In particular, we consider a simple model in which network owners have to make a concrete investment to establish a service at a location in the network. Network connections are owned by different players, and each player strives to establish a service point at different locations along her connections. These service points could be resting facilities at highways or caching, buffering, or amplification technology in telecommunication networks. We investigate the question of how

⋆ Supported by DFG Research Training Group 1042 “Explorative Analysis and Visualization of Large Information Spaces”.

P. Spirakis et al. (Eds.): WINE 2006, LNCS 4286, pp. 174–185, 2006.

the quality and density of these service locations changes when networks are owned privately vs. owned by a central authority. A player owning a set of connections has an incentive to cover all her connections with service. The motivation for this might either be economical or lawfully enforced. If at a location a service point is already established, the incident connections are covered. This might alter the motivation for some players to invest. Formally we model this interaction with a non-cooperative game, which we call the *vertex cover game* and analyze using notions from algorithmic game theory.

Our game is similar in spirit to the one considered in [2] for network creation. We assume that a number of k non-cooperative players have to create service points in a network. The network is modeled as a graph $G = (V, E)$, in which edges represent roads or connections and vertices represent possible service point locations. Each player i owns a subset $E_i \subseteq E$ of edges and strives to establish a service point at at least one endpoint of each edge in E_i, but with minimum investment. For establishing a service point at a vertex v, a cost $c(v)$ has to be paid, which can be shared among different players. A strategy for a player is an assignment of payments to vertices in V, and once a vertex is bought – that is, when a total amount of $c(v)$ is offered by the players for a vertex v, this vertex can be used by all players to cover any of their incident edges – no matter whether they contribute to the cost or not. In this game both the problem of finding the optimum strategy for a player and the problem of finding a centralized optimum cover for all edges of all players are the classic optimization problem of minimum weighted vertex cover.

We investigate our non-cooperative game in terms of stable solutions, which are the pure strategy Nash equilibria of the game. We do not consider mixed strategy equilibria, because our environment requires a concrete investment rather a randomized action, which would be the result of a mixed strategy. We consider the *price of anarchy* [14, 16], which measures the ratio of the cost of the worst Nash equilibrium over the cost of a minimum cost cover satisfying all requirements of all players for a game. In addition, we investigate the *price of stability* [1], which measures the best Nash equilibrium in terms of the optimum cost instead of the worst equilibrium. As in general both of these ratios are in $\Theta(k)$, we investigate the question how to derive cheap covers and cost distributions that provide low incentives to selfishly defect. We present an efficient algorithm with small constant approximation ratios and provide tightness results. In addition, we show that determining existence of Nash equilibria in the vertex cover game is NP-hard.

1.1 Related Work

The vertex cover problem is a classic optimization problem in graph theory and has been studied for decades. Recently, distributed variants of the problem have attracted interest in the area of algorithmic game theory. Specifically, a cooperative vertex cover game was studied in a more general context by Immorlica et al. [11]. In this coalitional game, each edge is an agent and each coalition of players is associated with a certain cost value - the cost of a minimum cover. In [11] cross-monotonic cost sharing schemes were investigated. For each coalition of players covered they distribute the cost to players in a way that every player is better off if the coalition expands. The authors showed that no more than $O(n^{-\frac{1}{3}})$ of the cost can be charged to the agents with a cross-monotonic scheme.

Closely related to cooperative games is the study of cost sharing mechanisms. Here a central authority distributes service to players and strives for their cooperation. Starting with [6] cost sharing mechanisms have been considered for a game based on set cover. Every player corresponds to a single item and has a private utility (i.e. a willingness to pay) for being in the cover. The mechanism asks each player for her utility value. Based on this information it tries to pick a subset of items to be covered, to find a minimum cost cover for the subset and to distribute costs to covered item players such that no coalition can be covered at a smaller cost. A strategyproof mechanism allows no player to lower her cost by misreporting her utility value. The authors in [6] presented strategyproof mechanisms for set cover and facility location games. For set cover games [18, 15] recently considered different social desiderata like fairness aspects and model formulations with items or sets being agents.

Cooperative games and the mechanism design framework are used to capture situations with selfish service receivers who can either cooperate to an offered cost sharing or manipulate. Players may also be excluded from the game depending on their utility. A major goal has been to derive good cost sharing schemes that guarantee truthfulness or budget balance. Our game, however, is strategic and non-cooperative in nature and allows players a much richer set of actions. In our game each player is motivated to participate in the game. We investigate distributed uncoordinated service installation scenarios rather than a coordinated environment with a mechanism choosing customers, providing service and charging costs. Our study is, however, related to these developments – especially the singleton games, which we study in Section 5.

Our analysis uses concepts developed for non-cooperative games in the area of algorithmic game theory, in particular prices of anarchy and stability characterizing worst- and best-case Nash equilibria. The price of anarchy has been studied in a large and diverse number of games, e.g. in areas like routing and congestion [14, 17, 3], network creation [2, 8], or wireless ad-hoc networks [7, 9]. The price of stability [1] has been introduced more recently and studied for instance in network creation games [1, 10] or linear congestion games [5]. Characterizing selfish improvement possibilities and social cost of a strategy combination in terms of multiplicative factors has been recently introduced in the study of network creation games [2, 10].

1.2 Outline and Contributions

We study our vertex cover game with respect to quality of pure strategy exact and approximate Nash equilibria. Throughout the paper we denote a feasible cover by $\mathcal{C}$ and the centralized optimum cover by $\mathcal{C}^*$. All proofs omitted in this extended abstract will be given in the full version of this paper. Our contributions are as follows.

- Section 2 presents the model and some initial observations. In Section 3 we show that the price of anarchy in the vertex cover game is k, even when the underlying graph is a tree. There exist simple unweighted and weighted games for two players without Nash equilibria. They can be used to prove that the price of stability can be arbitrarily close to $k - 1$. Determining existence of Nash equilibria for a given game is NP-hard, even for unweighted games or two players.

- In Section 4 we study a two-parameter optimization problem: Find covers that are cheap and allow low incentives for players to deviate. We formalize this notion as (x, y)-approximate Nash equilibria and propose a simple algorithm that finds (2,2)-approximate Nash equilibria for any vertex cover game. In addition to this algorithmic result, we show that in general there are games without a (x, y)-approximate equilibrium for $x < 2$. Recent progress on the complexity status of the minimum vertex cover problem can be used to reasonably conjecture that there can be no polynomial time algorithm with a better guarantee for the approximation ratio y as well. For planar games our argument extends to a lower bound of 1.5 on x, which can be increased close to 2 by forcing y to be close to 1 indicating a Pareto relationship between the ratios.
- Finally, in Section 5 we present games for which the price of stability is 1. For the class of *singleton games*, in which each player owns exactly one edge, we relate the results to recent work on mechanism design and cooperative game theory. For *bipartite games*, in which the graph is bipartite, our proof is based on the max-flow/min-cut technique for vertex cover. This provides new game-theoretic interpretations of classic results from graph theory and polynomial time algorithms to calculate cheap Nash equilibria.

2 The Model and Basic Results

The vertex cover game for k players is defined as follows. In an undirected graph $G = (V, E)$ with $n = |V|$ and $m = |E|$ each player i owns a set $E_i \subseteq E$ of edges. We denote by $G[E_i]$ the graph induced by the edges in E_i, and by $V(G[E_i])$ the set of vertices of $G[E_i]$. Each player strives to establish service at least one endpoint of each of her edge. For each vertex v there is a nonnegative cost $c(v)$ for establishing service at this vertex. A strategy for a player i is a function $p_i : V \to \mathbb{R}_0^+$ specifying an offer to costs of each vertex. The cost of a strategy p_i for player i is the sum of all money she offers to the vertices. Once the sum of offers of all players for vertex v exceeds its cost it is considered *bought*. Bought vertices can be used by all players to cover their incident edges. Each player strives to minimize her cost, but insists on covering her edges. A *payment scheme* is a vector $p = (p_1, \ldots, p_k)$ specifying a strategy for each player. A *Nash equilibrium* is a payment scheme such that no player i can unilaterally improve her payments by changing her strategy and still cover all her edges in E_i. A (x, y)-*approximate Nash equilibrium* is a payment scheme purchasing a cover $\mathcal{C}$ for which every player can improve her cost at most by a factor of x by switching to another strategy, and such that $c(\mathcal{C}) \leq yc(\mathcal{C}^*)$. We will refer to the factor y as the *approximation ratio*, and we term x as the *stability ratio*. The definitions of the approximation ratio and the stability ratio coincide for single-player games. Finally, we call a game *unweighted* if all vertices have equal costs, and *weighted* otherwise. We refer to games with a planar graph G as *planar games*.

The following observations can be used to simplify a game. Suppose an edge e is not included in any of the players edge sets. This edge is not considered by any player and has no influence on the game. Hence, in the following w.l.o.g. we will assume that $E = \bigcup_{i=1}^{k} E_i$.

For a player i assume the graph $G[E_i]$ induced by the players edge set E_i is not connected. The player has to cover edges in each component and her optimum strategy decomposes to cover both components independently at minimum cost. Hence, we can form an equivalent game in which the edges for each of the k_i components are owned by different subplayer $i_1, \ldots, i_{k_i}$. Then any approximate Nash equilibrium from this equivalent game can be translated to the original game, and eventually the stability ratio improves. Hence, for deriving approximate Nash equilibria we can assume that the edges of each player form only a single connected component.

Suppose an edge $e \in E$ is owned by a player i and a set of players J, i.e. $e \in E_i \cap (\bigcap_{j \in J} E_j)$. This is equivalent to one parallel edge for each player. Now consider a Nash equilibrium for an adjusted game in which there is only one edge e owned only by player i. In this equilibrium a player $j \in J$ has no better strategy to cover the edges in $E_j - e$. However, e is covered as well, potentially by a different player. If e is added to E_j again j has no incentive to deviate from her strategy as her covering requirement only increases. The Nash equilibrium for the adjusted game yields a Nash equilibrium in the original game. Hence, in the following we will assume that all edge sets E_i are mutually disjoint.

3 Quality and Existence of Nash Equilibria

In this section we consider the quality of pure Nash equilibria and the hardness of determining their existence. In general it is not possible to guarantee their existence, they can be hard to find or expensive. At first observe that the price of anarchy in the vertex cover game is k.

Theorem 1. *The price of anarchy in the vertex cover game is exactly k.*

Proof. Consider a star in which each vertex has cost 1 and each player owns a single edge. The centralized optimum cover $\mathcal{C}^*$ is the center vertex of cost 1. If each player purchases the vertex of degree 1 incident to her edge, we get a Nash equilibrium of cost k. Hence, the price of anarchy is at least k. On the other hand, k is a simple upper bound. If there is a Nash equilibrium $\mathcal{C}$ with $c(\mathcal{C}) > kc(\mathcal{C}^*)$, there is at least one player i that pays more than $c(\mathcal{C}^*)$. She could unilaterally improve by purchasing $\mathcal{C}^*$ all by herself. □

Note that the price of anarchy is k even for very simple games in which every player owns only one edge and G is a tree. Hence, we will in the following consider existence and quality of the best Nash equilibrium in a game.

Lemma 1. *There are planar games for two players without Nash equilibria.*

Proof. We consider the game for two players in Fig. 1(a) for an $\epsilon > 0$. For this game we examine four possible covers. A cover including all three vertices cannot be an equilibrium, because vertex u is not needed by any player to fulfill the covering requirement. Hence, any player contributing to the cost of u could feasibly improve by removing these payments. Suppose the cover representing an equilibrium includes v_1 and v_2. If player 1 contributes to v_1, she can remove these payments, because she only needs v_2

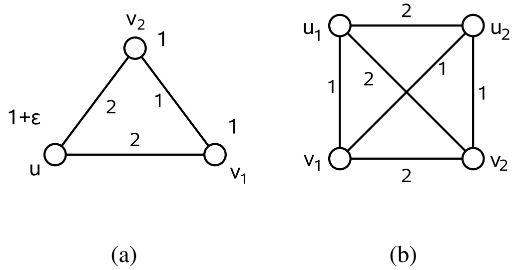

Fig. 1. Games for two players without Nash equilibria. (a) weighted game; (b) unweighted game. Edge type indicates player ownership. For the weighted game numbers at vertices indicate vertex costs.

to cover her edge. With the symmetric statement for v_2 we can see that in equilibrium player 1 could not pay anything. Player 2, however, cannot purchase both v_1 and v_2, because buying u offers a cheaper alternative to cover her edges. Finally, suppose u and v_1 are in the cover. In equilibrium player 1 will not pay anything for u. Player 2, however, cannot purchase u completely, because v_2 offers a cheaper alternative to cover the edge (u, v_2). With the symmetric observation for the cover of u and v_2, we see that there is no feasible cover that can be purchased by a Nash equilibrium. With similar arguments we can prove that the game on K_4 depicted in Fig. 1(b) has no pure Nash equilibria. This proves the lemma. □

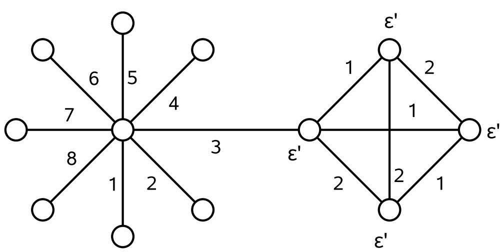

Fig. 2. A game with k=8, for which the cost of any Nash equilibrium is close to $(k-1)c(\mathcal{C}^*)$. Numbering of edges indicates player ownership. Indicated vertices have cost $\epsilon' \ll 1$, vertices without labels have cost 1.

Theorem 2. *For any $\epsilon > 0$ there is a weighted game in which the price of stability is at least $(k-1)-\epsilon$. There is an unweighted game in which the price of stability is $\frac{k+2}{4}$.*

Proof. Consider a game as depicted in Fig. 2. The centralized optimum cover includes the center vertex of the star and three vertices of the K_4-gadget yielding a total cost of $1 + 3\epsilon'$. If the center vertex of the star is in the cover and we assume to have a Nash equilibrium, no player can contribute anything to vertices of the K_4-gadget incident to edges of player 1 and 2. For this network structure, however, it is easy to note that players 1 and 2 cannot agree on a set of vertices covering their edges. Hence, to allow for a Nash equilibrium, the star center must not be picked which in turn requires all other adjacent star vertices to be in the cover. Under these conditions the best feasible cover includes the vertex that connects K_4 to the star yielding a cost of $k - 1 + 3\epsilon'$. Note that we can derive a Nash equilibrium purchasing this cover by assigning each player to purchase a star vertex - including the vertex that also belongs to K_4. Players 1 and 2 are assigned to purchase one of the additional K_4 vertices, respectively. With $\epsilon = \frac{3\epsilon'(k-2)}{1+3\epsilon'}$ the first part of the theorem follows. For the unweighted case we simply consider the game graph with all vertex costs equal to 1. A similar analysis delivers the stated bound and proves the second part of the theorem. □

Theorem 3. *It is NP-hard to determine whether (1) an unweighted vertex cover game or (2) a weighted vertex cover game for 2 players has a pure strategy Nash equilibrium, even if the graphs $G[E_i]$ are forests.*

4 Approximate Equilibria

In the previous section we saw that in general cheap pure Nash equilibria can be absent from the game. Hence, we study existence and algorithmic computation of solutions to a two-parameter optimization problem. Recall that (x, y)-approximate Nash equilibria are payment schemes that allow each player to reduce her payments by at most a factor of x and approximate $c(\mathcal{C}^*)$ to a factor of y.

Algorithm 1: (2,2)-approximate Nash equilibria

```
p_i(v) ← 0 for all players i and vertices v
γ_i(e) ← 0 for all players i and edges e
while there is an uncovered edge e = (u, v) ∈ E do
    Let i be the player owning edge e, and let γ_i(e) ← min(c(u), c(v))
    Increase payments: p_i(u) ← p_i(u) + γ_i(e) and p_i(v) ← p_i(v) + γ_i(e)
    Add all purchased vertices to the cover
    Reduce vertex costs: c(u) ← c(u) − γ_i(e) and c(v) ← c(v) − γ_i(e)
```

Theorem 4. *Algorithm 1 returns a (2,2)-approximate Nash equilibrium in $O(k(n+m))$ time.*

The algorithm is an adaption of the primal-dual algorithm for minimum vertex cover. It is also used to show that any socially optimum cover $\mathcal{C}^*$ can always be purchased by a $(2, 1)$-approximate Nash equilibrium.

Theorem 5. *For every game there is a (2,1)-approximate Nash equilibrium.*

For lower bounds on the ratios we note that any algorithm to find a (x, y)-approximate Nash equilibrium in the vertex cover game can be used as an approximation algorithm for minimum weighted vertex cover with approximation ratio $\min(x, y)$. The argument follows simply by considering a game with one player. This observation can be combined with recent conjectures on the complexity status of the minimum weighted vertex cover problem [13]. It suggests that if $P \neq NP$ and the unique games conjecture holds, there is no polynomial time algorithm delivering (x, y)-approximate Nash equilibria with $x < 2 - o(1)$ or $y < 2 - o(1)$. This bound applies only to polynomial time computability in general games. We now show that 2 is also a lower bound for the stability ratio, in a stronger sense.

Theorem 6. *For any $x < 2$ there is an unweighted game without (x, y)-approximate Nash equilibria for any $y \geq 1$.*

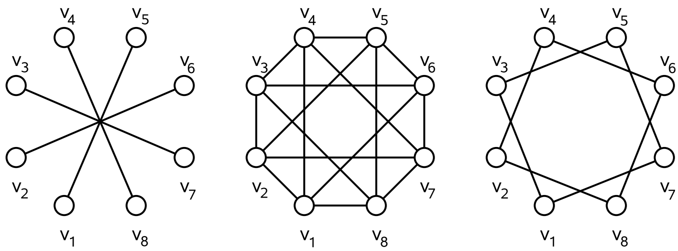

Fig. 3. From left to right the edges owned by the players in the first, second, and third classes of players for K_8. The first and second class consist of four players each, the third class of two players. Players in the first class own a single edge, while players in other classes own cycles of length 4.

Proof. The proof follows with a game on K_{4g} with $g \in \mathbb{N}$. We assume the vertices are numbered v_1 to v_{4g} and distribute the edges of the game to $2g^2 + g$ players in $g + 1$ classes as follows. In the first class there are $2g$ players. Every player i from this class owns only single edge (v_i, v_{2g+i}). Then, for each integer $j \in [1, g-1]$ there is another class of $2g$ players. A player i in one of the classes owns a cycle of four edges (v_i, v_{i+j}), (v_{i+j}, v_{2g+i}), (v_{2g+i}, v_{2g+i+j}) and (v_{2g+i+j}, v_i). Finally, there are g players in the last class. Each player i in this class also owns a cycle of four edges (v_i, v_{g+i}), (v_{g+i}, v_{2g+i}), (v_{2g+i}, v_{3g+i}) and (v_{3g+i}, v_i). See Fig. 3 for $g = 2$ and the distribution of the 10 players into 3 classes on K_8.

Any feasible vertex cover of a complete graph is composed of either all or all but one vertices. For a cover of all $4g$ vertices we can simply drop the payments to one vertex. This reduces the payment for at least one player. In addition, it increases the cost of some of the deviations as the players must now purchase the uncovered vertex in total. The stability ratio of the resulting payment scheme can only decrease. Hence, the minimum stability ratio is obtained by purchasing $4g - 1$ vertices.

So w.l.o.g. consider a cover of $4g - 1$ vertices including all but vertex v_{4g}. Note that some player subgraphs do not include v_{4g}, and there are only two types of player subgraphs - a single edge or a cycle of length 4. First, consider a player subgraph that consists of a single edge and both endvertices are covered. If the player contributes

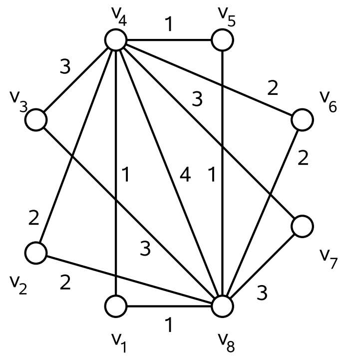

Fig. 4. Players that include v_8 in their subtree. Numbering of players as described in the text. Edge labels indicate player ownership.

to the cost of the incident vertices, she can drop the maximum of both contributions. Thus, if she contributes more than 0 to at least one of the vertices, her incentive to deviate is at least a factor of 2. Second, consider a player subgraph that consists of a cycle of length four. Label the four included vertices along a Euclidean tour with u_1, u_2, u_3 and u_4. Let the contributions of the player to u_j be x_j for $j = 1, 2, 3, 4$, resp. To optimally deviate from a given payment scheme, the player picks one of the possible minimum vertex covers $\{u_1, u_3\}$ or $\{u_2, u_4\}$ and removes all payments outside this cover. A factor of r bounding her incentives to deviate must thus obey the inequalities $\sum_{j=1}^{4} x_j \leq r(x_1 + x_3)$ and $\sum_{j=1}^{4} x_j \leq r(x_2 + x_4)$. In order to find the minimum r that is achievable we assume each player contributes only to vertices inside her subgraph. Summing the two inequalities yields $(2 - r) \sum_{j=1}^{4} x_j \leq \sum_{j=1}^{4} x_j$, so either her overall contribution is 0 or $r \geq 2$. Hence, to derive a payment scheme with stability ratio of less than 2, all $4g - 1$ vertices in the cover must be purchased by the $2g$ players whose subgraph includes v_{4g}.

For the rest of the proof we will concentrate on these $2g$ players. We will refer to player i, if she includes v_i in her subgraph, for $i = 1, \ldots, 2g - 1$. All these players own cycle subgraphs. The player that owns the edge (v_{2g}, v_{4g}) is labeled player $2g$. See Fig.4 for an example on K_8. We denote the contribution of player i to vertex v_j by p_{ij} for all $i = 1, \ldots, 2g$ and $j = 1, \ldots, 4g - 1$. Observe that for each player the set $\{v_{2g}, v_{4g}\}$ forms a feasible vertex cover. To achieve a stability ratio r, we must ensure that each player can only reduce her payments by a factor of at most r when switching to this cover. In the case of player $2g$ only $\{v_{2g}\}$ is needed, so we must ensure that she can reduce her payments by at most r when dropping all payments but $p_{2g,2g}$. As v_{4g} is not part of the purchased cover its cost of 1 must be purchased completely by a player that strives to use it in a deviation. This yields the following set of $2g$ inequalities: $\sum_{j=1}^{4g-1} p_{ij} \leq r(p_{i,2g} + 1)$, for $i = 1, \ldots, 2g - 1$ and $\sum_{j=1}^{4g-1} p_{2g,j} \leq r p_{2g,2g}$. We again strive to obtain the minimum ratio r that is possible. Note that in the minimum case no vertex gets overpaid, i.e. $\sum_{i=1}^{2g} p_{ij} = 1$ for all $j = 1, \ldots, 4g - 1$. Using this property in the sum of all the inequalities gives

$$4g-1=\sum_{j=1}^{4g-1}\sum_{i=1}^{2g}p_{ij}\leq r\left(2g-1+\sum_{i=1}^{2g}p_{i,2g}\right)\leq 2gr,$$

which finally yields $r \geq 2 - \frac{1}{2g}$. This proves that in the presented game no (x,y)-approximate Nash equilibrium with $x < 2 - \frac{1}{2g}$ exists. Thus, for every $\epsilon > 0$ we can pick $g \geq (2\epsilon)^{-1}$, which then yields a game without $(2-\epsilon, y)$-approximate Nash equilibria for any $y \geq 1$. □

It would be interesting to see, whether this lower bound is due to the integrality gap of vertex cover. Such a relation exists for approximate budget balanced core solutions in the cooperative game [12]. In a core solution each possible player coalition S contributes less than the cost of a minimum vertex cover for S. In our game, however, players make concrete strategic investments at the vertices, which alter the cost of the minimum cover for other players. In particular, our result is mainly due to the fact that the majority of players is sufficiently overcovered leaving only a small number of contributing players. This makes a relation to the integrality gap seem more complicated to establish.

Some classes of the vertex cover problem can be approximated to a better extent. For example, there is a PTAS for the vertex cover problem on planar graphs [4]. It is therefore natural to explore whether for planar games we can find covers with approximation and stability ratio arbitrarily close 1. The bad news is that in general there are also limits to the existence of cheap approximate Nash equilibria even on planar games. In particular, Theorem 6 provides a lower bound of 1.5 on the stability ratio for unweighted planar games. For weighted planar games there is an additional Pareto relationship between stability and approximation ratios that yields a stability ratio close to 2 for socially near-optimal covers.

Corollary 1. *There is a planar unweighted game without (x,y)-approximate Nash equilibria for any $x < 1.5$ and $y \geq 1$. For any $y < \frac{7}{6}$ there is a planar weighted game without (x,y)-approximate Nash equilibria for $x < 2/(2y-1)$.*

The better an algorithm is required to be in terms of social cost, the more it allows for selfish improvement by a factor close to 2. Note that all our lower bounds apply directly to any algorithm with or without polynomial running time.

5 Games with Cheap Nash Equilibria

In this section we present two classes of games that have cheap Nash equilibria: singleton games, in which each player owns only a single edge, and bipartite games, in which the graph is bipartite.

5.1 Singleton Games

An *exchange-minimal* vertex cover is a cover which cannot be improved by replacing a single vertex in the cover by a subset of its neighbors.

Lemma 2. *In singleton games every exchange-minimal vertex cover for G allows a distribution of vertex costs, such that no player can unilaterally improve her payments.*

Proof. Suppose we are given an exchange-minimal cover $\mathcal{C} \subset V$. For $v \in \mathcal{C}$ denote the neighbors outside the cover by $N_v(\mathcal{C}) = \{u \in V | (u,v) \in E, u \notin \mathcal{C}\}$. Suppose $c(N_v(\mathcal{C})) < c(v)$; then we can form a new cheaper feasible cover $\mathcal{C}'$ by replacing v with $N_v(\mathcal{C})$. This is a contradiction to $\mathcal{C}$ being exchange-minimal. Hence, for any $v \in \mathcal{C}$ it follows that $c(N_v(\mathcal{C})) \geq c(v)$.

This property allows a very simple algorithm to construct a Nash equilibrium from a given exchange-minimal cover $\mathcal{C}$. First initialize all payments of all players to 0. Then for each vertex $v \in \mathcal{C}$ iteratively consider all players owning an edge $e = (u,v)$ with $u \notin \mathcal{C}$. For player i set her contribution to $p_i(v) = \min(c(u), c(v) - \sum_{j \neq i} p_j(v))$. This leaves her no chance for improvement. In addition, by the previous argument every vertex $v \in \mathcal{C}$ gets paid for. □

Clearly, the centralized optimum cover $\mathcal{C}^*$ is an exchange-minimal cover, and hence there is a Nash equilibrium as cheap as $\mathcal{C}^*$. This proves that the price of stability in singleton games is 1. It does not prove, however, that a $(1,2)$-approximate Nash equilibrium can be found in polynomial time, since a 2-approximation algorithm for minimum vertex cover does not necessarily yield an exchange-minimal cover. We can devise an algorithm that starts from such an approximate cover and performs exchange operations to turn it into an exchange-minimal cover. In the weighted case, however, the number of exchange operations is not necessarily polynomial, and our algorithm could take exponential time. To circumvent this problem, we borrow a trick from Anshelevich et al. [2]. In the proposed algorithm each exchange operation guarantees a minimum improvement of the overall cost. The drawback is that we can only compute $(1+\epsilon, 2)$-approximate Nash equilibria, for any constant ϵ.

Theorem 7. *There is a polynomial time algorithm that finds $(1+\epsilon, 2)$-approximate Nash equilibria for weighted singleton games and $(1,2)$-approximate Nash equilibria for unweighted singleton games.*

Singleton games are similar in spirit to cooperative vertex cover games and mechanism design, as we assume that each edge is a single player. It is known that the core of the cooperative game contains only cost sharing functions that are at most 1/2 budget balanced. Our result states that once players have an intrinsic motivation to participate in the game and consider only selfish non-cooperative deviations, there is a cost-sharing function to distribute the full costs of an optimum cover. In this interpretation our game is close to a cooperative game that deals only with the global and singleton coalitions. Furthermore, our game is strategic, i.e. it specifies exactly to which vertex a player pays how much and in what way a player is motivated to reallocate her payments. This is a feature that is not considered in the cooperative framework.

5.2 Bipartite Games

Lemma 3. *In bipartite games there is an optimum vertex cover $\mathcal{C}^*$ for G which allows a distribution of vertex costs such that no player can unilaterally improve her payments.*

The proof relies on standard algorithmic techniques like maximum weight matching and max-flow/min-cut calculations. This allows to construct Nash equilibria with optimum social cost in polynomial time.

Theorem 8. *The price of stability in bipartite games is 1. Nash equilibria purchasing* $\mathcal{C}^*$ *can be found in polynomial time.*

References

1. E. Anshelevich, A. Dasgupta, J. Kleinberg, T. Roughgarden, É. Tardos, and T. Wexler. The price of stability for network design with fair cost allocation. In *Proc. 45th FOCS*, pages 295–304, 2004.
2. E. Anshelevich, A. Dasgupta, É. Tardos, and T. Wexler. Near-optimal network design with selfish agents. In *Proc. 35th STOC*, pages 511–520, 2003.
3. B. Awerbuch, Y. Azar, and A. Epstein. The price of routing unsplittable flow. In *Proc. 37th STOC*, pages 331–337, 2005.
4. B. Baker. Approximation algorithms for NP-complete problems on planar graphs. *Journal of the ACM*, 41(1):153–180, 1994.
5. G. Christodoulou and E. Koutsoupias. On the price of anarchy and stability of correlated equilibria of linear congestion games. In *Proc. 13th ESA*, pages 59–70, 2005.
6. N. Devanur, M. Mihail, and V. Vazirani. Strategyproof cost-sharing mechanisms for set cover and facility location problems. In *Proc. 4th EC*, pages 108–114, 2003.
7. S. Eidenbenz, A. Kumar, and S. Zust. Equilibria in topology control games for ad hoc networks. In *Proc. DIALM-POMC 2003*, pages 2–11, 2003.
8. A. Fabrikant, A. Luthera, E. Maneva, C. Papadimitriou, and S. Shenker. On a network creation game. In *Proc. 22nd PODC*, pages 347–351, 2003.
9. M. Halldórsson, J. Halpern, L. Li, and V. Mirrokni. On spectrum sharing games. In *Proc. 23rd PODC*, pages 107–114, 2004.
10. M. Hoefer and P. Krysta. Geometric network design with selfish agents. In *Proc. 11th COCOON*, pages 167–178, 2005.
11. N. Immorlica, M. Mahdian, and V. Mirrokni. Limitations of cross-monotonic cost sharing schemes. In *Proc. 16th SODA*, pages 602–611, 2005.
12. K. Jain and V. Vazirani. Applications of approximation algorithms to cooperative games. In *Proc. 33rd STOC*, pages 364–372, 2001.
13. S. Khot and O. Regev. Vertex cover might be hard to approximate within 2-ϵ. In *Proc. 18th CCC*, page 379, 2003.
14. E. Koutsoupias and C. Papadimitriou. Worst-case equilibria. In *Proc. 16th STACS*, pages 404–413, 1999.
15. X. Li, Z. Sun, and W. Wang. Cost sharing and strategyproof mechanisms for set cover games. In *Proc. 22nd STACS*, pages 218–230, 2005.
16. C. Papadimitriou. Algorithms, games and the internet. In *Proc. 33rd STOC*, pages 749–753, 2001.
17. T. Roughgarden and É.Tardos. How bad is selfish routing? *Journal of the ACM*, 49(2):236–259, 2002.
18. Z. Sun, X. Li, W. Wang, and X. Chu. Mechanism design for set cover games when elements are agents. In *Proc. 1st AAIM*, 2005.